# Global Stability of Relay Feedback Systems

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**Abstract**

For a large class of relay feedback systems (RFS) there will be limit cycle oscillations. Conditions to check existence and *local* stability of limit cycles for these systems are well known. *Global* stability conditions, however, are practically non-existent. This paper presents conditions in the form of linear matrix inequalities (LMIs) that guarantee *global* asymptotic stability of a limit cycle induced by a relay with hysteresis in feedback with an LTI stable system. The analysis is based on finding global quadratic Lyapunov functions for a Poincaré maps associated with the RFS. We found that most Poincaré map induced by an LTI flow between two hyperplanes can be represented as a linear transformation analytically parametrized by a scalar function of the state. Moreover, level sets of this function are convex. The search for globally quadratic Lyapunov functions is then done by solving a set of LMIs. Most RFS analyzed by the authors were proven to be globally stable. Systems analyzed include minimum-phase systems, systems of relative degree larger than one, and of high dimension. This leads us to believe that quadratic stability of associated Poincaré maps is common in RFS.

*Research supported in part by the NSF under grants ECS-9410531, ECS-9796099, and ECS-9796033, and by the AFOSR under grant F49620-96-1-0123

†Research supported in part by the NSF under grant ECS-9612558 and by the AFOSR under grant AFOSR F49620-95-0219.

# 1 Introduction

It is often possible to linearize a system, i.e., to obtain a linear representation of its behavior. That representation approximates the true dynamics well in a small region. For example, the true equations of the pendulum are never linear but, for very small deviations (a few degrees) they may be satisfactorily replaced by linear equations. In other words, for small deviations, the pendulum may be replaced by a harmonic oscillator. This ceases to hold, however, for large deviations and, in dealing with these, one must consider the nonlinear equation itself and not merely a linear substitute. In this work we are interested in a class of nonlinear systems known as *piecewise linear systems* (PLS). PLS are characterized by a finite number of linear dynamical models together with a set of rules for switching among these models. Therefore, this model description causes a partitioning of the state space into cells. These cells have distinctive properties in that the dynamics within each cell are described by linear dynamic equations. The boundaries of each cell are in effect switches between different linear systems. Those switches arise from the breakpoints in the piecewise linear functions of the model.

The reason why we are interested in studying this class of systems is to capture discontinuity actions in the dynamics from either the controller or system nonlinearities. On one hand, a wide variety of physical systems are naturally modeled this way due to real-time changes in the plant dynamics like collisions, friction, saturation, walking robots, etc. On the other hand, an engineer can introduce intentional nonlinearities to improve system performance, to effect economy in component selection, or to simplify the dynamic equations of the system by working with sets of simpler equations (e.g., linear) and switch among these simpler models (in order to avoid dealing directly with a set of nonlinear equations). Examples include control of inverted pendulums [3], control of anti-lock brake systems [12], control of missile autopilots [7], control of autopilot of aircrafts [14], auto-tuning of PID regulators using relays [4], etc.

Although widely used, very few results are available to analyze most PLS. More precisely, one typically cannot guarantee stability, robustness, and performance properties of PLS designs. Rather, any such properties are inferred from extensive computer simulations. However, in the absence of rigorous analysis tools, PLS designs come with no guarantees. In other words, complete and systematic analysis and design methodologies have yet to emerge.

In this paper, a new methodology to analyze PLS using Poincaré maps (also known as impact maps) is proposed. This methodology consists in computing global quadratic Lyapunov functions for Poincaré maps. The novelty of this work is based on expressing Poincaré maps induced by an LTI flow between two switching surfaces as linear transformations analytically parametrized by a scalar function of the state. Furthermore, level sets of this function are convex with dimension lower than the one of the switching surfaces. Global quadratic Lyapunov functions for Poincaré maps can this way be found. The search for these Lyapunov functions is done by solving a set of linear matrix inequalities which can be efficiently done using available computational tools.

To demonstrate the success of this methodology, we apply it to a simple yet very hard to analyze class of PLS known as relay feedback systems (RFS). Although the focus of this paper is on RFS, it is important to point out that most ideas behind the main results described here can be used in the analysis of more general PLS.

Analysis of this class of PLS, RFS, is in fact a classic field. The early work was motivated by relays in electromechanical systems and simple models of dry friction. Applications of relay feedback range from stationary control of industrial processes to control of mobile objects as used, for example, in space research. A vast collection of applications of relay feedback can be found in the first chapter of [15]. More recent examples include the delta-sigma modulator (as an alternative to conventional A/D converters) and the automatic tuning of PID regulators. In the delta-sigma modulator, a relay produces a bit stream output whose pulse density depends on the applied input signal amplitude (see, for example, [1]). Various methods were applied to the analysis of delta-sigma modulators. In most situations, however, none allowed to verify global stability of nonlinear oscillations. As for the automatic tuning of PID regulators, implemented in many industrial controllers, the idea is to determine some points on the Nyquist curve of a stable open loop plant by measuring the frequency of oscillation induced by a relay feedback (see, for example, [4]). One problem that needs to be solved here is the characterization of those systems that have unique global attractive unimodal[1] limit cycles. This problem is important because it gives the class of systems where relay tuning can be used.

Some important questions can be asked about RFS: do they have limit cycles? If so, are they locally stable or unstable? And if there exist a unique locally stable limit cycle, is it also globally stable? Over many years, researchers have been trying to answer these questions. [5] and [15] are references that survey a number of analysis methods. Rigorous results on existence and *local* stability of limit cycles of RFS can be found in [2, 10]. In [2], necessary and sufficient conditions for local stability of limit cycles are presented. [10] emphasizes fast switches and their properties. In [8], reasonably large regions of stability around limit cycles were characterized. For second-order systems, convergence analysis can be done in the phase-plane [13, 9]. Stable second-order non-minimum phase processes can in this way be shown to have a globally attractive limit cycle. In [11] it is proved that this also holds for processes having an impulse response sufficiently close, in a certain sense, to a second-order non-minimum phase process. Many important RFS, however, are not covered by this result. It is then clear that the problem of rigorous *global* analysis of relay-induced oscillations is still open.

In this paper, using the methodology described above, we give conditions for *global* stability of limit cycles of RFS. These conditions will be given in the form of linear matrix inequalities (LMIs) which can be efficiently solved using available computational tools.

The remainder of this paper is organized as follows. Section 2 starts by giving some mathematical preliminaries, including definitions of some standard concepts. Section 3 shows that most Poincaré maps induced by an LTI flow between a set and an hyperplane can be represented as a linear transformation analytically parametrized by a scalar function of the state. Section 4 formulates the problem of global asymptotic stability of limit cycles of RFS, and presents some relevant results from the literature. Section 5 presents global stability results of limit cycles of RFS together with some illustrative examples. Section 6 discusses computationally issues associated with the results of section 5. Finally, conclusions and future work are discussed in section 7.

## 2 Mathematical preliminaries

The purpose of this section is to introduce several mathematical concepts and tools that will be used throughout this paper. We start by introducing standard notation followed by

[1] A limit cycle is unimodal if it only switches twice per cycle.

some definitions and results of some useful mathematical tools like linear matrix inequalities and the S-procedure.

### 2.1 Standard notation

Let the field of *real* numbers be denoted by $\mathbb{R}$, the set of positive reals by $\mathbb{R}_+$, the set of $n \times 1$ *vectors* with elements in $\mathbb{R}$ by $\mathbb{R}^n$, and the set of all $n \times m$ *matrices* with elements in $\mathbb{R}$ by $\mathbb{R}^{n\times m}$. Superscript $(\cdot)^T$ denotes *transpose*. A matrix $D \in \mathbb{R}^{n\times m}$ is called *symmetric* if $D = D^T$, *square* if $m = n$, and *positive definite* (*positive semidefinite*) if $x^T Dx > 0$ ($x^T Dx \geq 0$) for all nonzero $x \in \mathbb{R}^n$. Let $I$ denote the identity matrix. The *2-norm* of $x \in \mathbb{R}^n$ is given by $\|x\|^2 = x^T x$. For some $D > 0$, define the *weighted Euclidean norm* of $x$ as $\|x\|_D^2 = \|D^{1/2}x\|^2 = x^T Dx$.

The *orthogonal complements* to a matrix $V$, denoted by $V^\perp$, are matrices with a maximal number of column vectors forming an orthonormal set such that $VV^\perp = 0$.

Consider a map $f : \mathbb{R}^n \to \mathbb{R}^n$ and a set $X \subset \mathbb{R}^n$. $X$ is said to be *invariant* under $f$ if $f(X) \subset X$. A set $X$ is said to be a *cone* if $x \in X$ implies $\lambda x \in X$ for any $\lambda \geq 0$.

### 2.2 Linear matrix inequalities and the S-procedure

A *linear matrix inequality* (LMI) has the form

$$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i > 0 \tag{1}$$

where $x \in \mathbb{R}^n$ is the variable and the symmetric matrices $F_i \in \mathbb{R}^{n\times n}$, $i = 0, 1, ..., n$ are given. The LMI (1) is a convex constraint on $x$, i.e., the set $\{x|\ F(x) > 0\}$ is convex. Expressing solutions to problems in terms of LMIs is a common practice these days. Mathematical and software tools available are capable of finding $x_i$ satisfying (1) efficiently. This is the idea behind this paper: express the problem of global analysis of relay-induced oscillations as LMIs.

One tool that will be useful later in the paper is the *S-procedure*. Here we describe a simple version of this tool. Let $\sigma_0(x) = x^T T_0 x$ and $\sigma_1(x) = x^T T_1 x$ be quadratic functions of the variable $x \in \mathbb{R}^n$, where $T_0 = T_0^T$ and $T_1 = T_1^T$. Assume there exists an $x$ such that $\sigma_1(x) > 0$. Then the following condition on $\sigma_0, \sigma_1$

$$\sigma_0(x) \geq 0 \text{ for all } x \text{ such that } \sigma_1(x) \geq 0$$

holds if and only if there exists a $\tau \geq 0$ such that

$$\sigma_0(x) - \tau\sigma_1(x) \geq 0$$

for all $x$. For more information on LMIs and the S-procedure the reader is referred, for example, to [6].

## 3 Poincaré maps

In this section, we show that most Poincaré maps induced by an LTI flow between a set and an hyperplane can be represented as a linear transformation analytically parametrized by a scalar function of the state.

Consider the following affine linear time-invariant (LTI) system

$$\begin{cases} \dot{x} &= Ax - B \\ y &= Cx \end{cases} \tag{2}$$

where $x \in \mathbb{R}^n$ and $A$ is Hurwitz. Let

$$\underline{S} = \{x \in \mathbb{R}^n : \ Cx = -d\}$$

and $U$ ($\underline{U}$) be the set of points such that $Cx > -d$ ($Cx < -d$). Assume all initial conditions $x_0 \in U$ and $CA^{-1}B \in \underline{U}$. This ensures the existence of at least one $t > 0$ such that $y(t) = -d$. Assume also that a certain trajectory $x^*(t)$ of (2) is known, with $x(0) = x_0^* \in U$, $x^*(t^*) = x_1^* \in \underline{S}$.

Consider now the solution of (2) with initial condition $x(0) = x^* + \Delta \in U$. Define $t_\Delta$ as the set of all times $t_i \geq 0$ such that $y(t_i) = -d$ and $y(t) \geq -d$ on $[0, t_i]$. For example, in the case of figure 1, $t_\Delta = \{t_1, t_2, t_3\}$.

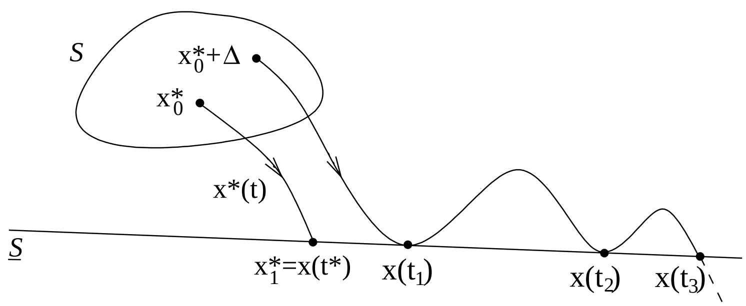


Figure 1: Definition of a Poincaré map

Let $x_1^* + \Delta_1 \in x(t_\Delta)$. As depicted in figure 1, for a given $\Delta$, there may be more than one solution for $\Delta_1$. Consider the multivalued Poincaré map $T_0 : U \to \underline{S}$ defined by $x_1^* + \Delta_1 \in T_0(x_0^* + \Delta)$. Since $x^*$ is fixed, the Poincaré map can be redefined as the map $T : U - x_0^* \to \underline{S} - x_1^*$ given by $\Delta_1 \in T(\Delta)$, where $T(\Delta) = T_0(x_0^* + \Delta) - x_1^*$. We have the following result.

**Theorem 3.1** *Consider the Poincaré map $T$ defined above. Let*

$$v_t = \left(e^{At} - e^{At^*}\right)\left(x_0^* - A^{-1}B\right)$$

*and $v = e^{At^*}\left(Ax_0^* - B\right)$, and assume $Cv_t \neq 0$ for $t > 0$ and $t \neq t^*$. Define*

$$H(t) = \begin{cases} \left(I - \frac{v_t C}{C v_t}\right) e^{At} & \text{if } t > 0 \text{ and } t \neq t^* \\ \left(I - \frac{vC}{Cv}\right) e^{At^*} & \text{if } t = t^* \end{cases}$$

*Then, for any $\Delta \in U - x_0^*$ and $\Delta_1 \in T(\Delta)$ there exists a $t > 0$ such that*

$$\Delta_1 = H(t)\Delta$$

This theorem says that any Poincaré map induced by an LTI flow between a set and an hyperplane can be represented as a linear transformation analytically parametrized by a scalar function of the state. The advantage of expressing the Poincaré map this way is to have all nonlinearities depending on only one parameter $t$. Although $t$ depends on $\Delta$, once $t$ is fixed, the Poincaré map becomes linear in $\Delta$. Note that $H(t)$ defined above is continuous in $t > 0$.

**Proof:** Let $x(0) = x_0 \in U$. Integrating the differential equation (2) gives

$$\begin{aligned} x(t) &= e^{At}x_0 - \int_0^t e^{A(t-\tau)}Bd\tau \\ &= e^{At}(x_0 - A^{-1}B) + A^{-1}B \end{aligned}$$

If $x(0) = x_0^*$ and $t = t^*$ then $x(t^*) = x_1^*$, i.e.,

$$x_1^* = e^{At^*}(x_0^* - A^{-1}B) + A^{-1}B \tag{3}$$

Now, let $x(0) = x_0^* + \Delta \in U$ and $\Delta_1 \in T(\Delta)$. Let also $t \in t_\Delta$ be the time it takes for the trajectory to go from $x_0^* + \Delta$ to $x_1^* + \Delta_1$. Then

$$x_1^* + \Delta_1 = e^{At}(x_0^* + \Delta - A^{-1}B) + A^{-1}B$$

Using (3), the last equality can be written as

$$\begin{aligned} \Delta_1 &= e^{At}(x_0^* - A^{-1}B + \Delta) - e^{At^*}(x_0^* - A^{-1}B) \\ &= e^{At}\Delta + v_t \end{aligned}$$

Since $x_1^* + \Delta_1 \in \underline{S}$, $C(x_1^* + \Delta_1) = -d$, or $C\Delta_1 = 0$, that is,

$$Ce^{At}\Delta + Cv_t = 0 \tag{4}$$

Therefore, it is also true that $v_t Ce^{At}\Delta + v_t Cv_t = 0$. Since by assumption $Cv_t \neq 0$ for $t > 0$ and $t \neq t^*$,

$$v_t = -\frac{v_t C}{Cv_t} e^{At}\Delta$$

for $t > 0$, $t \neq t^*$. Replacing above we get

$$\Delta_1 = \left(I - \frac{v_t C}{Cv_t}\right) e^{At}\Delta$$

for $t > 0$ and $t \neq t^*$, and

$$\Delta_1 = \left(I - \frac{vC}{Cv}\right) e^{At^*}\Delta$$

for $t = t^*$. ∎

## 4 Relay feedback systems

This section formulates the problem of global asymptotic stability of limit cycles of RFS, and presents some relevant results from the literature. First, we define RFS and talk about some of their properties. Then, in section 4.2, we recall a result on existence of limit cycles of RFS. Poincaré maps for RFS are introduced in section 4.3. Finally, local stability of limit cycles of RFS are also recalled.

## 4.1 Definitions and properties of RFS

As mentioned before, the focus of this paper is the study of stability of limit cycles generated by linear systems in relay feedback. Consider an LTI system satisfying the following linear dynamic equations

$$\left\{ \begin{array}{lcl} \dot{x} & = & Ax + Bu \\ y & = & Cx \end{array} \right. \tag{5}$$

where $x \in \mathbb{R}^n$ and $A$ is Hurwitz, in feedback with a relay (see figure 2) defined as

$$u(t) \in \left\{ \begin{array}{ll} \{-1\} & \text{if } y(t) > d, \text{ or } y(t) > -d \text{ and } u(t-0) = -1 \\ \{1\} & \text{if } y(t) < -d, \text{ or } y(t) < d \text{ and } u(t-0) = 1 \\ \{-1, 1\} & \text{if } y(t) = -d \text{ and } u(t-0) = -1, \text{ or } y(t) = d \text{ and } u(t-0) = 1 \end{array} \right. \tag{6}$$

where $d \geq 0$ is the hysteresis. By a solution of (5)-(6) we mean functions $(x, y, u)$ satisfying (5)-(6), where $u$ is piecewise constant. Note that existence of a solution is always guaranteed if $d > 0$, or if $d = 0$ and $CB < 0$, which are the cases we consider in this paper. $t$ is a switching time of a solution of (5)-(6) if $u$ is discontinuous at $t$. We say a trajectory of (5)-(6) switches at some time $t$ if $t$ is a switching time.

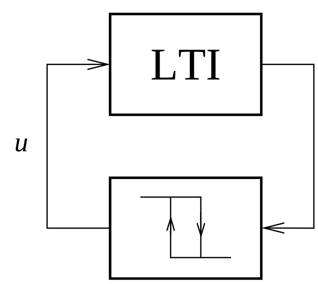


Figure 2: Relay Feedback System

The *switching surfaces* $S$ and $\underline{S}$ of the RFS are the surfaces of dimension $n-1$ where $y$ is equal to $d$ and $-d$, respectively. More precisely

$$S = \{x \in \mathbb{R}^n : \ Cx = d\}$$

and

$$\underline{S} = \{x \in \mathbb{R}^n : \ Cx = -d\}$$

Consider a subset $S_a$ of $S$ given by

$$S_a = \{x \in S : \ CAx + CB \geq 0\}$$

This set is important since it characterizes those points in $S$ that can be reached by any trajectory starting at $\underline{S}$. We call it the arrival set in $S$. Similarly, define $\underline{S_a}$ as

$$\underline{S_a} = \{x \in \underline{S} : \ CAx - CB \leq 0\}$$

It is easy to see that $S = -\underline{S}$ and $S_a = -\underline{S_a}$ where $-X$ stands for the set $\{-x | x \in X\}$.

Note that trajectories of $\dot{x} = Ax - B$ starting at any point $x_0 \in S$ will converge to the equilibrium point $A^{-1}B$. When connected in feedback with the relay, one of the following two possible scenarios will occur for a certain trajectory starting at $x_0$: it will either cross $\underline{S}$

at sometime, or it will never cross $\underline{S}$. The last situation is not interesting to us since it does not lead to limit cycle trajectories. One way to ensure a switch is to have $CA^{-1}B + d < 0$, although this is not a necessary condition for the existence of limit cycles. However, if we are looking for globally stable limit cycles, it is in fact necessary to have $CA^{-1}B + d < 0$. Otherwise a trajectory starting at $A^{-1}B$ would not converge to the limit cycle. Throughout the paper, it is assumed $CA^{-1}B + d < 0$.

An interesting property of linear systems in relay feedback is their symmetry around the origin.

**Proposition 4.1** *Consider a trajectory $x(t)$ of $\dot{x} = Ax - B$ starting at $x_0 \in S$. Then $-x(t)$ is a trajectory of $\dot{x} = Ax + B$ starting at $-x_0 \in \underline{S}$.*

**Proof:** Assume $x_0 \in S$. Since

$$\begin{aligned} -\dot{x}(t) &= -(Ax(t) - B) \\ &= A(-x(t)) + B \end{aligned}$$

$-x(t)$ is a trajectory of $\dot{x} = Ax + B$ starting at $-x_0 \in \underline{S}$. ■

## 4.2 Existence of limit cycles

As we mentioned before, for a large class of processes, there will be limit cycle oscillations. Let $\xi(t)$ be a nontrivial periodic solution of (5)-(6) with period $T$, and let $\gamma$ be the limit cycle defined by the trace of $\xi(t)$. The limit cycle $\gamma$ is called *symmetric* if $\xi(t+T/2) = -\xi(t)$. It is called *unimodal* if it only switches twice per cycle. A class of limit cycles we are particularly interested is the class of all symmetric unimodal limit cycles.

The next proposition, proven in [2], gives necessary and sufficient conditions for the existence of symmetric unimodal limit cycles.

**Proposition 4.2** *Consider the RFS (5)-(6). Assume there exists a symmetric unimodal limit cycle $\gamma$ with period $2t^*$. Then the following conditions hold*

$$g(t^*) = C(e^{At^*} + I)^{-1}(e^{At^*} - I)A^{-1}B - d = 0 \tag{7}$$

*and*

$$y(t) = C\left[e^{At}(x^* - A^{-1}B) + A^{-1}B\right] \geq -d \quad for \quad 0 \leq t < t^*$$

*Furthermore, the periodic solution $\gamma$ is obtained with the initial condition $x^* \in S_a$ given by*

$$x(0) = x^* = (e^{At^*} + I)^{-1}(e^{At^*} - I)A^{-1}B$$

## 4.3 Poincaré maps for RFS

The construction of a Poincaré map for RFS can be done in a similar way as in section 3. Consider a symmetric unimodal limit cycle $\gamma$, with period $2t^*$, obtained with the initial condition $x^* \in S_a$. This means that a trajectory $x(t)$ starting at $x^*$ crosses the switching surface $\underline{S}$ at $-x^* = x(t^*) \in \underline{S_a}$ (see figure 3).

To study the behavior of the system around the limit cycle we perturb $x^*$ by $\Delta$ such that $x^* + \Delta \in S_a$. Consider the solution of (5)-(6) with initial condition $x(0) = x^* + \Delta$. As in section 3, $t_\Delta = \{t_1, t_2, t_3\}$ in the example in figure 3. This means a switch can occur at

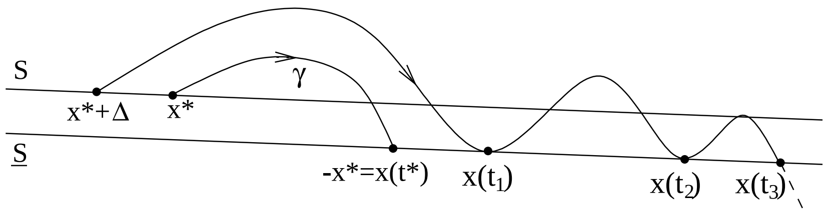


Figure 3: Definition of a Poincaré map for a RFS

$t = t_1$, $t = t_2$, or $t = t_3$. Obviously, if no switch occurred at $t = t_1$ or $t = t_2$, then a switch must occur at $t = t_3$.

Let $-x^* - \Delta_1 \in x(t_\Delta)$. Since $-x^* - \Delta_1 \in \underline{S_a}$ then $x^* + \Delta_1 \in S_a$. Consider the multivalued Poincaré map $T_0 : S_a \to S_a$ defined by $x^* + \Delta_1 \in T_0(x^* + \Delta)$. Since $x^*$ is fixed, the Poincaré map can be redefined as the map $T : S_a - x^* \to S_a - x^*$ given by $\Delta_1 \in T(\Delta)$, where $T(\Delta) = T_0(x^* + \Delta) - x^*$. In result, $\Delta = 0$ is an equilibrium point of the discrete-time system

$$\Delta_{k+1} \in T(\Delta_k) \tag{8}$$

### 4.4 Local stability

The following proposition, proven in [2], gives necessary and sufficient conditions for local stability of symmetric unimodal limit cycles. This result is based on the linearization of the Poincaré map around the origin.

**Proposition 4.3** *Consider the RFS (5)-(6). Assume there exists a symmetric unimodal limit cycle $\gamma$ with period $2t^*$, obtained with the initial condition $x^* \in S$. Assume the Poincaré map $T$ defined above is a function in some neighborhood of $\Delta = 0$. The Jacobian of the Poincaré map $T$ at $\Delta = 0$ is given by*

$$W = \left( \frac{vC}{Cv} - I \right) e^{At^*}$$

*where $v = -Ax^* - B$. The limit cycle $\gamma$ is locally stable if and only if $W$ has all its eigenvalues inside the unit disk.*

In this paper, we are interested in systems that have a unique locally stable unimodal limit cycle. For such systems, the idea is to find a global quadratic Lyapunov function for the associated Poincaré map. If this map is found to be globally quadratically stable then it follows that the limit cycle is globally asymptotically stable. The next section shows how to find global quadratic Lyapunov functions for a Poincaré map.

## 5 Global stability

Before we address the problem of finding global quadratic Lyapunov functions for a Poincaré map, we first need to express the map in a convenient way. This can be done using theorem 3.1 with some minor modifications. Let $v_t = \left(e^{At} - e^{At^*}\right)(x^* - A^{-1}B)$ and $v = e^{At^*}(Ax^* - B)$. Assuming $Cv_t \neq 0$ for $t > 0$ and $t \neq t^*$, $H(t)$ can be defined as

$$H(t) = \begin{cases} \left(\frac{v_t C}{C v_t} - I\right) e^{At} & \text{if } t > 0 \text{ and } t \neq t^* \\ \left(\frac{vC}{Cv} - I\right) e^{At^*} & \text{if } t = t^* \end{cases}$$

Then, for any $\Delta \in S_a - x^*$ and $\Delta_1 \in T(\Delta)$ there exists a $t > 0$ such that

$$\Delta_1 = H(t)\Delta$$

With this result, it is possible to represent most Poincaré maps induced by an LTI flow between two hyperplanes as a linear transformation analytically parametrized by a scalar function of the state. This result agrees with proposition 4.3. Using equality (3), $v$ can be written as $v = e^{At^*}(Ax^* - B) = -Ax^* - B$. This means $H(t^*)$ is exactly the Jacobian of the Poincaré map $T$ at $\Delta = 0$.

Since the domain of the Poincaré map is $S_a - x^*$, we now define what we mean by quadratic stability of $T$ in $S_a - x^*$.

**Definition 5.1** *Consider the discrete-time system (8). The origin is quadratically stable in $S_a - x^*$ if $S_a - x^*$ is invariant under $T$ and if there exists a symmetric matrix $P > 0$ such that*

$$T^T(\Delta)PT(\Delta) < \Delta^T P\Delta \ , \quad \forall \Delta \in S_a - x^*, \ \Delta \neq 0 \tag{9}$$

Due to the symmetry of the system around the origin (as seen in proposition 4.1) , it is not hard to see that if $S_a$ is a stable region, then $\underline{S_a}$ must also be a stable region. Therefore, all we need to do is find a $P > 0$ such that (9) is satisfied in order to show the limit cycle $\gamma$ is globally asymptotically stable.

An important remark is the fact that, although the vectors $\Delta$ and $\Delta_1$ are $n$-dimensional, the solution generated by the Poincaré map $T$ is restricted to the $(n-1)$-dimensional hyperplane $S$. Therefore, the map $T$ is actually a map from $\mathbb{R}^{n-1}$ to $\mathbb{R}^{n-1}$. Let $\Pi \in C^{\perp}$ be a map from $\mathbb{R}^{n-1}$ to $S$. It is then equivalent in (9) to solve for the symmetric $n \times n$ matrix $P > 0$ or for a symmetric $(n-1) \times (n-1)$ matrix $Q > 0$, where $Q = \Pi^T P \Pi$. Define also $F(t) = \Pi^T H(t)\Pi$.

**Proposition 5.1** *The limit cycle $\gamma$ is globally asymptotically stable if there exists a $Q > 0$ such that*

$$\alpha(t) = Q - F^T(t)QF(t) > 0 \ , \quad \forall t > 0 \tag{10}$$

**Proof:** If there exists a $P > 0$ such that

$$\Delta^T P\Delta - \Delta^T H^T(t)PH(t)\Delta > 0 \ , \quad \forall \Delta \in S, \ \Delta \neq 0, \ \forall t > 0$$

then (9) holds. Replacing $P = \Pi Q \Pi^T$ and $H(t) = \Pi F(t)\Pi^T$ in the last inequality, yields

$$\Delta^T \Pi \left(Q - F^T(t)QF(t)\right) \Pi^T \Delta > 0 \ , \quad \forall \Delta \in S, \ \Delta \neq 0, \ \forall t > 0$$

Let $\delta = \Pi^T \Delta \in \mathbb{R}^{n-1}$. The last inequality is then equivalent to

$$\delta^T \alpha(t)\delta > 0 \ , \quad \forall \delta \in \mathbb{R}^{n-1}, \ \delta \neq 0, \ \forall t > 0$$

which is the desired result. ■

**Example 5.1** Consider the following transfer function of an LTI system

$$P(s) = -\frac{s^2 + s - 4}{3(s+1)(s+2)(s+3)}$$

Assume this system is in relay feedback with $d = 0$. It is possible to pick such $d$ since any state-space realization results in $CB < 0$. Solving (7) for $t^* > 0$ we get $t^* \approx 1.4$. This corresponds to $x^* \approx [0.60 \quad -0.44 \quad 0.32]^T \in S_a$. Therefore, the closed loop system has a symmetric unimodal limit cycle obtained with the initial condition $x^* \in S_a$ and with period $2t^*$. We analyzed this same RFS in [8]. There, we characterized a reasonably large region of stability around the limit cycle. However, the following $Q > 0$

$$Q \approx \begin{bmatrix} 6.86 & 2.52 \\ 2.52 & 9.32 \end{bmatrix}$$

meets the conditions of proposition 5.1 proving that the limit cycle is actually globally asymptotically stable. ■

**Example 5.2** Consider the following $5th - order$ LTI system

$$P(s) = \frac{-3s^4 - 0.5s^3 + 7.5s^2 + 12.75s + 7}{(s+1)(s^2 + 2.5s + 2.25)(s^2 + s + 0.5)}$$

This system in relay feedback (with $d = 0$) has a symmetric unimodal limit cycle with period $2t^*$, where $t^* \approx 1.5$. The following $Q > 0$

$$Q \approx \begin{bmatrix} 0.565 & 0.496 & -0.267 & 0.08 \\ 0.496 & 3.216 & 0.174 & 0.124 \\ -0.268 & 0.174 & 2.893 & 0.298 \\ 0.08 & 0.124 & 0.298 & 3.119 \end{bmatrix}$$

meets the conditions of proposition 5.1, which means the limit cycle is globally asymptotically stable. ■

**Example 5.3** Consider the following $3rd - order$ minimum-phase LTI system

$$P(s) = \frac{s^2 + 3s + 10}{(s^2 + 4s + 2)(s + 3)}$$

Let $d = 0.1$. Then the RFS has a symmetric unimodal limit cycle with period $2t^*$, where $t^* \approx 0.22$. The following $Q > 0$

$$Q \approx \begin{bmatrix} 4.364 & -0.146 \\ -0.146 & 4.816 \end{bmatrix}$$

meets the conditions of proposition 5.1, which means the limit cycle is globally asymptotically stable. ■

**Example 5.4** Consider the following LTI system of relative degree 3

$$P(s) = \frac{1}{(s+1)^7}$$

Let $d = 0.1$. This system in relay feedback has a symmetric unimodal limit cycle with period $2t^*$, where $t^* \approx 6.89$. A $Q > 0$ (not shown here) satisfying the conditions of proposition 5.1 can be found, which means the limit cycle is globally asymptotically stable. However, if a smaller $d$ is chosen (say, $d = 0.01$) then no $Q > 0$ satisfies the conditions of proposition 5.1. ■

The fact that in the last example we could not find a $Q > 0$ satisfying (10) for small values of $d$ does not mean there does not exist a global quadratic Lyapunov function for $T$. The conditions of proposition 5.1 are conservative and can be improved. With that in mind, we take a closer look to what happens to the Poincaré map when $t$ is fixed. First, remember that the Poincaré map $T$ is a map from $\mathbb{R}^{n-1}$ to $\mathbb{R}^{n-1}$. When $t > 0$ is fixed, there are still $n-2$ more degrees of freedom to choose from. Therefore, $x^* + \Delta$ is restricted to a set in $S_a$ of dimension $n-2$.

Let $\tilde{S}_t$ be the set of points $x^* + \Delta \in S_a$ such that $t \in t_\Delta$ (see figure 4). In other words, a trajectory starting at $x_0 \in \tilde{S}_t$ satisfies both $y(t) \geq -d$ on $[0, t]$, and $y(t) = -d$. Let also $S_0$ be the boundary of $S_a$.

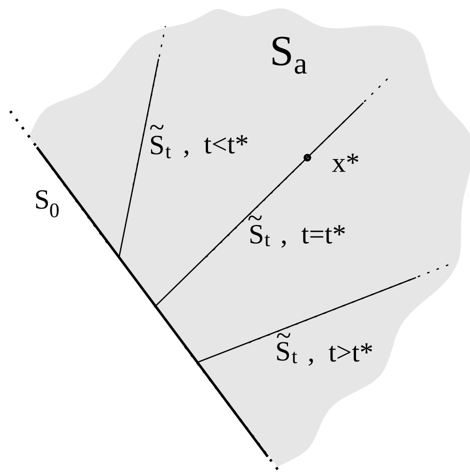


Figure 4: Example of several sets $\tilde{S}_t$ (for $n = 3$, $\tilde{S}_t$ are just segments of lines)

In proposition 5.1 we were looking for $Q > 0$ such that $\Delta^T \Pi \alpha(t) \Pi^T \Delta > 0$ for all $\Delta \in S$ and for all $t > 0$. However, this condition can be relaxed by allowing the inequality to be satisfied for $\Delta \in \tilde{S}_t - x^*$ for each $t > 0$. This is the idea behind the next result.

**Proposition 5.2** *The limit cycle $\gamma$ is globally asymptotically stable if there exists a $Q > 0$ such that*

$$\Delta^T \Pi \left( Q - F^T(t) Q F(t) \right) \Pi^T \Delta > 0 \ , \quad \forall \Delta \in \tilde{S}_t - x^* \text{ and } \forall t > 0 \qquad (11)$$

The problem with this result is that, in general, the sets $\tilde{S}_t$ are hard to characterize. An alternative is to consider a set $S_t \supset \tilde{S}_t$ obtained from equation (4), given by

$$S_t = \left\{ x^* + \Delta \in S_a : C e^{At} \Delta = -C v_t \right\}$$

To see the difference between $S_t$ and $\tilde{S}_t$, consider the examples in figure 5.

In the example on the left in figure 5, $t_\Delta = \{t_1, t_2\}$. This means $x^* + \Delta$ belongs to all the sets $S_{t_1}$, $S_{t_2}$, $\tilde{S}_{t_1}$, and $\tilde{S}_{t_2}$.

The example on the right side of figure 5 shows what would happen to $y(t)$ if the trajectory had not switched at $t = t_1$ (dashed curve). In that case, it would have intersected $S$ again at $= t_2$. Since $y(t)$ is negative for $t_1 < t < t_2$, a switch necessarily had to occur at

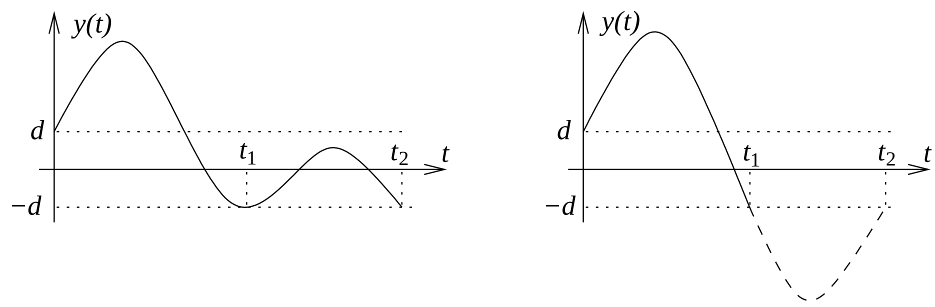


Figure 5: On the left: $y(t) \geq 0$ for $0 \leq t \leq t_2$; on the right: $y(t) \geq 0$ for $0 \leq t \leq t_1$

$t = t_1$. Although both $t_1$ and $t_2$ satisfy (4), only $t_1$ is a valid switching time, i.e., $t_\Delta = \{t_1\}$. Therefore, $x^* + \Delta$ belongs to $S_{t_1}$, $S_{t_2}$, and $\tilde{S}_{t_1}$, but it does not belong to $\tilde{S}_{t_2}$.

Since $\tilde{S}_t \subset S_t$, condition (11) holds if there exists a $Q > 0$ such that

$$\Delta^T \Pi \left( Q - F^T(t) Q F(t) \right) \Pi^T \Delta > 0 \ , \quad \forall \Delta \in S_t - x^* \text{ and } \forall t > 0 \tag{12}$$

Unfortunately, since quadratic functions are homogeneous[2], if $\Delta^T \Pi \alpha(t) \Pi^T \Delta > 0$ for all $\Delta \in S_t - x^*$, then $\Delta^T \Pi \alpha(t) \Pi^T \Delta > 0$ at least in the "smallest" cone centered at $x^*$ containing $S_t$ (see figure 6).

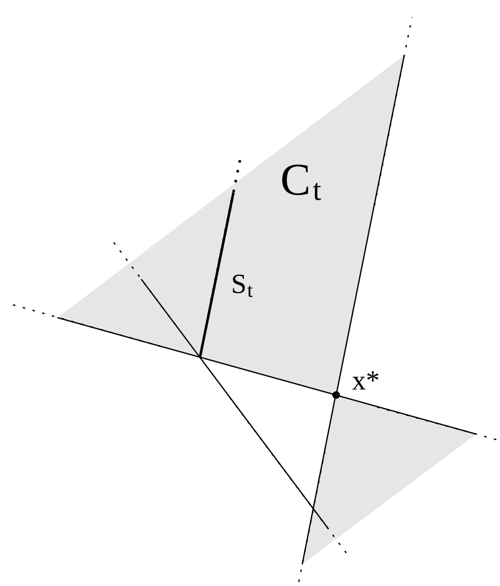


Figure 6: Cone $C_t$ where we want $\Delta^T \alpha(t) \Delta > 0$

To see this, assume $\Delta^T \Pi \alpha(t) \Pi^T \Delta > 0$ for all $\Delta \in S_t - x^*$. Let $x = \lambda \Delta$ where $\lambda \in \mathbb{R} \backslash \{0\}$ and $\Delta \in S_t - x^*$. Then $x^T \Pi \alpha(t) \Pi^T x = \lambda^2 \Delta^T \Pi \alpha(t) \Pi^T \Delta > 0$. This means that $\Delta^T \Pi \alpha(t) \Pi^T \Delta > 0$ at least in the cone centered at $x^*$ containing $S_t$. This cone is defined by two hyperplanes in $S$: one is the hyperplane parallel to $S_t$ containing $x^*$ and the other is the hyperplane defined by the intersection of $S_0$ and $S_t$, and the point $x^*$. This cone can be written as

$$C_t = \left\{ \Delta : \Delta^T \Pi \beta_t \Pi^T \Delta > 0 \right\}$$

[2]Meaning that if $x^T Q x > 0$ then $y = \lambda x$, $\lambda \neq 0$, also satisfies $y^T Q y > 0$.

where $\beta_t$ is a symmetric matrix obtained from those two hyperplanes (section 6.2 explains how such $\beta_t$ is constructed). The condition in the following result is then equivalent to condition (12).

**Corollary 5.1** *The limit cycle $\gamma$ is globally asymptotically stable if there exists a $Q > 0$ such that*

$$\Delta^T \Pi \left( Q - F^T(t) Q F(t) \right) \Pi^T \Delta > 0 \ , \quad \forall \Delta \in C_t \text{ and } \forall t > 0 \tag{13}$$

The goal in this paper is to derive computationally efficient conditions that can prove global stability of limit cycles. The conditions of the last result, however, are not computationally easy as they are. But, if the S-procedure is applied, these conditions can be transformed in LMIs (for each $t > 0$) which can then be solved using available efficient computational tools. Note that these LMIs conditions are equivalent to (12).

**Corollary 5.2** *The limit cycle $\gamma$ is globally asymptotically stable if there exists a $Q > 0$ and a scalar function $\tau_t > 0$ such that*

$$Q - F^T(t) Q F(t) - \tau_t \beta_t > 0 \ , \quad \forall t > 0 \tag{14}$$

**Proof:** Using the S-procedure, (13) is satisfied if and only if for each $t > 0$ there exists an $\tau_t > 0$ such that

$$\Delta^T \Pi \alpha(t) \Pi^T \Delta > \tau_t \Delta^T \Pi \beta(t) \Pi^T \Delta$$

for all $\Delta \in S$. ■

**Example 5.5** Consider again the relative degree 7 system analyzed in example 5.4. As we saw, for small values of $d > 0$ there was no $Q > 0$ satisfying the conditions of proposition 5.1. Using the results of corollary 5.2, however, a $Q > 0$ and a positive function $\tau_t$ satisfying (14) are known to exist for values of $d$ as small as 0.00404. This proves that the limit cycle is in fact globally asymptotically stable for small values of $d$. Note that for $0 < d < 0.00378$ there will be more than one limit cycle. ■

Before we conclude this section, it is important to point out that condition (14) can still be improved. Defining the cones $C_t$ to be the cones containing $\tilde{S}_t$ instead of $S_t$ can help in most cases the cones $C_t$ to become smaller, relaxing this way condition (14), and increasing the changes of finding global quadratic Lyapunov functions. How to characterize the sets $\tilde{S}_t$ is part of future research.

# 6 Computational issues

In this section we will talk about ways to find $Q > 0$. First, we will show that since $A$ is Hurwitz and $u = \pm 1$ is a bounded input, there is a bounded set such that any trajectory will eventually enter and stay there. This will lead to bounds on the difference between any two consecutive switching times. This way, the search for $Q > 0$ in (10) and (14) becomes restricted to $0 < t_- \leq t \leq t_+ < \infty$. Then, we will talk about the cones $C_t$ used in (14). In particular, it will be described how to construct $\beta_t$.

### 6.1 Bounds on switching times

For a fixed $t > 0$, condition (10) is an LMI with respect to $Q$, while (14) is an LMI with respect to $Q$ and $\tau_t$. In this section, we show that it is sufficient that conditions (10) or (14) are satisfied in some carefully chosen interval $[t_-, t_+]$, instead of being satisfied for all $t > 0$. In order to do so, one must guarantee there exists a $t_0$ such that the difference between any two consecutive switching times of a trajectory $x(t)$ for $t > t_0$ is higher than $t_-$ but lower than $t_+$. Before we find such bounds, we need to show there is a particular bounded set such that any trajectory will eventually enter and stay there (i.e., will not leave the set).

**Proposition 6.1** *Consider the system $\dot{x} = Ax + Bu$, $y = Fx$, where $A$ is Hurwitz, $u(t) = \pm 1$, and $F$ is a row vector. Then, for any fixed $\bar{t} \geq 0$,*

$$\limsup_{t\to\infty} |Fe^{A\bar{t}}x(t)| \leq \int_{\bar{t}}^{\infty} \left|Fe^{A\tau}B\right| d\tau \leq \|Fe^{At}B\|_1$$

**Proof:** At time $t$, $x(t)$ is given by

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Therefore

$$\begin{aligned}
\limsup_{t\to\infty} \left|Fe^{A\bar{t}}x(t)\right| &= \limsup_{t\to\infty} \left|Fe^{A\bar{t}}\left(e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right)\right| \\
&\leq \limsup_{t\to\infty} \left|Fe^{A\bar{t}}e^{At}x_0\right| + \limsup_{t\to\infty} \left|Fe^{A\bar{t}}\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right| \\
&\leq 0 + \limsup_{t\to\infty} \int_0^t \left|Fe^{A(t+\bar{t}-\tau)}Bu(\tau)\right| d\tau \\
&\leq \limsup_{t\to\infty} \int_0^t \left|Fe^{A(t+\bar{t}-\tau)}B\right| d\tau \\
&= \int_{\bar{t}}^{\infty} \left|Fe^{A\tau}B\right| d\tau \\
&\leq \int_0^{\infty} \left|Fe^{A\tau}B\right| d\tau
\end{aligned}$$

which is equal to $\|Fe^{At}B\|_1$. ∎

We now focus our attention in finding an upper bound for $t_+$. First, remember from the proof of theorem 3.1 that a trajectory $x(t)$ starting at $x_0 \in S_a$ is given by $x(t) = e^{At}(x_0 - A^{-1}B) + A^{-1}B$. Then the output $y(t) = Cx(t)$ is given by

$$y(t) = Ce^{At}(x_0 - A^{-1}B) + CA^{-1}B$$

By definition of $S_a$, $y(t) > -d$ at least in some interval $(0, \epsilon)$, where $\epsilon > 0$. However, since we are assuming $CA^{-1}B < -d$, and $A$ Hurwitz, it is easy to see that $y(t)$ cannot remain larger than $-d$ for all $t > 0$. For any initial condition $x_0$, $Ce^{At}(x_0 - A^{-1}B) \to 0$ as $t \to \infty$. Hence, since for sufficiently large time $t$, $x(t)$ is bounded (from the above proposition), an upper bound on $t_+$ can be obtained.

**Proposition 6.2** *Let $t_+ > 0$ be the smallest solution of*

$$\int_{t_+}^{\infty} \left|Ce^{A\tau}B\right| d\tau + |Ce^{At_+}A^{-1}B| \leq -(CA^{-1}B + d) \tag{15}$$

*If $t_a$ and $t_b$ are sufficiently large consecutive switching times then $|t_a - t_b| \leq t_+$.*

**Proof:** Assume that after a sufficiently large time the trajectory is at $x_0 \in S_a$. Without loss of generality, assume $x(0) = x_0$. Then $y(t)$ will be positive in some interval $(0, \epsilon)$. We are interested in finding an upper bound on the time it takes to switch. That is, we would like to find an upper bound $t_+ > 0$ of those $t > 0$ such that $y(t) = -d$, i.e.,

$$Ce^{At_+}(x_0 - A^{-1}B) = -(CA^{-1}B + d) > 0$$

Using proposition 6.1 with $F = c$ and $\bar{t} = t_+$, we can get a bound on the left side of the inequality

$$\begin{aligned} \left|Ce^{At_+}x_0 - Ce^{At_+}A^{-1}B\right| &\leq |Ce^{At_+}x_0| + |Ce^{At_+}A^{-1}B| \\ &\leq \int_{t_+}^{\infty} \left|Ce^{A\tau}B\right| d\tau + |Ce^{At_+}A^{-1}B| \end{aligned}$$

Therefore, $t_+ > 0$ must satisfy (15). ■

Remember that if $x_0 \in S_a$, $y(t)$ will be positive at least in some interval $(0, \epsilon)$. The next result shows that for $t$ large enough, $\epsilon$ cannot be made arbitrarily small. So, since for sufficiently large time $t$, $x(t)$ is bounded, an lower bound on on the time it takes between two consecutive switches can be obtained.

**Proposition 6.3** *Let $k_d = -2CB$, $k_{dd} = \|CA^2e^{At}B\|_1 + \max_{t\geq 0}|Ce^{At}AB|$, and $k_{dl} = \|CAe^{At}B\|_1 + \max_{t\geq 0}|Ce^{At}B|$ and define*

$$t_1 = \frac{k_d + \sqrt{k_d^2 + 4k_{dd}d}}{k_{dd}}$$

*and*

$$t_2 = \frac{2d}{k_{dl}}$$

*Also, let $t_- = \max\{t_1, t_2\}$. If $t_a$ and $t_b$ are sufficiently large consecutive switching times then $|t_a - t_b| \geq t_-$.*

**Proof:** Assume again that after a sufficiently large time the trajectory is at $x_0 \in S_a$. Without loss of generality, assume $x(0) = x_0$. We first find $t_1$. This means that right before the switch (at $t = 0^-$), $\dot{y}(0^-) \geq 0$, i.e., $CAx_0 + CB \geq 0$. Therefore, after the switch at $t = 0^+$, $\dot{y}(0^+) = CAx_0 - CB = CAx_0 + CB - 2CB \geq -2CB$. That is, $\dot{y}(0^+) \geq k_d$.

We also need bounds on the second derivative of $y$ for $t > 0$. From $y(t)$ we get $\dot{y}(t) = CAe^{At}(x_0 - A^{-1}B)$, and $\ddot{y}(t) = CA^2e^{At}(x_0 - A^{-1}B)$. This means that

$$\begin{aligned} |\ddot{y}(t)| &= \left|CA^2e^{At}(x_0 - A^{-1}B)\right| \\ &\leq |CA^2e^{At}x_0| + |Ce^{At}AB| \\ &\leq \|CA^2e^{At}B\|_1 + \max_{t\geq 0}|Ce^{At}AB| \\ &= k_{dd} \end{aligned}$$

So, $-k_{dd} \leq \ddot{y}(t) \leq k_{dd}$. In order to find a lower bound on the switching time, we consider the worst case scenario, that is, we consider the case when $\ddot{y}(t) = -k_{dd}$ and $\dot{y}(0) = k_d$. This implies that $\dot{y}(t) = -k_{dd}t + k_d$. Integrating once more and knowing that $y(0) = d$, yields

$$y(t) = -\frac{k_{dd}}{2}t^2 + k_d t + d$$

We are looking for values of $t = t_1$ such that $y(t_1) = -d$ and $t_1 > 0$. $y(t_1) = -d$ has two solutions

$$t_1 = \frac{k_d \pm \sqrt{k_d^2 + 4k_{dd}d}}{k_{dd}}$$

However, only one is positive (the one with the + sign) since $\ddot{y}(t) < 0$ for all $t$ and either $y(0) > 0$ (if $d > 0$) or $\dot{y}(0) > 0$ (if $d = 0$ and $CB < 0$).

To find $t_2$ we find a bound on the first derivative of $y$ for $t > 0$

$$\begin{aligned}
|\dot{y}(t)| &= \left|CAe^{At}(x_0 - A^{-1}B)\right| \\
&\leq |CAe^{At}x_0| + |Ce^{At}B| \\
&\leq \|CAe^{At}B\|_1 + \max_{t \geq 0}|Ce^{At}B| \\
&= k_{dl}
\end{aligned}$$

So, $-k_{dl} \leq \dot{y}(t) \leq k_{dl}$. The worst case scenario is the case when $\dot{y}(t) = -k_{dl}$ (with $y(0) = d$). Therefore, $y(t) = -k_{dl}t + d$. Again, we are looking for values of $t = t_2$ such that $y(t_2) = -d$ and $t_2 > 0$, i.e., the solution of $-k_{dl}t_2 + d = -d$. ■

## 6.2 Construction of the cones $C_t$

We now describe how to construct the cones $C_t$ introduced after proposition 5.2. Remember that for each $t > 0$, the cone is defined by two hyperplanes in $S$: one is the hyperplane parallel to $S_t$ containing $x^*$ and the other is the hyperplane defined by the intersection of $S_0$ and $S_t$, and the point $x^*$. Let $\Pi l_t$ and $\Pi s_t$, respectively, be vectors in $S$ perpendicular to each hyperplane. Once these vectors are known, the cone that we are interested can be easily found. The cone is composed of all the vectors $\Delta \in S - x^*$ such that $\Delta^T \Pi(s_t^T l_t + l_t^T s_t)\Pi^T \Delta \geq 0$. The matrix symmetric $\beta_t$ introduced in the definition of $C_t$ is just $\beta_t = s_t^T l_t + l_t^T s_t$. Remember that the cone is centered at $x^*$ and note that after $l_t$ is chosen, $s_t$ must have the right direction in order to guarantee $S_t \subset C_t$.

We first find $\Pi l_t$, the vector perpendicular to $S_t$. Looking back at the definition of $S_t$, $l_t$ is given by

$$l_t = -\frac{(Ce^{At}\Pi)^T}{\|Ce^{At}\Pi\|^2} C v_t$$

The derivation of $s_t$ is not as trivial as $l_t$. We actually need to introduce a few extra variables. The first one is $\Pi l_0$, the vector perpendicular to the set $S_0$ (remember that $S_0$ is the boundary of $S_a$). This set is given by all the vectors $\Delta \in S - x^*$ such that $CA(x^* + \Delta) + CB = 0$. Therefore,

$$l_0 = -\frac{(CA\Pi)^T}{\|CA\Pi\|^2} C(Ax^* + B)$$

**Proposition 6.4** *The hyperplane defined by the intersection of $S_0$ and $S_t$, and the point $x^*$ is perpendicular to the vector*

$$\frac{\Pi l_t}{\|l_t\|}\|l_0\| - \frac{\Pi l_0}{\|l_0\|}\|l_t\|$$

**Proof:** $S_0$ can be parameterize the following way

$$S_0 = \left\{x^* + \Delta \in S \mid \Delta = \Pi(l_0 + l_0^\perp u),\ u \in \mathbb{R}^{n-2}\right\}$$

and $S_t$

$$S_t = \left\{x^* + \Delta \in S_a \mid \Delta = \Pi(l_t + l_t^\perp w),\ w \in \mathbb{R}^{n-2}\right\}$$

The intersection of $S_0$ and $S_t$ occurs at points in $S$ such that $l_0 + l_0^\perp u = l_t + l_t^\perp w$. Multiplying on the left by $l_t^T$ we have $l_t^T l_0 + l_t^T l_0^\perp u = l_t^T l_t$ or

$$l_t^T l_0^\perp u = \|l_t\|^2 - l_t^T l_0 \tag{16}$$

We want to show that

$$\left(\frac{l_t}{\|l_t\|}\|l_0\| - \frac{l_0}{\|l_0\|}\|l_t\|\right)^T \left(l_0 + l_0^\perp u\right) = 0$$

Using (16) we have

$$\begin{aligned}
\left(\frac{l_t}{\|l_t\|}\|l_0\| - \frac{l_0}{\|l_0\|}\|l_t\|\right)^T \left(l_0 + l_0^\perp u\right) &= \frac{l_t^T l_0}{\|l_t\|}\|l_0\| + \frac{l_t^T l_0^\perp u}{\|l_t\|}\|l_0\| - \frac{l_0^T l_0}{\|l_0\|}\|l_t\| \\
&= \frac{l_t^T l_0}{\|l_t\|}\|l_0\| + \frac{\|l_t\|^2 - l_t^T l_0}{\|l_t\|}\|l_0\| - \|l_0\|\|l_t\| \\
&= 0
\end{aligned}$$

■

The characterization of $s_t$ is not complete yet. The orientation of $s_t$ must be carefully chosen to guarantee that the cone $C_t$ contains $S_t$.

**Proposition 6.5** *If*

$$s_t = C(Ax^* + B)\left(\frac{l_t}{\|l_t\|}\|l_0\| - \frac{l_0}{\|l_0\|}\|l_t\|\right)$$

*then the cone defined above contains $S_t$.*

The proof, omitted here, is based on taking a point $\Delta \in S_t$ and showing that $\Delta^T \beta_t \Delta \geq 0$.

## 7 Conclusion

This paper introduces the idea that global quadratic stability analysis of limit cycles of piecewise linear systems can be done using Poincaré maps. The development of stability conditions is based on expressing a Poincaré map induced by an LTI flow between two switching surfaces as linear transformations analytically parametrized by a scalar function of the state. By sampling carefully chosen values of this function, and taking advantage that level sets of this function are convex, it is possible to write a set of LMIs. A global quadratic Lyapunov function of the associated Poincaré map can be found by solving this set of LMIs using available and efficient computational tools,

To show how this approach can be powerful in the analysis of piecewise linear systems, we applied it to a simple, yet very hard to analyze, class of PLS known as relay feedback systems. We addressed the problem of global quadratic stability analysis of limit cycles for RFS with hysteresis. This is, in fact, a hard problem since very few results existed until now. However, with this new results, most RFS analyzed by the authors were proven to be globally stable. Systems analyzed include minimum-phase systems, systems of relative degree larger than one, and of high dimension. This leads us to believe that quadratic stability is common in RFS.

There are still many open problems following this work. An important extension of the results from this paper that authors are looking into is to find conditions that do not depend on $Q$ but guarantee its existence. Another extension that we are currently pursuing is finding how this new methodology can be applied to more general PLS.

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