Global Analysis of Relay Feedback Systems

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Outline

- Relay Feedback Systems (RFS)
  - Introduction and Motivation
  - Poincaré Maps for RFS
  - Quadratic Stability of Poincaré Maps
  - Computational Issues
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- Piecewise Linear systems
  - Introduction
  - Poincaré Maps for PLS
  - Quadratic constrains of Poincaré Maps for PLS

- Future Work and Conclusions

Relay Feedback Systems (RFS)

"Simple" class of Piecewise Linear Systems yet very hard to analyze .

Applications:

- Electromechanical systems
- Simple models of dry friction
- Delta-sigma modulators
- Automatic tuning of PID regulators

Property:

- RFS often tend to limit cycles

Relay Feedback Systems: Analysis

- Results on existence and *local* stability of limit cycles of RFS exist (Åström, Johansson et. al)

- Volume contraction (Johansson et. al)

- Gonçalves et. all characterized reasonably large regions of stability around limit cycles

- For second-order systems, global convergence analysis can be done in the phase-plane

- Megretski proved that this also holds for processes having an impulse response sufficiently close, in a certain sense, to a second-order non-minimum phase process

The problem of rigorous *global* analysis of relay-induced oscillations is still open

Relay Feedback Systems: Contributions

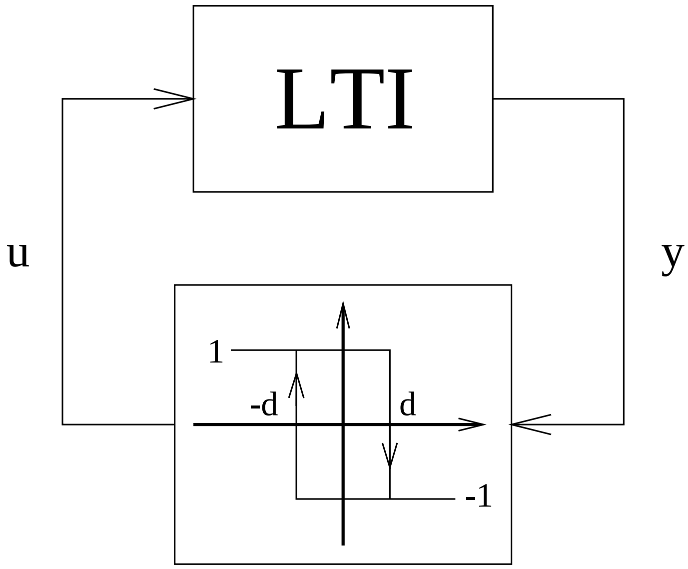
- INTEREST: prove global stability of symmetric unimodal limit cycles via quadratic stability of Poincaré maps

- DEVELOPED: representation of Poincaré maps of RFS as linear transformations parametrized by switching time $\rightarrow$ stability conditions given in the form of LMIs

- RESULT: global stability was proven for *all* examples we tried with a unique unimodal limit cycle and with no equilibrium points

- CONCLUSION: quadratic stability of Poincaré maps is common in RFS.

Relay Feedback System: Definition



where the LTI system is given by

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

where A is Hurwitz, and $d > 0$, or $d = 0$ if $CB < 0$. Also, $CA^{-1}B + d < 0$

The *switching surfaces* S and S_1 of the RFS are

$$S = \{x \in \mathbb{R}^n : \ Cx = d\}$$

and

$$S_1 = \{x \in \mathbb{R}^n : \ Cx = -d\}$$

Existence of Symmetric Unimodal Limit Cycles

- A limit cycle γ is called *symmetric* if $\gamma(t + T/2) = -\gamma(t)$. It is called *unimodal* if it only switches twice per cycle.

- Existence of symmetric unimodal limit cycles (from Åström):

  Assume there exists a symmetric unimodal limit cycle γ with period $2t^*$. Then the following conditions hold

  $$g(t^*) = C(e^{At^*} + I)^{-1}(e^{At^*} - I)A^{-1}B - d = 0$$

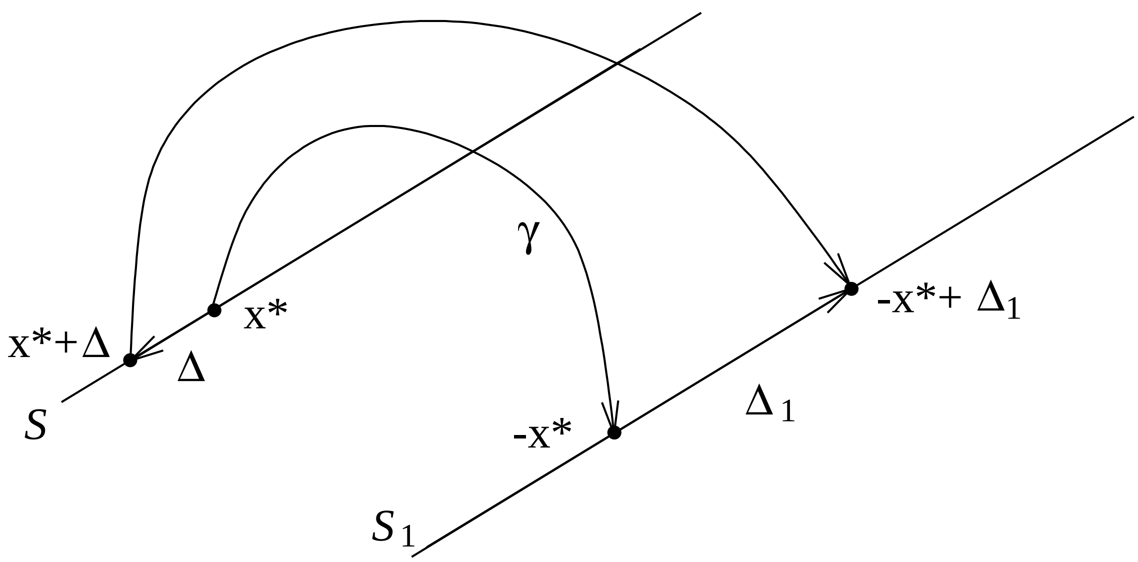
  and

  $$y(t) = C\left[e^{At}(x^* - A^{-1}B) + A^{-1}B\right] \geq -d$$

  $for \;\; 0 \leq t < t^*$. Furthermore, the periodic solution γ is obtained with the initial condition $x^* \in S$ given by

  $$x(0) = x^* = (e^{At^*} + I)^{-1}(e^{At^*} - I)A^{-1}B$$

Poincaré Maps for RFS



Consider the Poincaré map $T_1 : S \to S_1$

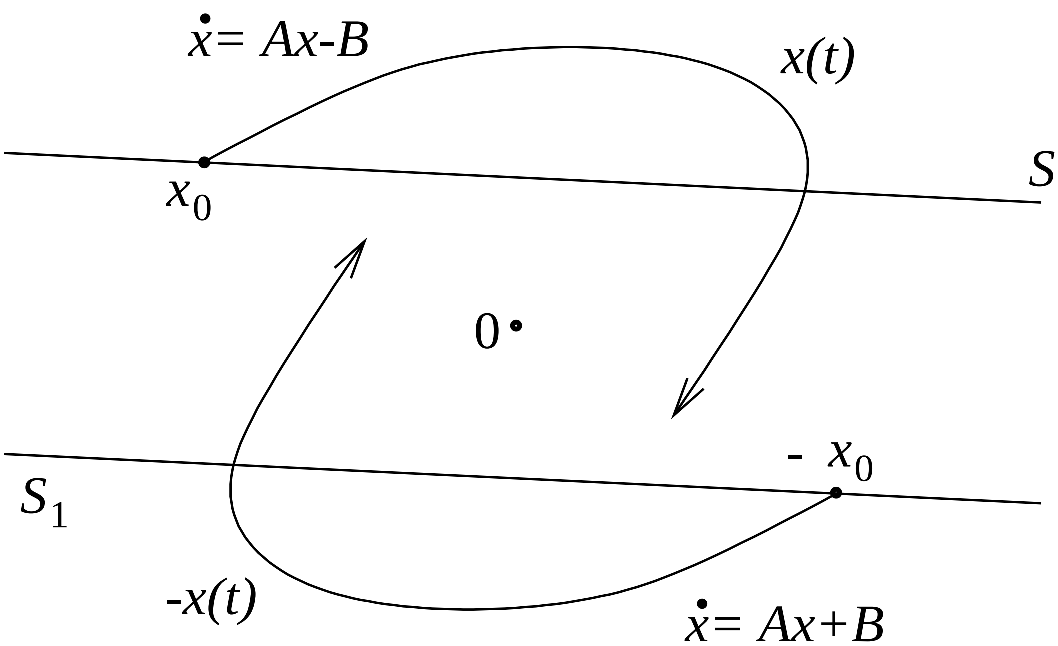
$$-x^* + \Delta_1 = T_1(x^* + \Delta)$$

or

$$\begin{aligned} \Delta_1 &= T_1\,(x^* + \Delta) - (-x^*) \\ &= T(\Delta) \end{aligned}$$

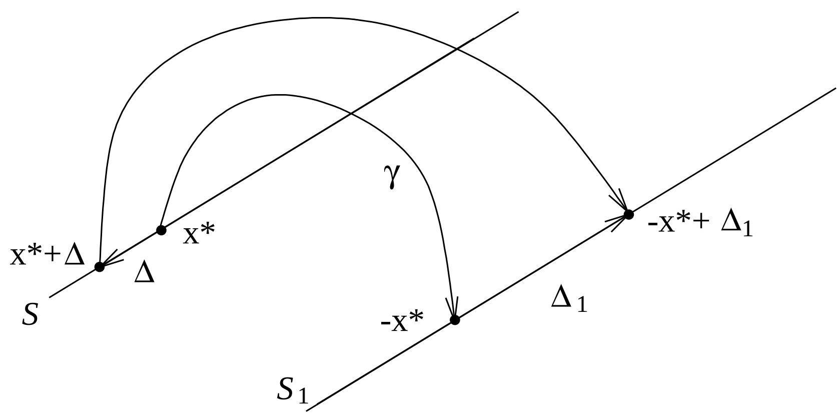
Relay Feedback System: Symmetry Property

Consider a trajectory $x(t)$ of $\dot{x} = Ax - B$ starting at $x_0 \in S$. Then $-x(t)$ is a trajectory of $\dot{x} = Ax + B$ starting at $-x_0 \in \underline{S}$.



$\Rightarrow$ There is only one Poincaré map to consider

Poincaré Maps for RFS



THEOREM: For any $\Delta \in S - x^*$ there exists a $t > 0$ such that

$$\Delta_1 = H(t)\Delta$$

where

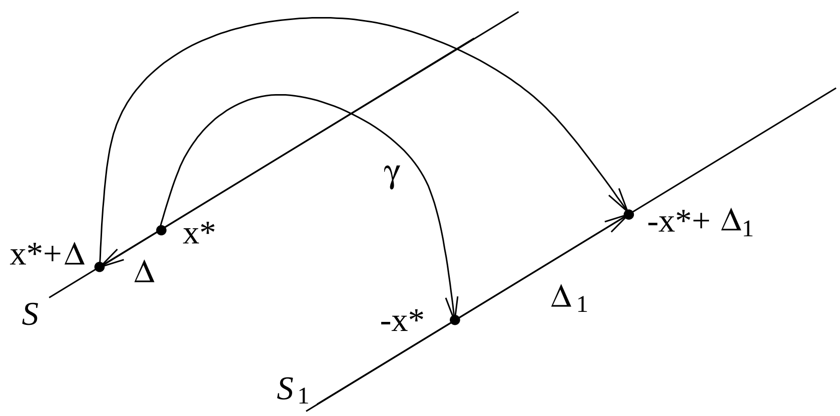
$$H(t) = \begin{cases} \left(I - \frac{v_t C}{C v_t}\right) e^{At} & \text{if } t > 0 \text{ and } t \neq t^* \\ \left(I - \frac{vC}{Cv}\right) e^{At^*} & \text{if } t = t^* \end{cases}$$

with

$$v_t = \left(e^{At} - e^{At^*}\right)\left(x^* - A^{-1}B\right)$$

and $v = e^{At^*}(Ax^* - B)$, and assuming $Cv_t \neq 0$ for $t > 0$ and $t \neq t^*$.

Poincaré Maps for RFS



For any $\Delta \in S - x^*$ there exists a $t > 0$ such that

$$\Delta_1 = H(t)\Delta$$

⇒ Most Poincaré maps induced by an LTI flow between a set and an hyperplane can be represented as a linear transformation analytically parametrized by a scalar function of the state.

⇒ Although t depends on Δ, once t is fixed, the Poincaré map becomes linear in Δ.

Quadratic Stability of Poincaré Maps of RFS

- $$\Delta_1 = H(t)\Delta$$

- Quadratic stability of Poincaré maps is guaranteed if there exists a $Q > 0$ such that

$$\Delta_1^T Q \Delta_1 < \Delta^T Q \Delta$$

$$\Uparrow$$

$$Q - H^T(t) Q H(t) > 0 \quad \text{on } S$$

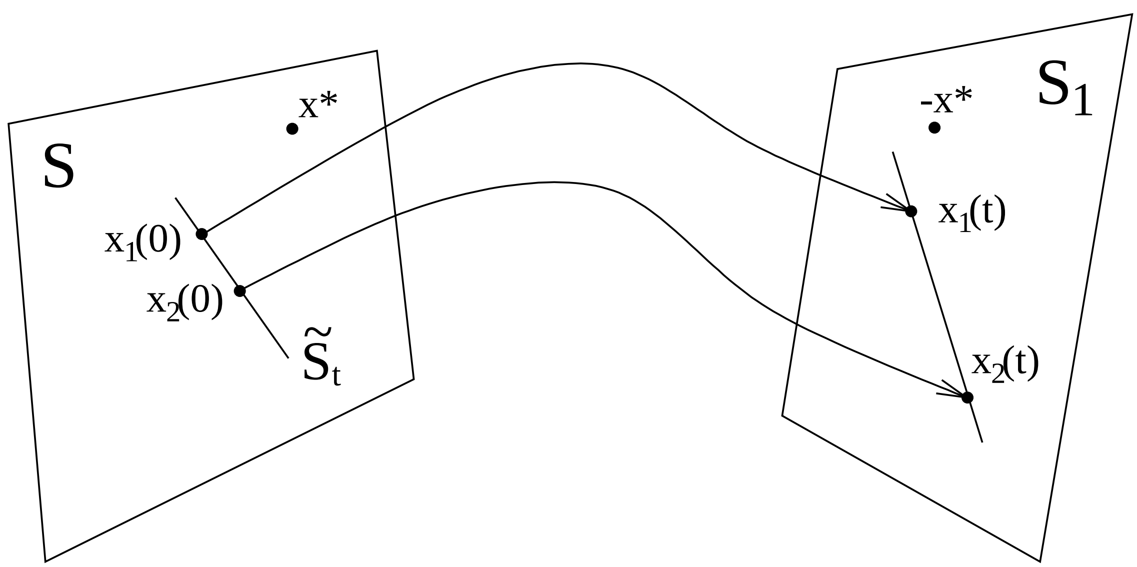
for all expected t

Quadratic Stability of Poincaré Maps of RFS

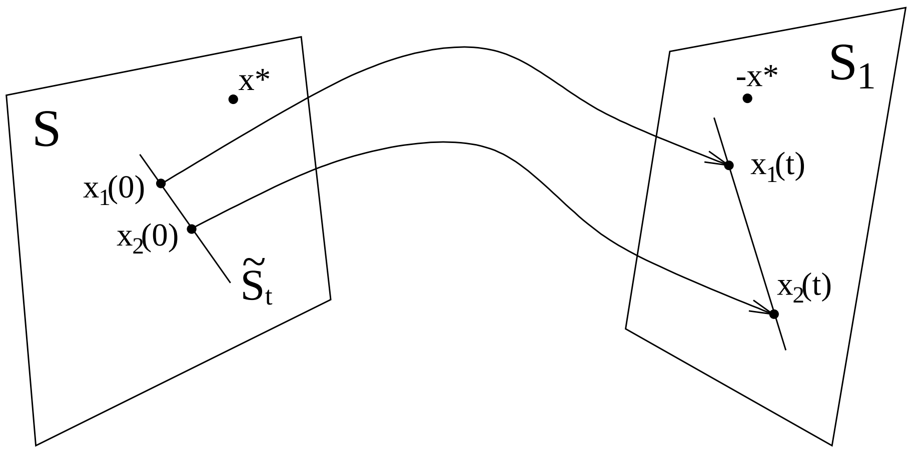
- $$\Delta_1 = H(t)\Delta$$

- Let $\tilde{S}_t \subset S$ be the set of all Δ which yield the next switch at t

- $\tilde{S}_t$ is an $n-2$ dimensional subspace



Quadratic Stability of Poincaré Maps of RFS



- Can improve stability condition

$$Q - H^T(t)QH(t) > 0 \quad \text{on } \tilde{S}_t$$

- Since $\Delta \in \tilde{S}_t$ satisfies a *conic* relation

$$\Delta^T \beta_t \Delta > 0$$

- can use the *S-Procedure*

$$Q - H^T(t)QH(t) - \tau_t \beta_t > 0 \quad \text{on } S$$

where $\tau_t > 0$

Computational Issues

- Since A is Hurwitz and $u = \pm 1$ is a bounded input, there is a bounded set such that any trajectory will eventually enter and stay there.

- In that set can find bounds on the difference between any two consecutive switching times. The difference between any two consecutive switching times of some trajectory is higher than t_- but lower than t_+.

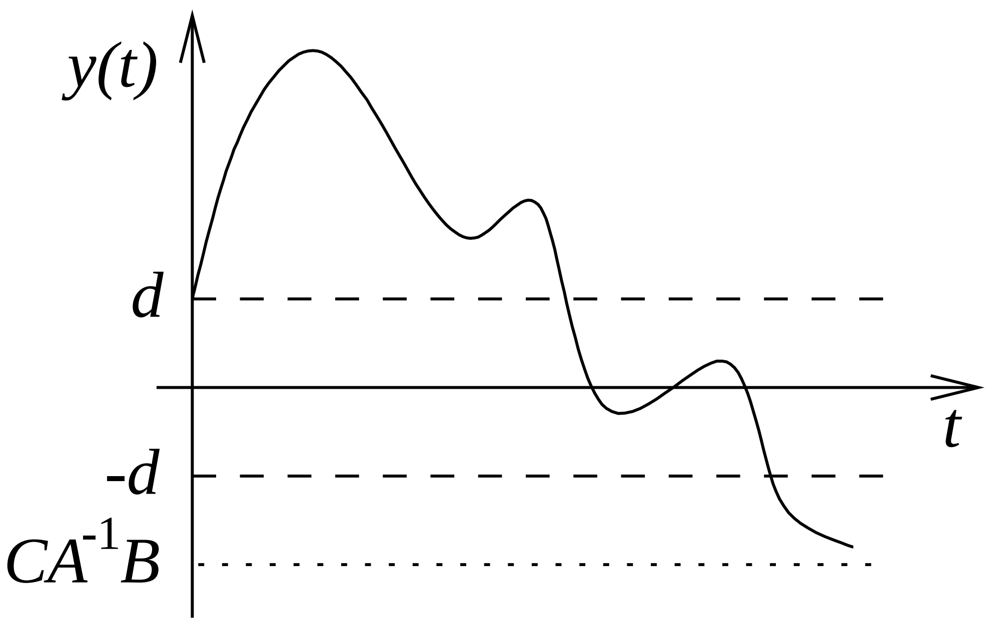
- This way, the search for $Q > 0$ in the stability conditions becomes restricted to $0 < t_- \leq t \leq t_+ < \infty$.

Upper Bound for t_+

- $y(t)$ is given by

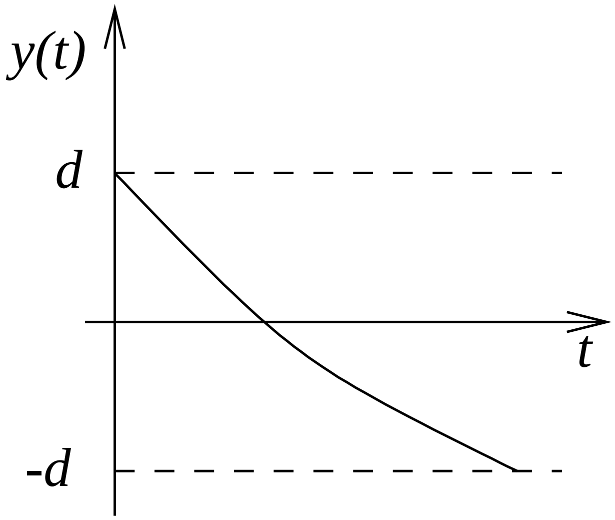
$$y(t) = Ce^{At}(x_0 - A^{-1}B) + CA^{-1}B$$

and $CA^{-1}B < -d$

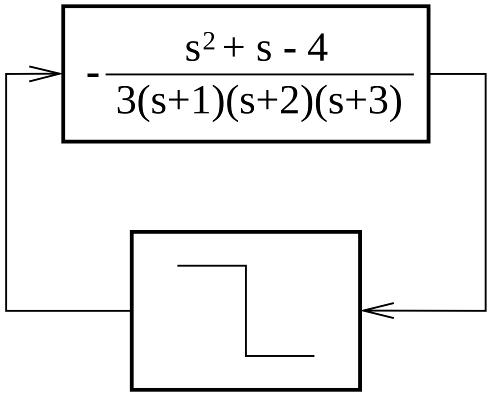


Lower Bound for t_-

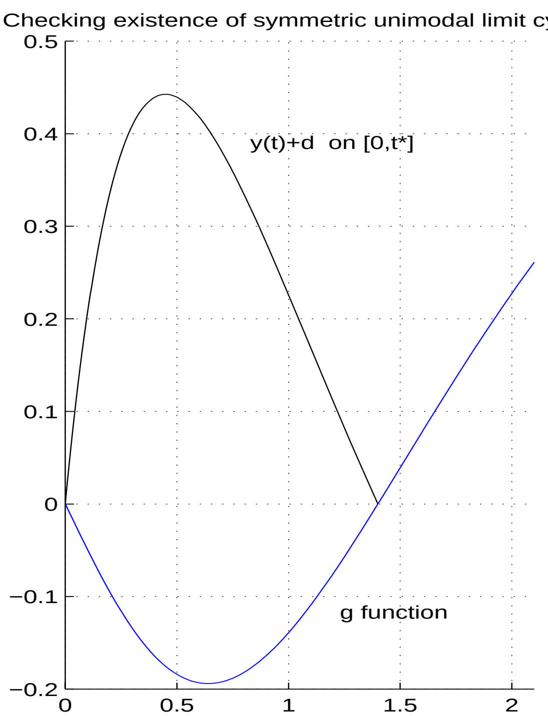
- $y(t)$ is positive at least in some interval $(0, \epsilon)$

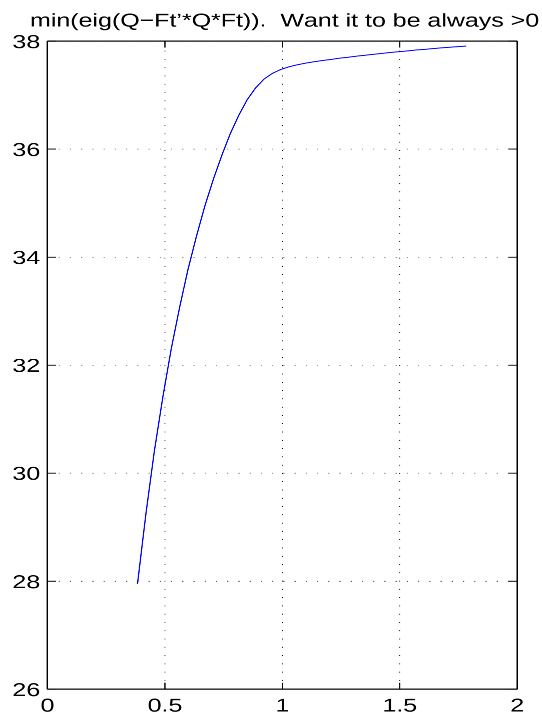


Example: $3^{rd} - Order$ Non-Minimum Phase System

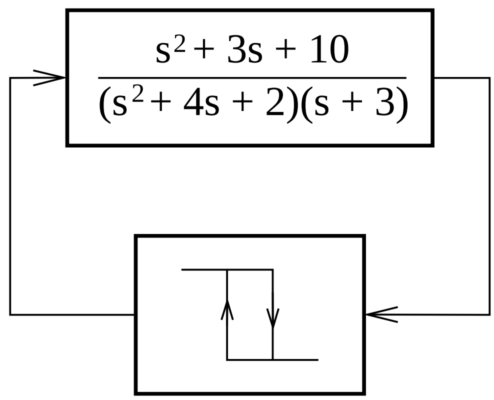


- Let $d = 0$ (possible since $CB < 0$). There is a unique symmetric unimodal limit cycle with half period $t^* \approx 1.4$ (*S-Procedure* was not used)

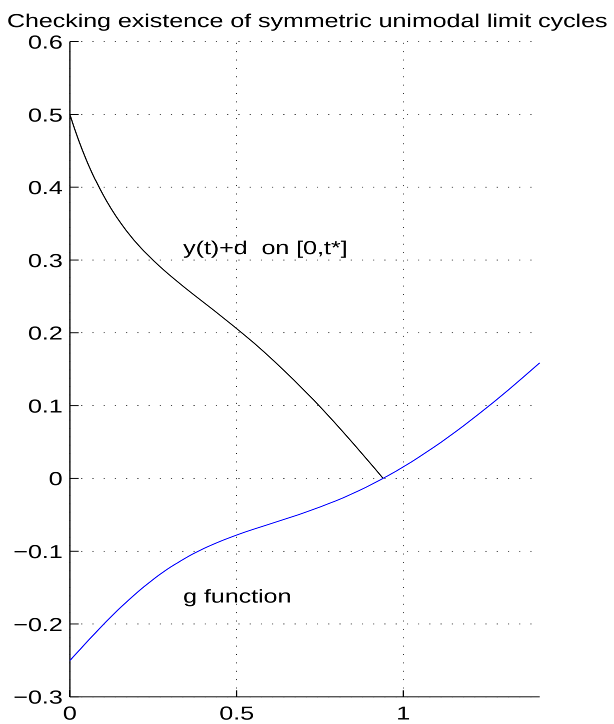


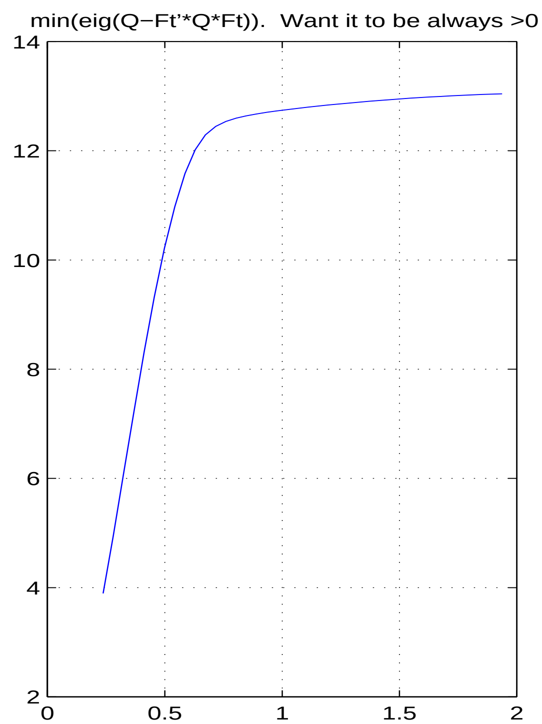


Example: $3^{rd} - Order$ Minimum Phase System

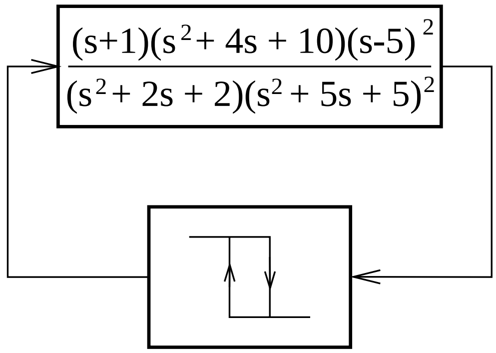


- Let $d = 0.25$. There is a unique symmetric unimodal limit cycle with half period $t^* \approx 0.94$ (*S-Procedure* was not used)

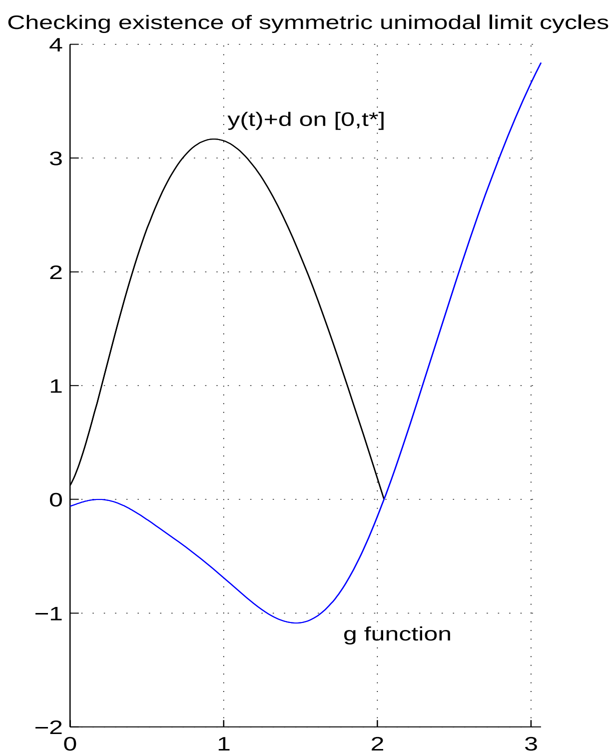


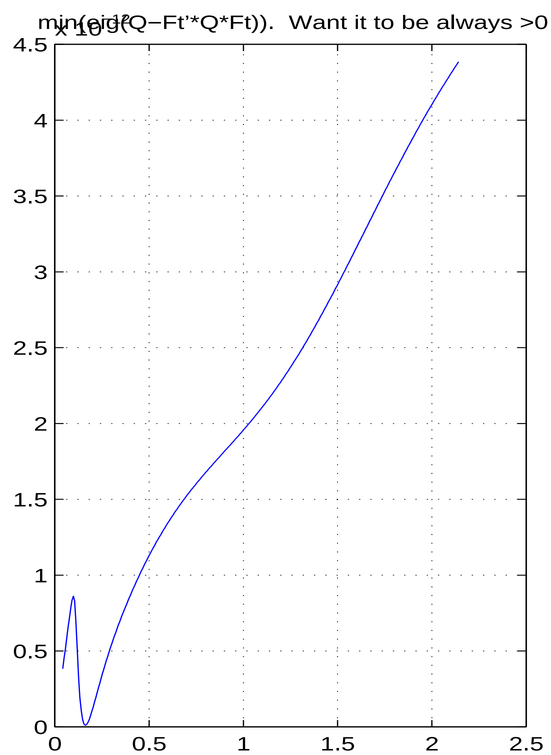


Example: 6^{rd} – *Order* System

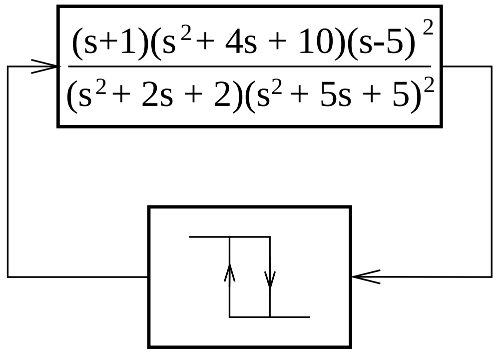


- Sliding modes occur if $d = 0$ ($CB = 1$). A $Q > 0$ is known to exist for d as low as 0.061 (see figure below). (*S-Procedure* was not used)

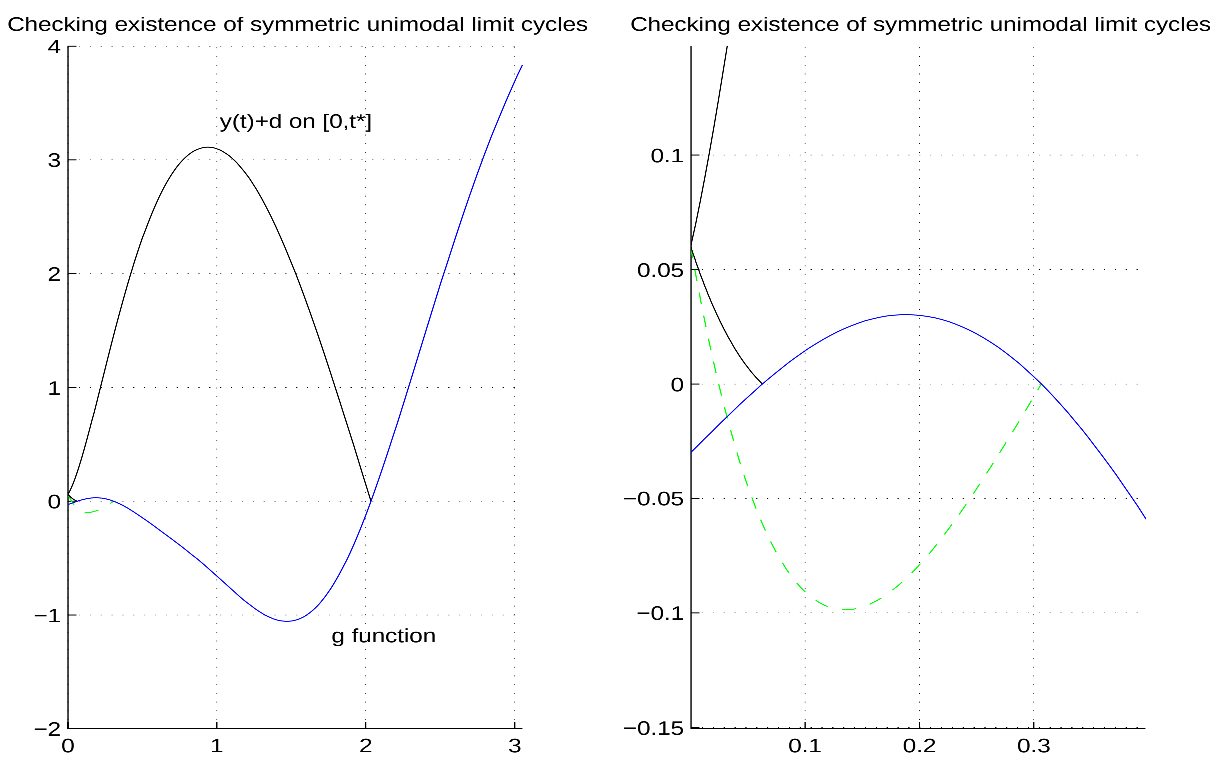




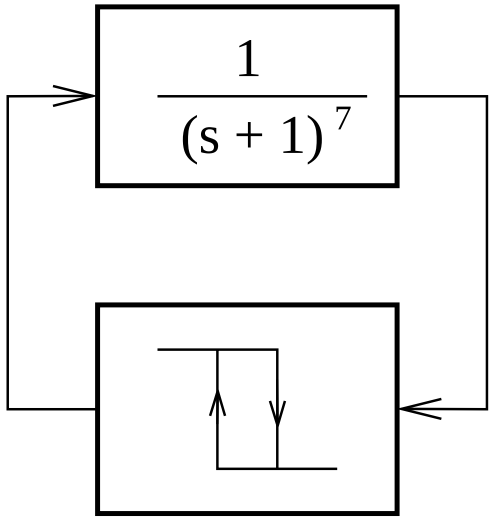
Example: $6^{rd} - Order$ **System**



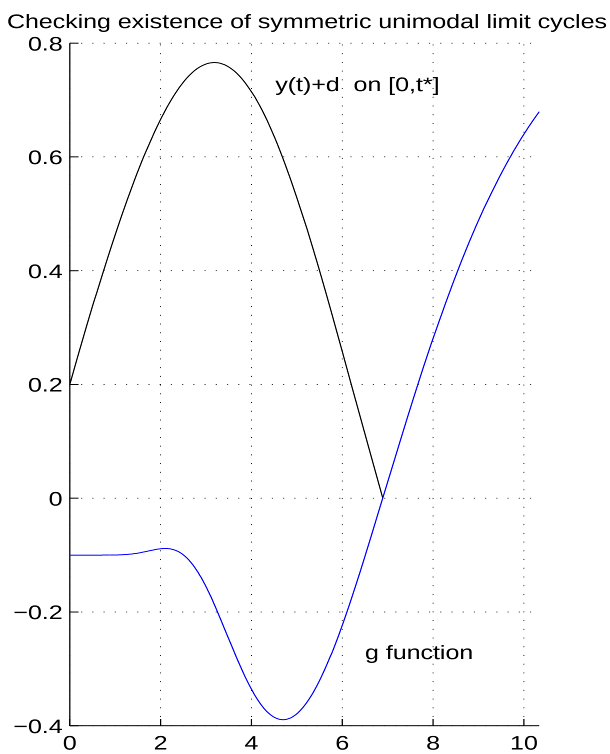
- Note that more than one limit cycle exists for $0 < d < 0.061$. For example, for $d = 0.03$:

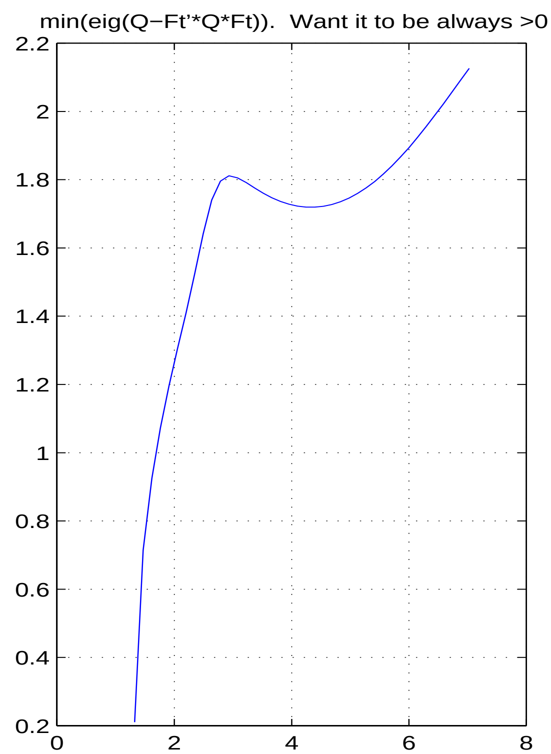


Example: $7^{rd}-Order$ System

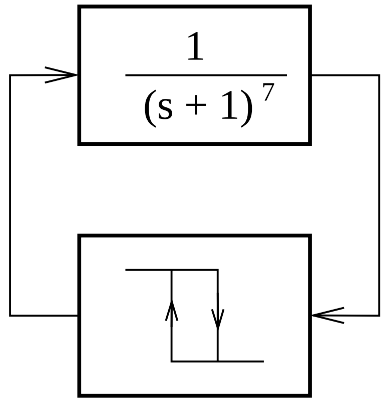


- Let $d = 0.1$. There is a unique symmetric unimodal limit cycle with half period $t^* \approx 6.89$ (*S-Procedure* was not used)

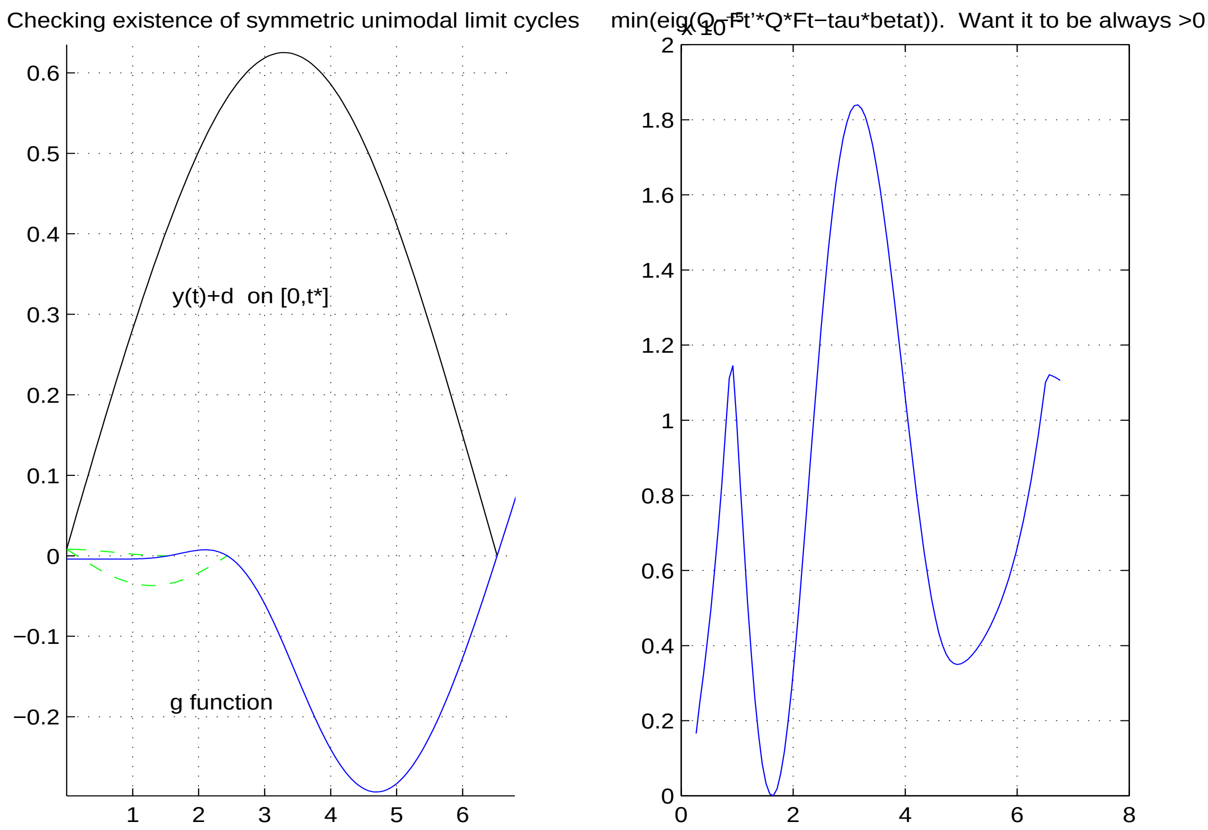




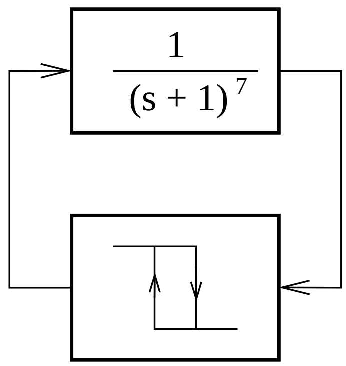
Example: $7^{rd} - Order$ **System (cont.)**



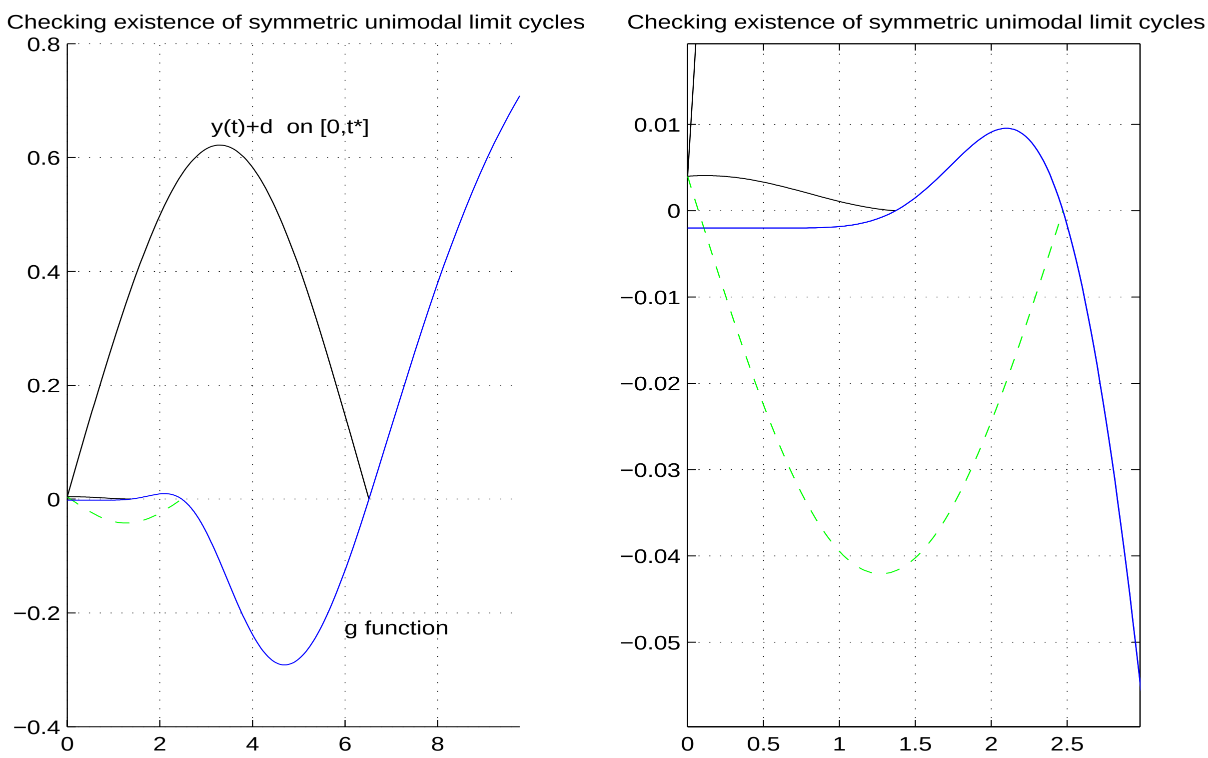
- Using the *S-Procedure* argument, global stability can be proven for d as low as 0.00404:



Example: $7^{rd} - Order$ System (cont.)



- For $0 < d < 0.00378$ there will be more than one limit cycle. For example, for $d = 0.002$:



Piecewise Linear Systems: Analysis

- DISCOVER: quadratic stability of Poincaré maps is common in RFS.

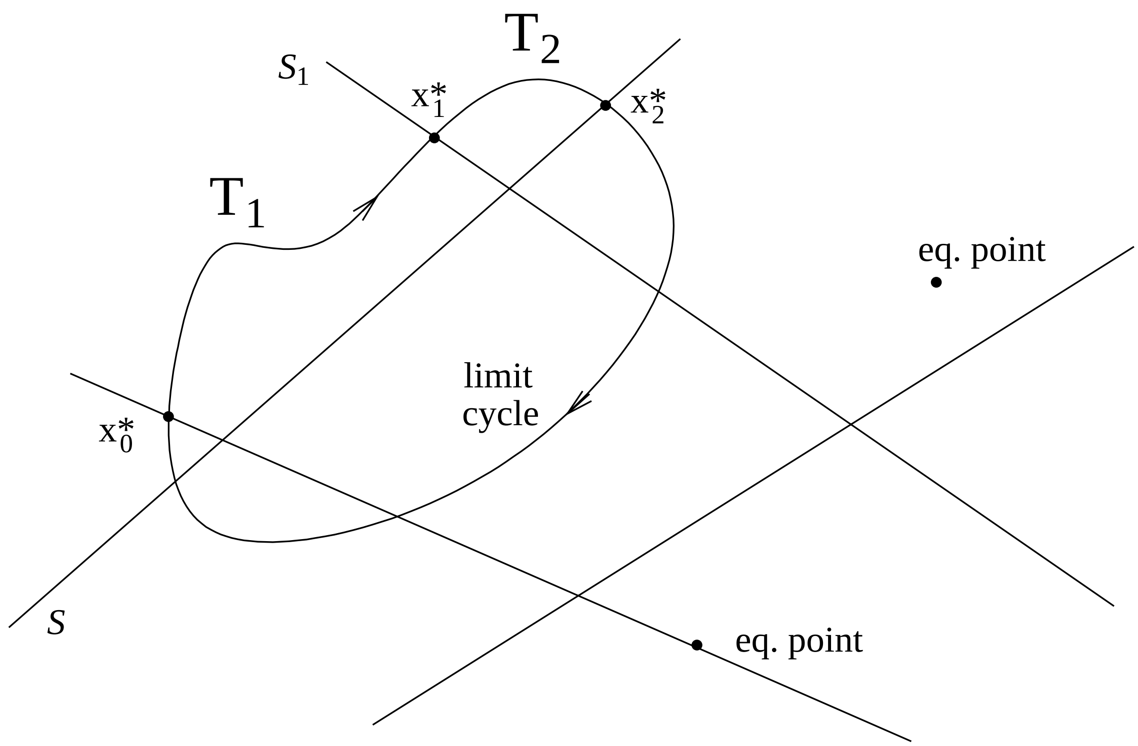
- SUSPECT: global stability of certain sets of trajectories can be proven using quadratic stability of Poincaré maps for PLS.

- HOW: express Poincaré maps induced by an LTI flow between two switching surfaces as linear transformations analytically parametrized by a scalar function of the state.

- IDEA: compute quadratic Lyapunov functions for Poincaré maps.

- The search for Lyapunov functions is done by solving a set of linear matrix inequalities.

Poincaré Maps for PLS

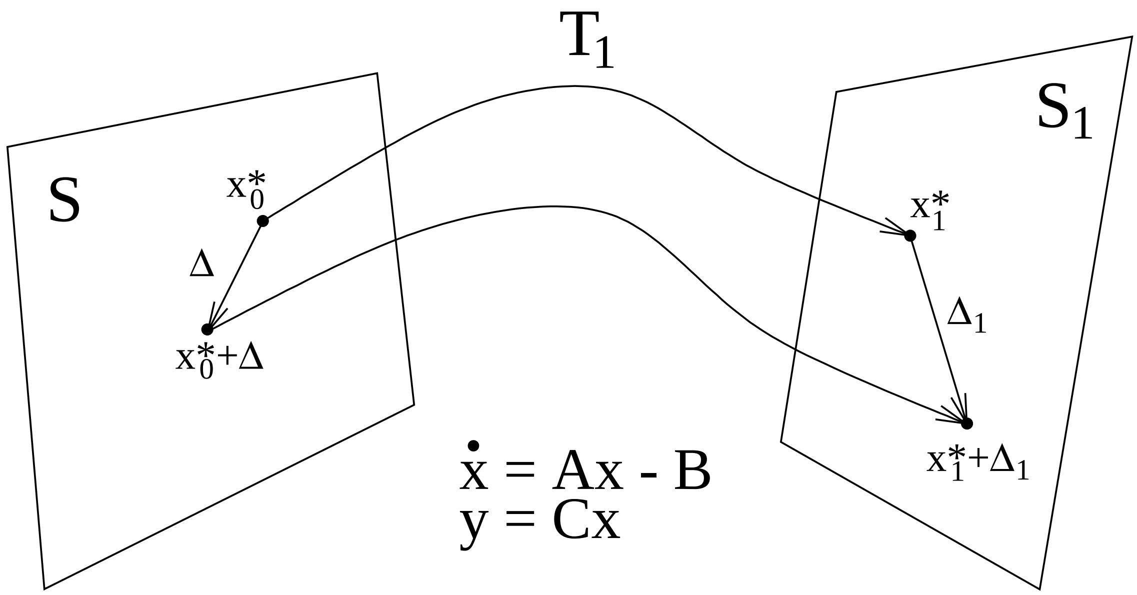


$$T_1(x_0^*) = x_1^*$$

$$T_2(x_1^*) = x_2^*$$

$$\vdots$$

Individual Poincaré Maps



$$S_1 = \{x \in \mathbb{R}^n : \; Cx + d = 0\}$$

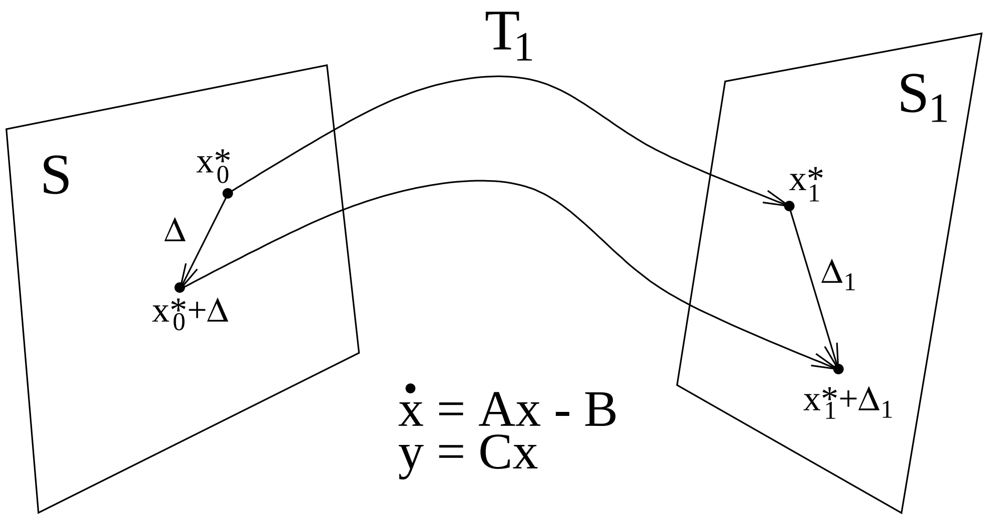
$$T_1 : S \to S_1$$

$$x_1^* = T_1(x_0^*)$$

or

$$\begin{aligned} \Delta_1 &= T_1(x_0^* + \Delta) - x_1^* \\ &= T(\Delta) \end{aligned}$$

Individual Poincaré Maps



THEOREM: For any $\Delta \in S - x^*$ there exists a $t > 0$ such that

$$\Delta_1 = H(t)\Delta$$

where

$$H(t) = \left(I - \frac{v_t C}{C v_t}\right) e^{At}$$

for $t > 0$ (for $t = t^*$, $H(t)$ is defined via continuation) with

$$v_t = x^*(t) - x_1^*$$

and assuming $Cv_t \neq 0$ for $t > 0$ and $t \neq t^*$.

Quadratic Constraints of Poincaré Maps of PLS

- $$\Delta_1 = H(t)\Delta$$

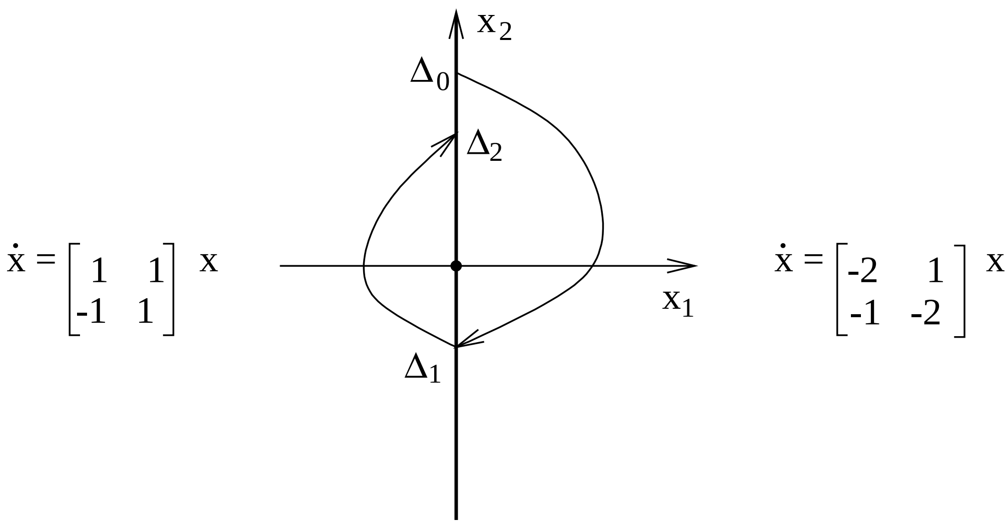
- Contraction in certain quadratic metrics can be checked

$$H^T(t)Q_1 H(t) < Q$$

for all expected t

- $\Rightarrow$ can compare quadratic function in S with quadratic function in S_1

Example: Unstable + Stable System



- Both maps can be expressed as

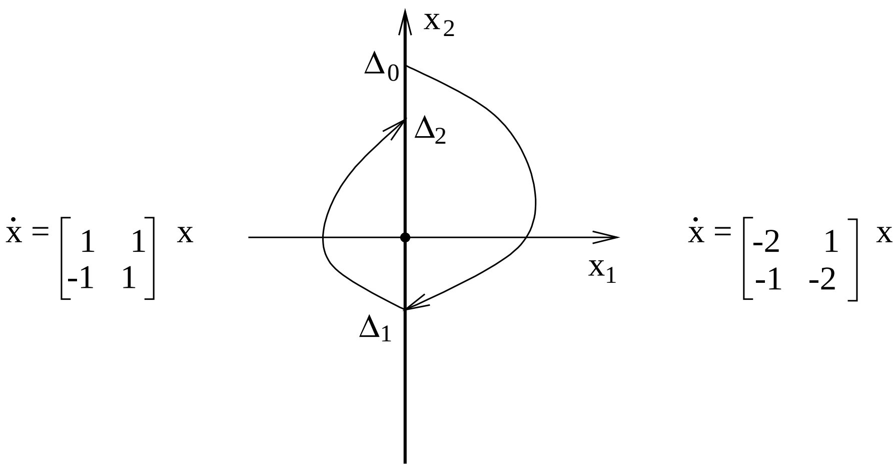
$$\begin{aligned} \Delta_1 &= H_1(t_1)\Delta_0 \\ \Delta_2 &= H_2(t_2)\Delta_1 \end{aligned}$$

- Stability follows if there exist $Q_0 > 0$ and $Q_1 > 0$ such that

$$H_1^T(t_1)Q_1H_1(t_1) < Q_0 \text{ for all expected } t_1$$
$$H_2^T(t_2)Q_0H_2(t_2) < Q_1 \text{ for all expected } t_2$$

  on the respective switching surfaces.

Example: Unstable + Stable System



- In our case

$$H_i(t_i) = -\begin{pmatrix} 0 & 1 \end{pmatrix} e^{A_i t_i} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Stability follows if there exists a $q > 0$ such that

$$(H_1(\pi))^2 \, q < 1$$
$$(H_2(\pi))^2 < q$$

- $q = 1000$

Conclusion

- We introduce the idea that global stability analysis of limit cycles of piecewise linear systems can be done using quadratic stability of Poincaré maps.

- Express a Poincaré map induced by an LTI flow between two switching surfaces as linear transformations analytically parametrized by a scalar function of the state.

- To show how this approach can be powerful in the analysis of piecewise linear systems, we applied it to a simple, yet very hard to analyze, class of PLS known as relay feedback systems.

- We addressed the problem of global quadratic stability analysis of limit cycles for RFS with hysteresis.

Conclusion

- Most RFS analyzed were proven to be globally stable. Systems analyzed include
  - Minimum-phase systems
  - Systems of relative degree larger than one
  - Systems of high dimension

- We *discovered* that quadratic stability of Poincaré maps is common in the analysis of RFS.

- We *suspect* that quadratic stability of Poincaré maps is also common in the analysis of many PLS. This is part of future work.

Paper and software (MATLAB) can be downloaded at

$$http://web.mit.edu/jmg/www/$$