Global Stability of Relay Feedback Systems

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Key Ideas

- Efficient analysis of piecewise linear systems can be done using *Quadratic Surface Lyapunov Functions*.
- Maps between switching surfaces induced by LTI flows can be represented as linear transformations analytically parametrized by a scalar function of the state.
- Successfully proved global asymptotic stability of Relay Feedback Systems, On/Off Systems, and Saturation Systems.

Relay Feedback Systems (RFS)

"Simple" class of Piecewise Linear Systems yet very hard to analyze .

Applications:

- Electromechanical systems
- Simple models of dry friction
- Delta-sigma modulators
- Automatic tuning of PID regulators

Property:

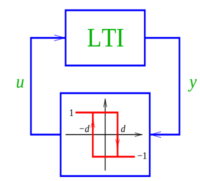
- RFS often exhibit limit cycles

Relay Feedback Systems: Analysis

- Survey of several analysis methods—Atherton'75, Guckenheimer and Holmes'83, Tsypkin'84
- Existence and local stability of limit cycles of RFS—Åström and Hagglund'84
- Properties of RFS—Johansson and Rantzer'99, Varigonda and Georgiou'00, Van der Schaft
- Global stability for processes with an impulse response sufficiently close, in a certain sense, to a second-order non-minimum phase process—Megretski'96

The problem of rigorous *global* analysis of relay-induced oscillations is still open

Relay Feedback System: Definition



The LTI system is given by

$$\begin{cases} \dot{x} = Ax + Bu\,, & A \text{ is a Hurwitz matrix} \\ y = Cx \end{cases}$$

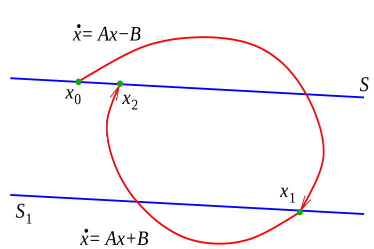
Relay Feedback System: Definition

The *switching surfaces* S and S_1 of the RFS are

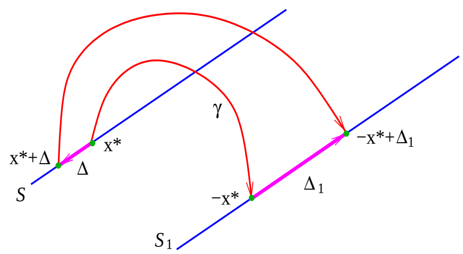
$$S = \{x \in \mathbb{R}^n : Cx = d\}$$

and

$$S_1 = \{x \in \mathbb{R}^n : Cx = -d\}$$



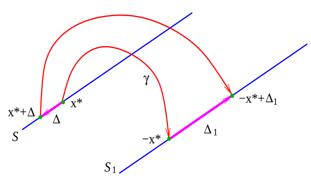
Map between switching surfaces of RFS



Consider the map

$$\Delta_1 = T(\Delta)$$

Global Stability of T



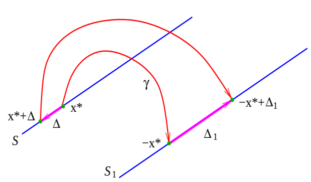
$$\Delta_1 = T(\Delta)$$

Global asymptotic stability of the equilibrium point of T:

$$T'(\Delta)QT(\Delta) \leq \gamma \Delta' Q \Delta$$

for some $Q > 0$, $0 \leq \gamma < 1$, all $\Delta \in S - x^*$.

Map between switching surfaces of RFS

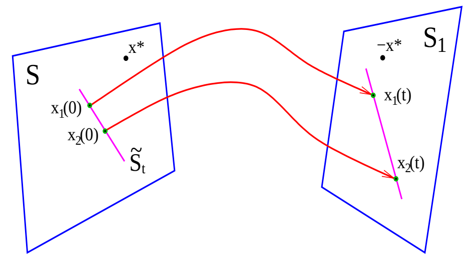


$$\begin{aligned} \Delta_1 &= T(\Delta) \\ &= H(t)\Delta \end{aligned}$$

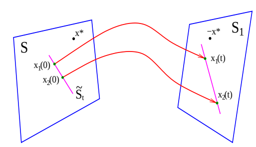
where t is the switching time associated with Δ, Δ_1 and $H(\cdot)$ is given explicitly.

Map between switching surfaces of RFS

- $$\Delta_1 = H(t)\Delta$$
- Let $\tilde{S}_t \subset S$ be the set of all Δ which yield the next switch at t
- $\tilde{S}_t$ is a linear manifold of dimension $n-2$



Quadratic Stability



- $$\Delta_1 = H(t)\Delta$$
- Quadratic stability is guaranteed if there exists a $Q > 0$ such that

$$\Delta_1' Q \Delta_1 < \Delta' Q \Delta$$

$$\Uparrow$$

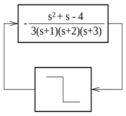
$$Q - H'(t)QH(t) > 0 \quad \text{on } \tilde{S}_t$$

for all expected t

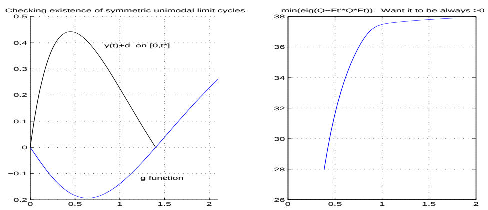
Technical Details

- Since A is Hurwitz and $u = \pm 1$ is a bounded input, there is a bounded set such that any trajectory will eventually enter and stay there.
- In that set can find bounds on the difference between any two consecutive switching times. The difference between any two consecutive switching times of some trajectory is higher than t_- but lower than t_+.
- It is sufficient to find $Q > 0$ that satisfies the stability conditions on $0 < t_- \leq t \leq t_+ < \infty$.
- We sample the stability conditions on $t \in [t_-, t_+]$ to obtain a finite set of LMIs, and solve for $Q > 0$.

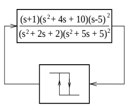
Example: 3^{rd} – *Order* Non-Minimum Phase System



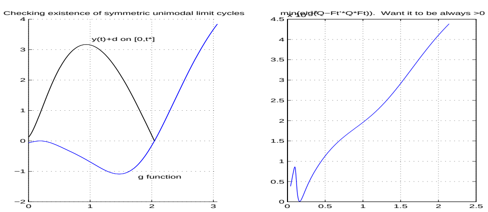
- Let $d = 0$ (possible since $CB < 0$). There is a unique symmetric unimodal limit cycle with half period $t^* \approx 1.4$



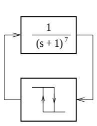
Example: 6^{th} – *Order* System



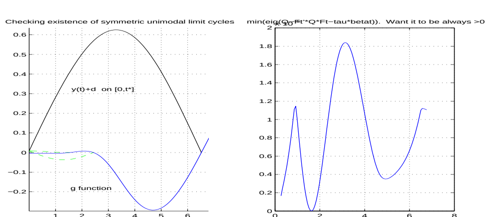
- Sliding modes occur if $d = 0$ ($CB = 1$). A $Q > 0$ is known to exist for d as low as 0.061



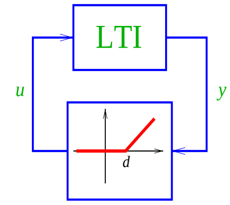
Example: 7^{rd} – *Order* System

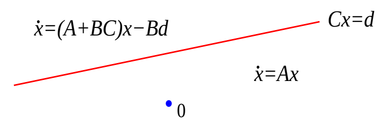


- Global stability can be proven for d as low as 0.00404

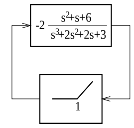


On/Off Systems

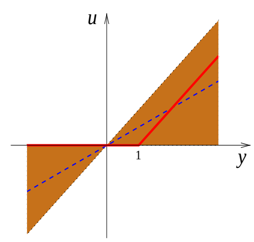




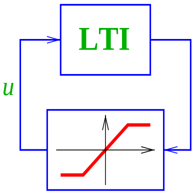
3^{rd}-order system with unstable nonlinearity sector

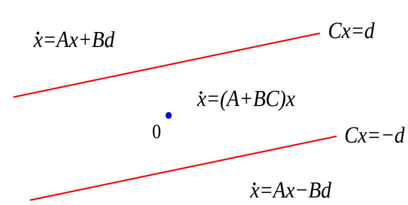


- System has an unstable nonlinearity sector

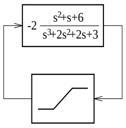


Saturation Systems

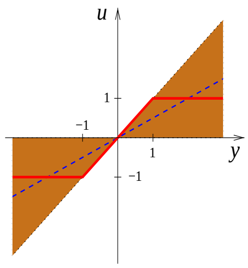




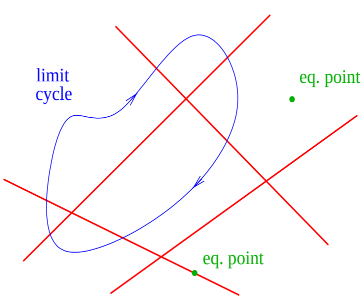
3^{rd}-order system with unstable nonlinearity sector



- System has an unstable nonlinearity sector



Piecewise Linear Systems



Conclusion

- We introduce the idea that global stability analysis of limit cycles and equilibrium points of piecewise linear systems can be done using *Quadratic Surface Lyapunov Functions*.
- Express maps between switching surfaces as linear transformations analytically parametrized by a scalar function of the state.
- The search of *quadratic Lyapunov functions on the switching surfaces* is done by solving a set of *LMIs*.

Conclusion

- Can prove global asymptotic stability of
  - limit cycles of Relay Feedback Systems with hysteresis
  - equilibrium points of On/Off Systems
  - equilibrium points of Saturation Systems
  - trajectories of large number of classes of Piecewise Linear Systems
- Paper and software (MATLAB) can be downloaded at

  `http://web.mit.edu/jmg/www/`