Stochastic Reachability, Sequential Monte Carlo and Air Traffic Conflict Detection

Oliver Watkins

ojw26@cam.ac.uk

University of Cambridge, UK

Presentation

- Conflict in ATC
- Stochastic Reachability
- Estimating Stochastic Reach Probability
- Alerting Logic
- Globally Effective Alerting

Conflict in ATC

The aim of Conflict Detection in Air Traffic Control is to detect conflict situations:

$$x_2 - x_1 \in \mathcal{T}$$

Where $\mathcal{T}$ is an exclusion zone around each aircraft.

If the trajectories of the two aircraft are known this is trivial, but control and tracking are imperfect.

Error Statistics

Experimental work analysing track data has suggested flight path deviation statistics.

$$x(t) \sim \mathcal{N}(\mu(t), V(t))$$

Where

$$\begin{aligned} V(t) &= \begin{bmatrix} \sigma_a^2 & 0 \\ 0 & \sigma_c^2 \end{bmatrix}, \\ \sigma_a^2 &= r_a^2 t^2, \\ \sigma_c^2 &= \min(r_c^2, \bar{\sigma}_c^2). \end{aligned}$$

A variety of discrete time stochastic difference equations have been proposed which replicate these statistics.

General form of CD

Assess the *criticality* $C(\cdot)$ of the current airspace configuration:

Algorithm 0.1 (General Conflict Detection)

When γ *changes*

Compute $C(\gamma)$

If $C(\gamma) > \bar{C}$

Issue an Alert

End

End

Many measures proposed for $C(\gamma)$, minimal evaluation performed.

Stochastic Reachability

Given an equation of the form

$$x_{k+1} = f(x_k) + g(x_k)w_k$$

Where $f(x_k)$ governs deterministic dynamics and $g(x_k)w_k$ represents stochastic innovations estimate

$$P\{\exists k \in [0,T] | x_k \in \mathcal{T}\}$$

i.e. The probability that the trajectory will enter a set $\mathcal{T}$ over the horizon $[0,T]$. Refer to this as P_c.

Signal Process

x_k evolves as a probability distribution with transition kernel $K(\cdot, \cdot)$ such that:

$$P\{X_k \in A | X_{k-1} = x_{k-1}\} = \int_A K(dx_k, x_{k-1}), A \in \mathcal{B}(\mathbb{R}^{n_x})$$

w_k is an i.i.d. sequence with:

$$P\{w_k \in \mathcal{C}\} = \int_{\mathcal{C}} P_w(dw) = \int_{\mathcal{C}} p_w(w) dw.$$

So we have:

$$K(x_k, x_{k-1}) = p_w\{g(x_k)^{-1}[x_k - f(x_{k-1})]\}$$

Reach Probability

We wish to estimate the expectation of some measurable function $\varphi(\cdot)$ of the signal trajectories:

$$P_c^t = \mathbb{E}[\varphi\left(X_{t:T+t}\right)]$$

Where:

$$\varphi\left(x_{t:T+t}\right) = \max_{k \in [t:T+t]} \left(I_k = \begin{cases} 1, \; x_k \in \mathcal{T} \\ 0, \; x_k \notin \mathcal{T} \end{cases} \right).$$

Which may be achieved thus:

$$P_c^t = \int \varphi(x_{t:T+t}) \delta_{y_t}(x_t) \prod_{k=t+1}^{t+T} K(x_k, x_{k-1}) dx_{t:t+T}$$

Monte Carlo Integration

The integration is not possible analytically. However if we sample

$$X_{0:T}^{(i)} \sim \delta_{y_0}(x_0) \prod_{k=1}^{T} K(x_k, x_{k-1})$$

M times, the probability may be approximated:

$$P_c^t = \frac{1}{M} \sum_{i=1}^{M} \varphi(X_{0:T}^{(i)})$$

With probabilistic convergence.

Convergence

Theorem 0.2 *Accuracy and confidence of Monte Carlo integration with M particles are given by:*

$$P\{|P_c^t - \tilde{P}_c^t| > \epsilon\} \leq 2e^{-2M\epsilon^2}$$

Where $P\{|P_c^t - \tilde{P}_c^t| > \epsilon\}$ is the probability of the error being more than ϵ.

This is the Chernoff bound, a well known probabilistic convergence law.

Computational Effort

If, every time an observation is received we have to repeat the MC integration, we have to sample new paths:

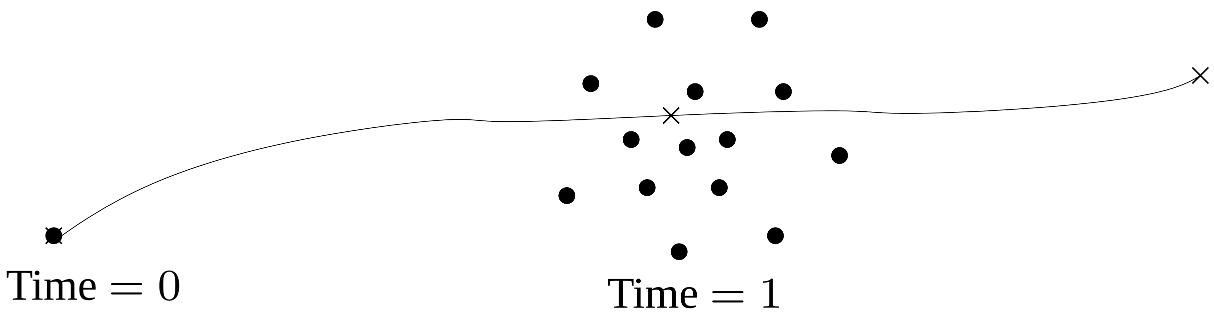
$$X^{(i)}_{t_1:T+t_1} \sim \delta_{y_{t_1}}(x_{t_1}) \prod_{k=t_1+1}^{t_1+T} K(x_k, x_{k-1})$$

Which, if $f(x_k)$ or $g(x_k)$ are particularly complex is highly computationally intensive.

So we use *Sequential Monte Carlo*.

SMC

If the process and signal models match, then the distribution of observations will match the distribution of simulated particles.



We can therefore (we hope) estimate $P_c^{t_1}$ using trajectories distributed according to:

$$\delta_{y_0}(x_0) \prod_{k=1}^{T} K(x_k, x_{k-1})$$

SMC

Theorem 0.3 *Conflict probability for an observation process initially at* y_{t_1} *is given by:*

$$P_c^{t_1} = \int \varphi(y_{t_1}, X_{t_1+1:T+t_1}) K(y_{t_1}, x_{t_1+1}) \delta_{y_0}(x_0) \prod_{k=1}^{T+t_1} K(x_k, x_{k-1}) dx_{0:T+t_1}.$$

Thus:

$$P_c^{t_1} = \sum_{i=1}^{M} \varphi\left(y_{t_1}, X_{t_1+1:T+t_1}\right) w^{(i)}$$

$$w^{(i)} \propto K\left(y_{t_1}, X_{t+1}^{(i)}\right), \sum w^{(i)} = 1.$$

Proofs are messy but relatively simple!

Accuracy problems

Recall the Chernoff bound:

$$P\{|P_c^t - \tilde{P}_c^t| > \epsilon\} \leq 2e^{-2M\epsilon^2}$$

Q: When weights have been applied, what is M?

A: $\hat{M}$, the effective sample size, is given by:

$$\hat{M} = \frac{1}{\sum_{i=1}^{M} \left(w^{(i)}\right) 2};$$

the equivalent number of i.i.d. samples that would given the same accuracy/confidence.

As $t - t_0$ increases, $\hat{M}$ decreases until the estimates are no longer reliable.

Simple Solution

When $\hat{M} < M_{crit}$ discard the distribution

$$\delta_{y_{t_0}}(x_{t_0}) \prod_{k=t_0+1}^{T} K(x_k, x_{k-1})$$

And resample trajectories:

$$\delta_{y_t}(x_t) \prod_{k=t+1}^{T+t} K(x_k, x_{k-1})$$

Essentially executing one iteration of simple Monte Carlo. Then $\hat{M} = M$ and Sequential Monte Carlo can continue.

This is just one of many versions of SMC and has been adapted to fit this specific situation.

Motivation

Compared to other methods of CD this all seems over compliated and unwieldy. But

- It does not rely on assumptions on the nature of conflict.
- It does not make excessive mathematical assumptions.
- Monte Carlo simulations can capture much more complex behaviour than other methods.

Does it pay off? How can we tell?

ATC

Air Traffic Conflict detection can be cast in exactly the framework described above.

- $\mathcal{T}$ represents an exclusion zone around each aircraft.
- $f(x_k)$ encodes the relative aircraft dynamics.
- $g(x_k)w_k$ encodes disturbances due to the wind.

The function of Conflict Detection is to alert ATC to impending conflict situations.

Alerting

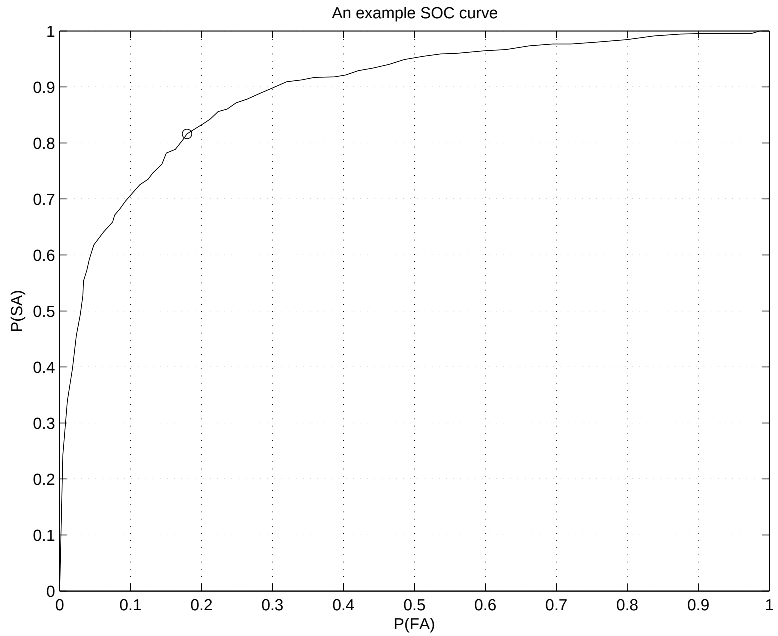
Assume we have accurate values for probability of conflict:

- We wish to alert to upcoming conflicts.
- Without missing alerts.
- While avoiding issuing false alerts.

If we alert when $P_c > \bar{C}$, the threshold $\bar{C} \in [0, 1]$ must be set to maximise probability of successful alerts, and minimize the probability of false alerts.

SOC Curves

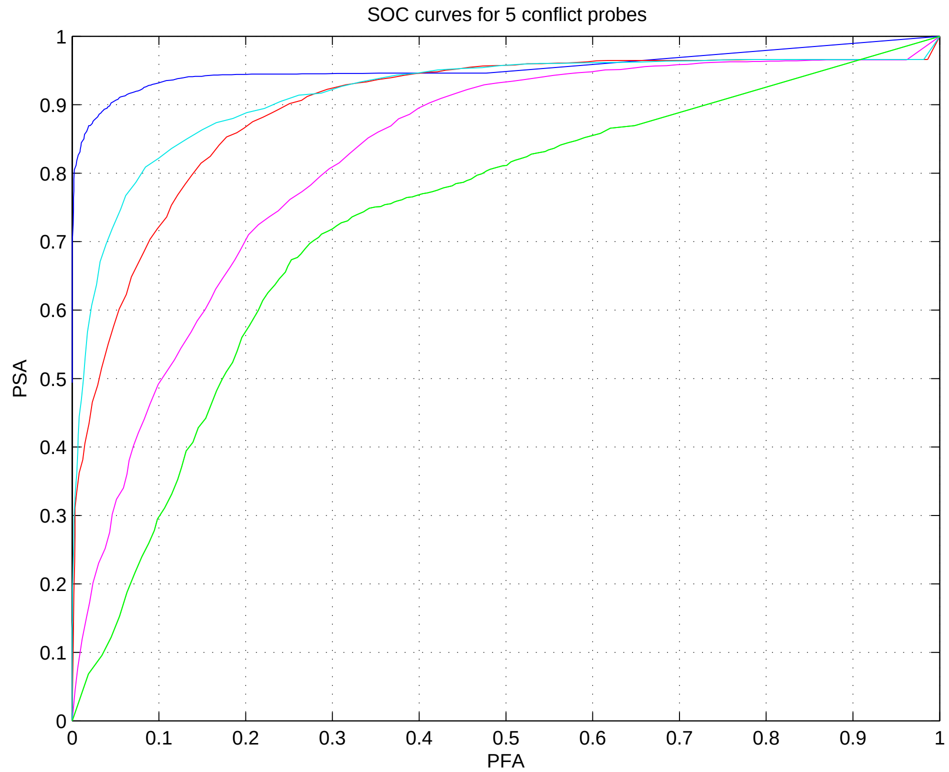
If we perform enough simulations we can calculate probability of false alert and probability of successful alert, parametrized by alert threshold.



Giving the SOC curve. The optimal threshold is that giving operation closest to $[0, 1]$.

Probe Evaluation

SOC curves for a variety of probes are shown below:



Which give a favourable comparison of SMC to more simple algorithms.

Conclusions

- Monte Carlo methods can be very useful for complex reachability problems.
- Sequential Monte Carlo can be used to accelerate MC.
- This problem has a direct application in ATC.
- It appears that taking the time to implement and run SMC pays dividends.