ROBUST CONTROL SYNTHESIS IN THE TIME DOMAIN

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ABSTRACT

This thesis investigates the synthesis of controllers for time varying systems in order to satisfy an induced 2-norm closed loop performance bound. This performance criterion is a generalisation of the well known $\mathscr{H}_\infty$ norm criterion used in the frequency domain for analysis and synthesis of linear time invariant control systems.

A number of different time varying system frameworks are considered, for which there are no frequency domain counterparts. One such class is that of aperiodic sampled-data systems, that is continuous time systems connected to a discrete controller via sampling and hold devices. Multiple generalized sampling and hold devices, which may be aperiodic and asynchronous, are permitted within the framework considered in this thesis. Using game theory, necessary and sufficient conditions are given for the existence of controllers satisfying a prespecified performance bound for such multi-rate sampled-data systems, and expressions for such a controller are given if one exists. In the state feedback case this result is generalised to include time varying systems with possible discontinuities in the state vector, of which sampled-data systems are a special case.

Another inherently time domain problem investigated in this work is the moving horizon $\mathscr{H}_\infty$ controller. The moving horizon problem was originally formulated as a method of stabilising time varying systems without requiring information about the system matrices over all future time, by minimizing a quadratic cost function up to some finite time ahead at each instant. In this thesis the method is generalized so that a differential game with quadratic cost function is solved up to a finite time ahead.

Explicit constructions are given in both the state and output feedback case corresponding to this min-max problem, based upon state estimation using information from a finite time in the past. It is shown that with the moving horizon controller it is possible to synthesize controllers which are not only stabilising but also satisfy some prespecified closed loop induced norm bound over all future time.

The moving horizon problem was originally formulated as a purely continuous time problem. In this thesis it is generalized also to the case when the controller is updated only on discrete, possibly overlapping, intervals. It is shown that, if the terminal weights satisfy certain conditions, then it is possible to synthesize controllers which are γ-feasible by solving separate dynamic games on each finite interval.

Keywords: sampled-data systems, robust control, time varying systems, moving horizon control.

PREFACE

Many people have helped me over the past three years to produce this thesis and it is a great pleasure for me to take this opportunity to express my gratitude to them all.

First and foremost, I would like to thank my supervisor, Professor Keith Glover, for providing me with the benefit of his insight and for giving his time generously to discuss the ideas that have made up this research.

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Finally, I would like to express my heartfelt thanks to my parents and my brother Dev for their continuing encouragement and support.

This research has been supported financially by the Science and Engineering Research Council of the United Kingdom, and by Cambridge University Engineering Department.

As required by University statute, I hereby declare that this dissertation is not substantially the same as any that I have submitted for a degree at any other university, is the result of my own work, and includes nothing which is the outcome of work done in collaboration.

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16th June 1995

CONTENTS

NOTATION

For the purpose of reference, the following glossary outlines the notation used in the text.

$\mathbb{Z}$	the integers
$\mathbb{Z}_+$	the non-negative integers
$\mathbb{R}$, $\mathbb{R}_+$	the real and non-negative real numbers
$\mathbb{R}^{p\times q}$	the real matrices with p rows and q columns
$\mathscr{L}_\infty^{p\times q}$	the set of matrix functions $M : \mathbb{R}_+ \to \mathbb{R}^{p\times q}$ such that there exists $\beta > 0$ such that $\bar{\sigma}(M(t)) < \beta$ for all $t \in \mathbb{R}_+$
LTI	linear time invariant
LTV	linear time varying
RDE	Riccati differential equation
RDI	Riccati differential inequality
LMI	linear matrix inequality
$:=$	right hand side defines the left hand side
$\mathscr{L}_2^n[t_0, t_1]$	the set of square integrable functions mapping $[t_0, t_1]$ to $\mathbb{R}^n$
$\mathscr{L}_2^n$	the set of square integrable functions mapping $[0, \infty)$ to $\mathbb{R}^n$
$\ell_2^n[k_0, k_1]$	sequences mapping $\{i \in \mathbb{Z} \mid k_0 \leq i \leq k_1\}$ to $\mathbb{R}^n$
ℓ_2^n	square summable sequences mapping $\mathbb{Z}_+$ to $\mathbb{R}^n$
$\lvert\cdot\rvert$	Euclidean norm on $\mathbb{R}^n$
$\lVert\cdot\rVert_2$	norm on $\mathscr{L}_2$ or ℓ_2
A'	transpose of real matrix A
G^*	adjoint of operator G
$\bar{\sigma}(A)$	maximum singular value of matrix A
$D_{12}^\perp$	the extension of D_{12} such that, if $D_{12}(t)'D_{12}(t) = I$ for all t, then $\begin{bmatrix} D_{12}(t) & D_{12}^\perp(t) \end{bmatrix}$ is orthogonal for all t.
$D_{21}^\perp$	the extension of D_{21} such that, if $D_{21}(t)D_{21}(t)' = I$ for all t, then $\begin{bmatrix} D_{21}(t)' & D_{21}^\perp(t)' \end{bmatrix}$ is orthogonal for all t.
$\pi_{[k_0,k_1]}$	projection operator mapping $\ell_2[j_0, j_1]$ to $\ell_2[k_0, k_1]$, where $[k_0, k_1] \subseteq [j_0, j_1]$, such that $(\pi_{[k_0,k_1]}u)(k) = u(k)$ for $k \in [k_0, k_1]$.
$u_{[k_0,k_1]}$	$\pi_{[k_0,k_1]}u$
$\mathcal{S}$	sampling operator
$\mathcal{H}$	hold operator

1. INTRODUCTION

This thesis is concerned with the synthesis of feedback controllers for uncertain systems. Given a mathematical model of a physical system, we would like a control methodology to provided answers to two basic questions: Can the specified performance be achieved? If so, which designs achieve it?

This task is further complicated by the fact that we only have an approximate mathematical model of the physical system to be controlled. We would like to construct a controller that gives good performance for the physical system itself, not just for the model. In order to achieve this, we assume that the physical system is in some specified set of mathematical models. In robust control theory, this set is usually specified by a nominal model and a set of possible perturbations.

The goals of feedback are to desensitize the control system to the effects of both unmodelled system dynamics and exogenous disturbances. We would also like to stabilize the system, since the presence of uncertainty means that we cannot control its inputs exactly.

Originally formulated by Zames [81] in 1981, $\mathscr{H}_\infty$ control theory has emerged as an extremely successful methodology for providing answers to the above design questions for complex multivariable systems. The motivation was based on frequency domain ideas, with most of the original results being for linear time invariant systems.

The fundamental result of robust control theory is the *small gain* theorem. One measure of the gain of a linear multivariable system $\mathcal{G}$ is given by

$$\|\mathcal{G}\| := \sup_{u \neq 0} \frac{\|\mathcal{G}u\|_2}{\|u\|_2}.$$

This quantity is known as the induced norm, or operator norm, of $\mathcal{G}$. The small gain theorem then states that, if in the feedback diagram of Figure 1.1 the product of the norms $\|\mathcal{G}\|\|\mathcal{K}\|$ is strictly less than 1, then the closed loop system will be stable.

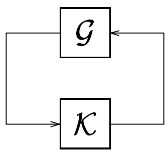


Figure 1.1: The small gain theorem

In the $\mathscr{H}_\infty$ control paradigm, the $\mathscr{H}_\infty$-norm of a system $\mathcal{G}$ is defined in terms of its transfer function G as

$$\|\mathcal{G}\|_\infty := \sup_{\omega} \bar{\sigma}(G(j\omega))$$

That is, it is the maximum singular value of the transfer function over all frequencies. It is a well-known result that the induced $\mathscr{L}_2$ norm is then equal to the $\mathscr{H}_\infty$-norm.

From the small gain framework, we are led to the *standard problem* setup of Figure 1.2. In this case, the goal is to find a controller $\mathcal{K}$ such that the closed loop is stable, and that the induced norm of the resulting closed loop operator from w to z is less than some prespecified level γ. We will refer to such controllers as γ-suboptimal or γ-feasible. We would also like to know when no such controller exists.

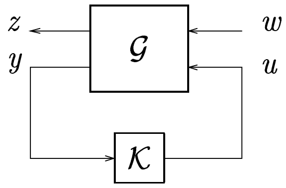


Figure 1.2: The $\mathscr{H}_\infty$ Standard Problem

This problem corresponds in a natural way to that of disturbance rejection, by regarding w as a disturbance input and z as a controlled output. Further, the small gain theorem immediately implies that with any system perturbation Δ connected between z and w, if the norm $\|\Delta\| < \gamma^{-1}$, then the closed loop system will remain stable. This property is known as *robust* stability.

There are many further refinements of this idea, to include structured system perturbations, and to include robust performance as well as robust stability. Reviews of this material can be found in the books by Dahleh [21] and Green and Limebeer [29].

The first approaches to the solution of this problem were made using frequency domain techniques. However, a major development in $\mathscr{H}_\infty$ control theory was the introduction of state space methods. Simple and intuitive solutions to the $\mathscr{H}_\infty$ synthesis problem were presented by Glover and Doyle [28] using these techniques.

This approach is particularly natural and appealing for the problem of $\mathscr{H}_\infty$ control of linear time varying systems, and of sampled-data systems. Although originally formulated in the frequency domain, the induced norm is still well-defined for linear time varying systems also, although the term "$\mathscr{H}_\infty$-norm" has become a commonly used misnomer for the induced norm of such systems.

After the state space solutions of Glover and Doyle [28] and Doyle, Glover, Khargonekar and Francis [22] were derived, connections were made to a number of other research areas, in particular that of risk sensitive control as developed by Whittle [78]. Further connections were made to dynamic game theory by Başar and Bernhard [7] and Limebeer et. al. [50]. In both these areas, solutions are developed in the time domain, using a *separation principle* to allow treatment of control and filtering problems separately. James [38] further develops this principle to give exact conditions for when the separation principle holds.

Dynamic games

The dynamic game formulation of $\mathscr{H}_\infty$ control is centred around the following simple observation. For the closed loop system with the controller $\mathcal{K}$ in place, the induced norm $\|\mathcal{T}_{zw}\| \leq \gamma$ if and only if

$$\|z\|_2^2 - \gamma^2\|w\|_2^2 \leq 0 \text{ for all } w \in \mathscr{L}_2$$

which is if and only if

$$\sup_{w \in \mathscr{L}_2} \left\{ \|z\|_2^2 - \gamma^2 \|w\|_2^2 \right\} \leq 0$$

We might therefore regard this as a suitable cost function for a controller synthesis procedure to try and minimize. If the resulting minimum is negative or zero, then the controller will achieve the induced norm bound given by γ. Further, if the resulting minimum is greater than zero, we know that there exists no controller achieving the desired induced norm bound. We therefore consider the problem

$$\inf_{\mathcal{K}} \sup_{w \in \mathscr{L}_2} \left\{ \|z\|_2^2 - \gamma^2 \|w\|_2^2 \right\}$$

which is a *dynamic game* problem. We can regard the controller $\mathcal{K}$ as trying to minimize the worst case cost function which the "malicious" disturbance w can produce.

The game theoretic formulation also allows us to consider *finite horizon* problems, that is problems in which we are concerned only with controlling the system for a finite time. In this case, the signal norms, and the induced norm, are defined on a finite interval $[t_0, t_1]$. In this case, a suitable cost function is of the form

$$\inf_{\mathcal{K}} \sup_{w \in \mathscr{L}_2} \int_{t_0}^{t_1} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt + x(t_1)'Q_1 x(t_1)$$

with Q_1 a fixed positive definite matrix weight on the terminal state. Clearly, with $x_0 = 0$, for any fixed $Q_1 \geq 0$, if there exists a minimax controller $\mathcal{K}$ then this will result in an induced norm less than or equal to γ on the finite horizon. That is,

$$\sup_{w \in \mathscr{L}_2[t_0, t_1]} \frac{\|\pi_{[t_0,t_1]} z\|}{\|w\|} \leq \gamma.$$

One particular application of finite horizon synthesis techniques has been in the area of moving horizon control. In this methodology, at each time t the controller is designed for the finite intervals $[t, t+T]$. The controller is then used on the possibly infinitesimal interval $[t, t+\delta]$ before being recomputed. This technique was motivated by Kleinman [44] and Kwon and Pearson [46] in the early 1970s for use with linear quadratic optimization problems. It was shown that the resulting controllers were stable when applied on an infinite horizon. This allowed the synthesis for linear time varying systems without full future knowledge of the plant parameters.

In this thesis, we extend the above moving horizon formulation to the game theoretic induced norm formulation. We show that moving horizon control with a minimax criterion not only gives stability, but also allows synthesis of γ-suboptimal controllers on an infinite horizon without explicit knowledge of the system parameters for more than a finite time ahead. We will consider both state feedback and output feedback synthesis techniques, and both continuous and discrete update of the control law.

Sampled-data systems

Physical systems generally evolve in continuous time. However, a controller will usually be implemented using digital hardware, and hence must consist of a discrete control law.

Such systems are described as hybrid, or sampled-data systems. A common approach to the synthesis problem has been to either start with a continuous model of the system, construct a continuous time controller using well-known techniques, and then construct a discrete controller which approximates the continuous controller. Alternatively, the designer might begin with a discrete model for the continuous physical system. Clearly, such a design process cannot take account of the behaviour of the system between the sampling instants.

Another reason for the construction of discrete controllers for continuous systems is that some systems can only provide information at discrete time instants. For example, in order to produce a measurement of the state of a reaction in a chemical process, it might be necessary to perform tests on a small sample of the reactants.

For these kind of systems, we would like to be able to explicitly construct discrete controllers for continuous time systems. We would like a synthesis procedure which, using a continuous time performance and robustness measure, gives us the achievable performance level using a discrete controller with a given sampling rate.

Hara and Kabamba [31], and Bamieh and Pearson [3], have given $\mathscr{H}_\infty$ synthesis procedures for sampled-data systems using the technique of *lifting*. In the case when the sample and hold operators are of fixed period and synchronized with each other, it is possible to transform the sampled-data system to a discrete system, whose inputs and outputs are at each time step in infinite dimensional spaces. It is then possible to construct a discrete system such that any γ-suboptimal discrete controller for the discrete system will also be γ-suboptimal for the original sampled-data system.

The technique of lifting fails, however, when the sample or hold operators are non periodic, or when they are not synchronized, or their rates not rationally related. The technique of lifting is extremely specialized to the particular periodic structure of the problem, since by a minor perturbation of the sampling and hold rates the problem becomes truly time varying and this technique no longer applies.

The direct time domain formulation of the dynamic game allows consideration of the multi-rate sampled-data problem, with non-periodic sample and hold devices. Further, we allow multiple sample and hold devices on different input and output channels, and construct finite horizon suboptimal $\mathscr{H}_\infty$ controllers for such systems.

1.1 Organisation of the thesis

Chapter 2: Preliminaries

Chapter 2 describes the basic signal and operator spaces used throughout the text, and assembles the state space mathematical descriptions we will use for linear time varying systems. We state the definitions of exponential stability, stabilizability and detectability that we will use throughout this thesis.

Chapter 3: State space $\mathscr{H}_\infty$ theory

In this chapter we describe the basic definitions of zero-sum static and dynamic games, and the idea of a saddle point. We show that the $\mathscr{H}_\infty$ problem is equivalent to a dynamic game problem over certain strategy spaces. In the output feedback case, we consider the separation theories of Whittle [78] and Başar and Bernhard [7], and also the use of information state for separation as developed by James, Baras and Elliott [40]. We give a new simple proof of the separation principle for the continuous time output feedback problem, and an explicit characterization of the discrete systems for which separation can be applied.

Chapter 4: Multi-rate sampled-data $\mathscr{H}_\infty$ control

We first consider the state feedback problem for a very general class of systems with possible jumps in the state vector. We then specialize to sampled-data systems, state the formulation of multi-rate sampled-data control we will use, and consider its solution by means of simple examples in the state feedback case. For finite horizon problems, we give necessary and sufficient conditions for existence of multi-rate sampled-data controllers given particular sample and hold operators, and if there exists such a controller we give an explicit construction technique. We show that the minimax controller requires knowledge of all future hold operations, but does not require knowledge of all future sampling operations.

Chapter 5: Riccati differential inequalities

In this chapter we present a solution of the continuous time infinite horizon linear time varying $\mathscr{H}_\infty$ synthesis problem, using the separation theory of Doyle, Glover, Khargonekar and Francis [22]. We extend the results of Khargonekar, Ravi and Nagpal [43] to replace the Riccati differential equations with Riccati differential inequalities, and remove a particular stability assumption on the solutions to these inequalities. We then obtain similar results using game theoretic techniques, by perturbing the past and future cost functions.

Chapter 6: Moving horizon $\mathscr{H}_\infty$ control

We now use the results of the previous chapter to construct moving horizon $\mathscr{H}_\infty$ controllers. We construct continuously updated moving horizon controllers in both the state and output feedback case, and show that for particular choices of initial and terminal state weights the resulting controllers are both stabilizing and γ-suboptimal. We show that this corresponds to a monotonicity condition in the linear time invariant case, and that the solutions are also solutions to the LMI formulation of $\mathscr{H}_\infty$ control.

Chapter 7: Discrete Interval Moving Horizon Control

This chapter shows how we can extend the continuously updated moving horizon controllers to the discretely updated case. It shows that we can construct $\mathscr{H}_\infty$ controllers piecemeal over time, by diving the time interval up into possibly overlapping segments and implementing different controllers in each segment. The boundary conditions obtained for this problem are related to the monotonicity conditions of Chapter 6.

Chapter 8: Conclusions

This chapter summarizes the contributions of this thesis, and outlines directions for future research into $\mathscr{H}_\infty$ control in the time domain.

2. PRELIMINARIES

2.1 Signals and Systems

We begin with this section which describes the basic continuous and discrete time systems we will encounter throughout this thesis. The following material is standard and can be found, for example, in [14].

Let $\mathbb{R}^n$ be n-dimensional Euclidean space, with norm $|\cdot|$ defined by $|x|^2 = \sum_{i=1}^n |x_i|^2$, where $x \in \mathbb{R}^n$ is given by $x = (x_1, \dots, x_n)$. The continuous signal spaces we will consider will be the Hilbert spaces $\mathscr{L}_2^n$ of Lebesgue measurable functions mapping $[0, \infty)$ to $\mathbb{R}^n$, defined by

$$\mathscr{L}_2^n := \left\{ u : [0, \infty) \to \mathbb{R}^n \mid \|u\|_2^2 := \int_0^\infty |u(t)|^2 \, dt < \infty \right\}.$$

We also define the $\mathscr{L}_2$ norm on a finite interval by

$$\mathscr{L}_2^n[t_0, t_1] := \left\{ u : [t_0, t_1] \to \mathbb{R}^n \mid \|u\|_2^2 := \int_{t_0}^{t_1} |u(t)|^2 \, dt < \infty \right\}$$

Often we will consider systems on a *finite horizon.* By this, we mean that the inputs and outputs will be in $\mathscr{L}_2[t_0, t_1]$, for some interval $[t_0, t_1]$. The extended space $\mathscr{L}_{2e}$ is defined to be the space of functions on $[0, \infty)$ such that $u \in \mathscr{L}_{2e}$ if for every $T > 0$, the restriction of u to the finite interval $[0, T]$ is a member of $\mathscr{L}_2[0, T]$. We also have the discrete time Hilbert space

$$\ell_2^n := \left\{ u : \mathbb{Z}_+ \to \mathbb{R}^n \mid \|u\|_2^2 := \sum_{i=0}^\infty |u(i)|^2 < \infty \right\}$$

and its finite horizon counterpart

$$\ell_2^n[j, k] := \{ u : [j, k] \to \mathbb{R}^n \} \quad \|u\|_2^2 := \sum_{i=j}^k |u(i)|^2.$$

Before discussing systems with inputs and outputs, we first consider the time varying linear differential equation

$$\dot{x}(t) = A(t)x(t) \qquad x(t_0) = x_0 \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ for $t \in \mathbb{R}_+$, and $A \in \mathscr{L}_\infty^{n \times n}$ is a bounded matrix function of time. Since A is bounded, the right hand side of this ordinary differential equation is a globally Lipschitz function of $x(t)$, and hence a solution always exists for all $t \geq t_0$. Proofs of these results can be found in, for example [61, 16]. Then define ϕ, the *transition matrix* of A by

$$\frac{\partial}{\partial t} \phi(t, s) = A(t)\phi(t, s) \qquad \phi(s, s) = I$$

The solution of (2.1) is then given by

$$x(t) = \phi(t, t_0)x(t_0)$$

Useful properties of ϕ. For all $s, t \in \mathbb{R}_+$,

a) $\phi(s,s) = I$

b) $\phi(s,t) = \phi(s,t_1)\phi(t_1,t)$

c) $\phi(s,t)^{-1} = \phi(t,s)$

d) $\frac{\partial}{\partial s}\phi(t,s) = -\phi(t,s)A(s)$

e) If A is time invariant, then $\phi(t,s) = e^{(t-s)A}$

f) $\det \phi(t,s) = e^{\int_s^t \text{trace } A(\tau)\, d\tau}$

g) if $\psi(t,s) = \phi(s,t)'$, then $\frac{\partial}{\partial t}\psi(t,s) = -A(t)'\psi(t,s)$

We will describe a finite dimensional linear time varying system $\mathcal{G}$ by a state space representation

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \qquad x(t_0) = x_0 \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \tag{2.2}$$

on either a finite horizon with $t \in [t_0, t_1]$ or an infinite horizon with $t \in \mathbb{R}_+$. Here $x(t) \in \mathbb{R}^n$ is the *state* of the system, and we assume $u \in \mathscr{L}_2^m$ and $y \in \mathscr{L}_2^p$. The system matrices satisfy $A \in \mathscr{L}_\infty^{n\times n}$, $B \in \mathscr{L}_\infty^{n\times m}$, $C \in \mathscr{L}_\infty^{p\times n}$, and $D \in \mathscr{L}_\infty^{p\times m}$. We will describe this system as *strictly proper* if $D \equiv 0$. In the infinite horizon case, this mapping may only be defined for particular inputs u and initial states x_0, since it is possible for an $\mathscr{L}_2$ input signal to give rise to an output signal y which does not have a finite 2-norm. We may also write this system as

$$\mathcal{G} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

For any initial state $x_0 \in \mathbb{R}^n$, the output of the system can be expressed in terms of the *variation of constants* formula

$$y(t) = C(t)\phi(t,t_0)x_0 + C(t)\int_{t_0}^{t} \phi(t,s)B(s)u(s)\, ds.$$

This formula can be verified easily by direct differentiation.

2.2 Linear Operators

Let $\mathcal{G}$ be a linear map from $\mathscr{X}$ to $\mathscr{Y}$, where $\mathscr{X}$, $\mathscr{Y}$ are normed spaces. Then the *induced norm* of $\mathcal{G}$ is defined by

$$||\mathcal{G}|| := \sup_{x\neq 0} \frac{||\mathcal{G}x||}{||x||}$$

If $||\mathcal{G}||$ is finite then we describe $\mathcal{G}$ as a *bounded* operator, and $||\mathcal{G}||$ is known as the operator norm.

For a matrix A, the induced 2-norm is given by

$$\|A\| := \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \bar{\sigma}(A)$$

where $\bar{\sigma}(A)$ is the maximum singular value of A. $\bar{\sigma}(A)$ is also the square root of the largest eigenvalue of AA^*. For a matrix function $X : \mathbb{R}_+ \to \mathbb{R}^{p \times q}$, $X \in \mathscr{L}_\infty^{p \times q}$, we will use the notation

$$\|X\|_\infty := \sup_{t \geq 0} \|X(t)\|$$

to indicate the maximum norm of X over all time. For a symmetric positive definite matrix X, $\|X\| < \beta$ if and only if $X < \beta I$.

2.3 Stability of LTV systems

Consider the system $\mathcal{G}$ defined by equations (2.2). Assume $x_0 = 0$, so that y is a linear function of u. We have the following definition.

Definition 2.1. *The system $\mathcal{G}$ is said to be **exponentially stable** if there exist $c_1, c_2 > 0$ such that $\|\phi(t_1, t_2)\| \leq c_1 e^{-c_2(t_1 - t_2)}$, where $\phi(t_1, t_2)$ is the transition matrix for $A(t)$.* □

Note that this definition depends on the homogeneous part of the system only, and we will use the phrase "A is stable" also to mean that $\mathcal{G}$ is stable. The following lemma is well known. Related results are given in [74, 1, 65].

Lemma 2.2. *If $\mathcal{G}$ is exponentially stable, and $x_0 = 0$, then the input output map $\mathcal{G} : \mathscr{L}_2 \to \mathscr{L}_2$ is a bounded linear operator, and its induced norm is defined as*

$$\|\mathcal{G}\| = \sup_{\|u\| \neq 0} \frac{\|y\|_2}{\|u\|_2}.$$

In order to state a converse result, we need the notions of stabilizability and detectability. The following definitions can be found in [58, 43].

Definition 2.3. *The system $\mathcal{G}$, or the pair (A, B), is said to be **stabilizable** if there exists a bounded matrix function F such that the system*

$$\dot{x}(t) = \big(A(t) - B(t)F(t)\big)x(t)$$

is exponentially stable. □

Definition 2.4. *The system $\mathcal{G}$, or the pair (A, C), is said to be **detectable** if there exists a bounded matrix function L such that the system*

$$\dot{x}(t) = \big(A(t) - L(t)C(t)\big)x(t)$$

is exponentially stable. □

We can now state the following result.

Lemma 2.5. *If $\mathcal{G}$ is stabilizable and detectable, then it is exponentially stable if and only if $\mathcal{G}$ is a bounded operator on $\mathscr{L}_2$.*

3. STATE SPACE $\mathscr{H}_\infty$ THEORY

3.1 $\mathscr{H}_\infty$ and differential games

Our goal in this chapter is to present some of the results that have been developed using time domain techniques to construct solutions to the $\mathscr{H}_\infty$, or induced 2-norm, synthesis problem. We will concentrate on the game theoretic formulations, and will use both the nonlinear concept of information state, first used in $\mathscr{H}_\infty$ theory by James and Baras [40], and the idea of worst case disturbances, as developed by Başar and Bernhard [7] and Whittle [78].

Throughout this thesis we will consider what has become known as the "standard" $\mathscr{H}_\infty$ synthesis problem. That is, we are given a causal linear system $\mathcal{G}$ mapping inputs w, u to outputs z, y. The inputs are assumed in $\mathscr{L}_2$ or ℓ_2, depending on whether the problem is continuous or discrete time. We regard z as the controlled output, w as the disturbance input, u as the controlled input and y as the measured output. Then we are looking for a causal controller $\mathcal{K}$ to connect between y and u so that the the induced norm of the closed loop map from w to z is minimized. This is also known as the disturbance rejection problem. If we partition the system $\mathcal{G}$ as

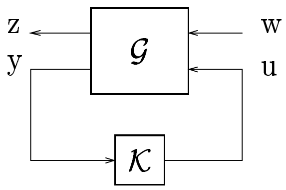


Figure 3.1: The $\mathscr{H}_\infty$ Standard Problem

$$\begin{pmatrix} z \\ y \end{pmatrix} = \begin{bmatrix} \mathcal{G}_{11} & \mathcal{G}_{12} \\ \mathcal{G}_{21} & \mathcal{G}_{22} \end{bmatrix} \begin{pmatrix} w \\ u \end{pmatrix}$$

and denote the closed loop map by $\mathcal{T}_{zw}$, then $\mathcal{T}_{zw} = \mathcal{G}_{11} + \mathcal{G}_{12}\mathcal{K}(I - \mathcal{G}_{22}\mathcal{K})^{-1}\mathcal{G}_{21}$. We would then like to choose $\mathcal{K}$ such that the induced norm

$$\|\mathcal{T}_{zw}\| := \sup_{w \in \mathscr{L}_2, w \neq 0} \frac{\|\mathcal{T}_{zw} w\|_2}{\|w\|_2}$$

is minimized. We also have the further requirement that the closed loop is well-posed and internally stable, that is, for every input signal, the signals y and u are well-defined.

Current approaches to this problem to the stated level of generality are via the *suboptimal* synthesis problem. Given a real number $\gamma > 0$, we look for a causal controller $\mathcal{K}$ such that $\|\mathcal{T}_{zw}\| < \gamma$, and such that the closed loop is well-posed and internally stable. We will also refer to such a controller as *γ-feasible*. Obviously, given such a synthesis procedure it is then possible to iterate on γ to achieve a controller that performs arbitrarily close to optimally. We will use game theory to construct such γ-feasible controllers.

In this chapter we will first describe the background game theory, for both static and dynamic games. We will show how the $\mathscr{H}_\infty$ problem is related to the theory of differential games, and make use of the Hamilton-Jacobi-Bellman equation to construct state feedback γ-feasible controllers. For measurement feedback problems, we give an overview of the current results on certainty equivalence, and provide a new direct derivation for the continuous time linear-quadratic case. We then solve the finite horizon time varying problem in the state and output feedback cases.

3.2 Zero-sum games

We will now describe a broad class of problems called zero-sum games. Most of the following material is standard and can be found in [8], for example. Let $\mathscr{U}$ and $\mathscr{W}$ be vector spaces, and $\mathscr{U}_1 \subseteq \mathscr{U}$, $\mathscr{W}_1 \subseteq \mathscr{W}$ be subsets. Let $J : \mathscr{U}_1 \times \mathscr{W}_1 \to \mathbb{R}$. Define

$$\begin{aligned} \bar{J} &:= \inf_{u\in\mathscr{U}_1} \sup_{w\in\mathscr{W}_1} J(u,w) \\ \underline{J} &:= \sup_{w\in\mathscr{W}_1} \inf_{u\in\mathscr{U}_1} J(u,w) \end{aligned}$$

We call $\bar{J}$ the *upper value* of the game, and $\underline{J}$ the *lower value* of the game. The function J is the cost function, or kernel, of the game and u and w are regarded as players. The objective of player u is to minimize J and the objective of player w is to maximize J, with neither player having access to the strategy of the other. Equivalently we might describe this as the players making their decisions simultaneously.

Remark 3.1. In the definition of $\bar{J}$, we allow the function $\sup_{w\in\mathscr{W}_1} J(u,w)$ to take infinite values for some u, and regard $\bar{J}$ as well defined provided that this function is bounded below with respect to u, and finite for some u. A similar statement holds for $\underline{J}$. □

A natural question to ask is, are these two values equal? That is, can we interchange the sup and inf operations in the above expressions? The answer in general, is no. The following is easily seen:

Lemma 3.2. *Suppose that $\underline{J}$ and $\bar{J}$ are well defined. Then*

$$\bar{J} = \inf_{u\in\mathscr{U}_1} \sup_{w\in\mathscr{W}_1} J(u,w) \geq \sup_{w\in\mathscr{W}_1} \inf_{u\in\mathscr{U}_1} J(u,w) = \underline{J}$$

Proof.

$$\begin{aligned} \inf_{u\in\mathscr{U}_1} J(u,w) &\leq J(u,w) && \forall u \in \mathscr{U}_1, w \in \mathscr{W}_1 \\ \sup_{w\in\mathscr{W}_1} \inf_{u\in\mathscr{U}_1} J(u,w) &\leq \sup_{w\in\mathscr{W}_1} J(u,w) && \forall u \in \mathscr{U}_1 \\ \sup_{w\in\mathscr{W}_1} \inf_{u\in\mathscr{U}_1} J(u,w) &\leq \inf_{u\in\mathscr{U}_1} \sup_{w\in\mathscr{W}_1} J(u,w) \end{aligned}$$

■

For any given strategy of the u player, the worst the w player can do is use a strategy which achieves

$$\sup_{w \in \mathscr{W}_1} J(u, w)$$

assuming that the maximum is achieved. So the u player would like to play a strategy which achieves the minimum in

$$\inf_{u \in \mathscr{U}_1} \sup_{w \in \mathscr{W}_1} J(u, w).$$

This is a *worst case* strategy for the u player, since no other strategy can guarantee a better outcome for player u against all possible moves of player w. The worst case cost for the u player is therefore $\overline{J}$. A similar philosophy holds for player w, and clearly the worst case outcome is given by $\underline{J}$ in this case.

In order to analyse the situation where both players choose strategies according to this scheme, we make the following definition.

Definition 3.3 (Saddle Point). *The game defined by the kernel J is said to have a saddle point if there exists $\hat{u} \in \mathscr{U}_1$, $\hat{w} \in \mathscr{W}_1$ such that*

$$J(\hat{u}, w) \leq J(\hat{u}, \hat{w}) \leq J(u, \hat{w})$$

for all $u \in \mathscr{U}_1$ and $w \in \mathscr{W}_1$. □

Lemma 3.4. *Suppose that $\underline{J}$ and $\overline{J}$ are well defined, and that the game has a saddle point at $\hat{u}$, $\hat{w}$. Then*

$$\inf_{u \in \mathscr{U}_1} \sup_{w \in \mathscr{W}_1} J(u, w) = J(\hat{u}, \hat{w}) = \sup_{w \in \mathscr{W}_1} \inf_{u \in \mathscr{U}_1} J(u, w)$$

Proof. Since the game has a saddle point, there exists $\hat{u} \in \mathscr{U}_1$, $\hat{w} \in \mathscr{W}_1$ such that

$$J(\hat{u}, w) \leq J(\hat{u}, \hat{w}) \leq J(u, \hat{w}).$$

for all $u \in \mathscr{U}_1$ and $w \in \mathscr{W}_1$. Therefore

$$\sup_{w \in \mathscr{W}_1} J(\hat{u}, w) = J(\hat{u}, \hat{w}) = \inf_{u \in \mathscr{U}_1} J(u, \hat{w}).$$

Also, clearly

$$\inf_{u \in \mathscr{U}_1} \sup_{w \in \mathscr{W}_1} J(u, w) \leq \sup_{w \in \mathscr{W}_1} J(\hat{u}, w)$$

and

$$\inf_{u \in \mathscr{U}_1} J(u, \hat{w}) \leq \sup_{w \in \mathscr{W}_1} \inf_{u \in \mathscr{U}_1} J(u, w).$$

Therefore

$$\inf_{u \in \mathscr{U}_1} \sup_{w \in \mathscr{W}_1} J(u, w) \leq J(\hat{u}, \hat{w}) \leq \sup_{w \in \mathscr{W}_1} \inf_{u \in \mathscr{U}_1} J(u, w).$$

From Lemma 3.2, the result follows. ■

Lemma 3.5. *Suppose that $\underline{J}$ and $\overline{J}$ are finite, and that $u_0 \in \mathscr{U}_1$ achieves the infimum in*

$$\inf_{u \in \mathscr{U}_1} \sup_{w \in \mathscr{W}_1} J(u,w)$$

and $w_0 \in \mathscr{W}_1$ achieves the supremum in

$$\sup_{w \in \mathscr{W}_1} \inf_{u \in \mathscr{U}_1} J(u,w).$$

Then if $\underline{J} = \overline{J}$, the strategy pair (u_0, w_0) is a saddle point.

Proof. Clearly

$$J(u_0, w_0) \leq \sup_{w \in \mathscr{W}_1} J(u_0, w) = \inf_{u \in \mathscr{U}_1} \sup_{w \in \mathscr{W}_1} J(u,w)$$

and

$$J(u_0, w_0) \geq \inf_{u \in \mathscr{U}_1} J(u, w_0) = \sup_{w \in \mathscr{W}_1} \inf_{u \in \mathscr{U}_1} J(u,w).$$

Hence, since $\underline{J} = \overline{J}$,

$$\inf_{u \in \mathscr{U}_1} \sup_{w \in \mathscr{W}_1} J(u,w) = J(u_0, w_0) = \sup_{w \in \mathscr{W}_1} \inf_{u \in \mathscr{U}_1} J(u,w).$$

Then

$$\begin{aligned} J(u_0, w_0) &= \sup_{w \in \mathscr{W}_1} \inf_{u \in \mathscr{U}_1} J(u,w) \\ &= \inf_{u \in \mathscr{U}_1} J(u, w_0) \\ &\leq J(u, w_0) \qquad \forall\ u \in \mathscr{U}_1 \end{aligned}$$

and a similar inequality holds for w, giving the required result. ■

If $\underline{J} = \overline{J}$ then we call this the *value* of the game and denote it by J^*. Note that there are games which have a value but not a saddle point, that is, although $\underline{J} = \overline{J}$ there is no u_0, w_0 for which $J(u_0, w_0) = J^*$, since the required infima and suprema are not achieved. There are also games which have neither value nor saddle point. For example, the zero sum game with cost function $J(u,w) = (u-w)^2$ defined for $u, w \in [0,2] \subset \mathbb{R}$ has $\overline{J} = 1$ and $\underline{J} = 0$, and hence does not have a value. An example of such a game in the dynamic case is given by Stoorvogel [63, p. 123]. The following result will be useful in the sequel.

Theorem 3.6. *Let*

$$\begin{aligned} u^*(w) &= \left\{ u_0 \in \mathscr{U}_1 \;\middle|\; \inf_{u \in \mathscr{U}_1} J(u,w) = J(u_0, w) \right\} \\ w^*(w) &= \left\{ w_0 \in \mathscr{W}_1 \;\middle|\; \sup_{w \in \mathscr{W}_1} J(u,w) = J(u, w_0) \right\} \end{aligned}$$

Then, $u_0 \in u^*(w_0)$ and $w_0 \in w^*(u_0)$, if and only if u_0, w_0 is a a saddle point solution.

Proof. (only if:) Suppose $u_0 \in u^*(w_0)$ and $w_0 \in w^*(u_0)$. Then

$$\inf_{u\in\mathscr{U}_1} J(u,w) = J(u_0,w)$$
$$\sup_{w\in\mathscr{W}_1} J(u,w) = J(u,w_0)$$

so $J(u_0,w) \le J(u_0,w_0) \le J(u,w_0)$ for all $u \in \mathscr{U}_1$, $w \in \mathscr{W}_1$.
(if:) Suppose u_0, w_0, are a saddle point solution for J. Then $J(u_0,w_0) \le J(u,w_0)$ for all $u \in \mathscr{U}_1$, and so $u_0 \in u^*(w_0)$. Similarly for w, giving the required result. ∎

Note that if, for each w, $u^*(w)$ is a single point, and for each u, $w^*(u)$ is a single point, then this result says that (u,w) is a saddle point if and only it is a fixed point of the map $(u,w) \mapsto (u^*(w), w^*(u))$.

3.3 Discrete time dynamic games with full state information

3.3.1 Nonlinear systems

Although in this thesis we will be concerned solely with linear systems and quadratic cost functions, many of the basic results we will use from the theory of dynamic games apply equally well to nonlinear systems. Because of this, we will first state the results in the more general nonlinear framework, and later apply them to linear systems.

In this section, we will consider finite dimensional nonlinear time varying systems on a finite horizon described by

$$x(k+1) = f(k, x(k), u(k), w(k)) \qquad x(0) = 0 \tag{3.1}$$
$$z(k) = g(k, x(k), u(k)) \tag{3.2}$$

Here the state $x(k) \in \mathbb{R}^n$, the disturbance $w \in \ell_2^{m_1}[0,K]$, and the control input $u \in \ell_2^{m_2}[0,K]$, with $0 \le k \le K$. Let

$$\mathscr{U}_{sf}^{n,p}[k,K] := \{\mu : [k,K] \times \mathbb{R}^n \to \mathbb{R}^p\}$$

be the space of all state feedback strategies on the discrete interval $[k,K]$. In the sequel we will usually omit the spatial dimensions. We will consider a problem where both players have access to the state of the system, that is

$$u(t) = \mu(t, x(t)), \qquad \mu \in \mathscr{U}_{sf}^{n,m_2}[0,K]$$
$$w(t) = \nu(t, x(t)), \qquad \nu \in \mathscr{U}_{sf}^{n,m_1}[0,K].$$

The cost function we will consider is

$$J(x(0),\mu,\nu) = \sum_{i=0}^{K} \Big\{ |z(i)|^2 - \gamma^2 |w(i)|^2 \Big\}.$$

This has obvious connections to the induced 2-norm of the system, since if the cost function $J(x(0),\mu,\nu) \le 0$ then clearly $\|z\|_2^2 \le \gamma^2 \|w\|_2^2$, although as yet we have not shown any direct relation between the γ-suboptimal synthesis problem and the dynamic game problem. The dynamic game problem is, given $x(0)$, to find a saddle point for the problem

$$\inf_{\mu\in\mathscr{U}_{sf}} \sup_{\nu\in\mathscr{U}_{sf}} J(x(0),\mu,\nu),$$

that is, to find $\mu_0 \in \mathscr{U}_{sf}$ and $\nu_0 \in \mathscr{U}_{sf}$ such that

$$J(\mu_0,\nu) \le J(\mu_0,\nu_0) \le J(\mu,\nu_0) \quad \forall \mu,\nu \in \mathscr{U}_{sf}. \tag{3.3}$$

for fixed $x(0)$. The following remark is crucial to the equivalence between this problem and the induced 2-norm synthesis problem.

Remark 3.7. If either μ_0 or ν_0 is a fixed feedback law in $\mathscr{U}_{sf}$, then choosing the other input in $\mathscr{U}_{sf}$ results in well-defined signals for the state and both players for any given initial state. The signals will be in ℓ_2 if the map from the inputs to the state is a bounded linear map. So without loss of generality we can replace strategies μ and ν in equation (3.3) with signals u and w in ℓ_2 for the linear case. □

This is expressed by the following proposition, where we abuse notation slightly to write J as a function of both strategies and signals.

Proposition 3.8. *The state feedback strategies $\mu_0,\nu_0 \in \mathscr{U}_{sf}$ satisfy*

$$J(\mu_0,\nu) \le J(\mu_0,\nu_0) \le J(\mu,\nu_0) \quad \forall \mu,\nu \in \mathscr{U}_{sf}$$

if and only if they satisfy

$$J(\mu_0,w) \le J(\mu_0,\nu_0) \le J(u,\nu_0) \quad \forall u,w \in \ell_2$$

Proof. We prove only one side of the inequality, and the other side follows similarly. That is, we show that, given $x_0 \in \mathbb{R}^n$, $\mu_0 \in \mathscr{U}_{sf}$ and $c \in \mathbb{R}$, then $J(x_0,\mu_0,\nu) \le c$ for all $\nu \in \mathscr{U}_{sf}$ if and only if $J(x_0,\mu,w) \le c$ for all $w \in \mathscr{L}_2$. But this is now obvious; by connecting an arbitary time varying state feedback around any given system we can span the entire input space to the system. ■

Note that it is assumed that μ_0 and ν_0 are members of $\mathscr{U}_{sf}$. Then we can write the lower and upper values of the game as

$$\begin{aligned}\bar{J} &= \inf_{\mu\in\mathscr{U}_{sf}} \sup_{w\in\ell_2} J(x(0),\mu,w)\\ \underline{J} &= \sup_{\nu\in\mathscr{U}_{sf}} \inf_{u\in\ell_2} J(x(0),u,\nu)\end{aligned}$$

so the upper value of the original differential game, where both players had full state information, is equal to the upper value of the game in which only the u player has full state information.

Since with $w=0$, the function $J(x(0),\mu,w) \ge 0$ for all μ, if for some $\mu \in \mathscr{U}_{sf}$ the function $J(x(0),\mu,w)$ is bounded with respect to w, then the upper value of the game is well defined. In which case, with the minimax controller μ_0 in place

$$\begin{aligned}\bar{J} &= \sup_{w\in\ell_2} J(x(0),\mu_0,w)\\ &= \sup_{w\in\ell_2}\Big\{\|z\|_2^2 - \gamma^2\|w\|_2^2\Big\}.\end{aligned}$$

In order to consider the induced norm of the closed loop system, defined by

$$\|\mathcal{T}_{zw}\| := \sup_{\substack{w \in \ell_2 \\ w \neq 0}} \frac{\|z\|_2^2}{\|w\|_2^2},$$

we must restrict the initial state of the system to be zero, that is $x(0) = 0$. It is easy to see that $\|\mathcal{T}_{zw}\| < \gamma$ if and only if there exists $\varepsilon > 0$ such that

$$\frac{\|z\|_2^2}{\|w\|_2^2} \leq \gamma^2 - \varepsilon \text{ for all } w \in \ell_2.$$

This holds if and only if

$$J(0, \mu, w) \leq -\varepsilon \|w\|_2^2 \text{ for all } w \in \ell_2.$$

Similarly, the nonstrict bound $\|\mathcal{T}_{zw}\| \leq \gamma$ is achieved if and only if

$$\|z\|_2^2 - \gamma^2 \|w\|_2^2 \leq 0 \text{ for all } w \in \ell_2.$$

This is equivalent to the condition that the feedback strategy μ for the u-player be such that $J(0, \mu, w) \leq 0$ for all $w \in \ell_2$. Clearly there will exist such a state feedback strategy μ if and only if

$$\inf_{\mu \in \mathscr{U}_{sf}} \sup_{w \in \ell_2} J(0, \mu, w) \leq 0$$

Hence the induced norm of the closed loop system, $\|\mathcal{T}_{zw}\|$, is less than or equal to γ if and only if the upper value of the game is less than or equal to zero. This same reasoning does not extend to the strictly suboptimal case, however. Obviously, for an observable system with $x_0 \neq 0$, the induced norm will not be finite, and the upper value of the game will therefore be strictly positive.

Remark 3.9. Note that the *upper value* can satisfy these criteria, and hence the strategy μ_0 be a solution to the suboptimal $\mathscr{H}_\infty$ problem, without there existing a saddle point for the differential game. That is, the lower value might be different from the upper value. □

Remark 3.10. The minimization of the induced norm is the motivation behind the $\mathscr{H}_\infty$ synthesis problem. However, we can still formulate the differential game problem when the initial state is non-zero. This in fact corresponds to synthesis problems which take account of the transient behaviour of the system also, and we will in the sequel study the more general case when the initial state is non-zero. □

Remark 3.11. In the infinite horizon case, or in the linear case with measurement feedback where G_{22} is not strictly proper, it is possible that the resulting differential equations describing the dynamics of the closed-loop do not have a well defined solution, given $\mu_0 \in \mathscr{U}_{sf}$, for all possible choices of $w \in \ell_2$. In this case we need to impose the further restriction that the controller μ is *internally stabilising*, and results in a well-posed closed loop. □

Written as it is above, the game is simply a static game, where the possible choices for u are defined by a class of functions. We might hope to be able to regard this as simply a static game with u and w both in ℓ_2. However, the existence of

$$\inf_{u\in\ell_2} \sup_{w\in\ell_2} J(x(0), u, w)$$

is not necessary for the existence of

$$\inf_{\mu\in\mathscr{U}_{sf}} \sup_{w\in\ell_2} J(x(0), \mu, w).$$

Hence we cannot necessarily achieve the same value of γ for these problems. Also, if $u \in \mathscr{L}_2$ is fixed, then the problem loses the induced norm interpretation, since $\|z\|$ will be non zero even for zero w, in which case

$$\frac{\|\mathcal{T}_{zw}w\|_2}{\|w\|_2}$$

is unbounded with respect to w.

Rather than viewing the above problem as a static game over function spaces, it is advantageous to view it as a sequence of static games over finite dimensional spaces $\mathbb{R}^{m_2}$ and $\mathbb{R}^{m_1}$, where $u(i) \in \mathbb{R}^{m_2}$ and $w(i) \in \mathbb{R}^{m_1}$. In this case, each successive game depends on the outcome of the previous one, and the functional dependence of u on the state is regarded as the information available to u at each time step about the previous actions of the players. Note that the future dynamics and hence future cost of the game can be determined entirely from the current state and future actions of the players. We can then specify other classes of problem by describing the information available to the players at any given time step, rather than describing the set of functions in which u or w must belong.

Using the standard arguments of dynamic programming, it is easy to see that

$$\begin{aligned}
&\inf_{\mu\in\mathscr{U}_{sf}[k,K]} \sup_{\nu\in\mathscr{U}_{sf}[k,K]} \sum_{i=k}^{K} \Big\{ |z(i)|^2 - \gamma^2 |w(i)|^2 \Big\} \\
&\qquad = \inf_{u(k)\in\mathbb{R}^{m_2}} \sup_{w(k)\in\mathbb{R}^{m_1}} \Bigg\{ z(k)'z(k) - \gamma^2 w(k)'w(k) \\
&\qquad\qquad + \inf_{\mu\in\mathscr{U}_{sf}[k+1,K]} \sup_{\nu\in\mathscr{U}_{sf}[k+1,K]} \sum_{i=k+1}^{K} \Big\{ |z(i)|^2 - \gamma^2 |w(i)|^2 \Big\} \Bigg\}
\end{aligned} \tag{3.4}$$

Since the last term on the right hand side of equation (3.4) is dependent solely on $x(k+1)$, we may define the *value function* as

$$V\big(k, x(k)\big) := \inf_{\mu\in\mathscr{U}_{sf}[k,K]} \sup_{\nu\in\mathscr{U}_{sf}[k,K]} \sum_{i=k}^{K} \Big\{ |z(i)|^2 - \gamma^2 |w(i)|^2 \Big\}$$

and write down the *Isaacs equation:*

$$V\big(k, x(k)\big) = \inf_{u(k)} \sup_{w(k)} \Big\{ z(k)'z(k) - \gamma^2 w(k)'w(k) + V\big(k+1, x(k+1)\big) \Big\} \tag{3.5}$$

with boundary condition $V(K+1, x(K+1)) = 0$. Here we have an expression of the form

$$V\big(k, x(k)\big) = \inf_{u(k)} \sup_{w(k)} F(x(k), u(k), w(k)) \tag{3.6}$$

hence the extremising $u(k)$ and $w(k)$ are functions of $x(k)$. The Isaacs equation is a special form of Hamilton-Jacobi-Bellman equation, derived using the principle of dynamic programming. It is a recursion, and may be calculated *backwards* in time. This is what we would expect, since with full state feedback the current decision depends only on the current state and the future cost. Hence, at time K, we know that the future cost is zero, and so we can calculate exactly the optimal current inputs as a function of the current state. Equation (3.5) was derived in the early 1950's by Rufus Isaacs [35] in continuous time. The following theorem can be found in [8].

Theorem 3.12. *Suppose there exists a function $V\big(k, x(k)\big)$ satisfying (3.5) for $0 \le k \le K$, and at each k there exists a unique saddle point solution to the static game described by equation (3.6). Then the state feedback strategies derived from this solution are a state feedback saddle point solution for the above dynamic game, and the value of the game is given by $V\big(0, x(0)\big)$.*

If the conditions of the theorem hold, then we can write the saddle point controller for the minimizing player as

$$\mu_0(k, x(k)) := \arg \inf_{u(k)} \sup_{w(k)} \Big\{ z(k)'z(k) - \gamma^2 w(k)'w(k) + V\big(k+1, x(k+1)\big) \Big\} \tag{3.7}$$

3.4 Continuous time dynamic games with full state information

Many of the arguments of the previous section regarding the correspondence of strategies and signals apply equally to this section, in particular Remark 3.7. We first describe the general results from dynamic programming in the nonlinear case, and then apply them to the linear problem.

3.4.1 Nonlinear continuous time dynamic games

We consider finite dimensional nonlinear systems as follows:

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t), w(t)) \qquad x(0) = x_0 \\ z(t) &= g(t, x(t), u(t)) \end{aligned}$$

The state $x(t) \in \mathbb{R}^n$, the disturbance $w \in \mathscr{L}_2^{m_1}[0, t_f]$, and the control input $u \in \mathscr{L}_2^{m_2}[0, t_f]$. Let

$$\mathscr{U}_{sf}^{n,k}[t, t_f] := \Big\{ v : [t, t_f] \times \mathbb{R}^n \to \mathbb{R}^k;\ \forall\ x \in \mathscr{L}_2^n[t, t_f],\ v(\cdot, x(\cdot)) \in \mathscr{L}_2^k[t, t_f] \Big\}$$

be the space of all continuous time state feedback strategies on $[t, t_f]$, where $\mu, \nu \in \mathscr{U}_{sf}$. The cost function is

$$J(x(0), \mu, \nu) = \int_0^{t_f} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt + x(t_f)'Q_f x(t_f)$$

Here Q_f is a symmetric positive semidefinite weighting matrix on the final state. We would like to find a saddle point for the problem

$$\inf_{\mu \in \mathscr{U}_{sf}} \sup_{\nu \in \mathscr{U}_{sf}} J(x(0), \mu, \nu).$$

Let $t_1 \in (0, t_f)$, then

$$\begin{aligned} &\inf_{\mu \in \mathscr{U}_{sf}[0,t_f]} \sup_{\nu \in \mathscr{U}_{sf}[0,t_f]} \left\{ \int_0^{t_f} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt + x(t_f)'Q_f x(t_f) \right\} \\ &\qquad = \inf_{\mu \in \mathscr{U}_{sf}[0,t_1]} \sup_{\nu \in \mathscr{U}_{sf}[0,t_1]} \left\{ \int_0^{t_1} z(s)'z(s) - \gamma^2 w(s)'w(s)\, ds \right. \\ &\qquad + \left. \inf_{\mu \in \mathscr{U}_{sf}[t_1,t_f]} \sup_{\nu \in \mathscr{U}_{sf}[t_1,t_f]} \int_{t_1}^{t_f} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt + x(t_f)'Q_f x(t_f) \right\} \end{aligned} \tag{3.8}$$

As in the discrete time case, we can now define the value function as

$$V(t, x(t)) = \inf_{\mu \in \mathscr{U}_{sf}[t,t_f]} \sup_{\nu \in \mathscr{U}_{sf}[t,t_f]} \int_t^{t_f} z(s)'z(s) - \gamma^2 w(s)'w(s)\, ds + x(t_f)'Q_f x(t_f). \tag{3.9}$$

Whereas in discrete time the value function satisfied a recursion, in continuous time it satisfies a partial differential equation. The continuous time counterpart of equation (3.5) is

$$-\frac{\partial}{\partial t} V(t, x(t)) = \inf_{u(t)} \sup_{w(t)} \left\{ z(t)'z(t) - \gamma^2 w(t)'w(t) + \frac{\partial}{\partial x} V(t, x(t)) \dot{x}(t) \right\} \tag{3.10}$$

with boundary condition $V(t_f, x(t_f)) = x(t_f)'Q_f x(t_f)$. The following theorem can be found in [7, 8].

Theorem 3.13. *Suppose there exists a continuously differentiable function $V(t, x(t))$ satisfying (3.10) for $t \in [0, t_f]$, and at each t there exists a saddle point solution to the corresponding static game in equation (3.10). Then the state feedback strategies derived from this solution are a full information saddle point solution for the above dynamic game, and the value of the game is given by $V(0, x(0))$.*

3.4.2 The linear-quadratic game

This game is referred to as "linear-quadratic" since it has linear dynamics and a quadratic cost function. We consider finite dimensional linear time varying systems as follows:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_1(t)w(t) + B_2(t)u(t) \\ z(t) &= C_1(t)x(t) + D_{12}(t)u(t) \end{aligned}$$

We make the simplifying assumptions that $C_1(t)'D_{12}(t) = 0$ and $D_{12}(t)'D_{12}(t) = I$ for all t. The state $x(t) \in \mathbb{R}^n$, the disturbance $w \in \mathscr{L}_2^{m_1}[0, t_f]$, and the control input

$u \in \mathscr{L}_2^{m_2}[0,t_f]$. We will try to solve the Isaacs equation with a value function of the form

$$V(t,x(t)) = x(t)'X_\infty(t)x(t). \tag{3.11}$$

Then equation (3.10) becomes

$$\begin{aligned} -x(t)'\dot{X}_\infty(t)x(t) = \inf_{u(t)}\sup_{w(t)}\Big\{ & z(t)'z(t) - \gamma^2 w(t)'w(t) \\ & + x(t)'X_\infty(t)\big[A(t)x(t) + B_1(t)w(t) + B_2(t)u(t)\big] \\ & + \big[A(t)x(t) + B_1(t)w(t) + B_2(t)u(t)\big]'X_\infty(t)x(t)\Big\}, \end{aligned}$$

which, by completing the square, becomes

$$\begin{aligned} -x(t)\dot{X}_\infty(t)x(t) = \inf_{u(t)}\sup_{w(t)}\Big\{ & x(t)'\Big\{X_\infty(t)A + A'X_\infty(t) + C_1'C_1 - \\ & X_\infty(t)(B_2B_2' - \gamma^{-2}B_1B_1')X_\infty(t)\Big\}x(t) \\ & + \big|u(t) + B_2'X_\infty(t)x(t)\big|^2 - \gamma^2\big|w(t) - \gamma^{-2}B_1'X_\infty(t)x(t)\big|^2\Big\} \end{aligned}$$

Since this equation is true for all initial conditions of the system, we arrive at the well known Riccati differential equation

$$-\dot{X}_\infty(t) = X_\infty(t)A + A'X_\infty(t) + C_1'C_1 - X_\infty(t)(B_2B_2' - \gamma^{-2}B_1B_1')X_\infty(t) \tag{3.12}$$

with boundary condition

$$X_\infty(t_f) = Q_f$$

and the min-max solutions for u and w are

$$\mu_0(t,x(t)) = -B_2(t)'X_\infty(t)x(t) \tag{3.13}$$

$$\nu_0(t,x(t)) = \gamma^{-2}B_1(t)'X_\infty(t)x(t) \tag{3.14}$$

Using Theorem 3.13, we see that this gives us a sufficient condition for existence of a saddle point to the continuous time dynamic game. If there exists a solution to the Riccati differential equation (3.12) on $[0,t_f]$, then there exists a saddle point to the dynamic game given by equations (3.13) and (3.14).

3.4.3 Completion of squares

We know that the above solution, since it is a saddle point, provides us with a solution to the $\mathscr{H}_\infty$ problem. We can also derive it directly as follows.

Theorem 3.14. *Suppose that equation* (3.12) *has a solution on* $[0,t_f]$. *Then*

$$J(x(0),\mu,\nu) = x(0)'X_\infty(0)x(0) + \|u - \mu_0\|_2^2 - \gamma^2\|w - \nu_0\|_2^2$$

for any controller μ and any input w, and the differential game has a unique saddle point solution at (μ_0,ν_0). Here u and w are the signals generated by μ and ν respectively, and we abbreviate $\mu_0(\cdot,x(\cdot))$ by μ_0. If $x(0)=0$, then with the controller $u(t)=\mu_0(t,x(t))$ in place, the closed loop norm satisfies $\|\mathcal{T}_{zw}\| < \gamma$.

Proof. Clearly

$$\begin{aligned}\int_0^{t_f} \frac{d}{dt} x' X_\infty x \, dt &= x(t_f)' X_\infty(t_f) x(t_f) - x(0)' X_\infty(0) x(0) \\ &= x(t_f)' Q_f x(t_f) - x(0)' X_\infty(0) x(0)\end{aligned}$$

Hence

$$\begin{aligned}J(x(0), \mu, \nu) &= \int_0^{t_f} z(t)' z(t) - \gamma^2 w(t)' w(t) \, dt + x(t_f)' Q_f x(t_f) \\ &= x(0)' X_\infty(0) x(0) + \int_0^{t_f} \left\{ z' z - \gamma^2 w' w + \frac{d}{dt} x' X_\infty x \right\} dt \\ &= x(0)' X_\infty(0) x(0) + \int_0^{t_f} \left\{ z' z - \gamma^2 w' w + \dot{x}' X_\infty x + x' X_\infty \dot{x} + x' \dot{X}_\infty x \right\} dt \\ &= x(0)' X_\infty(0) x(0) + \int_0^{t_f} \left\{ z' z - \gamma^2 w' w + (Ax + B_1 w + B_2 u)' X_\infty x \right. \\ &\qquad\qquad \left. + x' X_\infty (Ax + B_1 w + B_2 u) + x' \dot{X}_\infty x \right\} dt \\ &= x(0)' X_\infty(0) x(0) + \int_0^{t_f} \left\{ x' (C_1' C_1 + A' X_\infty + X_\infty A + \dot{X}_\infty) x + u' u \right. \\ &\qquad\qquad \left. - \gamma^2 w' w + (B_1 w + B_2 u)' X_\infty x + x' X_\infty (B_1 w + B_2 u) \right\} dt \\ &= x(0)' X_\infty(0) x(0) + \int_0^{t_f} \left\{ x' X_\infty (B_2 B_2' - \gamma^{-2} B_1 B_1') X_\infty x + u' u \right. \\ &\qquad\qquad \left. - \gamma^2 w' w + (B_1 w + B_2 u)' X_\infty x + x' X_\infty (B_1 w + B_2 u) \right\} dt \\ &= x(0)' X_\infty(0) x(0) + \int_0^{t_f} \left\{ (u + B_2' X_\infty x)' (u + B_2' X_\infty x) \right. \\ &\qquad\qquad \left. - \gamma^2 (w - \gamma^{-2} B_1' X_\infty x)' (w - \gamma^{-2} B_1' X_\infty x) \right\} dt \\ &= x(0)' X_\infty(0) x(0) + \|u - u^*\|_2^2 - \gamma^2 \|w - w^*\|_2^2\end{aligned}$$

So with $x(0) = 0$ and $u(t) = \mu_0(t, x(t))$, since

$$J(x(0), \mu, \nu) = \|z\|_2^2 - \gamma^2 \|w\|_2^2 + x(t_f)' Q_f x(t_f)$$

then

$$\|z\|_2^2 - \gamma^2 \|w\|_2^2 = -\gamma^2 \|w - \nu_0\|_2^2 - x(t_f)' Q_f x(t_f).$$

Since $Q_f \geq 0$, clearly $\|z\|_2^2 - \gamma^2 \|w\|_2^2 \leq 0$, hence $\|\mathcal{T}_{zw}\| \leq \gamma$. To show that this inequality is in fact strict, we need to show that there exists $\varepsilon > 0$ such that

$$\|z\|_2^2 - \gamma^2 \|w\|_2^2 \leq -\varepsilon \|w\|_2^2$$

for all $w \in \ell_2$.

With the controller in place, the closed loop dynamics satisfy

$$\begin{aligned}\dot{x} &= (A - B_2 B_2' X_\infty) x + B_1 w \\ w - \nu_0 &= -\gamma^{-2} B_1' X_\infty x + w.\end{aligned}$$

This gives an invertible finite dimensional state space relationship between w and $w-\nu_0$, so the induced norm both of this operator and its inverse is finite. Therefore there exists $\delta > 0$ such that

$$\|w\|_2^2 \leq \delta \|w - \nu_0\|_2^2 \qquad \forall w \in \mathscr{L}_2$$

and so the result follows. Note that this appears to contradict the possibility of w choosing $w(t) = \nu_0(t, x(t))$ as a strategy, which would give

$$\|z\|_2^2 - \gamma^2 \|w\|_2^2 = -x(t_f)' Q_f x(t_f)$$

which, if either $Q_f = 0$ or $x(t_f) = 0$ appears to imply that w can actually attain the induced norm bound. However, since $\nu_0(t, x(t)) = 0$ and $\mu_0(t, x(t)) = 0$ for $x(t) = 0$, with these strategies u, w and x are identically zero, hence z is identically zero also. ■

3.5 Continuous time games with imperfect measurement

3.5.1 Nonlinear systems

In this section we will describe the results that have been proved for differential games with partial information for nonlinear systems. The certainty equivalence principles that have been derived by various authors [78, 38, 7] do not depend on the linear character of the problem, and can be easily described in the nonlinear framework. We will then, in the next section, go on to specialize these results to the linear case. We will avoid describing the somewhat technical smoothness assumptions necessary to state the results precisely for general nonlinear systems, since in the rest of this thesis we will be concerned solely with linear systems.

Consider the system described by the following nonlinear differential equations on $[t_0, t_f]$

$$\begin{aligned} \dot{x}(t) &= f\big(t, x(t), u(t), w(t)\big) \qquad x(t_0) = x_0 \\ y(t) &= h\big(t, x(t), w(t)\big) \end{aligned} \tag{3.15}$$

Again the problem is a dynamic game, where the minimizing player is now required to be a causal function of the measured output y. That is, let

$$\mathscr{U}_{of}^{p_2,m_2}[t_0, t_f] := \big\{\mu : \mathscr{L}_2^{p_2}[t_0, t_f] \to \mathscr{L}_2^{m_2}[t_0, t_f] \;\big|\; \pi_{[t_0,t]} \mu \pi_{[t_0,t]} = \pi_{[t_0,t]} \mu \text{ for all } t \in [t_0, t_f]\big\}$$

where $\pi_{[t_0,t]}$ is the usual projection of $\mathscr{L}_2[t_0, t_f]$ onto $\mathscr{L}_2[t_0, t]$. The cost function will be

$$J(x_0, \mu, w) = \alpha(x_0) + \int_{t_0}^{t_f} g\big(t, x(t), u(t), w(t)\big)\, dt + \beta\big(x(t_f)\big)$$

and we would like to solve the differential game problem

$$\inf_{\mu \in \mathscr{U}_{of}} \sup_{w \in \mathscr{L}_2, x_0 \in \mathbb{R}^n} J(x_0, \mu, w)$$

Here we are considering the initial state x_0 of the system as part of the unknown disturbance, that is as part of the maximizer's strategy. This has a direct interpretation in the linear case, as we shall describe in the next section. Define the value of the game by

$$V(\tau,\xi) = \inf_{\mu\in\mathscr{U}_{sf}} \sup_{w\in\mathscr{L}_2} \int_\tau^{t_f} g\big(t,x(t),u(t),w(t)\big)\,dt + \beta\big(x(t_f)\big)$$

subject to $x(\tau)=\xi$ and the dynamics described by equations (3.15). Then, subject to certain technical assumptions, V will satisfy

$$-\frac{\partial}{\partial\tau}V(\tau,\xi) = \inf_{u(\tau)} \sup_{w(\tau)} \left\{ g\big(\tau,\xi,u(\tau),w(\tau)\big) + \frac{\partial}{\partial\xi}V(\tau,\xi) f\big(\tau,\xi,u(\tau),w(\tau)\big) \right\} \tag{3.16}$$

with boundary condition

$$V(t_f,x) = \beta(x) \qquad \forall x\in\mathbb{R}^n$$

Whittle [78] describes the quantity $V(\tau,\xi)$ as the (partially extremized) *future stress.* This is exactly the Isaacs equation for the state feedback differential game problem. Let $u(t)=\mu_0(t,x(t))$ be the optimal state feedback controller.

At time $\tau\in[t_0,t_f]$, the control law has been implemented as a signal on $[t_0,\tau]$. Similarly, we have measurements of the output y on this interval. We will denote these signals by u^τ and y^τ respectively. Similarly let w^τ be a possible past disturbance input.

Define the set of disturbances (x_0,w^τ) consistent with past information u^τ and y^τ by

$$\Omega_0(\tau) := \Big\{ (x_0,w^\tau)\ ; x_0\in\mathbb{R}^n,\ w^\tau\in\mathscr{L}_2[t_0,\tau],\ \text{such that } x(t_0)=x_0,$$
$$\left.\begin{aligned} \dot x(t) &= f\big(t,x(t),u^\tau(t),w^\tau(t)\big) \\ y^\tau(t) &= h\big(t,x(t),w^\tau(t)\big) \end{aligned} \qquad \text{for } t\in[t_0,\tau] \right\}.$$

and define those disturbances which are further consistent with the hypothesis that $x(\tau)=\xi$ by

$$\Omega(\tau,\xi) := \Big\{ (x_0,w^\tau)\ ; x_0\in\mathbb{R}^n,\ w^\tau\in\mathscr{L}_2[t_0,\tau],\ \text{such that } x(\tau)=\xi, x(t_0)=x_0,$$
$$\left.\begin{aligned} \dot x(t) &= f\big(t,x(t),u^\tau(t),w^\tau(t)\big) \\ y^\tau(t) &= h\big(t,x(t),w^\tau(t)\big) \end{aligned} \qquad \text{for } t\in[t_0,\tau] \right\}.$$

Now define the *information state* by

$$P(\tau,\xi) = \sup_{(x_0,w^\tau)\in\Omega(\tau,\xi)} \left\{ \alpha(x_0) + \int_{t_0}^{\tau} g\big(t,x(t),u^\tau(t),w^\tau(t)\big)\,dt \right\}$$

for all $t\in[t_0,\tau]$. The quantity $P(\tau,\xi)$ is the worst possible cost that could have been incurred so far if the current state were equal to ξ, consistent with the information given by knowledge of u^τ and y^τ. It is shown in [38] that the function P satisfies the Hamilton-Jacobi equation

$$\frac{\partial}{\partial\tau}P(\tau,\xi) = \sup_{\substack{w^\tau(\tau)\\ y^\tau(\tau)=h(\tau,\xi,w^\tau(\tau))}} \left\{ g\big(\tau,\xi,u(\tau),w(\tau)\big) - \frac{\partial}{\partial\xi}V(\tau,\xi) f\big(\tau,\xi,u(\tau),w(\tau)\big) \right\} \tag{3.17}$$

with the boundary condition

$$P(t_0, x) = \alpha(x) \qquad \forall x \in \mathbb{R}^n.$$

James and Baras [38] describe $P(\tau, \cdot)$ as an infinite dimensional state, and the above partial differential equation can then be viewed as the state evolution equation for this new state. For each τ, the information state is a functional on $\mathbb{R}^n$. Note that, for linear systems, the information state is a nonhomogeneous quadratic function of the ξ, see [37, 78]. Whittle [78] calls this function the *past stress*.

The certainty equivalence principle. The following theorem, the discrete version of which was first formulated by Whittle [78], and which was proved in this form for the continuous time problem by James and Baras [38] gives us the desired certainty equivalence principle.

Theorem 3.15. *Suppose V is a smooth solution of the Isaacs equation* (3.16) *with the infimising saddle point solution for u given by μ_0, and P is a smooth solution of the Hamilton-Jacobi equation* (3.17)*. Further, suppose that at each $\tau \in [t_0, t_f]$ there exists a unique maximizing $\bar{x}(\tau)$ such that*

$$\bar{x}(\tau) = \arg\max_{x \in \mathbb{R}^n} \{P(\tau, x) + V(\tau, x)\}$$

then the certainty equivalence controller

$$u(\tau) = \mu_0(\bar{x}(\tau)),$$

is minimizing for the game

$$\inf_{\mu \in \mathscr{U}_{of}} \sup_{\substack{w \in \mathscr{L}_2 \\ x_0 \in \mathbb{R}^n}} J(x_0, \mu, w).$$

where μ_0 is the state feedback saddle point controller.

This theorem is similar to one first introduced by Whittle [78] for the discrete time linear-quadratic problem. It is proved for the continuous time nonlinear problem in [38, 40] using arguments from dynamic programming, and there are several stringent technical assumptions regarding smoothness that must be made. It says that, if we assume that the u player will play optimally using state feedback in the future, to give a future cost of $V\big(\tau, x(\tau)\big)$, and consider the worst possible current state consistent with past optimization as a state estimate, then using this state estimate in the state feedback law is optimal for the problem. In this sense this is a *certainty equivalence* principle; we are using a worst case state as if it were the actual state.

It is also shown in [38] that the problem can be expressed in the following way. Let

$$G(\tau, x_0, u^\tau, w^\tau) = \alpha(x_0) + \int_{t_0}^{\tau} g\big(t, x(t), u^\tau(t), w^\tau(t)\big)\, dt + V(\tau, x(\tau)).$$

Also let

$$W(\tau, u^\tau, y^\tau) = \sup_{(x_0, w^\tau) \in \Omega_0(\tau)} G(\tau, x_0, u^\tau, w^\tau)$$

Consider the function

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^n}\big\{P(\tau,x)+V(\tau,x)\big\} = \\
&= \sup_{x\in\mathbb{R}^n}\left\{\sup_{(x_0,w^\tau)\in\Omega(\tau,x)}\left\{\alpha(x_0)+\int_{t_0}^{\tau} g\big(t,x(t),u^\tau(t),w^\tau(t)\big)\,dt\right\}+V\big(\tau,x(\tau)\big)\right\} \\
&= \sup_{x\in\mathbb{R}^n}\sup_{(x_0,w^\tau)\in\Omega(\tau,x)}\left\{\alpha(x_0)+\int_{t_0}^{\tau} g\big(t,x(t),u^\tau(t),w^\tau(t)\big)\,dt+V\big(\tau,x(\tau)\big)\right\} \\
&= \sup_{(x_0,w^\tau)\in\Omega_0(\tau)} G(\tau,x_0,u^\tau,w^\tau)
\end{aligned}$$

The following theorem is stated in [7].

Theorem 3.16 (Başar and Bernhard [7]). *Suppose V is a smooth solution of the Isaacs equation (3.16), and that μ_0 is the infimising state feedback control law for the u player. If, for every y^τ and every $\tau\in[t_0,t_f]$, the problem*

$$\sup_{(x_0,w^\tau)\in\Omega_0(\tau)}\left\{\alpha(x_0)+\int_{t_0}^{\tau} g\big(t,x(t),u^\tau(t),w^\tau(t)\big)\,dt+V(\tau,x(\tau))\right\}$$

has a unique maximum $(\hat{x}_0^\tau,\hat{w}^\tau)$ generating a state trajectory $\hat{x}^\tau$, then the control law $u(\tau)=\mu_0\big(\tau,\hat{x}^\tau(\tau)\big)$ is optimal for the differential game with output feedback information, and its value is given by

$$\inf_{\mu\in\mathscr{U}_{of}}\sup_{\substack{x_0\in\mathbb{R}^n\\ w\in\mathscr{L}_2}} J(x_0,\mu,w)=\max_{x_0\in\mathbb{R}^n}\big\{V(t_0,x_0)+\alpha(x_0)\big\}.$$

This theorem can be interpreted as follows. At any time τ, suppose that the minimizing player u will play optimal state feedback in the future, and calculate the worst case disturbances w^τ, x_0 which maximize the total cost. This worst case disturbance defines a unique worst case state trajectory, which should be used in the state feedback law as if it were the actual state.

In [7] the theorem is proved using techniques from dynamic programming. It is stated that the result holds not only for the linear quadratic case but also for nonlinear and nonquadratic problems.

Note that in discrete time, the corresponding results to Theorem 3.15 and Theorem 3.16 are not equivalent, and the certainty equivalence controller may not be optimal. Bernhardsson [12, p. 66] gives a counterexample to the discrete time nonlinear-nonquadratic case certainty equivalence principle. Also James [40] gives conditions for when the certainty equivalence principle is optimal in discrete time.

The certainty equivalence principle above can be proved in the following way:

Theorem 3.17. *Suppose that $\hat{\mu}\in\mathscr{U}_{of}$ is such that, for all $t_1,t_2\in[t_0,t_f]$ with $t_2\geq t_1$, and for all y, $W(t_2,u^{t_2},y^{t_2})\leq W(t_1,u^{t_1},y^{t_1})$. That is, W is a decreasing function of τ when the signal u is defined by the feedback law $\hat{\mu}$. Then*

$$\inf_{\mu\in\mathscr{U}_{of}}\sup_{\substack{x_0\in\mathbb{R}^n\\ w\in\mathscr{L}_2}} J(x_0,\mu,w)=\max_{x_0\in\mathbb{R}^n}\big\{V(t_0,x_0)+\alpha(x_0)\big\}$$

and the infimum is achieved by $\mu=\hat{\mu}$.

Proof. Clearly

$$J(x_0, \hat{\mu}, w) \leq W(t_f, u^{t_f}, y^{t_f}).$$

Since W is decreasing,

$$\begin{aligned} J(x_0, \hat{\mu}, w) &\leq W(t_0, u^{t_0}, y^{t_0}) \\ &= \max_{x_0 \in \mathbb{R}^n} \big\{ V(t_0, x_0) + \alpha(x_0) \big\} \qquad \forall w \in \mathscr{L}_2. \end{aligned}$$

Therefore, with this feedback strategy for the u player,

$$\sup_{x_0 \in \mathbb{R}^n, w \in \mathscr{L}_2} J(x_0, \mu, w) \leq \max_{x_0 \in \mathbb{R}^n} \big\{ V(t_0, x_0) + \alpha(x_0) \big\}$$

We therefore need to show that the for any other feedback strategy, the w player can make the cost function larger. That is, we wish to show that, for any strategy μ, there exists a disturbance w, x_0 such that

$$J(x_0, \mu, w) \geq \max_{x_0 \in \mathbb{R}^n} \big\{ V(t_0, x_0) + \alpha(x_0) \big\}.$$

Let

$$J_0(x_0, \mu, w) = \int_{t_0}^{t_f} g\big(t, x(t), u(t), w(t)\big)\, dt + \beta\big(x(t_f)\big).$$

Then for any $\mu \in \mathscr{U}_{of}$,

$$J_0(x_0, \mu, \nu_0) \geq V(t_0, x_0). \tag{3.18}$$

Here $\nu_0 \in \mathscr{U}_{sf}$ is the state feedback saddle point strategy for the maximizing player.

To be strictly accurate, we know that equation (3.18) holds for all possible signals generated by the minimizing player, rather than all possible feedback strategies. However, as in Remark 3.7, with ν_0 in place μ defines a unique $u \in \mathscr{L}_2$, and $J_0(x_0, \mu, \nu_0) = J_0(x_0, u, \nu_0)$. Since ν_0 is a saddle point strategy the inequality holds. Then, with $\mu = \hat{\mu}$, the feedback law ν_0 generates a unique signal $\hat{w} \in \mathscr{L}_2$ for which the inequality is satisfied also. Hence

$$\max_{x_0 \in \mathbb{R}^n} \big\{ J_0(x_0, \mu, \hat{w}) + \alpha(x_0) \big\} \geq \max_{x_0 \in \mathbb{R}^n} \big\{ V(t_0, x_0) + \alpha(x_0) \big\}$$

and therefore, for each $\mu \in \mathscr{U}_{of}$, there exists a $\hat{w}$ such that

$$J(x_0, \mu, \hat{w}) \geq \big\{ V(t_0, x_0) + \alpha(x_0) \big\}$$

and with $\mu = \hat{\mu}$

$$J(x_0, \hat{\mu}, w) \leq \max_{x_0 \in \mathbb{R}^n} \big\{ V(t_0, x_0) + \alpha(x_0) \big\} \qquad \forall w \in \mathscr{L}_2.$$

Hence result. ∎

Başar and Bernhard [7] claim that, for general linear systems, W is a decreasing function of τ for all y given uniqueness of the maximum of G for each τ. In the sequel we will give an explicit formula for $\frac{\partial W}{\partial \tau}$, and hence prove directly that the optimal state feedback with the certainty equivalence controller is the infimising solution to the upper value problem.

3.5.2 The linear case

We will consider the system described by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_1(t)w(t) + B_2(t)u(t) \qquad x(t_0) = x_0 \\ z(t) &= C_1(t)x(t) + D_{11}(t)w(t) + D_{12}(t)u(t) \\ y(t) &= C_2(t)x(t) + D_{21}(t)w(t) + D_{22}(t)u(t) \end{aligned}$$

on a finite time interval $[t_0, t_f]$. Under the assumption that $D_{12}(t)$ and $D_{21}(t)$ have full column rank and full row rank respectively for all $t \in [t_0, t_f]$, it is possible to show that we may transform the problem to an equivalent one in which

$$D_{21}(t)D_{21}(t)' = I \quad D_{12}(t)'D_{12}(t) = I \quad D_{11}(t) = 0 \quad D_{22}(t) = 0. \tag{3.19}$$

That is, if we can construct a controller for the problem with the above assumptions, then we may construct a controller for the original problem also. Further, if the original system matrices are continuous functions of time, then the transformed matrices will also be continuous. For details of these loopshifting transformations, see for example [50, 29]. Also, following [22] we will make the additional standing assumptions that

$$D_{21}(t)B_1(t)' = 0 \qquad\qquad D_{12}(t)'C_1(t) = 0 \tag{3.20}$$

for all $t \in [t_0, t_f]$. These assumptions are not without some loss of generality, but their inclusion reduces the complexity of the resulting formulae, and removing them would introduce no significant technical difficulties.

Again the problem is a dynamic game, where the minimizing player is now required to be a causal function of the measured output y, that is we will require that $u \in \mathscr{U}_{of}$. By similar arguments to those used in the state feedback case, we can consider w to be a signal in $\mathscr{L}_2$ without loss of generality.

It is also possible to consider x_0 to be part of the unknown disturbance, that is, part of the maximizing player's strategy. We can then define the cost by

$$J(x_0, \mu, w) = -\gamma^2 x_0' Q_0 x_0 + \int_{t_0}^{t_f} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt + x(t_f)'Q_f x(t_f)$$

where $Q_0 > 0$ and $Q_f > 0$ are symmetric positive definite weighting matrices. We would then like to solve the problem

$$\inf_{\mu \in \mathscr{U}_{of}} \sup_{w \in \mathscr{L}_2, x_0 \in \mathbb{R}^n} J(x_0, \mu, w)$$

If we define the norms

$$\begin{aligned} \|(z, x(t_f))\|^2 &:= \|z\|_2^2 + x(t_f)'Q_f x(t_f) \\ \|(w, x_0)\|^2 &:= \|w\|_2^2 + x_0' Q_0 x_0 \end{aligned}$$

then if we choose $\mu \in \mathscr{U}_{of}$ such that

$$J(x_0, \mu, w) \le 0 \qquad \forall\, w \in \mathscr{L}_2, x_0 \in \mathbb{R}^n$$

then

$$\|(z, x(t_f))\|^2 \le \gamma^2 \|(w, x_0)\|^2 \qquad \forall\, w \in \mathscr{L}_2, x_0 \in \mathbb{R}^n \tag{3.21}$$

Again, we can view this as an induced norm minimization, defining $\mathcal{T} : \mathscr{L}_2 \oplus \mathbb{R}^n \to \mathscr{L}_2 \oplus \mathbb{R}^n$ by $\mathcal{T} : (w, x_0) \mapsto (z, x(t_f))$, giving

$$\|\mathcal{T}\| \le \gamma \qquad \Longleftrightarrow \qquad J(x_0, \mu, w) \le 0 \quad \forall\, w \in \mathscr{L}_2, x_0 \in \mathbb{R}^n$$

This gives a formulation of $\mathscr{H}_\infty$ control taking transients into account, as described in [42]. Note that if $\|\mathcal{T}\| \le \gamma$, then $\|\mathcal{T}_{zw}\| \le \gamma$ also, since equation (3.21) holds for all x_0, including $x_0 = 0$. Define

$$G(\tau, x_0, u^\tau, w^\tau) := -\gamma^2 x_0' Q_0 x_0 + \int_{t_0}^{\tau} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt + V(\tau, x(\tau))$$

where V is the value of the game defined by equation (3.9). Then the worst case trajectory is given by solving the following maximization problem

$$W(\tau, u^\tau, y^\tau) := \sup_{(x_0, w) \in \Omega_0(\tau)} G(\tau, x_0, u^\tau, w^\tau). \tag{3.22}$$

For the linear quadratic problem, if X_∞ exists satisfying equation (3.12), then

$$V(t, x(t)) = x(t)' X_\infty(t) x(t)$$

from equation (3.11), where $X_\infty(t)$ is the solution to the control Riccati differential equation (3.12) introduced previously. We now explicitly construct the worst case trajectory and disturbance.

Lemma 3.18. *Suppose there exists a unique maximum to the constrained optimization problem in equation* (3.22)*. Then there exists a unique solution to the following two point boundary value problem, satisfied by the worst case trajectory $\hat{x}^\tau$ and worst case disturbance $(\hat{x}_0^\tau, \hat{w}^\tau)$.*

$$\begin{aligned}
\dot{\hat{x}}^\tau &= A\hat{x}^\tau + \gamma^{-2} B_1 B_1' \lambda^\tau + B_2 u & \hat{x}^\tau(t_0) &= \hat{x}_0^\tau \\
-\dot{\lambda}^\tau &= A'\lambda^\tau + C_1' C_1 \hat{x}^\tau + \gamma^2 C_2'(y^\tau - C_2 \hat{x}^\tau) & \lambda^\tau(\tau) &= X_\infty(\tau)\hat{x}^\tau(\tau) \\
\hat{w}^\tau &= \gamma^{-2} B_1' \lambda^\tau + D_{21}'(y^\tau - C_2 \hat{x}^\tau) & & \\
\hat{x}_0^\tau &= \gamma^{-2} Q_0^{-1} \lambda^\tau(t_0) & &
\end{aligned}$$

Proof. The proof is a standard application of the maximum principle to the constrained problem. Describe the system on $[t_0, \tau]$ by

$$\begin{pmatrix} x(\tau) \\ z^\tau \\ y^\tau \end{pmatrix} = \Theta \begin{pmatrix} x_0 \\ u^\tau \\ w^\tau \end{pmatrix}$$

Then

$$\begin{aligned}
G(\tau, x_0, u^\tau, w^\tau) = {} & \\
& \langle \Theta_{11} x_0 + \Theta_{12} u^\tau + \Theta_{13} w^\tau, X_\infty(\tau)(\Theta_{11} x_0 + \Theta_{12} u^\tau + \Theta_{13} w^\tau) \rangle \\
& \quad + \langle \Theta_{21} x_0 + \Theta_{22} u^\tau + \Theta_{23} w^\tau, \Theta_{21} x_0 + \Theta_{22} u^\tau + \Theta_{23} w^\tau \rangle \\
& \qquad - \gamma^2 \langle w^\tau, w^\tau \rangle - \gamma^2 \langle x_0, Q_0 x_0 \rangle
\end{aligned}$$

Let ρ be a Lagrange multiplier, then

$$\tilde{G} = G + \langle \rho, y^\tau \rangle - \langle \rho, \Theta_{31} x_0 + \Theta_{32} u^\tau + \Theta_{33} w^\tau \rangle$$

Write $\hat{w}^\tau$ for the maximizing w^τ, and $\hat{x}^\tau$ for the corresponding state trajectory for the problem at time τ. Maximization leads to

$$\begin{aligned} -\tfrac{1}{2}\Theta_{33}^* \rho + \Theta_{23}^* (\Theta_{21} x_0 + \Theta_{22} u^\tau + \Theta_{23} \hat{w}^\tau) & \\ + \Theta_{11}^* X_\infty(\tau)(\Theta_{11} \hat{x}_0^\tau + \Theta_{12} u^\tau + \Theta_{13} \hat{w}^\tau) - \gamma^2 \hat{w}^\tau &= 0 \end{aligned}$$

and

$$\begin{aligned} -\tfrac{1}{2}\Theta_{31}^* \rho + \Theta_{21}^* (\Theta_{21} x_0 + \Theta_{22} u^\tau + \Theta_{23} \hat{w}^\tau) & \\ + \Theta_{11}^* X_\infty(\tau)(\Theta_{11} \hat{x}_0^\tau + \Theta_{12} u^\tau + \Theta_{13} \hat{w}^\tau) - \gamma^2 Q_0 \hat{x}_0^\tau &= 0 \end{aligned}$$

We can rewrite these equations as

$$\begin{aligned} \gamma^2 \hat{w}^\tau &= -\tfrac{1}{2}\Theta_{33}^* \rho + \Theta_{23}^* \hat{z}^\tau + \Theta_{11}^* X_\infty(\tau) \hat{x}^\tau(\tau) \\ \gamma^2 Q_0 \hat{x}_0^\tau &= -\tfrac{1}{2}\Theta_{31}^* \rho + \Theta_{21}^* \hat{z}^\tau + \Theta_{11}^* X_\infty(\tau) \hat{x}^\tau(\tau) \end{aligned} \tag{3.23}$$

Let $\phi(t, r)$ be the transition matrix associated with $A(t)$. Then define the operator Φ on $\mathscr{L}_2[t_0, \tau]$ by

$$(\Phi z)(t) = \int_{t_0}^{t} \phi(t, r) z(r)\, dr.$$

Also define the operator $\boldsymbol{\phi} : \mathbb{R}^n \to \mathscr{L}_2[t_0, \tau]$ by

$$(\boldsymbol{\phi}\xi)(t) = \phi(t, t_0)\xi$$

and the linear operator $F : \mathscr{L}_2[t_0, \tau] \to \mathbb{R}^n$ by

$$Fz = z(\tau).$$

Then we can write

$$\Theta = \begin{bmatrix} F\boldsymbol{\phi} & F\Phi B_2 & F\Phi B_1 \\ C_1\boldsymbol{\phi} & C_1\Phi B_2 + D_{12} & C_1 \Phi B_1 \\ C_2\boldsymbol{\phi} & C_2\Phi B_2 & C_2\Phi B_1 + D_{21} \end{bmatrix}$$

and

$$\Theta^* = \begin{bmatrix} \boldsymbol{\phi}^* F^* & \boldsymbol{\phi}^* C_1' & \boldsymbol{\phi}^* C_2' \\ B_2' \Phi^* F^* & B_2' \Phi^* C_1' + D_{12}' & B_2' \Phi^* C_2' \\ B_1' \Phi^* F^* & B_1' \Phi^* C_1' & B_1' \Phi^* C_2' + D_{21}' \end{bmatrix}$$

Then let

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \Theta^* \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

and this operator has a realization given by

$$\begin{aligned} -\dot{\lambda}^\tau(t) &= A(t)'\lambda^\tau(t) + C_1(t)'\alpha_2(t) + C_2(t)'\alpha_3 \qquad \lambda^\tau(\tau) = \alpha_1 \\ \beta_3(t) &= B_1(t)'\lambda^\tau(t) + D_{21}(t)'\alpha_3(t) \\ \beta_2(t) &= B_2(t)'\lambda^\tau(t) + D_{12}(t)'\alpha_2(t) \\ \beta_1 &= \lambda^\tau(t_0) \end{aligned}$$

From equations (3.23), we choose

$$\begin{aligned} \alpha_1 &= X_\infty(\tau)\hat{x}^\tau(\tau) & \beta_1 &= \gamma^2 Q_0 \hat{x}_0^\tau \\ \alpha_2 &= \hat{z}^\tau & \beta_3 &= \gamma^2 \hat{w}^\tau \\ \alpha_3 &= -\tfrac{1}{2}\rho \end{aligned}$$

and hence the optimizing variables satisfy

$$\begin{aligned} -\dot{\lambda}^\tau &= A'\lambda^\tau + C_1'\hat{z}^\tau - \tfrac{1}{2}C_2'\rho & \lambda^\tau(\tau) &= X_\infty(\tau)\hat{x}^\tau(\tau) \\ -\dot{\hat{x}}^\tau &= A\hat{x}^\tau + B_1\hat{w}^\tau + B_2 u^\tau & \lambda^\tau(t_0) &= \gamma^2 Q_0 \hat{x}_0^\tau \\ \gamma^2\hat{w}^\tau &= B_1'\lambda^\tau - \tfrac{1}{2}D_{21}'\rho \\ \hat{z}^\tau &= C_1\hat{x}^\tau + D_{12}u^\tau \end{aligned}$$

The constraint is

$$y^\tau = C_2\hat{x}^\tau + D_{21}\hat{w}^\tau$$

Using our assumption that $D_{21}D_{21}' = I$ and that $D_{21}B_1' = 0$, combining this with the above equations, the result follows. ■

Clearly, as it stands Lemma 3.18 is not such a useful result for computation. We would like to construct a differential equation solution for the worst case state. As a first step towards this, we convert the two point boundary value problem into an initial value problem. Let

$$\begin{aligned} \dot{Y}_\infty &= AY_\infty + Y_\infty A' - Y_\infty(C_2'C_2 - \gamma^{-2}C_1'C_1)Y_\infty + B_1B_1' \\ Y_\infty(t_0) &= Q_0^{-1} \end{aligned} \tag{3.24}$$

and we repeat here the control Riccati equation

$$\begin{aligned} -\dot{X}_\infty &= A'X_\infty + X_\infty A - X_\infty(B_2B_2' - \gamma^{-2}B_1B_1')X_\infty + C_1'C_1 \\ X_\infty(t_f) &= Q_f \end{aligned} \tag{3.25}$$

The following proposition will be very useful in the sequel.

Proposition 3.19. *Let X_∞ satisfy equation (3.25). Then $X_\infty(t) \geq 0$ for all $t \in [t_0, t_f]$. If either $Q_f > 0$ or $C_1'C_1 > 0$, then $X_\infty(t) > 0$ for all $t \in [t_0, t_f]$. Similarly if Y_∞ satisfies (3.24), then $Y_\infty(t) \geq 0$ for all $t \in [t_0, t_f)$, and if either $Q_0 > 0$ or $B_1B_1' > 0$ then $Y_\infty(t) > 0$ for all $t \in [t_0, t_f]$*

Proof. Since $X_\infty(t)$ exists for all $t \in [t_0, t_f]$, there exist a transition matrix ψ for $A - (B_2B_2' - \gamma^{-2}B_1B_1')X_\infty/2$. We can rewrite the Riccati equation as

$$-\dot{X}_\infty = \left(A' - \tfrac{1}{2}X_\infty(B_2B_2' - \gamma^{-2}B_1B_1')\right)X_\infty + X_\infty\left(A - \tfrac{1}{2}(B_2B_2' - \gamma^{-2}B_1B_1')X_\infty\right) + C_1'C_1$$

and it is easy to show

$$X_\infty(t) = \psi(t_f,t)'Q_f\psi(t_f,t) + \int_t^{t_f} \psi(s,t)'C_1'(s)C_1(s)\psi(s,t)\,ds.$$

Since ψ is always invertible, the result for X_∞ follows. The result for Y_∞ follows by a similar argument. ∎

Applying Proposition 3.19 to equation (3.24) implies that $Y_\infty > 0$ for all $t \in [t_0, t_f]$, since $Q_0 > 0$, and similarly that $X_\infty > 0$ for all $t \in [t_0, t_f]$ since $Q_f > 0$. Define

$$\check{x}(t) := \hat{x}^\tau(t) - \gamma^{-2}Y_\infty(t)\lambda^\tau(t).$$

Differentiation gives

$$\dot{\check{x}} = A\check{x} + \gamma^{-2}Y_\infty C_1'C_1\check{x} + Y_\infty C_2'(y^\tau - C_2\check{x}) + B_2u$$

and the initial condition is given by

$$\begin{aligned}\check{x}(t_0) &= \hat{x}^\tau(t_0) - \gamma^{-2}Y_\infty(t_0)\lambda^\tau(t_0)\\ &= \gamma^{-2}Q_0^{-1}\lambda^\tau(t_0) - \gamma^{-2}Y_\infty(t_0)\lambda^\tau(t_0)\\ &= 0 \qquad \text{since we chose } Y_\infty(t_0) = Q_0^{-1}\end{aligned}$$

From these equations it is clear that $\check{x}(t)$ is independent of τ. Now we can replace the equation in λ^τ by the equation in $\check{x}$, giving

$$\begin{aligned}\dot{\hat{x}}^\tau &= A\hat{x}^\tau + B_1B_1'Y_\infty^{-1}\hat{x}^\tau - B_1B_1'Y_\infty^{-1}\check{x} + B_2u\\ \dot{\check{x}} &= A\check{x} + \gamma^{-2}Y_\infty C_1'C_1\check{x} + Y_\infty C_2'(y^\tau - C_2\check{x}) + B_2u \qquad \check{x}(t_0) = 0\end{aligned}$$

The boundary condition for $\hat{x}^\tau$ is now given by

$$\begin{aligned}\hat{x}^\tau(\tau) &= X_\infty^{-1}(\tau)\lambda^\tau(\tau)\\ &= \gamma^2X_\infty^{-1}(\tau)Y_\infty^{-1}(\tau)(\hat{x}^\tau(\tau) - \check{x}(\tau))\end{aligned}$$

hence

$$\hat{x}^\tau(\tau) = \left(I - \gamma^{-2}Y_\infty(\tau)X_\infty(\tau)\right)^{-1}\check{x}(\tau) \tag{3.26}$$

and the worst case disturbance is given by

$$\hat{x}_0^\tau = \hat{x}^\tau(t_0) \tag{3.27}$$

$$\hat{w}^\tau = B_1'Y_\infty^{-1}\hat{x}^\tau - B_1'Y_\infty^{-1}\check{x} + D_{21}'(y^\tau - C_2\hat{x}^\tau). \tag{3.28}$$

We can therefore express the worst case cost function (3.22) by

$$W(\tau, u^\tau, y^\tau) = \int_{t_0}^{\tau}\left\{\hat{x}^{\tau\prime}C_1'C_1\hat{x}^\tau + u'u - \gamma^2\hat{w}^{\tau\prime}\hat{w}^\tau\right\}dt + \hat{x}^\tau(\tau)'X_\infty(\tau)\hat{x}^\tau(\tau) - \gamma^2\hat{x}_0^{\tau\prime}Q_0\hat{x}_0^\tau \tag{3.29}$$

We now wish to consider the problem of when the constrained maximization problem defined by equation (3.22) has a finite supremum. This problem is a constrained maximization of a nonhomogeneous quadratic functional with a constraint restricting w to an affine subspace of $\mathscr{L}_2$. We will write it using the notation of Lemma 3.18 as follows. First note that $\Theta_{11} = \phi(\tau, t_0)$, and is therefore always invertible. Hence, there exists a bijection between the pairs (x_0, w^τ) and $(x(\tau), w^\tau)$. Therefore, $G(\tau, x_0, u^\tau, w^\tau)$ is bounded with respect to (x_0, w^τ) if and only if it is bounded with respect to $(x(\tau), w^\tau)$, when x_0 is replaced by $\Theta_{11}^{-1}(x(\tau) - \Theta_{13}w^\tau - \Theta_{12}u^\tau)$. We will therefore show that this latter problem is bounded. Define

$$\Xi = \begin{bmatrix} I & 0 \\ \Theta_{21}\Theta_{11}^{-1} & \Theta_{23} - \Theta_{21}\Theta_{11}^{-1}\Theta_{13} \\ 0 & I \\ \Theta_{11}^{-1} & -\Theta_{11}^{-1}\Theta_{13} \end{bmatrix}$$

so that

$$\begin{pmatrix} x(\tau) \\ z^\tau \\ w^\tau \\ x_0 \end{pmatrix} = \Xi \begin{pmatrix} x(\tau) \\ w^\tau \end{pmatrix}.$$

Then let

$$\Psi = \Xi^* \begin{bmatrix} X_\infty(\tau) & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & -\gamma^2 Q_0 \end{bmatrix} \Xi$$

and we will write

$$v = \begin{pmatrix} x(\tau) \\ w^\tau \end{pmatrix}$$

where v is in the product space $\mathbb{R}^n \oplus \mathscr{L}_2$. It is then easy to see that

$$G(\tau, x_0, u^\tau, w^\tau) = \langle v, \Psi v \rangle + < q, v > + c$$

where

$$q = \begin{bmatrix} 0 \\ \Theta_{22} - \Theta_{21}\Theta_{11}^{-1}\Theta_{12} \\ 0 \\ \Theta_{11}^{-1}\Theta_{12} \end{bmatrix} u^\tau$$

and c is a constant depending on u^τ only. We can also write the constraint that w^τ and x_0 be consistent with past measurements of y^τ as $\Lambda v = e$, where

$$\Lambda = \begin{bmatrix} \Theta_{31}\Theta_{11}^{-1} & \Theta_{33} - \Theta_{31}\Theta_{11}^{-1}\Theta_{13} \end{bmatrix} \qquad e = y^\tau + \Theta_{31}\Theta_{11}^{-1}\Theta_{12}u^\tau - \Theta_{32}u^\tau$$

The maximization problem (3.22) can then be written

$$W(\tau, u^\tau, y^\tau) = \sup_{v;\, \Lambda v = e} \left\{ \langle v, \Psi v \rangle + < q, v > + c \right\}$$

Lemma 3.20. *Suppose there exists $\varepsilon > 0$ such that $\langle v, \Psi v\rangle \le -\varepsilon\|v\|^2$ for all v such that $\Lambda v = 0$. Then $W(\tau, u^\tau, y^\tau)$ is finite for all u^τ and y^τ.*

Proof. We assume that there exists $v_2 \in \mathbb{R}^n \oplus \mathscr{L}_2$ such that $\Lambda v_2 = e$. Then $\Lambda v = e$ if and only if there exists $v_1 \in \ker \Lambda$ such that $v = v_1 + v_2$. For any $v \in \mathbb{R}^n \oplus \mathscr{L}_2$,

$$\begin{aligned}&\langle v, \Psi v\rangle + \langle q, v\rangle + c\\&\quad = \langle v_1, \Psi v_1\rangle + \langle q, v_1\rangle + \langle v_2, \Psi v_2\rangle + 2\langle v_2, \Psi v_1\rangle + \langle q, v_2\rangle + c\\&\quad = \langle v_1, \Psi v_1\rangle + \langle q + \Psi v_2, v_1\rangle + d\end{aligned}$$

where d depends on e and u^τ only. Hence if

$$\sup_{v_1 \in \ker \Lambda} \left\{\langle v_1, \Psi v_1\rangle + \langle q + \Psi v_2, v_1\rangle\right\}$$

is finite, then so is W. Using the Cauchy-Schwartz inequality, we have

$$\langle v_1, \Psi v_1\rangle + \langle q + \Psi v_2, v_1\rangle \le \langle v_1, \Psi v_1\rangle + \|q + \Psi v_2\|\|v_1\|$$

and, writing $v_1 = \lambda v_0$, where $\|v_0\| = 1$,

$$\begin{aligned}\langle v_1, \Psi v_1\rangle + \langle q + \Psi v_2, v_1\rangle &\le \lambda^2\langle v_0, \Psi v_0\rangle + \lambda\|q + \Psi v_2\|\\&\le \sup_{v_0 \in \ker \Lambda;\, \|v_0\|=1} -\frac{\|q + \Psi v_2\|}{2\langle v_0, \Psi v_0\rangle}\end{aligned}$$

since the previous equation is a quadratic in λ. Therefore, if $\langle v_0, \Psi v_0\rangle \le -\varepsilon\|v_0\|^2$ for all $v_0 \in \ker \Lambda$, then

$$\langle v_1, \Psi v_1\rangle + \langle q + \Psi v_2, v_1\rangle \le \frac{1}{2\varepsilon}\|q + \Psi v_2\|$$

for all $v_1 \in \ker \Lambda$, and hence $\langle v, \Psi v\rangle + \langle q, v\rangle + c$ is bounded for all v satisfying $\Lambda v = e$. ■

We can now make use of this Lemma. If we set $y^\tau = 0$ and $u^\tau = 0$, then $e = 0$ and $q = 0$. Hence, according to the Lemma, if there exists $\varepsilon > 0$ such that

$$G(\tau, x_0, u^\tau, 0) < -\varepsilon(|x(\tau)|^2 + \|w^\tau\|_2^2)$$

with $x_0 = \Theta_{11}^{-1}(x(\tau) - \Theta_{13}w^\tau - \Theta_{12}u^\tau)$, for all x_0 and w^τ which satisfy $\Lambda \begin{pmatrix} x(\tau) \\ w^\tau \end{pmatrix} = 0$, then $W(\tau, u^\tau, y^\tau)$ is finite for all y^τ and u^τ. This leads to the following result.

Lemma 3.21. *Suppose there exists a matrix function Y_∞ satisfying the Riccati differential equation* (3.24) *on $[t_0, \tau]$, and a matrix function X_∞ satisfying the Riccati differential equation* (3.25) *on $[\tau, t_f]$ such that $X_\infty(\tau) - \gamma^2 Y_\infty^{-1}(\tau) < 0$. Then there exists a unique maximum to the constrained maximization problem defined by equation* (3.22).

Proof. Applying Lemma 3.20, we need only consider the case when $u^\tau = 0$ and $y^\tau = 0$. Adding and subtracting $\gamma^2 x' Y_\infty^{-1} x$ to the cost function (3.29), and using (3.28) gives

$$\begin{aligned} G(\tau, x_0, u^\tau, w^\tau) = \int_{t_0}^{\tau} \Big\{ \gamma^2 \frac{d}{dt}(x' Y_\infty^{-1} x) + u^{\tau\prime} u^\tau - \gamma^2 w^{\tau\prime} w^\tau + 2\gamma^2 w^{\tau\prime} B_1' Y_\infty^{-1} x \\ - \gamma^2 x' Y_\infty^{-1} B_1 B_1' Y_\infty^{-1} x - 2\gamma^2 u^{\tau\prime} B_2 Y_\infty^{-1} x + \gamma^2 x' C_2' C_2 x \Big\} \, dt \\ + x(\tau)' \big(X_\infty(\tau) - \gamma^2 Y_\infty^{-1}(\tau) \big) x(\tau). \end{aligned}$$

Since $y = 0$, for any $t \in [t_0, \tau]$ we know that $0 = C_2(t)x(t) + D_{21}w(t)$. Then we can parametrize all $w(t)$ satisfying this equation by

$$w(t) = -D_{21}' C_2 x(t) + D_{21}^{\perp\prime} r(t) \qquad \forall v(t) \in \mathbb{R}^{m_3}.$$

where $m_3 < m_1$ if $D_{21} \neq I$, and the set of v satisfying $\Lambda v = 0$ is parametrized by x_0 and $r \in \mathscr{L}_2$. Hence, setting $u = 0$ as well, we can write the cost function as

$$\begin{aligned} G(\tau, x_0, 0, w^\tau) &= \int_{t_0}^{\tau} -\gamma^2 \big| D_{21}^{\perp\prime} r(t) - B_1' Y_\infty^{-1} x(t) \big|^2 \, dt + x(\tau)' \big(X_\infty(\tau) - \gamma^2 Y_\infty^{-1}(\tau) \big) x(\tau) \\ &= \int_{t_0}^{\tau} -\gamma^2 \big| r(t) - D_{21}^{\perp} B_1' Y_\infty^{-1} x(t) \big|^2 \, dt + x(\tau)' \big(X_\infty(\tau) - \gamma^2 Y_\infty^{-1}(\tau) \big) x(\tau). \\ &= -\gamma^2 \| \mathcal{L} r \|_2^2 + x(\tau)' \big(X_\infty(\tau) - \gamma^2 Y_\infty^{-1}(\tau) \big) x(\tau) \end{aligned}$$

where $\mathcal{L}$ is given by

$$\mathcal{L} = \left[\begin{array}{c|c} A & B_1 D_{21}^{\perp\prime} \\ \hline -D_{21}^{\perp} B_1' Y_\infty^{-1} & I \end{array} \right].$$

and its inverse has realization

$$\mathcal{L}^{-1} = \left[\begin{array}{c|c} A + B_1 D_{21}^{\perp\prime} D_{21}^{\perp} B_1' Y_\infty^{-1} & B_1 D_{21}^{\perp\prime} \\ \hline D_{21}^{\perp} B_1' Y_\infty^{-1} & I \end{array} \right].$$

Since any finite dimensional linear time varying operator on a finite horizon is bounded, we know that $\|r\| \leq \|\mathcal{L}^{-1}\| \|\mathcal{L} r\|$, and hence

$$G(\tau, x_0, 0, w^\tau) \leq -\gamma^2 \frac{1}{\|\mathcal{L}^{-1}\|^2} \|r\|^2 + x(\tau)' \big(X_\infty(\tau) - \gamma^2 Y_\infty^{-1}(\tau) \big) x(\tau)$$

Since $X_\infty(\tau) - \gamma^2 Y_\infty^{-1}(\tau) < 0$, and is finite dimensional, it is automatically less than $-\varepsilon I$ for some $\varepsilon > 0$. The result then follows applying the previous Lemma. ∎

A direct proof of certainty equivalence. In the following Lemma we give an explicit construction of the rate of change of the worst case cost. This will allow us to immediately show that use of the certainty equivalence controller guarantees that W is decreasing with τ, and hence satisfies the assumptions of Lemma 3.17.

Lemma 3.22. *If $X_\infty(\tau) - \gamma^2 Y_\infty^{-1}(\tau) < 0$, then the unique maximum $W(\tau, u^\tau, y^\tau)$ for the problem described by equation* (3.22) *satisfies*

$$\frac{d}{d\tau} W(\tau, u^\tau, y^\tau) = -\gamma^2 \big| y(\tau) - C_2(\tau)\bar{x}(\tau) \big|^2 + \big| u(\tau) + B_2(\tau)' X_\infty(\tau) \bar{x}(\tau) \big|^2$$

where $\bar{x}(\tau) = \hat{x}^\tau(\tau)$*. Hence, if* $u(\tau) = -B_2(\tau)' X_\infty(\tau)\bar{x}(\tau)$ *for all* $\tau \in [t_0, t_f]$*, then*

$$\frac{d}{d\tau} W(\tau, u^\tau, y^\tau) \leq 0$$

for all y^τ*. Further,* $\bar{x}$ *is the solution of*

$$\dot{\bar{x}} = A\bar{x} + \gamma^{-2} B_1 B_1' X_\infty \bar{x} + Z_\infty C_2'(y - C_2\bar{x}) + \gamma^{-2} Z_\infty X_\infty B_2 B_2' X_\infty \bar{x} + Z_\infty Y_\infty^{-1} B_2 u$$

with initial condition $\bar{x}(t_0) = 0$.

Proof. We present here a direct proof of this. Substituting into equation (3.29) from equation (3.28) gives

$$W(\tau, u^\tau, y^\tau) = \int_{t_0}^{\tau} \left\{ \hat{x}^{\tau\prime} C_1' C_1 \hat{x}^\tau + u'u - \gamma^2 \left| B_1' Y_\infty^{-1}(\check{x} - \hat{x}^\tau)\right|^2 - \gamma^2 \left| y^\tau - C_2 \hat{x}^\tau \right|^2 \right\} dt \\ + \hat{x}^\tau(\tau)' X_\infty(\tau) \hat{x}^\tau(\tau) - \gamma^2 \hat{x}_0^{\tau\prime} Q_0 \hat{x}_0^\tau$$

We now add and subtract $\gamma^2 x' Y_\infty^{-1} x$ to give

$$W(\tau, u^\tau, y^\tau) = \int_{t_0}^{\tau} \left\{ \gamma^2 \frac{d}{dt}(\hat{x}^{\tau\prime} Y_\infty^{-1} \hat{x}^\tau) + \hat{x}^{\tau\prime} C_1' C_1 \hat{x}^\tau + u'u \right. \\ \left. - \gamma^2 \left| B_1' Y_\infty^{-1}(\check{x} - \hat{x}^\tau)\right|^2 - \gamma^2 \left| y^\tau - C_2 \hat{x}^\tau \right|^2 \right\} dt \\ + \hat{x}^\tau(\tau)' \big(X_\infty(\tau) - \gamma^2 Y_\infty^{-1}(\tau) \big) \hat{x}^\tau(\tau)$$

and after some manipulation,

$$W(\tau, u^\tau, y^\tau) = \int_{t_0}^{\tau} \left\{ u^{\tau\prime} u^\tau - \gamma^2 y^{\tau\prime} y^\tau + 2\gamma^2 u^{\tau\prime} B_2' Y_\infty^{-1} \hat{x}^\tau \right. \\ \left. - \gamma^2 \check{x}' Y_\infty^{-1} B_1 B_1' Y_\infty^{-1} \check{x} + 2\gamma^2 y^{\tau\prime} C_2 \hat{x}^\tau \right\} dt \\ + \hat{x}^\tau(\tau)' \big(X_\infty(\tau) - \gamma^2 Y_\infty^{-1}(\tau) \big) \hat{x}^\tau(\tau)$$

Now note that

$$\frac{d}{dt}(\hat{x}^{\tau\prime} Y_\infty^{-1} \check{x}) = u^{\tau\prime} B_2' Y_\infty^{-1} \check{x} - \check{x}' Y_\infty^{-1} B_1 B_1' Y_\infty^{-1} \check{x} + y^{\tau\prime} C_2 \hat{x}^\tau + \hat{x}^{\tau\prime} Y_\infty^{-1} B_2 u^\tau$$

Therefore, since $\check{x}(t_0) = 0$,

$$W(\tau, u^\tau, y^\tau) = \\ \int_{t_0}^{\tau} \left\{ -2\gamma^2 \frac{d}{dt}(\hat{x}^{\tau\prime} Y_\infty^{-1} \check{x}) + u^{\tau\prime} u^\tau - \gamma^2 y^{\tau\prime} y^\tau \right. \\ \left. + 2\gamma^2 u^{\tau\prime} B_2' Y_\infty^{-1} \hat{x}^\tau - \gamma^2 \check{x}' Y_\infty^{-1} B_1 B_1' Y_\infty^{-1} \check{x} + 2\gamma^2 y^{\tau\prime} C_2 \hat{x}^\tau \right\} dt \\ + \hat{x}^\tau(\tau)' \big(X_\infty(\tau) - \gamma^2 Y_\infty^{-1}(\tau) \big) \hat{x}^\tau(\tau) + 2\gamma^2 \hat{x}^\tau(\tau)' Y_\infty^{-1}(\tau) \check{x}(\tau)$$

From equation (3.26), this becomes

$$W(\tau, u^\tau, y^\tau) = \int_{t_0}^{\tau} \Big\{ -2\gamma^2 \frac{d}{dt}(\hat{x}^{\tau\prime} Y_\infty^{-1} \check{x}) + u^{\tau\prime} u^\tau - \gamma^2 y^{\tau\prime} y^\tau + 2\gamma^2 u^{\tau\prime} B_2' Y_\infty^{-1} \hat{x}^\tau - \gamma^2 \check{x}' Y_\infty^{-1} B_1 B_1' Y_\infty^{-1} \check{x} + 2\gamma^2 y^{\tau\prime} C_2 \hat{x}^\tau \Big\}\, dt - \hat{x}^\tau(\tau)' \big(X_\infty(\tau) - \gamma^2 Y_\infty^{-1}(\tau)\big) \hat{x}^\tau(\tau)$$

and hence

$$W(\tau, u^\tau, y^\tau) = \int_{t_0}^{\tau} \Big\{ u^{\tau\prime} u^\tau - \gamma^2 y^{\tau\prime} y^\tau + \gamma^2 \check{x}' Y_\infty^{-1} B_1 B_1' Y_\infty^{-1} \check{x} - 2\gamma^2 u^{\tau\prime} B_2' Y_\infty^{-1} \check{x} \Big\}\, dt - \hat{x}^\tau(\tau)' \big(X_\infty(\tau) - \gamma^2 Y_\infty^{-1}(\tau)\big) \hat{x}^\tau(\tau)$$

Note that we had to make two separate substitutions here since equation (3.26) only holds at time $t = \tau$. The advantage of this form for the worst case cost function is that all the terms within the integral are independent of τ, except at the endpoint of the interval. Hence

$$\frac{d}{d\tau} W(\tau, u^\tau, y^\tau) = \frac{d}{d\tau} \Big\{ \hat{x}^\tau(\tau)' \big(\gamma^2 Y_\infty^{-1}(\tau) - X_\infty(\tau)\big) \hat{x}^\tau(\tau) \Big\} + u(\tau)' u(\tau) - \gamma^2 y(\tau)' y(\tau) + \gamma^2 \check{x}(\tau)' Y_\infty^{-1}(\tau) B_1(\tau) B_1(\tau)' Y_\infty^{-1}(\tau) \check{x}(\tau) - 2\gamma^2 u(\tau)' B_2(\tau)' Y_\infty^{-1}(\tau) \check{x}(\tau)$$

Define the matrix $Z_\infty(t)$ by

$$Z_\infty(t) = Y_\infty(t)(I - \gamma^{-2} X_\infty(t) Y_\infty(t))^{-1}. \tag{3.30}$$

Note that, by the assumptions stated in the Lemma, $Z_\infty(t)$ is well defined for t in some neighbourhood of τ, so $Z_\infty(\tau)$ satisfies

$$\dot{Z}_\infty = (A + \gamma^{-2} B_1 B_1' X_\infty) Z_\infty + Z_\infty (A + \gamma^{-2} B_1 B_1' X_\infty)' - Z_\infty (C_2' C_2 - \gamma^{-2} X_\infty B_2 B_2' X_\infty) Z_\infty + B_1 B_1'.$$

Now let $\bar{x}(\tau) = \hat{x}^\tau(\tau)$ for all τ. Then

$$\begin{aligned} \bar{x} &= (I - \gamma^{-2} Y_\infty X_\infty)^{-1} \check{x} \\ &= Z_\infty Y_\infty^{-1} \check{x} \end{aligned}$$

Again note that $\check{x}$ is independent of τ. Therefore

$$\dot{\bar{x}} = A\bar{x} + \gamma^{-2} B_1 B_1' X_\infty \bar{x} + Z_\infty C_2'(y - C_2 \bar{x}) + \gamma^{-2} Z_\infty X_\infty B_2 B_2' X_\infty \bar{x} + Z_\infty Y_\infty^{-1} B_2 u$$

and

$$\begin{aligned} \frac{d}{dt}\big(\bar{x}'(\gamma^2 Y_\infty^{-1} - X_\infty)\bar{x}\big) = {}& 2\bar{x}'(\gamma^2 Y_\infty^{-1} - X_\infty)\gamma^{-2} B_1 B_1' X_\infty \bar{x} + 2\gamma^2 \bar{x}' C_2'(y - C_2 \bar{x}) \\ &+ 2\bar{x}' X_\infty B_2 B_2' X_\infty \bar{x} + 2\gamma^2 \bar{x}' Y_\infty^{-1} B_2 u - \bar{x}' X_\infty B_2 B_2' X_\infty \bar{x} \\ &+ \gamma^{-2} \bar{x}' X_\infty B_1 B_1' X_\infty \bar{x} + \gamma^2 \bar{x}' C_2' C_2 \bar{x} - \gamma^2 \bar{x}' Y_\infty^{-1} B_1 B_1' Y_\infty^{-1} \bar{x} \end{aligned}$$

Hence,

$$\frac{d}{d\tau}W(\tau, u^\tau, y^\tau) = -\gamma^2|y(\tau) - C_2(\tau)\bar{x}(\tau)|^2 + u(\tau)'u(\tau) \\ + 2u(\tau)'B_2(\tau)'X_\infty(\tau)\bar{x}(\tau) + \bar{x}(\tau)X_\infty(\tau)B_2(\tau)B_2(\tau)'X_\infty(\tau)\bar{x}(\tau)$$

which gives

$$\frac{d}{d\tau}W(\tau, u^\tau, y^\tau) = -\gamma^2\big|y(\tau) - C_2(\tau)\bar{x}(\tau)\big|^2 + \big|u(\tau) + B_2(\tau)'X_\infty(\tau)\bar{x}(\tau)\big|^2$$

■

Corollary 3.23. *Suppose that, for all $t \in [t_0, t_f]$, there exist matrices $X_\infty(t)$ and $Y_\infty(t)$ satisfying equations (3.25) and (3.24), and that for all $t \in [t_0, t_f]$ these solutions satisfy $X_\infty(t) - \gamma^2 Y_\infty^{-1}(t) < 0$. Then define the feedback law $\hat{\mu}$ by*

$$u(t) = -B_2(t)'X_\infty(t)\bar{x}(t).$$

That is, it is the state feedback law with the worst case state estimate. Then

$$\sup_{x_0 \in \mathbb{R}^n, w \in \mathscr{L}_2} J(x_0, \hat{\mu}, w) = 0.$$

Proof. This follows directly from Theorem 3.17 and the previous Lemma. ■

Note that, since $Y(t) > 0$, the condition that $X_\infty(t) - \gamma^2 Y_\infty^{-1}(t) < 0$ for all $t \in [t_0, t_f]$ is equivalent to the spectral radius condition $\rho\big(X_\infty(t)Y_\infty(t)\big) < \gamma$. This is straightforward from

$$\begin{aligned} & X_\infty - \gamma^2 Y_\infty^{-1} < 0 \\ \iff \quad & Y_\infty^{\frac{1}{2}} X_\infty Y_\infty^{\frac{1}{2}} - \gamma^2 I < 0 \\ \iff \quad & \rho(Y_\infty^{\frac{1}{2}} X_\infty Y_\infty^{\frac{1}{2}}) < \gamma \\ \iff \quad & \rho(X_\infty Y_\infty) < \gamma. \end{aligned}$$

We have now presented a direct derivation of the certainty equivalence principle for continuous time linear time varying systems on a finite horizon. We will make use of this in the sequel in constructing moving horizon $\mathscr{H}_\infty$ control laws for linear time varying systems on an infinite horizon.

3.6 Discrete time games with imperfect measurement

3.6.1 The information state approach

We consider finite dimensional nonlinear time varying systems on $[k_0, k_f]$ described by

$$\begin{aligned} x(k+1) &= f(k, x(k), u(k), w(k)) \qquad x(k_0) = x_0 \\ y(k) &= h(k, x(k), w(k)) \end{aligned}$$

Here the state $x(k) \in \mathbb{R}^n$, the disturbance $w \in \ell_2^{m_1}[k_0, k_f]$, and the control input $u \in \ell_2^{m_2}[k_0, k_f]$. Let the cost function be defined by

$$J(x_0, \mu, w) := \alpha(x_0) + \sum_{i=k_0}^{k_f} g\big(i, x(i), u(i), w(i)\big) + \beta\big(x(k_f+1)\big), \tag{3.31}$$

where the final state weight is at time $k_f + 1$ so that the final inputs $u(k_f)$ and $w(k_f)$ are included in the cost function, and so that, if used in a moving horizon problem, the closed loop tends to be stable. We will now give a review of the work done by James, Baras and Elliott [40] on the information state approach to constructing solutions to the problem

$$\inf_{\mu \in \mathscr{U}_{of}} \sup_{\substack{w \in \ell_2 \\ x_0 \in \mathbb{R}^n}} J(x_0, \mu, w)$$

where the set of strictly causal discrete time output feedback strategies, $\mathscr{U}_{of}$, is defined by

$$\mathscr{U}_{of}^{p,m}[k_0, k_1] = \big\{\mu : \ell_2^p[k_0, k_1] \to \ell_2^m[k_0, k_1] \;\big|\; \pi_{[k_0,k]}\mu\pi_{[k_0,k-1]} = \pi_{[k_0,k]}\mu \text{ for all } k \in [k_0, k_1]\big\}.$$

Definition 3.24. *Define the Value of the game for $k_0 \le j \le k_f$ by*

$$V\big(j, x(j)\big) := \inf_{\mu \in \mathscr{U}_{sf}[j,k_f]} \sup_{w \in \ell_2[j,k_f]} \left\{ \sum_{i=j}^{k_f} g\big(i, x(i), u(i), w(i)\big) + \beta\big(x(k_f+1)\big) \right\} \tag{3.32}$$

□

The Value function is that used in the standard technique of dynamic programming, hence we do not add $\alpha(x_0)$ when $j = k_0$. The following Lemma gives the dynamic programming result.

Lemma 3.25. *The value function V satisfies the dynamic programming recursion*

$$V\big(j, x(j)\big) = \inf_{u(j) \in \mathbb{R}^{m_2}} \sup_{w(j) \in \mathbb{R}^{m_1}} \left\{ g\big(j, x(j), u(j), w(j)\big) + V\big(j+1, x(j+1)\big) \right\} \tag{3.33}$$

with boundary condition

$$V\big(k_f + 1, x(k_f+1)\big) = \beta\big(x(k_f+1)\big) \tag{3.34}$$

Proof. This is straightforward by induction. ■

Again, assume that at time j, we have measurements of y and u on $[k_0, j-1]$, which we will denote by $y^{j-1} \in \ell_2[k_0, j-1]$ and $u^{j-1} \in \ell_2[k_0, j-1]$. We now define the following disturbance class, which is the set of all possible (w, x_0) disturbances consistent with y^j and u^j.

$$\Omega_0(j) := \Big\{ (x_0, w)\ ; x_0 \in \mathbb{R}^n,\ w \in \ell_2[k_0, k_f] \text{ such that } \begin{array}{c} x(i+1) = Ax(i) + B_1 w(i) + B_2 u^j(i) \\ y^j(i) = C_2 x(i) + D_{21} w(i) \end{array} \quad \text{for } k_0 \le i \le j \Big\}$$

and the following subset which is further consistent with the hypothesis that $x(j+1) = \xi$,

$$\Omega(j,\xi) := \Big\{(x_0,w)\ ; x_0 \in \mathbb{R}^n,\ \ w \in \ell_2[k_0,k_f] \text{ such that, for } k_0 \le i \le j,$$
$$x(i+1) = Ax(i) + B_1 w(i) + B_2 u^j(i)$$
$$y^j(i) = C_2 x(i) + D_{21} w(i)$$
$$\text{and } x(j+1) = \xi\Big\}.$$

Note that these disturbance sets actually depend on past values of u and y, although we omit them to reduce notation.

Definition 3.26. *Define the information state $P\big(j,x(j)\big)$ by*

$$P\big(j,x(j)\big) := \sup_{(w,x_0)\in\Omega(j-1,x(j))} \Big\{\alpha(x_0) + \sum_{i=k_0}^{j-1} g\big(i,x(i),u(i),w(i)\big)\Big\} \tag{3.35}$$

□

Note that $P\big(j,x(j)\big)$ is actually a function of y^{j-1} and u^{j-1} also, but again we omit them from the arguments to reduce notation. We use the convention that the supremum over an empty set equals $-\infty$. It is easy to show by induction the following result.

Lemma 3.27. *The information state satisfies the dynamic programming recursion*

$$P\big(j+1,x(j+1)\big) = \sup_{\substack{x(j),w(j)\\ x(j+1)=f(j,x(j),u^j(j),w(j))\\ y^j(j)=h(j,x(j),w(j))}} \Big\{P\big(j,x(j)\big) + g\big(j,x(j),u(j),w(j)\big)\Big\} \tag{3.36}$$

with the boundary condition $P\big(k_0,x_0\big) = \alpha(x_0)$. ■

Write $p(j) = P(j,\cdot)$. That is, $p(j) : \mathbb{R}^n \to \mathbb{R}$. Then the above dynamic programming recursion can be written as

$$p(j+1) = F(p(j),u(j),y(j))$$

and this describes the evolution of the infinite dimensional 'state' p. This new infinite dimensional system has 'inputs' u and y. We shall show that p can be regarded as the state for a new dynamic game, in which case equation (3.36) is the evolution equation for this system.

Define the class $\mathscr{U}_p[i,j]$ of strategies for the minimizing player which feedback the information state as the set of controllers of the form $u(j) = \mu\big(p(i),\dots,p(j)\big)$. Note that here μ is a functional. Each $p(j)$ could be written as $p(j) = P(j,\cdot,u^{j-1},y^{j-1})$, dependent on the past inputs and measurements of the system. Therefore a controller μ in $\mathscr{U}_p[k_0,j]$ is also a member of $\mathscr{U}_{of}$.

We now define a cost function for the new game with state p. This cost function, unlike J in equation (3.31), will only have a final state weight; the corresponding functions to g and α will be zero. The cost function is

$$J\big(p(k_0),\mu,y\big) = B\big(p(k_f+1)\big).$$

where the final weight on the information state is

$$\mathsf{B}(\mathsf{p}(k_f+1)) = \sup_{x(k_f+1)} \Big\{ P\big(k_f+1, x(k_f+1)\big) + \beta\big(x(k_f+1)\big) \Big\}.$$

We have written B here to indicate the correspondence to the final state weight β for the original finite dimensional problem. We will define the new game with the infinite dimensional state p as

$$\inf_{\mu\in\mathscr{U}_p[k_0,k_f]} \sup_{y\in\ell_2[k_0,k_f]} \mathsf{J}\big(\mathsf{p}(k_0), \mu, y\big).$$

Then, in exactly the same way as in Definition 3.24, we can define a Value function for the new game with this cost function. This new Value function is simply

$$\mathsf{V}\big(j, \mathsf{p}(j)\big) := \inf_{\mu\in\mathscr{U}_p[j,k_f]} \sup_{y\in\ell_2[j,k_f]} \mathsf{B}\big(\mathsf{p}(k_f+1)\big) \tag{3.37}$$

since there are no terms in $\mathsf{p}(i)$ for $k_0 \le i \le k_f$ in this new cost function. Since there is only a final term, the corresponding equation to (3.33) is very simple. It is

$$\mathsf{V}\big(j, \mathsf{p}(j)\big) = \inf_{u(j)\in\mathbb{R}^{m_2}} \sup_{y(j)\in\mathbb{R}^{p_2}} \mathsf{V}\big(j+1, \mathsf{p}(j+1)\big) \tag{3.38}$$

and again this is straightforward to show by induction. Clearly, $\mathsf{V}\big(j, \mathsf{p}(j)\big)$ is actually a function of y^{j-1} and u^{j-1} also. We then have the following useful Lemma.

Lemma 3.28. *The following equality holds:*

$$\sup_{y\in\ell_2[j,k_f]} \mathsf{J}\big(\mathsf{p}(k_0), \mu, y\big) = \sup_{(x_0,w)\in\Omega_0(j)} J(x_0, \mu, w).$$

Proof. The left hand side is equal to

$$\sup_{y\in\ell_2[j,k_f]} \sup_{x(k_f+1)} \left\{ \sup_{(w,x_0)\in\Omega(k_f,x(k_f+1))} \Big\{ \alpha(x_0) + \sum_{i=k_0}^{k_f} g\big(i, x(i), u(i), w(i)\big) \Big\} + \beta\big(x(k_f+1)\big) \right\}$$

and the result follows, since the maximizations with respect to w and x_0 are performed subject to constraints on w defined by the values of $y_{[j,k_f]}$, and subject to $x(k_f+1)$ being fixed. But for every given w and x_0, there exists a corresponding y and $x(k_f+1)$, and we are maximizing with respect to $y_{[j,k_f]}$ and $x(k_f+1)$. ■

Corollary 3.29.

$$\sup_{y\in\ell_2[k_0,k_f]} \mathsf{J}\big(\mathsf{p}(k_0), \mu, y\big) = \sup_{w\in\ell_2[k_0,k_f]} \sup_{x_0\in\mathbb{R}^n} J(x_0, \mu, w) \tag{3.39}$$

Corollary 3.30. *The function $\mathsf{V}\big(k_0, \mathsf{p}(k_0)\big)$ satisfies*

$$\mathsf{V}\big(k_0, \mathsf{p}(k_0)\big) = \inf_{\mu\in\mathscr{U}_{of}} \sup_{w\in\ell_2[k_0,k_f]} \sup_{x_0\in\mathbb{R}^n} J(x_0, \mu, w).$$

Proof. The above statement follows from Corollary 3.29 when $\mathscr{U}_{of}$ is replaced by $\mathscr{U}_p$. However, since V satisfies the dynamic programming recursion (3.38), the optimal μ minimizing the left hand side of equation (3.39) must be given by $\mu \in \mathscr{U}_p$, and hence the optimal μ for the original problem is also given by $\mu \in \mathscr{U}_p$. ■

Since we know that V satisfies the dynamic programming equation (3.38), if we can solve this recursion then we can find $u(j)$ at each $j \in [k_0, k_f]$, as a function of $\mathsf{p}(j)$. That is, the resulting control law will be a feedback law dependent on the information state.

This result is quite general. It holds for a wide class of nonlinear systems, and in principle gives us a method for recursively generating the optimal solution to the problem of minimizing J. The problem is that in general $\mathsf{p}(j)$ is described by an infinite dimensional evolution equation. Then V is a difference equation defined on a space of functions, which appears extremely difficult to solve at present.

However, Başar and Bernhard [7], give a *separation principle* for this problem, which replaces the infinite dimensional recursion (3.38) for V by the finite dimensional recursion for V, the value of the game. Bernhardsson [12] gives a counterexample to this principle in the general nonlinear case. Following James [37], we shall now derive this separation principle from the information state formulation, and see that a particular saddle point condition is necessary for it to hold.

Theorem 3.31. *Suppose that, for all* $j \in [k_0, k_f]$,

$$\mathsf{V}\big(j, \mathsf{p}(j)\big) = \sup_{x(j)\in\mathbb{R}^n} \Big\{ P\big(j, x(j)\big) + V\big(j, x(j)\big) \Big\} \tag{3.40}$$

Then let

$$\bar{x}(k) \in \arg \max_{x(j)\in\mathbb{R}^n} \Big\{ P\big(j, x(j)\big) + V\big(j, x(j)\big) \Big\}$$

then

$$u(j) = \mu_0\big(j, \bar{x}(j)\big)$$

is an optimal control law for the dynamic game problem defined above, where μ_0 *defined by equation* (3.7) *is the optimal state feedback control law. Further, for each* $j \in [k_0, k_f]$, *condition* (3.40) *holds if and only if*

$$\begin{aligned} &\inf_{u(j)\in\mathbb{R}^{m_2}} \sup_{x(j)\in\mathbb{R}^n} \sup_{w(j)\in\mathbb{R}^{m_1}} \Big\{ P\big(j, x(j)\big) + g\big(j, x(j), u(j), w(j)\big) + V\big(j+1, x(j+1)\big) \Big\} \\ &= \sup_{x(j)\in\mathbb{R}^n} \inf_{u(j)\in\mathbb{R}^{m_2}} \sup_{w(j)\in\mathbb{R}^{m_1}} \Big\{ P\big(j, x(j)\big) + g\big(j, x(j), u(j), w(j)\big) + V\big(j+1, x(j+1)\big) \Big\} \end{aligned} \tag{3.41}$$

Proof. Let

$$\Omega_x\big(x(j+1), u^j(j), w(j)\big) = \Big\{ x(j)\ ;\ x(j+1) = f\big(j, x(j), u^j(j), w(j)\big) \Big\}$$
$$\Omega_w\big(x(j), y^j(j)\big) = \Big\{ w(j)\ ;\ y^j(j) = h\big(j, x(j), w(j)\big) \Big\}.$$

Suppose condition (3.40) holds at time $j+1$. Then

$$\begin{aligned}
\mathsf{V}\big(j+1,\mathsf{p}(j+1)\big) &= \sup_{x(j+1)\in\mathbb{R}^n}\Big\{P\big(j+1,x(j+1)\big)+V\big(j+1,x(j+1)\big)\Big\}\\
&= \sup_{x(j+1)\in\mathbb{R}^n}\left\{\sup_{x(j)\in\Omega_x}\sup_{w(j)\in\Omega_w}\Big\{P\big(j,x(j)\big)\right.\\
&\qquad\left.+\,g\big(j,x(j),u(j),w(j)\big)\Big\}+V\big(j+1,x(j+1)\big)\right\}\\
&= \sup_{x(j+1)\in\mathbb{R}^n}\sup_{x(j)\in\Omega_x}\sup_{w(j)\in\Omega_w}\Big\{P\big(j,x(j)\big)\\
&\qquad+\,g\big(j,x(j),u(j),w(j)\big)+V\big(j+1,x(j+1)\big)\Big\}
\end{aligned}$$

Since we are maximizing with respect to $x(j)$ such that $x(j+1)=f(j,x(j),u^j(j),w(j))$ and then maximizing with respect to $x(j+1)$, this implies

$$\begin{aligned}
\mathsf{V}\big(j+1,\mathsf{p}(j+1)\big) = \sup_{x(j)\in\mathbb{R}^n}\sup_{w(j)\in\Omega_w}\Big\{&P\big(j,x(j)\big)\\
&+g\big(j,x(j),u(j),w(j)\big)+V\big(j+1,x(j+1)\big)\Big\}
\end{aligned}$$

Then from the dynamic programming recursion (3.38),

$$\begin{aligned}
\mathsf{V}\big(j,\mathsf{p}(j)\big) = &\inf_{u(j)\in\mathbb{R}^{m_2}}\sup_{y(j)\in\mathbb{R}^{p_2}}\sup_{x(j)\in\mathbb{R}^n}\sup_{w(j)\in\Omega_w}\\
&\Big\{P\big(j,x(j)\big)+g\big(j,x(j),u(j),w(j)\big)+V\big(j+1,x(j+1)\big)\Big\}
\end{aligned}$$

Then, since we are maximizing with respect to $w(k)$ such that $y(j)=h(j,x(j),w(j))$ and then maximizing with respect to $y(j)$, this implies

$$\begin{aligned}
\mathsf{V}\big(j,\mathsf{p}(j)\big) = &\inf_{u(j)\in\mathbb{R}^{m_2}}\sup_{x(j)\in\mathbb{R}^n}\sup_{w(k)\in\mathbb{R}^{m_1}}\\
&\Big\{P\big(j,x(j)\big)+g\big(j,x(j),u(j),w(j)\big)+V\big(j+1,x(j+1)\big)\Big\}
\end{aligned}$$

If condition (3.40) holds at time j, then

$$\mathsf{V}\big(j,\mathsf{p}(j)\big) = \sup_{x(j)\in\mathbb{R}^n}\Big\{P\big(j,x(j)\big)+V\big(j,x(j)\big)\Big\} \tag{3.42}$$

$$\begin{aligned}
= \sup_{x(j)\in\mathbb{R}^n}\Big\{P\big(j,x(j)\big)+\inf_{u(j)\in\mathbb{R}^{m_2}}\sup_{w(j)\in\mathbb{R}^{m_1}}\Big\{&g\big(j,x(j),u(j),w(j)\big)\\
&+V\big(j+1,x(j+1)\big)\Big\}\Big\}
\end{aligned} \tag{3.43}$$

$$\begin{aligned}
= &\sup_{x(j)\in\mathbb{R}^n}\inf_{u(j)\in\mathbb{R}^{m_2}}\sup_{w(j)\in\mathbb{R}^{m_1}}\\
&\Big\{P\big(j,x(j)\big)+g\big(j,x(j),u(j),w(j)\big)+V\big(j+1,x(j+1)\big)\Big\}
\end{aligned}$$

Comparing this equation with the previous one proves the last part of the theorem. Further, we know that the maximizing $x(j)$ in equation (3.42) is $\bar{x}(j)$, and the minimizing $u(t)$ in equation (3.43) is given by $u(j) = \mu_0(j, x(j)) = \mu_0(j, \bar{x}(j))$. ■

3.6.2 The linear case

We now specialize the previous theory to the case where the system is linear, and the cost function is quadratic. Specifically, we consider the system as described on $[k_0, k_f]$ by

$$\begin{aligned} x(k+1) &= A(k)x(k) + B_1(k)w(k) + B_2(k)u(k) \qquad x(k_0) = x_0 \\ z(k) &= C_1(k)x(k) + D_{12}(k)u(k) \\ y(k) &= C_2(k)x(k) + D_{21}(k)w(k) \end{aligned}$$

Here the state $x(k) \in \mathbb{R}^n$, the disturbance $w \in \ell_2^{m_1}[k_0, k_f]$, and the control input $u \in \ell_2^{m_2}[k_0, k_f]$. By similar arguments to those used in the continuous time case, we limit the system without loss of generality to the case when $D_{11}(k) = D_{22}(k) = 0$ for all $k \in [k_0, k_f]$. We also make the standard assumptions

$$D_{21}(k)D_{21}(k)' = I \quad D_{12}(k)'D_{12}(k) = I \quad D_{21}(k)B_1(k)' = 0 \quad D_{12}(k)'C_1(k) = 0.$$

for all $k \in [k_0, k_f]$. Again, let the cost function be defined by

$$J(x_0, \mu, w) := -\gamma^2 x_0' Q_0 x_0 + \sum_{i=k_0}^{k_f} \Big\{ |z(i)|^2 - \gamma^2 |w(i)|^2 \Big\} + x(k_f+1)' Q_f x(k_f+1).$$

where $Q_f > 0$, $Q_0 > 0$. It is straightforward to show that there exists a saddle point for the state feedback dynamic game if and only if the solution to the Riccati difference equation

$$\tilde{X}_\infty(k) = A(k)'\big(\tilde{X}_\infty(k+1)^{-1} + B_2(k)B_2(k)' - \gamma^{-2}B_1(k)B_1(k)'\big)^{-1}A(k) + C_1(k)'C_1(k) \tag{3.44}$$

with boundary condition

$$\tilde{X}_\infty(k_f+1) = Q_f.$$

satisfies

$$\gamma^2 I - B_1(k)'\tilde{X}_\infty(k)B_1(k) > 0$$

for all $k \in [k_0, k_f]$ (see for example [7]). Then the value function is given by

$$V\big(k, x(k)\big) = x(k)'\tilde{X}_\infty(k)x(k)$$

and the optimal state feedback control law is given by

$$\begin{aligned} u(k) &= \mu_0(k, x(k)) \\ &= -B_2(k)'\big(\tilde{X}_\infty(k+1)^{-1} + B_2(k)B_2(k)' - \gamma^{-2}B_1(k)B_1(k)'\big)^{-1}A(k)x(k) \end{aligned}$$

Also, the information state is given by

$$P\big(k, x(k)\big) = -\gamma^2\big(x(k) - \check{x}(k)\big)'\tilde{Y}_\infty(k)^{-1}\big(x(k) - \check{x}(k)\big) + \Psi(k)$$

where $\Psi(k)$ contains terms independent of $x(k)$, (depending on $\check{x}$, y^{j-1} and u^{j-1}), and $\tilde{Y}_\infty(k)$ satisfies the Riccati difference equation

$$\tilde{Y}_\infty(k+1) = A(k)\Delta(k)^{-1}A(k)' + B_1(k)B_1(k)' \qquad \tilde{Y}_\infty(0) = Q_0^{-1}$$

with

$$\Delta(k) = \tilde{Y}_\infty(k)^{-1} + C_2(k)'C_2(k) - \gamma^2 C_1(k)'C_1(k).$$

and

$$\begin{aligned}\check{x}(k+1) = {} & A(k)\check{x}(k) + B_2(k)u(k) \\ & + A(k)\Delta(k)^{-1}\big(\gamma^{-2}C_1(k)'C_1(k)\check{x}(k) + C_2(k)'\big(y(k) - C_2(k)\check{x}(k)\big)\big).\end{aligned}$$

A sufficient condition for a saddle point. In the following Lemma, we give a simple sufficient condition for the existence of a saddle point for a static quadratic game defined on finite dimensional spaces. We will make use of this to show that the linear discrete dynamic game with a quadratic cost function satisfies the saddle point condition required in order to use the certainty equivalence principle. We will also use it in Chapter 4 when deriving the corresponding result for the multi-rate sampled-data problem.

Lemma 3.32. *Suppose $J(u,w)$ is the cost function defined by*

$$J(u,w) = \begin{pmatrix} u \\ w \end{pmatrix}' \begin{bmatrix} R_{11} & R_{12} \\ R_{12}' & -R_{22} \end{bmatrix} \begin{pmatrix} u \\ w \end{pmatrix} + 2u'r_1 + 2w'r_2$$

for the static game

$$\min_{u\in\mathbb{R}^{m_2}} \max_{w\in\mathbb{R}^{m_1}} f(u,w)$$

Then, if $R_{11} > 0$ and $R_{22} > 0$, the game has a unique saddle point.

Proof. Since $R_{11} > 0$ and $R_{22} > 0$, for each fixed u there exists a unique maximizing w denoted by $w^*(u)$, and for each fixed w there exists a unique minimising u denoted by $u^*(w)$. By differentiation, these satisfy

$$\begin{aligned} R_{11}u^*(w) + R_{12}w + r_1 &= 0 \\ R_{12}'u - R_{22}w^*(u) + r_2 &= 0 \end{aligned}$$

Suppose there exists u_0, w_0 such that $u_0 = u^*(w_0)$ and $w_0 = w^*(u_0)$, then these will satisfy

$$J(u_0,w) \le J(u_0,w_0) \le J(u,w_0)$$

for all $u \in \mathbb{R}^{m_2}$, $w \in \mathbb{R}^{m_1}$, and hence will be a saddle point solution. Conversely, if $J(u_0,w_0) \le J(u,w_0)$ for all $u \in \mathbb{R}^{m_2}$, then $u_0 = u^*(w_0)$, and if $J(u_0,w) \le J(u_0,w_0)$ for all $w \in \mathbb{R}^{m_1}$, then $w_0 = w^*(u_0)$. Hence u_0, w_0 are a saddle point for this game if and only if they are a solution to

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{12}' & -R_{22} \end{bmatrix} \begin{pmatrix} u_0 \\ w_0 \end{pmatrix} = -\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

Since R_{11} is nonsingular, the matrix on the left hand side of this equation nonsingular if and only if $-R_{22} - R_{12}'R_{11}^{-1}R_{12}$ is nonsingular, which by the positive definiteness assumptions is true. Therefore there exists a unique saddle point solution. ■

We now wish to check that, for the discrete time linear quadratic problem, the conditions of Theorem 3.31 hold. We have

$$
\begin{aligned}
& g\big(k, x(k), u(k), w(k)\big) + V\big(k+1, x(k+1)\big) \\
&\quad = |z(k)|^2 - \gamma^2 |w(k)|^2 + x(k+1)'\tilde{X}_\infty(k+1)x(k+1) \\
&\quad = \begin{pmatrix} x(k+1) \\ z(k) \\ w(k) \end{pmatrix}' \begin{bmatrix} \tilde{X}_\infty & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -\gamma^2 I \end{bmatrix} \begin{pmatrix} x(k+1) \\ z(k) \\ w(k) \end{pmatrix} \\
&\quad = \begin{pmatrix} w(k) \\ u(k) \\ x(k) \end{pmatrix}' T' \begin{bmatrix} \tilde{X}_\infty & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -\gamma^2 I \end{bmatrix} T \begin{pmatrix} w(k) \\ u(k) \\ x(k) \end{pmatrix} \\
&\quad = \begin{pmatrix} w(k) \\ u(k) \\ x(k) \end{pmatrix}' \begin{bmatrix} B_1'\tilde{X}_\infty B_1 - \gamma^2 I & B_1'\tilde{X}_\infty B_2 & B_1'\tilde{X}_\infty A \\ B_2'\tilde{X}_\infty B_1 & B_2'\tilde{X}_\infty B_2 + I & B_2'\tilde{X}_\infty A \\ A'\tilde{X}_\infty B_1 & A'\tilde{X}_\infty B_2 & A'\tilde{X}_\infty A \end{bmatrix} \begin{pmatrix} w(k) \\ u(k) \\ x(k) \end{pmatrix}
\end{aligned}
$$

where the system matrices A, B_1, B_2, C_1, D_{12} are evaluated at time k, and $\tilde{X}_\infty$ is evaluated at $k+1$, and we have written

$$
T = \begin{bmatrix} B_1 & B_2 & A \\ 0 & D_{12} & C_1 \\ I & 0 & 0 \end{bmatrix}.
$$

We wish to check that the problem

$$
\inf_{u(k)\in\mathbb{R}^{m_2}} \sup_{x(k)\in\mathbb{R}^n} \sup_{w(k)\in\mathbb{R}^{m_1}} \Big\{ P\big(k, x(k)\big) + g\big(k, x(k), u(k), w(k)\big) + V\big(k+1, x(k+1)\big) \Big\} \tag{3.45}
$$

has a saddle point in $u(k)$, $x(k)$. Let

$$
J(k) = (B_1(k)'\tilde{X}_\infty(k+1)B_1(k) - \gamma^2 I)^{-1}
$$

then it is clear that the maximum with respect to w has a unique solution if and only if $J(k) < 0$ for all $k \in [k_0, k_f]$. Note that this is also a necessary condition for the existence of a saddle point in the state feedback problem also, see for example [7, Theorem 3.2]. If this condition holds, then let

$$
L = \begin{bmatrix} I & JB_1'\tilde{X}_\infty B_2 & JB_1'\tilde{X}_\infty A \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}
$$

$$
\begin{bmatrix} R_{11} & R_{12} \\ R_{12}' & R_{22} \end{bmatrix} = \begin{bmatrix} B_2'\tilde{X}_\infty B_2 + I & B_2'\tilde{X}_\infty A \\ A'\tilde{X}_\infty B_2 & A'\tilde{X}_\infty A + C_1'C_1 \end{bmatrix} - \begin{bmatrix} B_2'\tilde{X}_\infty B_1 \\ A'\tilde{X}_\infty B_1 \end{bmatrix} J \begin{bmatrix} B_1'\tilde{X}_\infty B_2 & B_1'\tilde{X}_\infty A \end{bmatrix}
$$

then, since $P\big(k, x(k)\big) = -\gamma^2\big(x(k) - \check{x}(k)\big)'\tilde{Y}_\infty(k)^{-1}\big(x(k) - \check{x}(k)\big) + \Psi(k)$, with $\Psi(k)$ independent of $x(k)$, using the Schur complement formula, we arrive at

$$
\begin{aligned}
&\sup_{w(k)\in\mathbb{R}^{m_1}} \Big\{P\big(k, x(k)\big) + g\big(k, x(k), u(k), w(k)\big) + V\big(k+1, x(k+1)\big)\Big\} \\
&= \sup_{w(k)\in\mathbb{R}^{m_1}} \begin{pmatrix} w(k) \\ u(k) \\ x(k) \end{pmatrix}' L' \begin{bmatrix} J & 0 & 0 \\ 0 & R_{11} & R_{12} \\ 0 & R_{12}' & R_{22} \end{bmatrix} L \begin{pmatrix} w(k) \\ u(k) \\ x(k) \end{pmatrix} \\
&\qquad - \gamma^2\big(x(k) - \check{x}(k)\big)'\tilde{Y}_\infty(k)^{-1}\big(x(k) - \check{x}(k)\big) - \Psi(k) \\
&= \begin{pmatrix} u(k) \\ x(k) \end{pmatrix}' \begin{bmatrix} R_{11} & R_{12} \\ R_{12}' & R_{22} \end{bmatrix} \begin{pmatrix} u(k) \\ x(k) \end{pmatrix} \\
&\qquad - \gamma^2\big(x(k) - \check{x}(k)\big)'\tilde{Y}_\infty(k)^{-1}\big(x(k) - \check{x}(k)\big) - \Psi(k).
\end{aligned}
$$

Then by the previous Lemma, the minimax problem in equation (3.41) has a saddle point if

$$
\begin{aligned}
R_{11} &> 0 \\
R_{22} - \gamma^2\tilde{Y}_\infty(k)^{-1} &< 0
\end{aligned}
$$

and algebra gives

$$R_{11} = B_2'\tilde{X}_\infty(k+1)B_2 + I - B_2'\tilde{X}_\infty(k+1)B_1 J(k)^{-1}B_1'\tilde{X}_\infty(k+1)B_2 \tag{3.46}$$

$$R_{22} = A'\tilde{X}_\infty(k+1)A + C_1'C_1 - A'\tilde{X}_\infty(k+1)B_1 J(k) B_1\tilde{X}_\infty(k+1)A \tag{3.47}$$

Since $J(k) < 0$, the first condition that $R_{11} > 0$ is automatically satisfied. The second condition that $R_{22} - \gamma^2\tilde{Y}_\infty(k)^{-1} < 0$ is a generalization of the usual *coupling condition* that occurs in the case when $u(k)$ is allowed to depend on measurements of $y(k)$ also. In the latter case, the condition is that $\tilde{X}_\infty(k) - \gamma^2\tilde{Y}_\infty(k)^{-1} \le 0$ for all $k \in [k_0, k_f]$. In this case, we require that u is a strictly causal function of y. Condition (3.47) is in fact stronger than the standard condition, and implies that condition.

If $B_2(k)$ were equal to zero, then condition (3.47) would be equivalent to the standard condition at time k. This is exactly what we expect for the one step delayed measurement problem, since during that one step u has no control, and so the problem during that time step is simply a quadratic maximization with respect to w. This kind of condition is repeated throughout the time domain derivations of various measurement problems, notably the sampled-data problem, when a necessary condition is that the maximum with respect to w of the cost function in between samples is bounded.

If these conditions are satisfied, we can calculate $\bar{x}$ using

$$\bar{x}(k) \in \arg\max_{x(j)\in\mathbb{R}^n} \Big\{P\big(j, x(j)\big) + V\big(j, x(j)\big)\Big\}$$

and we know

$$
\begin{aligned}
&P\big(j, x(j)\big) + V\big(j, x(j)\big) \\
&\quad = -\gamma^2(x(k) - \check{x}(k))'\tilde{Y}_\infty(k)^{-1}(x(k) - \check{x}(k)) + \Psi(k) + x(k)'\tilde{X}_\infty(k)x(k) \\
&\quad = \begin{pmatrix} x \\ \check{x} \end{pmatrix}' \begin{bmatrix} -\gamma^2\tilde{Y}_\infty^{-1} + \tilde{X}_\infty & \gamma^2\tilde{Y}_\infty^{-1} \\ \gamma^2\tilde{Y}_\infty^{-1} & -\tilde{Y}_\infty^{-1} \end{bmatrix} \begin{pmatrix} x \\ \check{x} \end{pmatrix} + \Psi \\
&\quad = \begin{pmatrix} x \\ \check{x} \end{pmatrix}' T' \begin{bmatrix} \tilde{X}_\infty - \gamma^2\tilde{Y}_\infty^{-1} & 0 \\ 0 & -\gamma^2\tilde{Y}_\infty^{-1} - \gamma^2\tilde{Y}_\infty^{-1}(\gamma^{-2}\tilde{X}_\infty - \tilde{Y}_\infty^{-1})^{-1}\tilde{Y}_\infty^{-1} \end{bmatrix} T \begin{pmatrix} x \\ \check{x} \end{pmatrix}
\end{aligned}
$$

where

$$T = \begin{bmatrix} I & (\gamma^{-2}\tilde{Y}_\infty \tilde{X}_\infty - I)^{-1} \\ 0 & I \end{bmatrix}$$

By assumption, condition (3.47) is satisfied, and hence $\tilde{X}_\infty - \gamma^2 \tilde{Y}_\infty^{-1} < 0$. Therefore this maximization problem has a unique solution, given by

$$\bar{x}(k) = (I - \gamma^{-2}\tilde{Y}_\infty(k)\tilde{X}_\infty(k))^{-1}\check{x}(k)$$

and the optimal controller is therefore

$$\begin{aligned} u(k) &= \mu_0(k, \bar{x}(k)) \\ &= -B_2(k)'\left(\tilde{X}_\infty(k+1)^{-1} + B_2(l)B_2(k)' - \gamma^{-2}B_1(k)B_1(k)'\right)^{-1} \\ &\qquad .A(k)(I - \gamma^{-2}\tilde{Y}_\infty(k)\tilde{X}_\infty(k))^{-1}\check{x}(k) \end{aligned}$$

3.7 Summary

In this chapter we began with the basic results of static games, and then showed that finite horizon induced 2-norm problems can be cast into the dynamic game formulation, which can be regarded as a sequence of coupled static games in the discrete case, and a differential equation dependent on a dynamic game in the continuous time case.

We have given an overview of the certainty equivalence principle derived by Whittle [78] and Başar and Bernhard [7], and given a new explicit derivation of this for the linear-quadratic continuous time finite horizon output feedback $\mathscr{H}_\infty$ problem.

In the discrete problem, we have used the information state ideas of James, Baras and Elliott [40] to construct a solution to the one-step-delayed output feedback problem. This derivation is not new, but the ideas involved will be used heavily in Chapter 4 to construct solutions to the multi-rate sampled-data problem.

4. MULTI-RATE SAMPLED-DATA $\mathscr{H}_\infty$ CONTROL

4.1 Motivation

In this chapter we will consider the disturbance rejection problem for multi-rate sampled-data systems. By sampled-data systems we mean systems where a continuous signal from a plant is sampled to give measurements at discrete intervals. Further, control is exerted on the plant through a discrete output from the controller, which is then converted to a continuous signal using a hold device.

The requirement for sampled-data systems arises naturally in engineering problems in two main ways. The first is through the desire to implement modern controllers on digital hardware. This means that the controller has to be implemented as a discrete algorithm. Often, the assumption is made that by sampling at a fast enough rate it is possible to simply approximate a desired continuous controller with such a discrete controller. This natural assumption has been shown to be justified in practice, and is also given some theoretical justification by [41]. However, we aim in this chapter to give a systematic design procedure for such systems. That is, we wish to know exactly how fast it is necessary to sample in order to achieve some specified level of performance.

The second major justification for the sampled-data approach to controller design is that some processes only provide discrete information to the controller. A simple example is that of a chemical process, where in order to find out how far a particular reaction has progressed, it is necessary to remove a sample of the reactants and perform some kind of analysis on it. This would normally provide information only at discrete intervals. Similarly, in biomedical applications such as anaesthetic control measurements such as heart-rate can only be obtained discretely. Again, for such a system we are looking for a systematic method to design robust controllers.

Several authors have given synthesis techniques for construction of $\mathscr{H}_\infty$ suboptimal controllers for sampled-data systems, in particular solutions to the infinite horizon time invariant problem are given by Hara and Kabamba [31] and Bamieh and Pearson [3], by constructing equivalent discrete time systems. Further simplifications to this solution are given by Chen and Francis in [19]. These results are derived using lifting techniques, where the original hybrid system is transformed to one on a discrete operator-valued space. The finite horizon time varying problem has also been considered, notably by Başar and Bernhard [7]. Further solutions to this problem are considered in [6, 67, 64, 11].

The above results all consider systems with single rate periodic sampling and hold devices synchronized with each other. In this chapter we will be considering sampled-data systems which are both *multi-rate* and *asynchronous*. In this case the controller is connected to the plant through multiple separate sampling and holding devices, which are assumed to be asynchronous. Further, we will not assume any periodicity of the sampling and hold operators.

The motivation for this kind of synthesis result comes both from physical systems and technical reasons. One example of a physical system where the design of such a controller is a significant problem is in the control of automobile engines. In this case the system has a natural almost-periodic behaviour, and also different kinds of measurements and control are available at different timings in the engine cycle. Also, the rate of the engine is not fixed, but varies according to a command signal from the driver.

We give necessary and sufficient conditions for the existence of a controller satisfying a given performance bound with given asynchronous sampling and hold devices, and give a realisation for such a controller if one exists. This allows a comparison of the achievable performance for systems with different sampling rates.

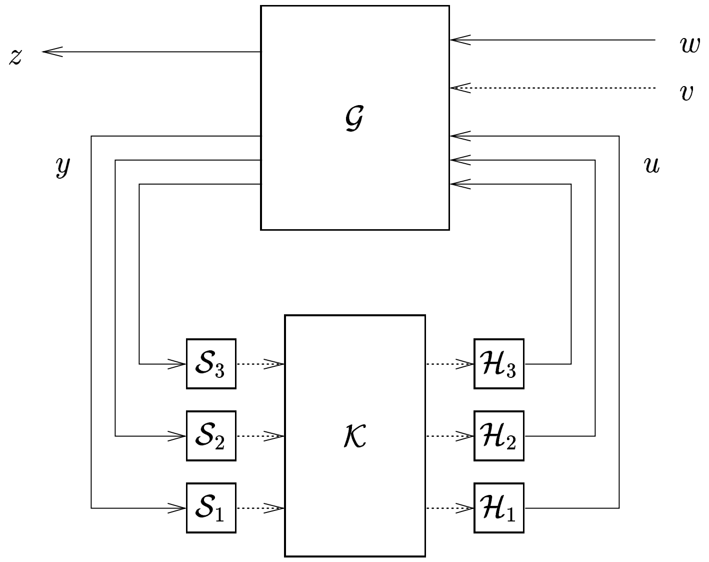


Figure 4.1: The Multi-rate Sampled-data Problem

The general sampled-data setup we will use is shown in Figure 4.1. Again we will use as performance measure an induced 2-norm. It will be natural for us to consider some disturbances as discrete and others as continuous.

Multi-rate systems have been considered by several authors, notably Chen and Qiu [20] and Voulgaris [76]. In these papers, the sampling and hold devices are assumed to have rational rates, and to be synchronized with each other, so that the system can be lifted up to either the fastest or the slowest commensurate period.

We will not use lifting techniques in this chapter, and so we will avoid the need for overall periodicity, and hence shift invariance when lifted to some appropriate level, of the overall system. We will use the results of Chapter 3 here to derive solutions in the time domain, by separating the problem into the state feedback and estimation cases, and then showing that the separation theory holds for this problem.

Firstly, we will describe a very general class of systems, that of hybrid systems with discrete jumps in the state vector. We will derive necessary and sufficient conditions for existence of a state feedback controller for such systems. We then specialise these results to the multi-rate sampled-data problem, and derive an expression for the information state for the measurement feedback problem. We will then be able to recombine these results to derive a solution to the output feedback problem.

4.2 State feedback

4.2.1 Systems with jumps

We will first consider controller synthesis for a wide class of systems, those with discrete jumps in the state vector. Let $t_0 < t_1 < \cdots < t_f$ be a fixed known sequence of times. We will consider systems of the form

$$\begin{aligned} \dot{x} &= \hat{A}(t)x(t) + \hat{B}_1(t)w(t) \qquad t \notin \{t_0, t_1, \ldots, t_f\} \qquad x(t_0^-) = x_0 \\ x(t_i) &= \tilde{A}[t_i]x(t_i^-) + \hat{B}_2[t_i]\tilde{u}[t_i] \\ z(t) &= \hat{C}_1(t)x(t) \end{aligned} \tag{4.1}$$

for $t_0 \leq t \leq t_f$ and $t_i \in \{t_0, t_1, \ldots, t_f\}$. The system matrices $\hat{A}$, $\hat{B}_1$ and $\hat{C}_1$ are bounded functions of the continuous interval $[t_0, t_f]$, and $\tilde{A}$, $\hat{B}_2$ are bounded discrete functions defined on the set $\{t_0, t_1, \ldots, t_f\}$.

Here w is a continuous disturbance input, z is the continuous controlled output, and $\tilde{u}$ is a discrete control input. The state x is a right continuous function of time, but it may be left discontinuous with finite jumps at times t_i. We will denote by $x(t_i^-)$ the value of $x(t)$ just before time t_i; that is $x(t_i^-) := \lim_{\varepsilon>0, \varepsilon\to 0} x(t_i - \varepsilon)$.

We will show that, if we wish to synthesize a controller for a sampled-data system, it is possible to express the continuous system combined with generalized sample and hold operators in this form.

This class of systems was developed by Sun, Nagpal and Khargonekar [64], who used quite different techniques to construct solutions to $\mathscr{H}_\infty$ synthesis problems for particular special cases of this general form of system. In this chapter we solve explicitly the general problem of finding a state feedback controller $\tilde{u}[t_i] = \mu\big(t_i, x(t_i^-)\big)$ such that the $\mathscr{L}_2$-induced norm from w to z is less than a prespecified level γ. We will then specialize this result to both single and multi-rate sampled-data systems, and go on to construct the controller in the measurement feedback case, using the information state theory of Chapter 3.

4.2.2 Single hold

First, to illustrate the approach, we will consider synthesis with a single rate discrete input $\tilde{u}$. Define the space of causal and memoryless state feedback strategies by

$$\mathscr{U}_{sf}^{n,p}[t_i, t_j] := \big\{\mu : \{t_i, \ldots, t_j\} \times \mathbb{R}^n \to \mathbb{R}^p\big\}$$

with the control signal defined by $\tilde{u}[t_i] = \mu\big(t_i, x(t_i^-)\big)$ for $\mu \in \mathscr{U}_{sf}$. Then we would like to find the minimizing controller for the following dynamic game problem

$$\inf_{\mu\in\mathscr{U}_{sf}} \sup_{w\in\mathscr{L}_2} \left\{ \int_{t_0}^{t_f} z(t)'z(t) - \gamma^2 w(t)'w(t)\,dt + x(t_f^-)'Q_f x(t_f^-) \right\}.$$

where Q_f is a positive semidefinite weighting matrix. As in the purely discrete problem of Chapter 3, we can apply dynamic programming to give

$$\begin{aligned}
&\inf_{\mu\in\mathscr{U}_{sf}[t_0,t_f]}\sup_{w\in\mathscr{L}_2}\left\{\int_{t_0}^{t_f} z(t)'z(t)-\gamma^2 w(t)'w(t)\,dt + x(t_f)'Q_f x(t_f)\right\}\\
&\quad=\inf_{\tilde u[t_0]\in\mathbb{R}^{m_2}}\sup_{w\in\mathscr{L}_2[t_0,t_1]}\left\{-\gamma^2 x_0'Q_0x_0+\int_{t_0}^{t_1} z(t)'z(t)-\gamma^2 w(t)'w(t)\,dt\right.\\
&\qquad+\inf_{\tilde u[t_1]\in\mathbb{R}^{m_2}}\sup_{w\in\mathscr{L}_2[t_1,t_2]}\left\{\int_{t_1}^{t_2} z(t)'z(t)-\gamma^2 w(t)'w(t)\,dt+\cdots\right.\\
&\qquad+\inf_{\tilde u[t_{f-1}]\in\mathbb{R}^{m_2}}\sup_{w\in\mathscr{L}_2[t_{f-1},t_f]}\left\{\int_{t_{f-1}}^{t_f} z(t)'z(t)-\gamma^2 w(t)'w(t)\,dt\right.\\
&\qquad\qquad\left.\left.\left.+\,x(t_f^-)'Q_f x(t_f^-)\right\}\right\}\right\}.
\end{aligned}$$

Then as before we can define the Value of the game

$$V\big(t_i,x(t_i^-)\big)=\inf_{\mu\in\mathscr{U}_{sf}[t_i,t_f]}\sup_{w\in\mathscr{L}_2[t_i,t_f]}\int_{t_i}^{t_f} z(t)'z(t)-\gamma^2 w(t)'w(t)\,dt+x(t_f^-)'Q_f x(t_f^-).$$

It is then straightforward to show by induction that the Value function satisfies the dynamic programming recursion

$$\begin{aligned}
&V\big(t_i,x(t_i^-)\big)=\\
&\quad\inf_{\tilde u[t_i]\in\mathbb{R}^{m_2}}\sup_{w\in\mathscr{L}_2[t_i,t_{i+1}]}\left\{\int_{t_i}^{t_{i+1}} z(t)'z(t)-\gamma^2 w(t)'w(t)\,dt+V\big(t_{i+1},x(t_{i+1}^-)\big)\right\}
\end{aligned}\tag{4.2}$$

with boundary condition

$$V\big(t_f,x(t_f^-)\big)=x(t_f^-)'Q_f x(t_f^-).$$

Note now that on each subinterval $[t_i,t_{i+1}]$, the $\tilde u$ player has only *open loop* information, that is $\tilde u$ only has knowledge of $x(t_i^-)$. Further, according to Remark 3.7 in Chapter 3, we need only consider w disturbances in $\mathscr{L}_2$. Therefore, each subproblem is tantamount to a *static* linear quadratic game. This recursion is thus very similar to that derived in the discrete $\mathscr{H}_\infty$ problem, only we have $\tilde u[t_i]\in\mathbb{R}^{m_2}$ and $w\in\mathscr{L}_2[t_i,t_{i+1}]$.

We will now show by induction that the Value function has the form $V\big(t_i,x(t_i^-)\big)=x(t_i^-)'X[t_i]x(t_i^-)$ for some matrix function X. Clearly, this holds at time t_f, since $V\big(t_f,x(t_f^-)\big)=x(t_f^-)'Q_f x(t_f^-)$. Now assume that it holds at time t_{i+1}.

Theorem 4.1. *Suppose that $V\big(t_{i+1},x(t_{i+1}^-)\big)=x(t_{i+1}^-)'X[t_{i+1}]x(t_{i+1}^-)$. Then, if there exists a bounded solution to Riccati differential equation*

$$-\dot Z_i=\hat A'Z_i+Z_i\hat A+\gamma^{-2}Z_i\hat B_1\hat B_1'Z_i+\hat C_1\hat C_1' \qquad Z_i(t_{i+1})=X[t_{i+1}]\tag{4.3}$$

on the interval $[t_i,t_{i+1}]$, then the value at time t_i is given by

$$V(t_i,x(t_i^-))=x(t_i^-)'X[t_i]x(t_i^-),$$

where

$$X[t_i] = \tilde{A}[t_i]'Z_i(t_i)\tilde{A}[t_i] - \tilde{A}[t_i]'Z_i(t_i)\hat{B}_2[t_i]\big(\hat{B}_2[t_i]'Z_i(t_i)\hat{B}_2[t_i]\big)^{\dagger}\hat{B}_2[t_i]'Z_i(t_i)\tilde{A}[t_i].$$

Further, one possible minimizing strategy for $\tilde{u}[t_i]$ *is given by*

$$\tilde{u}[t_i] = -\big(\hat{B}_2[t_i]'Z_i(t_i)\hat{B}_2[t_i]\big)^{\dagger}\hat{B}_2[t_i]'Z_i(t_i)\tilde{A}[t_i]x(t_i^-) \tag{4.4}$$

and the corresponding worst case disturbance $w \in \mathscr{L}_2[t_i, t_{i+1}]$ *is given by*

$$w(t) = \gamma^{-2}\hat{B}_1(t)'Z_i(t)x(t).$$

Proof. Define the cost function on the subinterval $[t_i, t_{i+1}]$

$$J_s\big(x(t_i^-), \tilde{u}[t_i], w\big) = \int_{t_i}^{t_{i+1}} z(t)'z(t) - \gamma^2 w(t)'w(t)\,dt + x(t_{i+1}^-)'X[t_{i+1}]x(t_{i+1}^-)$$

Then

$$V\big(t_i, x(t_i^-)\big) = \inf_{\tilde{u}[t_i]\in\mathbb{R}^{m_2}} \sup_{w\in\mathscr{L}_2[t_i,t_{i+1}]} J_s\big(x(t_i^-), \tilde{u}[t_i], w\big). \tag{4.5}$$

We first solve the maximization problem. Straightforward completion of the square gives

$$J_s\big(x(t_i^-), \tilde{u}[t_i], w\big) = x(t_i)'Z_i(t_i)x(t_i) - \int_{t_i}^{t_{i+1}} \gamma^2|w(t) - \gamma^{-2}\hat{B}'(t)Z_i(t)x(t)|^2\,dt \tag{4.6}$$

which gives the above expression for the worst case w. Note that, although this is expressed as a feedback law, for any fixed $\tilde{u}[t_i]$ it corresponds to a unique signal in $\mathscr{L}_2$.

We then have to solve the minimization problem

$$\begin{aligned}
J_s\big(x(t_i^-), \tilde{u}[t_i], w\big) &= \inf_{\tilde{u}[t_i]\in\mathbb{R}^{m_2}} \sup_{w\in\mathscr{L}_2[t_i,t_{i+1}]} J_s\big(x(t_i^-), \tilde{u}[t_i], w\big)\\
&= \inf_{\tilde{u}[t_i]\in\mathbb{R}^{m_2}} x(t_i)'Z_i(t_i)x(t_i)\\
&= \inf_{\tilde{u}[t_i]\in\mathbb{R}^{m_2}} \big(\tilde{A}[t_i]x(t_i^-) + \hat{B}_2[t_i]\tilde{u}[t_i]\big)'Z_i(t_i)\big(\tilde{A}[t_i]x(t_i^-) + \hat{B}_2[t_i]\tilde{u}[t_i]\big)\\
&= \inf_{\tilde{u}[t_i]\in\mathbb{R}^{m_2}} \begin{pmatrix} x(t_i^-)\\ \tilde{u}[t_i]\end{pmatrix}' \begin{bmatrix} \tilde{A}[t_i]'Z_i(t_i)\tilde{A}[t_i] & \tilde{A}[t_i]'Z_i(t_i)\hat{B}_2[t_i]\\ \hat{B}_2[t_i]'Z_i(t_i)\tilde{A}[t_i] & \hat{B}_2[t_i]'Z_i(t_i)\hat{B}_2[t_i]\end{bmatrix} \begin{pmatrix} x(t_i^-)\\ \tilde{u}[t_i]\end{pmatrix}.
\end{aligned}$$

Since $Q_f \geq 0$, Proposition 3.19 implies that $Z_i(t_i) \geq 0$, and hence applying the Schur complement formula gives

$$\begin{bmatrix} \tilde{A}[t_i]'Z_i(t_i)\tilde{A}[t_i] & \tilde{A}[t_i]'Z_i(t_i)\hat{B}_2[t_i]\\ \hat{B}_2[t_i]'Z_i(t_i)\tilde{A}[t_i] & \hat{B}_2[t_i]'Z_i(t_i)\hat{B}_2[t_i]\end{bmatrix} = T\begin{bmatrix} S & 0\\ 0 & \hat{B}_2[t_i]'Z_i(t_i)\hat{B}_2[t_i]\end{bmatrix}T'$$

where

$$\begin{aligned}
T &= \begin{bmatrix} I & \tilde{A}[t_i]'Z_i(t_i)\hat{B}_2[t_i]\big(\hat{B}_2[t_i]'Z_i(t_i)\hat{B}_2[t_i]\big)^{\dagger}\\ 0 & I\end{bmatrix}\\
S &= \tilde{A}[t_i]'Z_i(t_i)\tilde{A}[t_i] - \tilde{A}[t_i]'Z_i(t_i)\hat{B}_2[t_i]\big(\hat{B}_2[t_i]'Z_i(t_i)\hat{B}_2[t_i]\big)^{\dagger}\hat{B}_2[t_i]'Z_i(t_i)\tilde{A}[t_i]
\end{aligned}$$

and the result then follows. ∎

This result shows that, if $V\big(t_{i+1}, x(t_{i+1}^-)\big)$ is quadratic in x, then so is $V\big(t_i, x(t_i^-)\big)$, and it gives us a recursive formula for $X[t_i]$. Hence by induction we can conclude that, if there exists a solution to the Riccati equation (4.3) on each interval $[t_i, t_{i+1}]$, then $V\big(t_i, x(t_i^-)\big)$ is quadratic for all t_i.

If $x(t_0^-) = 0$, then $V\big(t_0, x(t_0^-)\big) = 0$, and hence

$$0 = \inf_{\mu \in \mathscr{U}_{sf}} \sup_{w \in \mathscr{L}_2} \left\{ \int_{t_0}^{t_f} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt + x(t_f^-)'Q_f x(t_f^-) \right\}.$$

Hence the state feedback controller defined by equation (4.4) achieves the induced $\mathscr{L}_2$ norm bound $\|\mathcal{T}_{zw}\| \leq \gamma$.

We now derive the necessary conditions for existence of a bounded upper value for the dynamic game, and hence those for existence of a γ-feasible state feedback controller.

4.2.3 Necessary conditions for state feedback control

We will make use of the following result, from Başar and Bernhard [7, Theorem 8.3].

Theorem 4.2. *Let A, B_1 and Q be bounded matrix functions on the interval $[t_0, t_f]$, and suppose $Q(t) > 0$ for all $t \in [t_0, t_f]$. Consider the Riccati equation*

$$-\dot{S} = A'S + SA + \gamma^{-2} S B_1 B_1' S + Q \qquad S(t_f) = Q_f. \tag{4.7}$$

Define

$$\hat{\gamma} = \inf\{\tilde{\gamma}\ ;\ \text{ for all } \gamma > \tilde{\gamma} \text{ the RDE (4.7) has a bounded solution on } [t_0, t_f]\}.$$

Then, for any fixed $x_0 \in \mathbb{R}^n$, the quadratic optimization

$$\sup_{w \in \mathscr{L}_2[t_0, t_f]} \int_{t_0}^{t_f} x(t)'Q(t)x(t) - \gamma^2 w(t)'w(t)\, dt + x(t_f)Q_f x(t_f).$$

subject to the dynamics

$$\dot{x} = Ax + B_1 w \qquad x(t_0) = x_0$$

has a finite supremum if $\gamma > \hat{\gamma}$, and only if $\gamma \geq \hat{\gamma}$.

Theorem 4.3. *Suppose there exists a control law $\tilde{u}[t_i] = \mu\big(t_i, x(t_i^-)\big)$ for $i = 1, \ldots, f$ which results in*

$$\sup_{w \in \mathscr{L}_2} \left\{ \int_{t_0}^{t_f} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt + x(t_f^-)'Q_f x(t_f^-) \right\} \leq 0$$

for all $w \in \mathscr{L}_2[t_0, t_f]$, with initial state $x(t_0^-) = 0$. Define $\hat{\gamma}$ as the infimum of the set of γ such that the Riccati differential equation (4.3) has a bounded solution on $[t_i, t_{i+1}]$ for all $i = 1, \ldots, f$. Then $\gamma \geq \hat{\gamma}$.

Proof. Suppose that $\gamma < \hat{\gamma}$. Then, for some i, γ is strictly less than the infimum of the set of γ such that the Riccati differential equation (4.3) has a bounded solution on $[t_i, t_{i+1}]$. Therefore, applying Lemma 4.2, we know that the maximization problem given by equation (4.5) is unbounded with respect to w for all $x(t_i)$. Hence the upper value of the game is unbounded. ■

4.3 Single-rate sampled-data control

We now specialize the results of the previous section to the full state information sampled data problem, with a single zero-order hold input. Suppose the continuous time system has the form

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_1(t)w(t) + B_2(t)u(t) \qquad x(t_0) = x_0 \\ z(t) &= C_1(t)x(t) + D_{12}(t)u(t) \end{aligned} \tag{4.8}$$

on the interval $[t_0, t_f]$, with the system matrices bounded functions of time. We again make the following assumptions about the system matrices:

$$D_{12}(t)'C_1(t) = 0 \qquad\qquad D_{12}(t)'D_{12}(t) = I$$

for all $t \geq 0$. The hold operator $\mathcal{H} : \ell_2^{m_2}[0, f] \to \mathscr{L}_2^{m_2}[t_0, t_f]$ is defined by

$$u(t) = \tilde{u}[t_k] \qquad \forall\, t \in [t_k, t_{k+1}]).$$

We may then write the following system with jumps describing the relationship between the discrete input $\tilde{u}$, the continuous input w and the continuous output z. Let

$$\begin{aligned} \hat{A} &= \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\ \hat{B}_1 &= \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \qquad \hat{B}_2 = \begin{bmatrix} 0 \\ I \end{bmatrix} \\ \hat{C}_1 &= \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \end{aligned} \tag{4.9}$$

then it is easy to see that the system

$$\begin{aligned} \dot{p} &= \hat{A}(t)p(t) + \hat{B}_1(t)w(t) \qquad t \notin \{t_0, t_1, \ldots, t_f\} \qquad p(t_0^-) = x_0 \\ p(t_i) &= \tilde{A}[t_i]p(t_i^-) + \hat{B}_2[t_i]\tilde{u}[t_i] \\ z(t) &= \hat{C}_1(t)p(t) \end{aligned} \tag{4.10}$$

is a realization for the sampled-data system also. Then the following theorem is a straightforward consequence of Theorem 4.1.

Theorem 4.4. *Consider the system described by equations* (4.10). *Let*

$$X[t_f] = \begin{bmatrix} Q_f & 0 \\ 0 & 0 \end{bmatrix}$$

and suppose there exists a bounded solution to the Riccati differential equation

$$-\dot{Z}_i = \hat{A}'Z_i + Z_i\hat{A} + \gamma^{-2}Z_i\hat{B}_1\hat{B}_1'Z_i + \hat{C}_1\hat{C}_1' \qquad Z_i(t_{i+1}) = X[t_{i+1}]$$

on each interval $[t_i, t_{i+1}]$ *for* $0 \leq i < f$*, where* $X[t_i]$ *is defined recursively by*

$$X[t_i] = \begin{bmatrix} Z_{i11}(t_i) - Z_{i12}(t_i)Z_{i22}^{-1}(t_i)Z_{i21}(t_i) & 0 \\ 0 & 0 \end{bmatrix}$$

for $0 \leq i < f$*. Then the controller*

$$u(t_i) = -Z_{i22}(t_i)^{-1}Z_{i12}(t_i)'x(t_i)$$

results in $\|\mathcal{T}_{zw}\| \leq \gamma$.

Proof. All we need to do is apply the formulae of Theorem 4.1 to the above special case, which directly gives the above expressions. The only thing we have to show is that $Z_{i22}(t_i)$ is invertible, so that we can replace the pseudo-inverse by the usual inverse. This holds because equation (4.6) implies

$$\begin{aligned} p(t_i)'Z_i(t_i)p(t_i) &\geq J_s\big(p(t_i^-), \tilde{u}[t_i], w\big) \\ &= \int_{t_i}^{t_{i+1}} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt + p(t_{i+1}^-)'X[t_{i+1}]p(t_{i+1}^-) \end{aligned}$$

and this holds for all $\tilde{u}[t_i]$, w and $p(t_i^-)$. Choosing $w = 0$ and $p(t_i) = 0$, and using

$$\begin{aligned} p(t_i) &= \tilde{A}[t_i]p(t_i^-) + \hat{B}_2[t_i]\tilde{u}[t_i] \\ &= \begin{pmatrix} x(t_i) \\ \tilde{u}[t_i] \end{pmatrix} \end{aligned}$$

we have

$$\tilde{u}[t_i]'Z_{i22}(t_i)\tilde{u}[t_i] \geq \int_{t_i}^{t_{i+1}} z(t)'z(t)\, dt + p(t_{i+1}^-)'X[t_{i+1}]p(t_{i+1}^-)$$

and since $z(t)'z(t) = x(t)'C_1(t)'C_1(t)x(t) + u(t)'u(t) > 0$ for all $u \neq 0$, this implies that $Z_{i22}(t_i) > 0$. ■

We have now shown that, provided a solution to equation (4.3) exists on $[t_i, t_{i+1}]$, then the *upper value* for the static game on this subinterval is bounded. Hence there will exist a solution to the the recursion (4.2) and a well defined minimax control. However, we have not shown that this is a saddle point for the game. Clearly if on each subinterval the upper value exists, then the lower value, given by

$$\sup_{w \in \mathscr{L}_2[t_i, t_{i+1}]} \inf_{\tilde{u}[t_i] \in \mathbb{R}^{m_2}} J_s\big(x(t_i), \tilde{u}[t_i], w\big).$$

is finite, since the quadratic terms in u and x are positive, which implies that the lower value is bounded below, and the lower value is bounded above by the upper value. However, in order for a saddle point to exist we need the lower value to equal the upper value. Note however that the existence of a saddle point is not necessary for the existence of a state feedback controller satisfying the induced 2-norm criterion.

Generalized hold functions. A wide class of systems can be described in the form of equations (4.1). This clearly includes both discrete and continuous time systems as special cases. We have seen above that the sampled-data system, combined with a zero-order hold, fits into this framework. It is easily possible to describe also sampled-data systems with any hold operator describable by a finite dimensional linear time varying system. For example, if given $\tilde{u}[t_i]$, the hold operator with output $u(t)$ on the interval $[t_i, t_{i+1}]$ is represented by the continuous LTI system

$$\begin{aligned} \dot{x}_2 &= Fx_2 \qquad x_2(t_i) = G\tilde{u}[t_i] \\ u(t) &= Hx_2(t) \end{aligned}$$

then if this hold operator is connected to the continuous time system (4.8), the corresponding system with jumps has coefficient matrices

$$\hat{A} = \begin{bmatrix} A & B_2 H \\ 0 & F \end{bmatrix} \qquad \tilde{A} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$
$$\hat{B}_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \qquad \hat{B}_2 = \begin{bmatrix} 0 \\ G \end{bmatrix}$$
$$\hat{C}_1 = \begin{bmatrix} C_1 & D_{12} H \end{bmatrix}.$$

4.4 Multi-rate controller synthesis

4.4.1 State feedback with a multi-rate hold: A simple example

In this section we proceed with an example of state feedback with a multi-rate hold, where the input consists of two channels u_1 and u_2, which can change as shown in Figure 4.2. We revert to the continuous time system description given by equations (4.8), hence the state x is now a continuous function, and we do not need to distinguish between $x(t_i^-)$ and $x(t_i)$. In the next section we will solve the more general multi-rate state feedback sampled-data problem with an arbitrary number of inputs. In Figure 4.2,

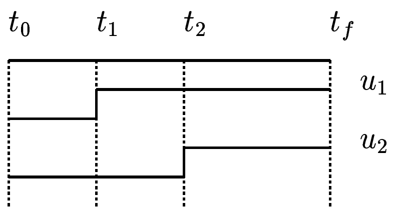


Figure 4.2: Input signals generated by multi-rate hold

u_1 is fixed on $[t_1, t_f)$ and also on $[t_a, t_1)$, for some $t_a < t_0$. Also, u_2 is required to be fixed on $[t_0, t_2)$ and on $[t_2, t_f)$. We therefore use $\{t_0, t_1, t_2, t_f\}$ as the set of points on which to break up the interval. These choices can be summed up as

$$\begin{array}{cl} \text{At time} & \qquad \text{Choose} \\ t_0 & u_2 \text{ for the interval } [t_0, t_2) \\ t_1 & u_1 \text{ for the interval } [t_1, t_f) \\ t_2 & u_2 \text{ for the interval } [t_2, t_f) \end{array} \tag{4.11}$$

The first step is to solve the last stage of the dynamic programming recursion (4.2), on the interval $[t_2, t_f]$. Using the same technique as before, first we solve the corresponding maximization problem

$$\sup_{w \in \mathscr{L}_2[t_2, t_f]} J_s\big(x(t_2), \tilde{u}[t_2], w\big)$$

where

$$J_s\big(x(t_2), \tilde{u}[t_2], w\big) = \int_{t_2}^{t_f} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt + x(t_f)' \begin{bmatrix} Q_f & 0 \\ 0 & 0 \end{bmatrix} x(t_f).$$

If there exists a solution to the Riccati differential equation

$$-\dot{Z} = \hat{A}'Z + Z\hat{A} + \gamma^{-2}Z\hat{B}\hat{B}'Z + \hat{C}'\hat{C} \qquad Z(t_f) = \begin{bmatrix} Q_f & 0 \\ 0 & 0 \end{bmatrix}$$

on the interval $[t_2, t_f]$, partitioning $\tilde{u}[t_2]$ into $\begin{pmatrix} \tilde{u}_1[t_2] \\ \tilde{u}_2[t_2] \end{pmatrix}$ we have

$$\sup_{w \in \mathscr{L}_2[t_2,t_1]} J\big(x(t_2), \tilde{u}[t_2], w\big) = \begin{pmatrix} x(t_2) \\ \tilde{u}_1[t_2] \\ \tilde{u}_2[t_2] \end{pmatrix}' \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{12}' & Z_{22} & Z_{23} \\ Z_{13}' & Z_{23}' & Z_{33} \end{bmatrix} (t_2) \begin{pmatrix} x(t_2) \\ \tilde{u}_1[t_2] \\ \tilde{u}_2[t_2] \end{pmatrix}.$$

We can use the Schur complement formula again to write

$$\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{12}' & Z_{22} & Z_{23} \\ Z_{13}' & Z_{23}' & Z_{33} \end{bmatrix} = T' \begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} & 0 \\ \tilde{Z}_{12}' & \tilde{Z}_{22} & 0 \\ 0 & 0 & Z_{33}^{-1} \end{bmatrix} T$$

where

$$T = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ Z_{33}^{-1}Z_{13}' & Z_{33}^{-1}Z_{23}' & I \end{bmatrix}$$

$$\tilde{Z} = \begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} \\ \tilde{Z}_{12}' & \tilde{Z}_{22} \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}' & Z_{22} \end{bmatrix} - \begin{bmatrix} Z_{13} \\ Z_{23} \end{bmatrix} Z_{33}^{-1} \begin{bmatrix} Z_{13}' Z_{23}' \end{bmatrix}.$$

We can now solve the minimization problem

$$\inf_{\tilde{u}_2[t_2]} \sup_{w \in \mathscr{L}_2[t_2,t_f]} J_s\big(x(t_2), \tilde{u}_2[t_2], w\big)$$

and the minimizing $\tilde{u}_2[t_2]$ is given by

$$\tilde{u}_2[t_2] = -Z_{33}^{-1}(t_2)\big(Z_{13}(t_2)'x(t_2) + Z_{23}(t_2)'\tilde{u}_1[t_2]\big).$$

As we might expect, the solution for $\tilde{u}_2[t_2]$ depends on the value of $\tilde{u}_1[t_2]$, which will have been chosen at time t_1. In order to determine which value we should choose for $\tilde{u}_1[t_1]$, we need to solve the next step of the recursion on the interval $[t_1, t_2]$.

The Value of the game at time t_2 is given by

$$V\big(t_2, x(t_2), \tilde{u}_1[t_1]\big) = \begin{pmatrix} x(t_2) \\ \tilde{u}_1[t_1] \end{pmatrix}' \tilde{Z} \begin{pmatrix} x(t_2) \\ \tilde{u}_1[t_1] \end{pmatrix} \tag{4.12}$$

and hence the final weight for the next subproblem, and therefore also the boundary condition for the Riccati differential equation on $[t_1, t_2]$ is given by

$$Z(t_2^-) = \begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} & 0 \\ \tilde{Z}_{12}' & \tilde{Z}_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that if we substitute the value of $\tilde{u}_1[t_1]$ into equation (4.12), this gives the value of the game from time t_2 onwards. Also, if we need to calculate the value of the game at a time at which no hold operator becomes active, this can be done by simply omitting the optimization with respect to u, since u is fixed at some previous hold time(s).

We now know $\tilde{u}_2[t_2]$ in terms of the state $x(t_2)$ and the value of $\tilde{u}_1[t_2]$. The next subproblem, in a similar way, will give $\tilde{u}_1[t_1]$ in terms of $x(t_1)$ and $\tilde{u}_2[t_1]$. This chain of optimizations continues until we reach a time when none of the channels are constrained, that is when we have choice over all spatial components of u. This will occur at the initial time, but may also occur at later times also. Note also that when the controller is in place, at any time we need to choose some components of u all the other components of u have definite values, rather than being functions of past states. Hence we can always calculate the components we need in terms of the current state.

4.4.2 The general multi-rate case for jump systems

In this section, we will return to the problem of synthesizing state feedback controller for systems with jumps, as in Section 4.2.1. However, we will consider the multi-rate problem; the discrete inputs $\tilde{u}$ will be divided into 'channels', each of which can have a hold operator that becomes active at different times.

At each time t_i, we will describe which channels of $\tilde{u}$ can be chosen in the following way. Let $\bar{R}[t_k] \in \mathbb{R}^{m_2 \times m_2}$ be a permutation matrix, that is, a nonsingular matrix in which each column contains exactly one element equal to 1, with all the other elements in that column zero. Then partition $\bar{R}[t_k]$ as

$$\bar{R}[t_k] = \begin{bmatrix} R^{\perp}[t_k] \\ R[t_k] \end{bmatrix}.$$

At time t_k, we may partition $\tilde{u}[t_k]$ as

$$\tilde{u}[t_k] = \bar{R}[t_k]' \begin{pmatrix} R^{\perp}[t_k]\tilde{u}[t_k] \\ R[t_k]\tilde{u}[t_k] \end{pmatrix}.$$

If we now make the restriction that $\tilde{u}$ must satisfy

$$R^{\perp}[t_k]\tilde{u}[t_{k-1}] = R^{\perp}[t_k]\tilde{u}[t_k]$$

then the choice of R determines the components of $\tilde{u}$ which may change at any time. Only those spatial components of $\tilde{u}[t_k]$ in the kernel of $R^{\perp}[t_k]$ may be chosen at time t_k. We will use this construction to define the multi-rate property of the hold operator. Naturally, the controller will have knowledge of $R[t_k]$ at least each time t_k, and in fact we shall see that this is all the knowledge of R that the controller needs in order to minimize the upper value of the dynamic game. We assume that $R[t_0] = I$ so that all the hold channels are well defined at time t_0. Then the Isaacs recursion becomes

$$V\big(t_i, x(t_i^-), R^{\perp}[t_i]\tilde{u}[t_i]\big) = \inf_{R[t_i]\tilde{u}[t_i]} \sup_{w \in \mathscr{L}_2[t_i, t_{i+1}]} \left\{ \int_{t_i}^{t_{i+1}} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt + V\big(t_{i+1}, x(t_{i+1}^-), R^{\perp}[t_{i+1}]\tilde{u}[t_{i+1}]\big) \right\}$$

We now show, as in the single rate case, that $V\big(t_i, x(t_i^-), R^{\perp}[t_i]\tilde{u}[t_i]\big)$ is a quadratic for all i, $0 \le i \le f$, and give a recursive formula for it.

Theorem 4.5. *Suppose that*

$$V\big(t_{i+1}, x(t_{i+1}^-), R^\perp[t_{i+1}]\tilde{u}[t_{i+1}]\big) = \begin{pmatrix} x(t_{i+1}^-) \\ R^\perp[t_{i+1}]\tilde{u}[t_{i+1}] \end{pmatrix}' X[t_{i+1}] \begin{pmatrix} x(t_{i+1}) \\ R^\perp[t_{i+1}]\tilde{u}[t_{i+1}] \end{pmatrix}. \quad (4.13)$$

Then, if there exists a bounded solution to Riccati differential equation

$$-\dot{Z}_i = \hat{A}'Z_i + Z_i\hat{A} + \gamma^{-2}Z_i\hat{B}_1\hat{B}_1'Z_i + \hat{C}_1\hat{C}_1' \quad (4.14)$$

with boundary condition

$$Z_i(t_{i+1}) = \begin{bmatrix} I & 0 \\ 0 & R^\perp[t_{i+1}] \end{bmatrix}' X[t_{i+1}] \begin{bmatrix} I & 0 \\ 0 & R^\perp[t_{i+1}] \end{bmatrix} \quad (4.15)$$

on the interval $[t_i, t_{i+1}]$, *then the value at time* t_i *is given by*

$$V\big(t_i, x(t_i^-), R^\perp[t_i]\tilde{u}[t_i]\big) = \begin{pmatrix} x(t_i^-) \\ R^\perp[t_i]\tilde{u}[t_i] \end{pmatrix}' X[t_i] \begin{pmatrix} x(t_i^-) \\ R^\perp[t_i]\tilde{u}[t_i] \end{pmatrix}$$

where

$$\begin{aligned} X[t_i] = &\begin{bmatrix} \tilde{A}'Z\tilde{A} & \tilde{A}'Z\hat{B}_2R^{\perp\prime} \\ R^\perp\hat{B}_2'Z\tilde{A} & R^\perp\hat{B}_2'Z\hat{B}_2R^{\perp\prime} \end{bmatrix} \\ &- \begin{bmatrix} \tilde{A}'Z\hat{B}_2R' \\ R^\perp\hat{B}_2'Z\hat{B}_2R' \end{bmatrix} (R\hat{B}_2'Z\hat{B}_2R')^\dagger \begin{bmatrix} R\hat{B}_2Z\tilde{A} & R\hat{B}_2'Z\hat{B}_2R^{\perp\prime} \end{bmatrix} \end{aligned} \quad (4.16)$$

where all matrices are evaluated at time t_i. *Further, one possible minimizing strategy for* $R[t_i]\tilde{u}[t_i]$ *is given by*

$$Ru = -(R\hat{B}_2'Z\hat{B}_2R')^\dagger\big(R\hat{B}_2Z\tilde{A}x(t_i^-) + R\hat{B}_2'Z\hat{B}_2R^{\perp\prime}R^\perp u\big) \quad (4.17)$$

and the corresponding worst case disturbance $w \in \mathscr{L}_2[t_i, t_{i+1}]$ *is given by*

$$w(t) = \gamma^{-2}\hat{B}_1(t)'Z_i(t)x(t).$$

Proof. We could now apply directly Theorem 4.1, since clearly the multi-rate problem for a system with jumps is in fact a special case of the original synthesis problem for systems with jumps. However, the direct proof is straightforward, and illustrates the relationship to the previous special case with two inputs, so we include it here. Define the cost function on the subinterval $[t_i, t_{i+1}]$

$$\begin{aligned} J_s\big(x(t_i^-), \tilde{u}[t_i], w\big) = &\int_{t_i}^{t_{i+1}} z(t)'z(t) - \gamma^2 w(t)'w(t)\,dt \\ &+ \begin{pmatrix} x(t_{i+1}^-) \\ R^\perp[t_{i+1}]\tilde{u}[t_{i+1}] \end{pmatrix}' X[t_{i+1}] \begin{pmatrix} x(t_{i+1}) \\ R^\perp[t_{i+1}]\tilde{u}[t_{i+1}] \end{pmatrix}. \end{aligned} \quad (4.18)$$

Then the Isaacs recursion becomes, at time t_i,

$$V\big(t_i, x(t_i^-), R^\perp[t_i]\tilde{u}[t_i]\big) = \inf_{R[t_i]\tilde{u}[t_i]} \sup_{w\in\mathscr{L}_2[t_i,t_{i+1}]} J_s\big(x(t_i^-), \tilde{u}[t_i], w\big). \quad (4.19)$$

As in the previous example, we first solve the maximization with respect to w for fixed $\tilde{u}[t_i]$. Straightforward completion of the square gives

$$J_s\big(x(t_i^-), \tilde{u}[t_i], w\big) = x(t_i)'Z_i(t_i)x(t_i) - \int_{t_i}^{t_{i+1}} \gamma^2 |w(t) - \gamma^{-2}\hat{B}'(t)Z_i(t)x(t)|^2 \, dt \quad (4.20)$$

which gives the above expression for the worst case w. We then have to solve the minimization problem

$$\begin{aligned} J_s\big(x(t_i^-), \tilde{u}[t_i], w\big) &= \inf_{R[t_i]\tilde{u}[t_i]} \sup_{w \in \mathscr{L}_2[t_i, t_{i+1}]} J_s\big(x(t_i^-), \tilde{u}[t_i], w\big) \\ &= \inf_{R[t_i]\tilde{u}[t_i]} x(t_i)'Z_i(t_i)x(t_i) \\ &= \inf_{R[t_i]\tilde{u}[t_i]} \big(\tilde{A}[t_i]x(t_i^-) + \hat{B}_2[t_i]\tilde{u}[t_i]\big)' Z_i(t_i) \big(\tilde{A}[t_i]x(t_i^-) + \hat{B}_2[t_i]\tilde{u}[t_i]\big) \\ &= \inf_{R[t_i]\tilde{u}[t_i]} \begin{pmatrix} x(t_i^-) \\ \tilde{u}[t_i] \end{pmatrix}' \begin{bmatrix} \tilde{A}[t_i]'Z_i(t_i)\tilde{A}[t_i] & \tilde{A}[t_i]'Z_i(t_i)\hat{B}_2[t_i] \\ \hat{B}_2[t_i]'Z_i(t_i)\tilde{A}[t_i] & \hat{B}_2[t_i]'Z_i(t_i)\hat{B}_2[t_i] \end{bmatrix} \begin{pmatrix} x(t_i^-) \\ \tilde{u}[t_i] \end{pmatrix}. \end{aligned}$$

Since we only wish to minimize with respect to the channel of $\tilde{u}[t_i]$ which we are allowed to change at time t_i, we now separate $\tilde{u}$ into a sum of those channels which can be changed and those that cannot, using

$$\tilde{u}[t_i] = \begin{bmatrix} R^\perp[t_i] \\ R[t_i] \end{bmatrix}' \begin{pmatrix} R^\perp[t_i]\tilde{u}[t_i] \\ R[t_i]\tilde{u}[t_i] \end{pmatrix}.$$

Hence

$$J_s\big(x(t_i^-), \tilde{u}[t_i], w\big) = \inf_{R[t_i]\tilde{u}[t_i]} \begin{pmatrix} x(t_i^-) \\ R^\perp[t_i]\tilde{u}[t_i] \\ R[t_i]\tilde{u}[t_i] \end{pmatrix}' W \begin{pmatrix} x(t_i^-) \\ R^\perp[t_i]\tilde{u}[t_i] \\ R[t_i]\tilde{u}[t_i] \end{pmatrix}$$

where W is given by

$$\begin{aligned} W &= \begin{bmatrix} I & 0 \\ 0 & \begin{bmatrix} R^\perp[t_i] \\ R[t_i] \end{bmatrix} \end{bmatrix} \begin{bmatrix} \tilde{A}'Z_i\tilde{A} & \tilde{A}'Z_i\hat{B}_2 \\ \hat{B}_2'Z_i\tilde{A} & \hat{B}_2'Z_i\hat{B}_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \begin{bmatrix} R^\perp[t_i] \\ R[t_i] \end{bmatrix}' \end{bmatrix} \\ &= \begin{bmatrix} \tilde{A}'Z_i\tilde{A} & \tilde{A}'Z_i\hat{B}_2R^{\perp\prime} & \tilde{A}'Z_i\hat{B}_2R' \\ R^\perp\hat{B}_2'Z_i\tilde{A} & R^\perp\hat{B}_2'Z_i\hat{B}_2R^{\perp\prime} & R^\perp\hat{B}_2'Z_i\hat{B}_2R' \\ R\hat{B}_2'Z_i\tilde{A} & R\hat{B}_2'Z_i\hat{B}_2R^{\perp\prime} & R\hat{B}_2'Z_i\hat{B}_2R' \end{bmatrix} \end{aligned}$$

where all matrices are evaluated at time t_i. Applying the Schur complement formula to perform the minimization directly gives the desired result. ■

Note that, in the above formula for the minimax controller given by equation (4.17), we have $R[t_i]\tilde{u}[t_i]$ given in terms of both $x(t_i^-)$ and $R^\perp[t_i]\tilde{u}[t_i]$. This is as we expect, since $R^\perp[t_i]\tilde{u}[t_i]$ has a fixed value at time t_i, since it has already been chosen at previous hold times and cannot be changed at time t_i.

Theorem 4.6. *Suppose there exists a multi-rate state feedback control law given by* $\tilde{u}[t_i] = \mu\big(t_i, x(t_i^-)\big)$ *satisfying the condition that* $R^{\perp}[t_k]\tilde{u}[t_{k-1}] = R^{\perp}[t_k]\tilde{u}[t_k]$*, for* $i = 1, \ldots, f$ *which results in*

$$\sup_{w \in \mathscr{L}_2} \left\{ \int_{t_0}^{t_f} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt + x(t_f^-)'Q_f x(t_f^-) \right\} \leq 0$$

for all $w \in \mathscr{L}_2[t_0, t_f]$*, with initial state* $x(t_0^-) = 0$*. Define* $\hat{\gamma}$ *as the infimum of the set of* γ *such that the Riccati differential equation* (4.15) *with boundary conditions* (4.14) *has a bounded solution on* $[t_i, t_{i+1}]$ *for all* $i = 1, \ldots, f$*. Then* $\gamma \geq \hat{\gamma}$*.*

Proof. Suppose that $\gamma < \hat{\gamma}$. Then, for some i, γ is strictly less than the infimum of the set of γ such that the Riccati differential equation (4.14) has a bounded solution on $[t_i, t_{i+1}]$. Therefore, applying Lemma 4.2, we know that the maximization problem given by equation (4.5) is unbounded with respect to w. Hence the upper value of the game is unbounded. ■

4.4.3 The general multi-rate case for sampled-data systems

We now return to the problem of finding a state feedback sampled-data for the multi-rate problem. We will use the same system setup as in Section 4.3, with the multi-rate restriction of the previous section that $\tilde{u}$ must satisfy

$$R^{\perp}[t_k]\tilde{u}[t_{k-1}] = R^{\perp}[t_k]\tilde{u}[t_k].$$

Again, we need only apply Theorem 4.5 to the special case of the system specified by equations (4.8). This gives the following result.

Theorem 4.7. *Consider the system described by equations* (4.10)*. Let*

$$X[t_f] = \begin{bmatrix} Q_f & 0 \\ 0 & 0 \end{bmatrix}$$

and suppose there exists a bounded solution to the Riccati differential equation

$$-\dot{S}_i = A'S_i + S_iA + \gamma^{-2}S_iB_1B_1'S_i + C_1'C_1 \qquad S_i(t_{i+1}) = X_{11}[t_{i+1}] \tag{4.21}$$

on the interval $[t_i, t_{i+1}]$*. Then there exists a bounded solution to the Riccati differential equation*

$$-\dot{Z}_i = \hat{A}'Z_i + Z_i\hat{A} + \gamma^{-2}Z_i\hat{B}_1\hat{B}_1'Z_i + \hat{C}_1\hat{C}_1' \qquad Z_i(t_{i+1}) = X[t_{i+1}] \tag{4.22}$$

on each interval $[t_i, t_{i+1}]$ *for* $0 \leq i < f$*, where* $X[t_i]$ *is defined recursively by*

$$X[t_i] = \begin{bmatrix} Z_{i11} & Z_{i12}R^{\perp\prime} \\ R^{\perp}Z_{i21} & R^{\perp}Z_{i22}R^{\perp\prime} \end{bmatrix} - \begin{bmatrix} Z_{i12}R' \\ R^{\perp}Z_{i22}R' \end{bmatrix} (RZ_{i22}R')^{-1} \begin{bmatrix} RZ_{i21} & RZ_{i22}R^{\perp\prime} \end{bmatrix}. \tag{4.23}$$

with all matrices evaluated at time t_i*. Then the multi-rate sampled-data state feedback controller*

$$R[t_i]\tilde{u}[t_i] = -\big(RZ_{i22}R'\big)^{-1}\big(RZ_{i12}x(t_i)' + RZ_{i22}R^{\perp\prime}R^{\perp}\tilde{u}[t_i]\big) \tag{4.24}$$

results in $\|\mathcal{T}_{zw}\| \leq \gamma$*.*

Proof. We can explicitly write out equation (4.22) as

$$\begin{aligned}
-\dot{Z}_{i11} &= A'Z_{i11} + Z_{i11}A + \gamma^{-2}Z_{i11}B_1B_1'Z_{i11} + C_1'C_1 & Z_{i11}(t_{i+1}) &= X_{11}[t_{i+1}] \\
-\dot{Z}_{i22} &= B_2'Z_{i11} + Z_{i12}'B_2 + I & Z_{i22}(t_{i+1}) &= R^{\perp\prime}[t_{i+1}]X_{12}R^{\perp}[t_{i+1}] \\
-\dot{Z}_{i12} &= A'Z_{i12} + Z_{i11}B_2 + \gamma^{-2}Z_{i11}B_1B_1'Z_{i12}[t_{i+1}] & Z_{i12}(t_{i+1}) &= X_{12}[t_{i+1}]R^{\perp}[t_{i+1}]
\end{aligned}$$

using the standard fact (see for example [59]) that solutions to this form of Riccati differential equation are symmetric. The first of these equations is simply equation (4.21), and by assumption this has a bounded solution. The next two are then linear equations with bounded time varying coefficients, and hence have well defined solutions on any finite interval, see for example [30]. Therefore, existence of a solution to (4.21) is sufficient for existence of a solution to (4.22).

We now need to apply Theorem 4.5 to the special case of the system specified by equations (4.8). The expressions for the controller and $X[t_i]$ are given by direct substitution of equations (4.9) into equations (4.16) and (4.17).

All that remains of the proof is to show that $RZ_{i22}R'$ is invertible, so that we can replace the pseudo inverse with the actual inverse. We can show that Z_{i22} is invertible in exactly the same was as in the proof of Theorem 4.4. Then, since R is full row rank, the result follows. ∎

4.5 One step delayed output feedback

4.5.1 Problem formulation

For the remainder of this chapter, we restrict attention to the sampled-data problem, rather than considering general systems with jumps, and concentrate on the multi-rate case. We will consider the system described by the following equations on the time interval $[t_0, t_f] \subset \mathbb{R}_+$.

$$\begin{aligned}
\dot{x}(t) &= A(t)x(t) + B_1(t)w(t) + B_2(t)u(t) \qquad x(t_0) = x_0 \\
z(t) &= C_1(t)x(t) + D_{12}(t)u(t) \\
y(t) &= C_2x(t)
\end{aligned}$$

The system matrices are assumed to be bounded functions of time. The times $t_0 \leq t_1 \cdots \leq t_f$ are those at which either a hold operator or a sampling operator becomes active. That is, at each t_k, $0 \leq k \leq f$, we either receive new information about some spatial components of $y(t_k)$, or we must decide some spatial components of the signal u on $[t_k, t_{k+j}]$ for some $j > 0$. The spatial components here correspond to different 'channels' in Figure 4.1. The set of possible choices of u will be determined by the matrix R as before.

The discrete measured output is then

$$\tilde{y}[t_k] = \Gamma[t_k](\mathcal{S}y)[t_k] + \Gamma[t_k]D_{21}[t_k]v[t_k]$$

where $\Gamma[t_k]$ is a square diagonal matrix whose elements are either 1 or 0, and the operator $\mathcal{S} : \mathscr{L}_2^{p_2}[t_0, t_f] \to \ell_2^{p_2}[0, f]$ is a sampling operator. The signal v is a discrete noise input. If the ith component of $\Gamma[t_k]$ is a 1, this indicates that the ith component of $\tilde{y}[t_k]$ is available for measurement at that time. The actual sampling operator $\mathcal{S}$ need not be diagonal for the details here to work, although it clearly makes more physical sense that it be so. The sampling operator we will use here will be defined by

$$(\mathcal{S}y)[t_k] = y(t_k)$$

Note that, since we are sampling the output of the system, and there is no direct feedthrough from w to y, this operation is well-defined for $\mathscr{L}_2$ inputs u and w. Also, at each time t_k, the controller knows $\Gamma[t_k]$, and we shall see that the controller does not in fact need advance knowledge of Γ. We again make the following assumptions about the system matrices:

$$D_{12}(t)'C_1(t) = 0 \qquad D_{12}(t)'D_{12}(t) = I \qquad D_{21}[t_k]B_1'(t_k) = 0 \qquad D_{21}[t_k]D_{21}[t_k] = I$$

for all t and t_k. The hold operator $\mathcal{H} : \ell_2^{m_2}[0, f] \to \mathscr{L}_2^{m_2}[t_0, t_f]$ is defined by

$$u(t) = \tilde{u}[t_k] \qquad \forall\ t \in [t_k, t_{k+1}].$$

The disturbance has a discrete component v as well as a continuous component w, and we will also include the initial state x_0 as part of the disturbance. We therefore consider the disturbance as an element of the product space $\mathscr{L}_2 \oplus \ell_2 \oplus \mathbb{R}^n$, and shall write the elements of this space as (w, v, x_0). Then the norm of the disturbance (w, v, x_0) is given by

$$\|(w, v, x_0)\|^2 = \|w\|_2^2 + \|v\|_2^2 + x_0'Q_0x_0.$$

where $Q_0 > 0$ is some given weighting matrix. Let $\mathcal{T} : (w, v, x_0) \mapsto z$ be the closed loop linear operator with the controller $\tilde{u} = \mu\tilde{y}$. Our aim is to construct a causal controller μ such that

$$\|\mathcal{T}\| := \sup_{(w,v,x_0)} \frac{\|\mathcal{T}(w, v, x_0)\|}{\|(w, v, x_0)\|} \leq \gamma$$

which occurs if and only if

$$\|z\|_2^2 - \gamma^2\|w\|_2^2 - \gamma^2\|v\|_2^2 - \gamma^2 x_0'Q_0x_0 \leq 0$$

for all disturbances (w, v, x_0). We therefore try to solve the differential game

$$\inf_{\mu \in \mathscr{U}_{mrfb}} \sup_{\substack{w \in \mathscr{L}_2 \\ v \in \ell_2 \\ x_0 \in \mathbb{R}^n}} \Bigg\{ -\gamma^2 x_0'Q_0'x_0 + \int_{t_0}^{t_f} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt \\ - \gamma^2 \sum_{i=0}^{f-1} v[t_i]'v[t_i] + x(t_f)'Q_f x(t_f) \Bigg\} \tag{4.25}$$

where $\mathscr{U}_{mrfb}$ is the set of all causal multi-rate sampled-data controllers mapping $\tilde{y}$ to $\tilde{u}$.

4.5.2 An expression for the information state

We would now like to construct output feedback controllers for the sampled data problem using the information state approach. The development in this section parallels that in Section 3.6. The problem we shall solve is to find the minimax controller for the above problem when $\tilde{u}[t_j]$ depends on $\Gamma[t_i]\tilde{y}[t_i]$ for i strictly less than j. That is, the controller has access to past samples of the measured output, but not access to samples which occur at time t_j. As before, the cost function is

$$J(x_0, \mu, w, v) = -\gamma^2 x_0' Q_0' x_0 + \int_{t_0}^{t_f} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt \\ - \gamma^2 \sum_{i=0}^{f-1} v[t_i]'v[t_i] + x(t_f)'Q_f x(t_f). \quad (4.26)$$

It is possible to regard the sampled-data system as a discrete time system, using the technique of *lifting*. That is, at each time t_j, we regard the signals w and z as taking values in $\mathscr{L}_2[t_j, t_{j+1}]$. This technique is used by several authors for both analysis and synthesis in sampled-data problems, when the system has single periodic synchronized sample and hold devices. We will not use this technique here, although it may make comparisons with Section 3.6 easier to see to regard $w(j)$ and $z(j)$ in that section as corresponding to $w_{[t_j, t_{j+1}]}$ and $z_{[t_j, t_{j+1}]}$ in this section.

Define the information state for the sampled-data problem by

$$P\big(t_j, \eta\big) = \sup_{w \in \mathscr{L}_2[t_0, t_j]} \sup_{v \in \ell_2[t_0, t_{j-1}]} \sup_{x_0 \in \mathbb{R}^n} \Big\{ -\gamma^2 x_0' Q_0 x_0 \\ + \int_{t_0}^{t_j} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt - \gamma^2 \sum_{i=0}^{j-1} v[t_i]'v[t_i] \Big\}$$

subject to the constraints

$$\begin{aligned} \eta &= x(t_j) \\ \tilde{y}[t_i] &= \Gamma[t_i] C_2(t_i) x(t_i) + \Gamma[t_i] D_{21}[t_i] v[t_i] \qquad \text{for all } i = 0, \dots, j-1. \end{aligned}$$

It is then easy to show by induction that P satisfies the dynamic programming recursion

$$P\big(t_{j+1}, \eta\big) = \sup_{x(t_j) \in \mathbb{R}^n} \sup_{w \in \mathscr{L}_2[t_j, t_{j+1}]} \sup_{v[t_j] \in \mathbb{R}^{m_3}} \Big\{ P\big(t_j, x(t_j)\big) \\ + \int_{t_j}^{t_{j+1}} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt - \gamma^2 v[t_j]'v[t_j] \Big\}$$

subject to the constraints that

$$\begin{aligned} \eta &= x(t_{j+1}) \\ \dot{x}(t) &= A(t)x(t) + B_1(t)w(t) + B_2(t)\tilde{u}[t_j] \\ \tilde{y}[t_j] &= \Gamma[t_j] C_2(t_j) x(t_j) + \Gamma[t_j] D_{21}[t_j] v[t_j] \end{aligned}$$

on the interval $[t_j, t_{j+1}]$, and with the boundary condition

$$P\big(t_0, x_0\big) = -\gamma^2 x_0' Q_0 x_0$$

Note that $P\big(t_j, x(t_j)\big)$ is actually a function of $\tilde{y}_{[t_0,t_{j-1}]}$ and $\tilde{u}_{[t_0,t_{j-1}]}$ also, but we omit these arguments to reduce notation. We will now show by induction that P takes the form

$$P\big(t_j, x(t_j)\big) = -\gamma^2\big(x(t_j) - \xi(t_j)\big)' L(t_j)\big(x(t_j) - \xi(t_j)\big) + c[t_j] \tag{4.27}$$

and give recursive formulae for L and ξ. In this expression $c[t_j]$ is independent of $x(t_j)$, and depends solely on $y_{[t_0,t_{j-1}]}$ and $u_{[t_0,t_{j-1}]}$.

Firstly, note that the above equation holds at time t_0, with $L(t_0) = Q_0$, $\xi(t_0) = 0$ and $c[t_0] = 0$. Now suppose it holds at time t_j. Then

$$\begin{aligned} P\big(t_{j+1}, \eta\big) = \sup_{x(t_j)\in\mathbb{R}^n}\ \sup_{w\in\mathscr{L}_2[t_j,t_{j+1}]}\ \sup_{v[t_j]\in\mathbb{R}^{m_3}} \Big\{ -\gamma^2\big(x(t_j) - \xi(t_j)\big)' L(t_j)\big(x(t_j) - \xi(t_j)\big) \\ + \int_{t_j}^{t_{j+1}} z(t)'z(t) - \gamma^2 w(t)'w(t)\,dt - \gamma^2 v[t_j]'v[t_j] \Big\} + c[t_j] \end{aligned}$$

subject to the constraints that

$$\begin{aligned} \eta &= x(t_{j+1}) \\ \dot{x}(t) &= A(t)x(t) + B_1(t)w(t) + B_2(t)\tilde{u}[t_j] \\ \tilde{y}[t_j] &= \Gamma[t_j]C_2(t_j)x(t_j) + \Gamma[t_j]D_{21}[t_j]v[t_j] \end{aligned}$$

on the interval $[t_j, t_{j+1}]$.

We now solve this maximization problem directly. Denote the signal w on $[t_j, t_{j+1}]$ by w_j, and similarly for z_j. Then we can write the system equations on $[t_j, t_{j+1}]$ as

$$\begin{pmatrix} x(t_j) \\ z_j \\ \tilde{y}[t_j] \end{pmatrix} = \Psi \begin{pmatrix} x(t_{j+1}) \\ \tilde{u}[t_j] \\ w_j \\ v[t_j] \end{pmatrix} \tag{4.28}$$

Here Ψ is a linear operator between Cartesian product spaces. Then we can write

$$\begin{aligned} P\big(t_{j+1}, \eta\big) = \sup_{x(t_j)\in\mathbb{R}^n}\ \sup_{w_j\in\mathscr{L}_2[t_j,t_{j+1}]}\ \sup_{v[t_j]\in\mathbb{R}^{m_3}} \\ \big\{-\gamma^2\langle(x(t_j) - \xi(t_j)), L(t_j)(x(t_j) - \xi(t_j))\rangle + \langle z_j, z_j\rangle - \gamma^2\langle w_j, w_j\rangle - \gamma^2\langle v[t_j], v[t_j]\rangle\big\} \end{aligned}$$

again subject to the above constraints. Then, using similar notation to that of Section 3.5.2, we can write the operator Ψ as

$$\Psi = \begin{bmatrix} \hat{F}\hat{\phi} & -\hat{F}\hat{\Phi}B_2H & -\hat{F}\hat{\Phi}B_1 & 0 \\ C_1\hat{\phi} & -C_1\hat{\Phi}B_2H + D_{12}H & -C_1\hat{\Phi}B_1 & 0 \\ \Gamma[t_j]C_2(t_j)\hat{F}\hat{\phi} & -\Gamma[t_j]C_2(t_j)\hat{F}\hat{\Phi}B_2H & -\Gamma[t_j]C_2(t_j)\hat{F}\hat{\Phi}B_1 & \Gamma[t_j]D_{21}[t_j] \end{bmatrix}.$$

where

$$\begin{aligned}
&\hat{F} : \mathscr{L}_2[t_j, t_{j+1}] \to \mathbb{R}^n && \hat{F}u := u(t_j)\\
&\hat{\boldsymbol{\phi}} : \mathbb{R}^n \to \mathscr{L}_2[t_j, t_{j+1}] && (\hat{\boldsymbol{\phi}}\xi)(t) := \phi(t, t_{j+1})\xi\\
&\hat{\Phi} : \mathscr{L}_2[t_j, t_{j+1}] \to \mathscr{L}_2[t_j, t_{j+1}] && (\hat{\Phi}u)(t) := \int_t^{t_{j+1}} \phi(t,s)u(s)\,ds\\
&H : \mathbb{R}^n \to \mathscr{L}_2[t_j, t_{j+1}] && (H\xi)(t) := \xi
\end{aligned}$$

with $\phi(t,s)$ the transition matrix for A. We use B_1 and B_2 as multiplication operators also without distinguishing this.

Since $\Psi_{11} = \phi(t_{j+1}, t_j)$, and a transition matrix is always invertible, for every fixed w_j and $v[t_j]$ there is a bijection between the states $x(t_j)$ and $x(t_{j+1})$. Therefore, fixing $x(t_{j+1}) = \eta$, there is no need to maximize with respect to $x(t_j)$ in the expression for P. Therefore,

$$\begin{aligned}
P\big(t_{j+1}, x(t_{j+1})\big) = \sup_{w_j \in \mathscr{L}_2[t_j, t_{j+1}]} \sup_{v[t_j] \in \mathbb{R}^{m_3}}\\
\big\{-\gamma^2\langle\big(x(t_j) - \xi(t_j)\big), L(t_j)\big(x(t_j) - \xi(t_j)\big)\rangle + \langle z_j, z_j\rangle - \gamma^2\langle w_j, w_j\rangle - \gamma^2\langle v[t_j], v[t_j]\rangle\big\}.
\end{aligned}$$

Introduce the Lagrange multiplier $\rho \in \mathbb{R}^{p_2}$, then let

$$\begin{aligned}
\tilde{P} = -\gamma^2\langle x(t_j) - \xi(t_j), L(t_j)\big(x(t_j) - \xi(t_j)\big)\rangle + \langle z_j, z_j\rangle - \gamma^2\langle w_j, w_j\rangle - \gamma^2\langle v[t_j], v[t_j]\rangle\\
+ \langle \rho, \tilde{y}[t_j]\rangle - \langle \rho, \Psi_{31}x(t_{j+1}) + \Psi_{32}\tilde{u}[t_j] + \Psi_{33}w_j + \Psi_{34}v[t_j]\rangle.
\end{aligned}$$

We can now substitute the expressions for z_j, and $x(t_j)$ from equation (4.28). On an inner product space, if the maximum of $\langle v, Av\rangle + \langle b, v\rangle$ exists, then the optimal v satisfies $2Av + b = 0$. Therefore, if we denote the maximizing, or worst case, inputs for this subproblem by $\hat{w}$ and $\hat{v}$, with corresponding trajectory written as $\hat{x}_j$, they must satisfy

$$\begin{aligned}
-\gamma^2\hat{w} &= \gamma^2\Psi_{13}^*L\big(x(t_j) - \xi(t_j)\big) - \Psi_{23}^*z_j + \tfrac{1}{2}\Psi_{33}^*\rho\\
-\gamma^2\hat{v} &= \gamma^2\Psi_{14}^*L\big(x(t_j) - \xi(t_j)\big) - \Psi_{24}^*z_j + \tfrac{1}{2}\Psi_{34}^*\rho.
\end{aligned} \tag{4.29}$$

We must now calculate the dual operator, Ψ^*. The following easily derived lemma will be useful for this purpose.

Lemma 4.8. *Consider the system*

$$\begin{aligned}
\dot{x} &= Ax + Bq_2 \qquad x(t_1) = q_1\\
p_1 &= x(t_0)\\
p_2 &= Cx + Dq_2
\end{aligned}$$

and denote this operator by

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \Psi \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

then if we write the dual operation

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \Psi^* \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

then Ψ^* *has realization*

$$\begin{aligned} -\dot{\lambda} &= A'\lambda - C'\alpha_2 \qquad \lambda(t_0) = \alpha_1 \\ \beta_1 &= \lambda(t_1) \\ \beta_2 &= -B'\lambda + D'\alpha_2. \end{aligned}$$

■

Suppose that

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \Psi^* \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}. \tag{4.30}$$

Here $\beta_1 \in \mathbb{R}^n$, $\beta_2 \in \mathbb{R}^{m_2}$, $\beta_3 \in \mathscr{L}_2[t_j, t_{j+1}]$, $\beta_4 \in \mathbb{R}^{m_3}$, and $\alpha_1 \in \mathbb{R}^n$, $\alpha_2 \in \mathscr{L}_2[t_j, t_{j+1}]$, $\alpha_3 \in \mathbb{R}^{p_2}$. In the same way as for the above lemma, the dual operator has realization

$$\begin{aligned} -\dot{\lambda}_j(t) &= A(t)'\lambda_j(t) - C_1(t)'\alpha_2(t) \qquad \lambda_j(t_j) = \alpha_1 + C_2(t_j)'\Gamma[t_j]\alpha_3 \\ \beta_1 &= \lambda_j(t_{j+1}) \\ \beta_2 &= -H^*B_2(t)'\lambda_j(t) + H^*D_{12}(t)'\alpha_2(t) \\ \beta_3(t) &= -B_1(t)'\lambda_j(t) \\ \beta_4 &= D_{21}[t_j]'\Gamma[t_j]\alpha_3. \end{aligned}$$

Comparing equations (4.29) with equation (4.30), if we choose

$$\begin{aligned} \alpha_1 &= \gamma^2 L(t_j)\big(\hat{x}_j(t_j) - \xi(t_j)\big) \\ \alpha_2 &= -z_j \\ \alpha_3 &= \tfrac{1}{2}\rho \end{aligned}$$

then equations (4.29) imply that

$$\begin{aligned} -\gamma^2\hat{w} &= \beta_3 \\ -\gamma^2\hat{v} &= \beta_4. \end{aligned}$$

Hence we can realize equations (4.29) as

$$\begin{aligned} \dot{\hat{x}}_j &= A\hat{x}_j(t) + \gamma^{-2}B_1B_1'\lambda_j + B_2u \\ -\dot{\lambda}_j &= A'\lambda_j(t) + C_1'C_1\hat{x}_j \\ \hat{w} &= \gamma^{-2}B_1'\lambda_j \\ \hat{v}[t_j] &= -\tfrac{1}{2}\gamma^{-2}D_{21}[t_j]'\Gamma[t_j]\rho \end{aligned}$$

with boundary condition

$$\lambda_j(t_j) = \gamma^2 L(t_j)\big(\hat{x}_j(t_j) - \xi(t_j)\big) + \tfrac{1}{2}C_2(t_j)\Gamma[t_j]\rho$$

and where $\hat{x}_j(t_{j+1}) = x(t_{j+1})$ is given. Then the constraint can be written as

$$\begin{aligned} \tilde{y}[t_j] &= \Gamma[t_j]C_2(t_j)\hat{x}_j(t_j) + \Gamma[t_j]D_{21}[t_j]v[t_j] \\ &= \Gamma[t_j]C_2(t_j)\hat{x}_j(t_j) - \tfrac{1}{2}\gamma^{-2}\Gamma[t_j]\rho \end{aligned}$$

which gives an expression for the Lagrange multiplier

$$\tfrac{1}{2}\Gamma[t_j]\rho = -\gamma^2\big(\tilde{y}[t_j] - \Gamma[t_j]C_2(t_j)\hat{x}_j(t_j)\big)$$

and hence the worst case disturbance is

$$\hat{v}[t_j] = D_{21}[t_j]'\big(\tilde{y}[t_j] - \Gamma[t_j]C_2(t_j)\hat{x}_j(t_j)\big).$$

Now let Y_j be the solution to the Riccati differential equation defined on $[t_j, t_{j+1}]$

$$\dot{Y}_j = AY_j + Y_jA' + \gamma^2 Y_jC_1'C_1Y_j + B_1B_1'$$

and let

$$\check{x}_j(t) = \hat{x}_j(t) - \gamma^{-2}Y_j(t)\lambda_j(t).$$

We now substitute $\check{x}_j$ and Y_j for λ_j. This gives

$$\begin{aligned}
\dot{\hat{x}}_j &= A\hat{x}_j + B_1B_1'Y_j^{-1}(\hat{x}_j - \check{x}_j) + B_2u \\
\dot{\check{x}}_j &= A\check{x}_j + \gamma^{-2}Y_jC_1'C_1\check{x}_j + B_2u \\
\hat{w} &= B_1'Y_j^{-1}(\hat{x}_j - \check{x}_j) \\
\hat{v}[t_j] &= D_{21}[t_j]'\big(\tilde{y}[t_j] - \Gamma[t_j]C_2(t_j)x(t_j)\big).
\end{aligned}$$

on $[t_j, t_{j+1}]$. The boundary conditions at t_j are given by

$$\check{x}_j(t_j) = \hat{x}_j(t_j) - Y_j(t_j)L(t_j)\big(\hat{x}_j(t_j) - \xi(t_j)\big) + \gamma^2 Y_j(t_j)C_2'(t_j)\big(\tilde{y}(t_j) - C_2\hat{x}_j(t_j)\big).$$

We would like to choose this $\check{x}_j(t_j)$ and $Y_j(t_j)$ so that $\check{x}_j$ is independent of $\hat{x}_j$. In order to do this, we choose $\check{x}_j(t_j)$ and $Y_j(t_j)$ so that the above equation is true for all $\hat{x}_j(t_j)$. This gives

$$Y_j^{-1}(t_j) = L(t_j) + C_2(t_j)'\Gamma[t_j]C_2(t_j) \tag{4.31}$$

$$\check{x}_j(t_j) = Y_j\big(C_2'\Gamma y(t_j) + L\xi(t_j)\big). \tag{4.32}$$

Note that the latter of these conditions can also be written as

$$\check{x}_j(t_j) = \xi(t_j) + Y_jC_2'\Gamma\big(y(t_j) - C_2\xi(t_j)\big)$$

We now wish to evaluate $P\big(t_{j+1}, x(t_{j+1})\big)$. In order to do this, let $e_j = \hat{x}_j - \check{x}_j$. Then

$$\dot{e_j} = Ae_j - \gamma^{-2}Y_jC_1'C_1\check{x}_j + B_1B_1'Y_j^{-1}e_j$$

and consider $e_j'Y_j^{-1}e_j$. Differentiating this gives

$$\begin{aligned}
\frac{d}{dt}e_j'Y_j^{-1}e_j &= e_j'Y_j^{-1}B_1B_1'Y_j^{-1}e_j - 2\gamma^{-2}\check{x}_j'C_1'C_1e_j - \gamma^{-2}e_j'C_1'C_1e \\
&= -\gamma^{-2}z'z + \hat{w}'\hat{w} + \gamma^{-2}\check{x}_j'C_1'C_1\check{x}_j
\end{aligned}$$

Then

$$
\begin{aligned}
&P\big(t_{j+1}, x(t_{j+1})\big) \\
&\quad = -\gamma^2\big(\hat{x}_j(t_j) - \xi(t_j)\big)' L(t_j)\big(\hat{x}_j(t_j) - \xi(t_j)\big) \\
&\qquad\qquad + \int_{t_j}^{t_{j+1}} \Big\{ z(t)'z(t) - \gamma^2 \hat{w}(t)'\hat{w}(t) \Big\}\, dt - \gamma^2 v[t_j]'v[t_j] \\
&\quad = -\gamma^2\big(\hat{x}_j(t_j) - \xi(t_j)\big)' L(t_j)\big(\hat{x}_j(t_j) - \xi(t_j)\big) - \gamma^2 e(t_{j+1})'Y_j^{-1}(t_{j+1})e(t_{j+1}) \\
&\qquad\qquad + \gamma^2 e(t_j)'Y_j^{-1}(t_j)e(t_j) - \gamma^2 v[t_j]'v[t_j] \\
&\qquad\qquad + \int_{t_j}^{t_{j+1}} \Big\{ z(t)'z(t) - \gamma^2 w(t)'w(t) + \gamma^2 \frac{d}{dt} e_j' Y_j^{-1} e_j \Big\}\, dt \\
&\quad = -\gamma^2\big(\hat{x}_j(t_j) - \xi(t_j)\big)' L(t_j)\big(\hat{x}_j(t_j) - \xi(t_j)\big) + \gamma^2 e(t_j)'Y_j^{-1}(t_j)e(t_j) - \gamma^2 v[t_j]'v[t_j] \\
&\qquad\qquad - \gamma^2 e(t_{j+1})'Y_j^{-1}(t_{j+1})e(t_{j+1}) + \int_{t_j}^{t_{j+1}} \Big\{ \check{x}_j' C_1' C_1 \check{x}_j + u(t)'u(t) \Big\}\, dt
\end{aligned}
$$

We make use of equation (4.31), and evaluate all expressions at t_j in the following manipulations:

$$
\begin{aligned}
&\gamma^2(\hat{x}_j - \check{x}_j)'Y_j^{-1}(\hat{x}_j - \check{x}_j) - \gamma^2(\hat{x}_j - \xi)'L(\hat{x}_j - \xi) \\
&\qquad = \gamma^2(\hat{x}_j - \check{x}_j)'(L + C_2'\Gamma C_2)(\hat{x}_j - \check{x}_j) - \gamma^2(\hat{x}_j - \xi)'L(\hat{x}_j - \xi) \\
&\qquad = \gamma^2 e_j' C_2' \Gamma C_2' e_j - 2\gamma^2 e_j' L(\check{x}_j - \xi) - \gamma^2(\check{x}_j - \xi)'L(\check{x}_j - \xi)
\end{aligned}
$$

Now note that combining equations (4.31) and (4.32) gives

$$
C_2'\Gamma\tilde{y} - C_2'\Gamma C_2 \check{x}_j = L(\check{x}_j - \xi)
$$

so

$$
\begin{aligned}
&\gamma^2(\hat{x}_j - \check{x}_j)'Y_j^{-1}(\hat{x}_j - \check{x}_j) - \gamma^2(\hat{x}_j - \xi)L(\hat{x}_j - \xi) \\
&\qquad = \gamma^2 e_j' C_2' \Gamma C_2 e_j - 2\gamma^2 e_j' C_2' \Gamma \tilde{y} + 2\gamma^2 e_j' C_2' \Gamma C_2 \check{x}_j - \gamma^2(\check{x}_j - \xi)'L(\check{x}_j - \xi).
\end{aligned}
$$

Then

$$
\begin{aligned}
\hat{v}'\hat{v} &= \tilde{y}'\tilde{y} + \check{x}_j' C_2' \Gamma C_2 \check{x}_j - 2\tilde{y}'\Gamma C_2 \check{x}_j + e_j' C_2' C_2 e_j - 2\tilde{y}'\Gamma C_2 e_j + 2\check{x}_j' C_2' \Gamma C_2 e_j \\
&= \tilde{y}'\tilde{y} + \check{x}_j' C_2' \Gamma C_2 \check{x}_j - 2\tilde{y}'\Gamma C_2 \check{x}_j + (\check{x}_j - \xi)'L(\check{x}_j - \xi) \\
&\qquad\qquad + (\hat{x}_j - \check{x}_j)'Y_j^{-1}(\hat{x}_j - \check{x}_j) - (\hat{x}_j - \xi)'L(\hat{x}_j - \xi).
\end{aligned}
$$

Therefore we arrive at

$$
\begin{aligned}
&P\big(t_{j+1}, x(t_{j+1})\big) = \\
&\qquad - \gamma^2\big(x(t_{j+1}) - \check{x}_j(t_{j+1})\big)Y_j^{-1}(t_{j+1})\big(x(t_{j+1}) - \check{x}_j(t_{j+1})\big) \\
&\qquad + \Big[-\gamma^2\tilde{y}'\tilde{y} - \gamma^2\check{x}_j' C_2' \Gamma C_2 \check{x}_j + 2\gamma^2\tilde{y}'\Gamma C_2 \check{x}_j - \gamma^2(\check{x}_j - \xi)'L(\check{x}_j - \xi)\Big](t_j) \\
&\qquad + \int_{t_j}^{t_{j+1}} u(t)'u(t) + \check{x}_j' C_1' C_1 \check{x}_j \, dt + c[t_j]
\end{aligned}
$$

Hence $P\big(t_{j+1}, x(t_{j+1})\big)$ has the form of equation (4.27), with

$$\begin{aligned} L(t_{j+1}) &= Y_j^{-1}(t_{j+1}) \\ \xi(t_{j+1}) &= \check{x}_j(t_{j+1}) \end{aligned}$$

giving the recursive formulae

$$Y_j(t_j) = \big(Y_{j-1}^{-1}(t_j) + C_2(t_j)'\Gamma[t_j]C_2(t_j)\big)^{-1} \qquad Y_{-1}(t_0) = Q_0^{-1} \tag{4.33}$$

$$\dot{Y}_j = AY_j + Y_jA' + \gamma^2 Y_j C_1' C_1 Y_j + B_1 B_1' \tag{4.34}$$

and

$$\begin{aligned} \check{x}_j(t_j) &= \check{x}_{j-1}(t_j) + Y_j(t_j)C_2(t_j)'\Gamma[t_j]\big(\tilde{y}[t_j] - C_2\check{x}_{j-1}(t_j)\big) \qquad \check{x}_{-1}(t_0) = 0 \\ \dot{\check{x}}_j &= A\check{x}_j + \gamma^{-2}Y_jC_1'C_1\check{x}_j + B_2u \end{aligned}$$

Note that $c[t_{j+1}]$ depends only on $y_{[t_0,t_j]}$ and $u_{[t_0,t_j]}$ as expected.

Proposition 4.9. *As defined above, for each $j = 0, \ldots, f$, the matrix $Y_j(t)$ is strictly positive definite for all $t \in [t_j, t_{j+1}]$.*

Proof. This is easily demonstrated by induction. Since $Q_0 > 0$, $Y_0(t_0) > 0$. Proposition 3.19 implies that, if $Y_j(t_j) > 0$, then $Y_j(t) > 0$ for all $t \in [t_j, t_{j+1}]$. Further, if $Y_j(t_{j+1}) > 0$, then equation (4.33) implies $Y_{j+1}(t_{j+1}) > 0$ also. ■

4.5.3 Recoupling

For the multi-rate sampled-data problem, we can define the partial information value function as

$$W\big(t_j, P(t_j,\cdot)\big) := \inf_{\mu \in \mathscr{U}_{mp}[t_j,t_{f-1}]} \sup_{\tilde{y}\in \ell_2[t_j,t_{f-1}]} \sup_{x(t_f)\in\mathbb{R}^n} \Big\{P\big(t_f, x(t_f)\big) + x(t_f)'Q_f x(t_f)\Big\}.$$

Note that the arguments W are of different types; the first is t_j, a real number, and the second is $P(t_j,\cdot)$ which is a function. Thus, with respect to its second argument, W is a functional. Also here $\mathscr{U}_{mp}$ is the subset of $\mathscr{U}_p$ for which $R^\perp[t_i]\tilde{u}[t_i] = R^\perp[t_i]\tilde{u}[t_{i-1}]$, that is the set of information state feedback control laws satisfying the multi-rate hold constraints. Then the following is the sampled-data analogue of Corollary 3.29.

Lemma 4.10. *The following equality holds:*

$$\sup_{\substack{w\in\mathscr{L}_2[t_0,t_f] \\ v\in\ell_2[t_0,t_f]}} \sup_{x_0\in\mathbb{R}^n} J(x_0,\mu,w,v) = \sup_{\tilde{y}\in\ell_2[t_0,t_{f-1}]} \sup_{x(t_f)\in\mathbb{R}^n} \Big\{P\big(t_f, x(t_f)\big) + x(t_f)'Q_f x(t_f)\Big\}.$$

Proof. The proof follows exactly the lines of that of Corollary 3.29, so is omitted. ■

It is straightforward to verify that the results Lemma 3.28 to Theorem 3.31 from our discussion on the discrete time problem hold with the minor changes necessary for the multi-rate sampled data problem, since the proofs of that succession simply involve successive maximizations and minimizations, and there are no order changes. The function W satisfies the dynamic programming recursion

$$W\big(t_j, P(t_j,\cdot)\big) = \inf_{R[t_j]\tilde{u}[t_j]} \sup_{\tilde{y}[t_j]} W\big(t_{j+1}, P(t_{j+1},\cdot)\big)$$

with boundary condition

$$W\big(t_f, P(t_f,\cdot)\big) = \sup_{x(t_f)} \Big\{ P\big(t_f, x(t_f)\big) + x(t_f)'Q_f x(t_f) \Big\}.$$

Certainty equivalence. In order to apply the certainty equivalence principle to this problem, we would now like to verify condition the saddle point condition (3.41) of Theorem 3.31. Let J_1 be defined by

$$\begin{aligned} J_1(t_j) &:= P\big(t_j, x(t_j)\big) \\ &\quad + \int_{t_j}^{t_{j+1}} z(t)'z(t) - \gamma^2 w(t)'w(t)\,dt + V\big(t_{j+1}, x(t_{j+1}^-), R^{\perp}[t_{j+1}]\tilde{u}[t_{j+1}]\big) \end{aligned} \tag{4.35}$$

In fact, $J_1(t_j)$ can be written with arguments as $J_1(t_j, x(j), u_{[t_0,t_j]}, y_{[t_0,t_j]}, w_{[t_j,t_{j+1}]}, v[t_j])$, but we omit these for notational convenience. Condition (3.41) can be written

$$\inf_{R[t_j]\tilde{u}[t_j]\in\mathbb{R}^{m_2}} \sup_{x(t_j)\in\mathbb{R}^n} \sup_{v[t_j]} \sup_{w\in\mathscr{L}_2[t_j,t_{j+1}]} J_1(t_j) = \sup_{x(t_j)\in\mathbb{R}^n} \inf_{R[t_j]\tilde{u}[t_j]\in\mathbb{R}^{m_2}} \sup_{v[t_j]} \sup_{w\in\mathscr{L}_2[t_j,t_{j+1}]} J_1(t_j)$$

We can now use equation (4.20) to substitute into this equation the expression for $\sup_{w\in\mathscr{L}_2[t_j,t_{j+1}]} J_1(t_j)$. However, equation (4.20) is for the system with jumps; therefore the state in that equation is in fact given by $\big(x(t_i)' \quad \tilde{u}[t_i]'\big)'$ in terms of the state of the sampled-data system here, as can be seen from equation (4.9). This implies that

$$\begin{aligned} \sup_{v[t_j]} \sup_{w\in\mathscr{L}_2[t_j,t_{j+1}]} J_1(t_j) &= P\big(t_j, x(t_j)\big) + \begin{pmatrix} x(t_j) \\ \tilde{u}[t_j] \end{pmatrix}' Z_j(t_j) \begin{pmatrix} x(t_j) \\ \tilde{u}[t_j] \end{pmatrix} \\ &= -\gamma^2 \big(x(t_j) - \check{x}_{j-1}(t_j)\big)' Y_{j-1}^{-1}(t_j) \big(x(t_j) - \check{x}_{j-1}(t_j)\big) + c[t_j] \\ &\quad + \begin{pmatrix} x(t_j) \\ \tilde{u}[t_j] \end{pmatrix}' Z_j(t_j) \begin{pmatrix} x(t_j) \\ \tilde{u}[t_j] \end{pmatrix} \end{aligned}$$

since this is an *unconstrained* optimization. This is consistent with the one step delayed information pattern, and corresponds to the fact that during that one step the disturbance can attempt to maximize the cost function with the controller having to wait until the next time t_{j+1} to act.

Applying Lemma 3.32, we see that a sufficient condition for the saddle point condition between $u(t_i)$ and $x(t_i)$ is that

$$\begin{aligned} -\gamma^2 Y_{i-1}^{-1}(t_i) + Z_{i11}(t_i) &< 0 \\ R[t_i] Z_{i22}(t_i) R[t_i]' &> 0 \end{aligned}$$

The latter of these conditions follows from Theorem 4.7, using the fact that $R[t_i]$ is full row rank. The former condition is the *coupling* condition.

Lemma 4.11. *If* $-\gamma^2 Y_{i-1}^{-1}(t_i) + Z_{i11}(t_i) < 0$, *then* $-\gamma^2 Y_{i-1}^{-1}(t_i) + X_{11}[t_i] < 0$.

Proof. From equation (4.23),

$$X_{11}[t_i] = \left[Z_{i11} - Z_{i12}R'(RZ_{i22}R')^{-1}RZ_{i12}'\right](t_i)$$

where we have use the fact that Z_i is symmetric. Since $Z_{i22} > 0$, the result follows. ∎

Remark 4.12. Note that the above coupling condition is analogous to that in the one step delayed discrete time problem. The matrix Z_{i11} is the quadratic coefficient of $x(t_i)$ in the value function *before* the minimization with respect to $u(t_i)$ takes place for that iteration of the Isaacs recursion. This might be expected, since during the 'one step' w has the opportunity to maximize the cost function with u only having open loop control. Similar conditions occur also in the $\mathscr{H}_\infty$ problem with delayed measurements, see for example [54]. □

Theorem 4.13. *Suppose, for each* $i = 0, \dots, f$, *there exists a bounded solution to each of the Riccati differential equations* (4.34) *on the interval* $[t_i, t_{i+1}]$, *with* $Y_{-1}(t_0) = Q_0$. *Suppose also that the conditions of Theorem 4.7, are satisfied, and that for each* $i = 0, \dots, f$ *the coupling condition*

$$-\gamma^2 Y_{i-1}^{-1}(t_i) + Z_{i11}(t_i) < 0$$

holds. Let $\bar{x}$ *be defined by*

$$\bar{x}(t_i) = \left(I + \gamma^{-2}Y_{i-1}(t_i)X_{11}[t_i]\right)^{-1}\left(\check{x}_{i-1}(t_i) + \gamma^{-2}Y_{i-1}(t_i)X_{12}[t_i]R^{\perp}[t_i]\tilde{u}[t_i]\right)$$

and the controller given by

$$R[t_i]\tilde{u}[t_i] = -\left(RZ_{i22}R'\right)^{-1}\left(R[t_i]Z_{i12}\bar{x}(t_i) + Z_{i22}R^{\perp\prime}R^{\perp}\tilde{u}[t_i]\right).$$

Then this controller is optimal for the differential game problem (4.25). *Further, this controller gives the closed loop induced norm bound* $\|\mathcal{T}\| \le \gamma$.

Proof. Applying Theorem 3.31, let

$$\bar{x}(t_i) = \arg \max_{x(t_i)\in\mathbb{R}^n} \left\{P\left(t_j, x(t_j)\right) + V\left(t_j, x(t_j^-), R^{\perp}[t_j]\tilde{u}[t_j]\right)\right\}.$$

Then, since the multi-rate version of condition (3.41) holds, we can use $\bar{x}$ as if it were the actual state in the state feedback control law, and the resulting output feedback controller will be optimal. We have

$$\bar{x}(t_i) = \arg\max_x \left\{\gamma^2\left(x - \check{x}_{i-1}\right)'Y_{i-1}^{-1}\left(x - \check{x}_{i-1}\right) + c[t_i] + \begin{pmatrix} x \\ R^{\perp}\tilde{u}\end{pmatrix}' X \begin{pmatrix} x \\ R^{\perp}\tilde{u}\end{pmatrix}\right\} \tag{4.36}$$

where all quantities on the right hand side are evaluated at t_i. Using Lemma 4.11, we can differentiate the above expression to give

$$-\gamma^2 Y_{i-1}^{-1}(\bar{x} - \check{x}_{i-1}) + X_{11}\bar{x} + X_{12}R^{\perp}\tilde{u} = 0$$

hence

$$\bar{x} = (I + \gamma^{-2} Y_{i-1} X_{11})^{-1} (\check{x}_{i-1} + \gamma^{-2} Y_{i-1} X_{12} R^{\perp} \tilde{u}).$$

This then gives the optimal controller. Corollary 3.30 then gives the optimal value. Since $\check{x}_{-1}(t_0) = 0$, $R^{\perp}[t_0] = 0$, and $Y_{-1}^{-1}(t_0) = Q_0$, we have, with the optimal control in place

$$\begin{aligned} &\sup_{w \in \mathscr{L}_2[t_0,t_f]} \sup_{v \in \ell_2[0,f]} \sup_{x_0 \in \mathbb{R}^n} J(x_0, \mu, w, v) \\ &\quad = W\big(t_0, P(t_0, \cdot)\big) \\ &\quad = \sup_{x_0 \in \mathbb{R}^n} \Big\{ P\big(t_0, x(t_0)\big) + V\big(t_0, x(t_0^-), R^{\perp}[t_0]\tilde{u}[t_0]\big) \Big\} \\ &\quad = \sup_{x_0 \in \mathbb{R}^n} \Big\{ -\gamma^2 x_0' Q_0 x_0 + x_0' X_{11}[t_0] x_0 \Big\} \\ &\quad = 0. \end{aligned}$$

■

4.5.4 Necessity

Theorem 4.14. *Suppose there exists j, $0 \le j \le f$, such that*

$$-\gamma^2 Y_{j-1}^{-1}(t_j) + Z_{i11}(t_j) > 0.$$

Then

$$\sup_{w \in \mathscr{L}_2[t_0,t_f]} \sup_{v \in \ell_2[t_0,t_f]} \sup_{x_0 \in \mathbb{R}^n} J(x_0, \mu, w, v)$$

is infinite for all controllers μ.

Proof. With J_1 defined as in equation (4.35), we have

$$\begin{aligned} &W\big(t_j, P(t_j, \cdot)\big) \\ &\quad = \inf_{R[t_j]\tilde{u}[t_j]} \sup_{\tilde{y}[t_j]} W\big(t_{j+1}, P(t_{j+1}, \cdot)\big) \\ &\quad = \inf_{R[t_j]\tilde{u}[t_j] \in \mathbb{R}^{m_2}} \sup_{x(t_j) \in \mathbb{R}^n} \sup_{v[t_j]} \sup_{w \in \mathscr{L}_2[t_j,t_{j+1}]} J_1(t_j) \\ &\quad = \inf_{R[t_j]\tilde{u}[t_j] \in \mathbb{R}^{m_2}} \sup_{x(t_j) \in \mathbb{R}^n} \Big\{ -\gamma^2 \big(x(t_j) - \check{x}_{j-1}(t_j)\big)' Y_{j-1}^{-1}(t_j) \big(x(t_j) - \check{x}_{j-1}(t_j)\big) + c[t_j] \\ &\qquad\qquad + \begin{pmatrix} x(t_j) \\ \tilde{u}[t_j] \end{pmatrix}' Z_j(t_j) \begin{pmatrix} x(t_j) \\ \tilde{u}[t_j] \end{pmatrix} \Big\} \end{aligned}$$

Hence, if

$$-\gamma^2 Y_{i-1}^{-1}(t_i) + Z_{i11}(t_i) > 0$$

then $W\big(t_J, P(t_J, \cdot)\big)$ will be unbounded at time j. Therefore, the upper value of the game will be unbounded for all choices of $\tilde{u}[t_j]$. ■

Theorem 4.15. *Suppose, for some j, $0 \le j \le f$, the Riccati equation (4.34) is unbounded on $[t_j, t_{j+1}]$. Then for any controller μ,*

$$\sup_{w \in \mathscr{L}_2[t_0,t_f]} \sup_{v \in \ell_2[t_0,t_f]} \sup_{x_0 \in \mathbb{R}^n} J(x_0, \mu, w, v)$$

is infinite.

Proof. If for some i, the Riccati differential equation (4.34) is unbounded on $[t_i, t_{i+1}]$, then for all y the information state $P\big(t_i, x(t_i)\big)$ will be unbounded above also. Then, since the upper value satisfies the forward dynamic programming recursion (3.36), we know that $P\big(t_{t_f}, x(t_{t_f})\big)$ will be unbounded also, and the result follows. ∎

It is interesting to consider the Riccati equations as continuous equations with jumps. At each time t_j,

$$Y_j(t_j) = \big(Y_{j-1}^{-1}(t_j) + C_2(t_j)'\Gamma[t_j]C_2(t_j)\big)^{-1} \tag{4.37}$$

so, writing $Q(t) = Y_j^{-1}(t)$ for $t \in (t_j, t_{j+1})$, then

$$-\dot{Q} = QA + A'Q + QB_1B_1' + \gamma^{-2}C_1'C_1 + \delta_Q C_2'RC_2$$

where

$$\delta_Q(t) = \sum_j \delta(t - t_j)$$

and δ is the Dirac delta function. Here we can see that the Riccati equation has jumps corresponding to the gain of information at each t_j. Also, the $\check{x}$ equation becomes

$$\dot{\check{x}}_j = A\check{x} + \gamma^{-2}Y_jC_1'C_1\check{x} + B_2u + \delta_Q Y_j(t_j)C_2(t_j)'\Gamma[t_j]\big(\tilde{y}[t_j] - C_2\check{x}(t_j)\big) \tag{4.38}$$

and similarly has jumps at t_i. In this sense the sampling operation is causing discrete changes in the information state at each time t_i, and these changes do in fact correspond to a discretization of the variation in the information state caused by continuously available measurements.

4.6 Summary

In this chapter we have given necessary and sufficient conditions for the existence of a γ-feasible controller for the finite horizon induced 2-norm problem for multi-rate sampled-data systems. We have also given a construction technique for such controllers. For the more general problem of a system with jumps, we have also given necessary and sufficient conditions for existence of a γ-feasible state feedback controller, and an expression for one such controller.

Using the techniques of game theory and dynamic programming, we have shown that the controller which minimizes the upper value of the game does not need to know future information about the times at which the sampled-data information becomes available. But it does require complete future information about the times at which the hold operator becomes active. This appears to be a significant obstacle to extending this formulation to the infinite time problem.

In some sense, given the dynamic formulation we are using, and in comparison with the corresponding results in the continuous time problem, we expect this kind of result. In the continuous time case, the controller Riccati differential equation for X_∞ is solved backwards in time from an endpoint condition, and the filtering Riccati differential equation for Y_∞ is integrated forwards in time from an initial condition.

But this also would indicate that the special nature of the minimization we are performing has imposed the rather strict requirement of full future knowledge of the hold times. Indeed, we know that, in the continuous time case, the controller which minimizes the upper value of the differential game is far from the only γ-suboptimal $\mathscr{H}_\infty$ controller. Indeed, the "central" controller of [22] is not this controller.

One viewpoint on this nonuniqueness is that, for the induced 2-norm problem, we do not really need to minimize the upper value of the differential game. All we need to do is ensure boundedness of the upper value. This corresponds to the idea that we do not need to ensure that $X_\infty(t_0)$ in the continuous time case is small, we simply need to ensure that it is finite. In the sequel, this idea will lead us onto the idea of Riccati differential inequalities, which correspond to the LMI formulation of $\mathscr{H}_\infty$ control in the infinite horizon case.

This suggests a future approach to the multi-rate sampled-data problem, which would enable us to tackle infinite horizon problems, essentially by the combination of the results in this chapter and those in the subsequent chapters. We will not pursue this further here, but suggest it as an opportunity for future research.

5. RICCATI DIFFERENTIAL INEQUALITIES

The solution to the $\mathscr{H}_\infty$ problem for the time invariant infinite horizon case by Glover and Doyle [28] was characterised in terms of two algebraic Riccati equations. The subsequent solution of the time varying problem was similarly characterised by Khargonekar, Ravi, Nagpal [43] in terms of two Riccati differential equations. However, in the time varying case these solutions are not so straightforward to apply. The control Riccati differential equation for X_∞ must be integrated over all future time, and in the nonsingular problem it is required that $A - (B_2B_2' - \gamma^{-2}B_1B_1')X_\infty$ is exponentially stable. For a general time varying system it is difficult both to construct X_∞ and to test if this condition holds.

The solution to the time varying problem has also been expressed in terms of Riccati differential inequalities, or linear matrix inequalities with a differential term, in the parameter varying case by Wu, Yang, Packard and Becker [79]. However, in this solution the resulting controller depends explicitly on $\dot{X}_\infty$ and hence also on the rate of change of the parameter. This solution is derived by algebraic manipulation of the bounded real lemma applied to the closed loop system, and it is difficult to see how the rate dependence might be removed.

In this chapter, we derive a solution to the time varying $\mathscr{H}_\infty$ suboptimal synthesis problem in terms of two Riccati differential inequalities also. However, we do not require the above a-priori condition that $A - (B_2B_2' - \gamma^{-2}B_1B_1')X_\infty$ be exponentially stable, and the controller formulae do not depend explicitly on $\dot{X}_\infty$. We make use of the separation theory of Doyle, Glover, Khargonekar and Francis [22], also known as DGKF, and we also separately derive a solution in terms of inequalities using the game theory of Chapter 3. We will apply the results of this chapter in Chapter 6 to the moving horizon problem, whereby we will further remove the restriction that the solution for X_∞ be integrated over all future time.

5.1 Preliminaries

We define a finite dimensional time varying linear system $\mathcal{P}$ by its state space realization

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \qquad x(0) = x_0 \\ y(t) &= C(t)x(t) + D(t)u(t). \end{aligned}$$

The system is defined for $t \in \mathbb{R}_+$, and, at time t, the state $x(t) \in \mathbb{R}^n$, and the input and output signals have dimensions $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ respectively. The system matrices A, B, C and D are bounded functions of time. We will write $\phi(t_1, t_2)$ for the transition matrix of A. The following lemmas are proved in [58].

Lemma 5.1. *Suppose the pair (A, C) is detectable, and the matrix function P satisfies, for all $t \geq 0$,* *a)* $\dot{P}(t) + A(t)'P(t) + P(t)A(t) \leq -C(t)'C(t)$

b) There exists $\beta > 0$ *such that* $P(t) \leq \beta I$

c) $P(t) \geq 0$

then the system $\dot{x}(t) = A(t)x(t)$ *is exponentially stable.*

Lemma 5.2. *Suppose the pair* (A, B) *is stabilizable, and the matrix function* Q *satisfies, for all* $t \geq 0$,

a) $\dot{Q}(t) - A(t)Q(t) - Q(t)A(t)' \geq B(t)B(t)'$

b) There exists $\beta > 0$ *such that* $Q(t) \leq \beta I$

c) $Q(t) \geq 0$

then the system $\dot{x}(t) = A(t)x(t)$ *is exponentially stable.*

5.2 Inequalities via DGKF separation

In this section we use the separation theory of DGKF [22] to give a solution to the infinite horizon time varying $\mathscr{H}_\infty$ problem. In particular, our solution technique is essentially the same as that of Khargonekar, Ravi and Nagpal [43]. However, we show that the controller can be constructed using any solution to a pair of Riccati differential inequalities, and remove particular stability requirements on their solutions.

5.2.1 The bounded real lemma

Lemma 5.3. *Suppose* $\mathcal{G} : \mathscr{L}_2 \to \mathscr{L}_2$ *is an exponentially stable linear time varying system with* $\mathcal{G} : u \mapsto y$ *defined by*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \qquad\qquad x(0) = 0 \\ y(t) &= C(t)x(t) \end{aligned}$$

where $A \in \mathscr{L}_\infty^{n\times n}$, $B \in \mathscr{L}_\infty^{n\times m}$ *and* $C \in \mathscr{L}_\infty^{p\times n}$. *Then, given* $\gamma > 0$,

a) if there exists $X \in \mathscr{L}_\infty^{n\times n}$ *such that*

$$\dot{X} + A'X + XA + \gamma^{-2}XBB'X + C'C \leq 0 \qquad \forall\, t \in \mathbb{R}_+ \tag{5.1}$$

then $\|\mathcal{G}\| \leq \gamma$.

b) if, in addition to the conditions of a) above, the matrix function $A + \gamma^{-2}BB'X$ *is exponentially stable, then* $\|\mathcal{G}\| < \gamma$.

Proof. a) A simple completion of squares gives

$$\frac{d}{dt}x(t)'X(t)x(t) \leq -y(t)'y(t) + \gamma^2 u(t)'u(t) - \gamma^2|u(t) - \gamma^{-2}B'(t)X(t)x(t)|^2.$$

Integrating this expression over $[0, \infty)$, and using $x(0) = 0$ and exponential stability, we arrive at

$$\|y\|_2^2 \leq \gamma^2 \|u\|_2^2 - \gamma^2 \|u - \gamma^{-2} B' X x\|_2^2 \tag{5.2}$$

Since the induced norm of $\mathcal{G}$ is defined by

$$\|\mathcal{G}\|^2 := \sup_{u \in \mathscr{L}_2} \frac{\|y\|_2^2}{\|u\|_2^2}$$

this implies that $\|\mathcal{G}\| \leq \gamma$.

For b), note that $\|\mathcal{G}\| < \gamma$ if and only if there exists $\varepsilon > 0$ such that $\|y\|_2^2 - \gamma^2 \|u\|_2^2 \leq -\varepsilon \|u\|_2^2$ for all $u \in \mathscr{L}_2$. Let $v = u - \gamma^{-2} B' X x$, then the map from v to u is given by

$$\left[\begin{array}{c|c} A + \gamma^{-2} B B' X & B \\ \hline \gamma^{-2} B' X & I \end{array}\right]$$

hence, if $A + \gamma^{-2} B B' X$ is exponentially stable, this is a bounded linear operator and has a finite induced norm. Therefore there exists $\delta > 0$ such that $\|u\|_2^2 \leq \delta \|u - \gamma^{-2} B' X x\|_2^2$, and equation (5.2) becomes

$$\|y\|_2^2 \leq \gamma^2 \|u\|_2^2 - \frac{\gamma^2}{\delta} \|u\|_2^2$$

giving the desired result. ■

The converse of this result can be stated as follows.

Lemma 5.4. *Suppose $\mathcal{G}$ is an exponentially stable linear time varying system defined as in Lemma 5.3. Then if $\|\mathcal{G}\| < \gamma$, there exists $X \in \mathscr{L}_\infty^{n \times n}$, with $X \geq 0$, such that*

$$\dot{X} + A' X + X A + \gamma^{-2} X B B' X + C' C \leq 0 \qquad \forall\, t \in \mathbb{R}_+$$

Proof. See Tadmor [65]. ■

Remark 5.5. Note that the above results can be shown without assuming differentiability of X, or even finite dimensionality of the system matrices. In this case, the Riccati differential equations are replaced by integral equations. In the sequel, we will write the Riccati equations in the differential form. □

Remark 5.6. Even if we find or construct X which satisfies inequality (5.1) strictly, this alone does not guarantee that $\|\mathcal{G}\|$ is strictly less than γ. However, we would like a simpler condition than stability of $A + \gamma^{-2} B B' X$, since this appears to be an untestable condition for arbitrary X. □

Remark 5.7. The bounded real lemma as stated above makes the important assumption that the system is exponentially stable. Often we would like to use this in a synthesis framework, with a view to constructing an X satisfying (5.1). In order to do this, we would like to be able to use this as a test for a system whose stability is unknown. The next result gives a condition for this purpose. □

Lemma 5.8. *Suppose $\mathcal{G} : \mathscr{L}_{2e} \to \mathscr{L}_{2e}$ is a linear time varying system, not a priori known to be stable, with $\mathcal{G} : u \mapsto y$ defined by*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \qquad\qquad x(0) = 0 \\ y(t) &= C(t)x(t) \end{aligned}$$

where $A \in \mathscr{L}_\infty^{n\times n}$, $B \in \mathscr{L}_\infty^{n\times m}$ and $C \in \mathscr{L}_\infty^{p\times n}$. Then, given $\gamma > 0$, if there exists a positive definite $X \in \mathscr{L}_\infty^{n\times n}$ and $\varepsilon > 0$ such that

$$\dot{X} + A'X + XA + \gamma^{-2}XBB'X + C'C \leq -\varepsilon I \qquad \forall\, t \in \mathbb{R}_+$$

then $\mathcal{G}$ is exponentially stable, and $\|\mathcal{G}\| < \gamma$.

Proof. As in Lemma 5.3, we easily arrive at

$$\|y\|_2^2 \leq \gamma^2\|u\|_2^2 - \gamma^2\|u - \gamma^{-2}B'Xx\|_2^2 - \varepsilon\|x\|_2^2.$$

Since C is a bounded matrix function, there exists $\delta > 0$ such that $\|y\|_2^2 \leq \delta\|x\|_2^2$, which implies

$$\left(1 + \frac{\varepsilon}{\delta}\right)\|y\|_2^2 \leq \gamma^2\|u\|_2^2$$

which therefore implies

$$\begin{aligned} \|\mathcal{G}\|^2 &\leq \gamma^2\left(1 + \frac{\varepsilon}{\delta}\right)^{-1} \\ &< \gamma^2. \end{aligned}$$

The system $\mathcal{G}$ is exponentially stable since X satisfies

$$\dot{X} + A'X + XA \leq -\varepsilon I \qquad \forall\, t \in \mathbb{R}_+$$

and since $(A, \varepsilon^{1/2}I)$ is detectable, stability follows from Lemma 5.1. ■

The case when $\mathcal{G}$ has a direct feedthrough term is not significantly more difficult to derive, but results in more complicated expressions, as follows.

Lemma 5.9. *Suppose $\mathcal{G} : \mathscr{L}_{2e} \to \mathscr{L}_{2e}$ is a linear time varying system with $\mathcal{G} : u \mapsto y$ defined by*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \qquad\qquad x(0) = 0 \\ y(t) &= C(t)x(t) + D(t)u(t). \end{aligned}$$

Suppose also $\|D\|_\infty < \gamma$. Then

a) if $\mathcal{G}$ is exponentially stable, then if there exists $X \in \mathscr{L}_\infty^{n\times n}$ satisfying

$$\begin{aligned} &\dot{X} + A'X + XA + \gamma^{-1}C'C \\ &\qquad + \gamma(XB + \gamma^{-1}C'D)(\gamma^2 I - D'D)^{-1}(B'X + \gamma^{-1}D'C) \leq 0 \end{aligned} \tag{5.3}$$

then $\|\mathcal{G}\| \leq \gamma$.

b) If, in addition to the conditions for a), the time-varying matrix function $A + B(\gamma^2 - D'D)^{-1}(D'C + B'X)$ is exponentially stable, then $\|\mathcal{G}\| < \gamma$.

c) If $\mathcal{G}$ is not known a priori to be stable, then if (A, C) is detectable and there exists $X \in \mathscr{L}_\infty^{n\times n}$ satisfying equation (5.3) with $X \geq 0$ then $\|\mathcal{G}\| \leq \gamma$ and $\mathcal{G}$ is exponentially stable. Note that it is easy to construct a similar result for (A, B) stabilizable by consideration of $\|\mathcal{G}^*\|$.

d) If $\mathcal{G}$ is not known a priori to be stable, then if the inequality (5.3) is uniformly satisfied by $X \in \mathscr{L}_\infty^{n\times n}$ with $X \geq 0$ then $\mathcal{G}$ is exponentially stable and $\|\mathcal{G}\| < \gamma$. By uniformly here we mean that the left hand side is less than $-\varepsilon I$ for some constant $\varepsilon > 0$,

Proof. The proof follows exactly that of Lemmas 5.3 and 5.8 so is omitted. ■

Remark 5.10. Note that the X used in equation (5.3) is not the same as that used in equation (5.1). Substituting $D = 0$ into (5.3) gives

$$\dot{X} + A'X + XA + \gamma^{-1}XBB'X + \gamma^{-1}C'C \leq 0$$

which is equivalent to

$$\dot{\hat{X}} + A'\hat{X} + \hat{X}A + \gamma^{-2}\hat{X}BB'\hat{X} + C'C \leq 0$$

with $\hat{X} = \gamma X$. □

Remark 5.11. An alternative form for the Riccati differential inequality (5.3) is

$$\begin{aligned}\dot{X} + (A + B(\gamma^2 I - D'D)^{-1}D'C)'X + X(A + B(\gamma^2 I - D'D)^{-1}D'C) \\ + \gamma XB(\gamma^2 I - D'D)^{-1}B'X + \gamma^{-1}C'(I + D(\gamma^2 I - D'D)^{-1}D')C \leq 0.\end{aligned}$$

□

Remark 5.12. Another alternative form for the Riccati differential inequality (5.3) is

$$\dot{X} + A'X + XA + \begin{bmatrix} B'X \\ C \end{bmatrix}' \begin{bmatrix} \gamma I & -D' \\ -D & \gamma I \end{bmatrix}^{-1} \begin{bmatrix} B'X \\ C \end{bmatrix} \leq 0 \tag{5.4}$$

□

Remark 5.13. Yet another alternative form for the *strict* version of the Riccati differential inequality (5.3) is

$$\begin{bmatrix} \dot{X} + A'X + XA & XB & C' \\ B'X & -\gamma I & D' \\ C & D & -\gamma I \end{bmatrix} < 0$$

This is the Linear Matrix Inequality formulation. Applying the Schur complement formula allows conversion to the form (5.4), from which another application of the Schur formula results in the form (5.3).

Note that the nonstrict version of this inequality is not equivalent to the nonstrict Riccati form. In fact, it is easy to see [15, p. 28] that

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \leq 0$$

if and only if

$$R \leq 0 \qquad Q - SR^{\dagger}S' \leq 0 \qquad S(I - RR^{\dagger}) = 0$$

and so in general the nonstrict LMI is a stronger condition than the nonstrict Riccati inequality. In the case where $\|D\|_\infty < \gamma$, the matrix R is invertible and the two forms are equivalent. Note however that we have not said anything about the case when $\|D\|_\infty = \gamma$. □

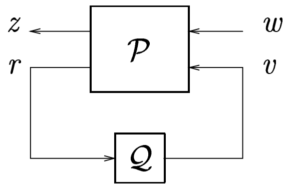


Figure 5.1: Feedback configuration for Lemma 5.14

We will need the following lemma.

Lemma 5.14. *Let $\mathcal{P}$ be a system partitioned as*

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix}$$

and let $\mathcal{Q}$ be another system connected to $\mathcal{P}$ as in Figure 5.1. Assume that $\mathcal{P}$ and $\mathcal{Q}$ are exponentially stable. Suppose that $\mathcal{P}$ satisfies

$$\gamma^2\|r\|_2^2 + \|z\|_2^2 \leq \gamma^2\|w\|_2^2 + \|v\|_2^2 \qquad \text{for all } w, v \in \mathscr{L}_2. \tag{5.5}$$

Then if $\|\mathcal{Q}\| < \gamma$, the closed loop system is exponentially stable and $\|\mathcal{T}_{zw}\| \leq \gamma$. Further, if $\mathcal{P}_{21}$ has an exponentially stable inverse, then $\|\mathcal{T}_{zw}\| < \gamma$.

Proof. From (5.5) with $w = 0$, the norm $\|\mathcal{P}_{22}\| \leq \gamma^{-1}$, and hence $\|\mathcal{Q}\| < \gamma$ implies that the closed loop is internally stable, since $\mathcal{P}$ and $\mathcal{Q}$ are both stable. The nonstrict inequality then follows, since

$$\|z\|_2^2 - \gamma^2\|w\|_2^2 \leq \|v\|_2^2 - \gamma^2\|r\|_2^2 \leq 0.$$

For the strict inequality, first note that $\|\mathcal{Q}\| < \gamma$ implies there exists $\varepsilon > 0$ such that $\|v\|_2^2 - \gamma^2\|r\|_2^2 \leq -\varepsilon\|r\|_2^2$. Then if $\mathcal{P}_{21}$ has an exponentially stable inverse, then $w = \mathcal{P}_{21}^{-1}(I - \mathcal{P}_{22}\mathcal{Q})r = \mathcal{T}_{wr}r$, with $\mathcal{T}_{wr}$ a bounded operator. Therefore $\|w\|_2^2 \leq \|\mathcal{T}_{wr}\|^2\|r\|_2^2$ giving

$$\begin{aligned} \|v\|_2^2 - \gamma^2\|r\|_2^2 &\leq -\varepsilon\|r\|_2^2 \\ &\leq -\frac{\varepsilon}{\|\mathcal{T}_{wr}\|^2}\|w\|_2^2. \end{aligned}$$

Thus
$$\|z\|_2^2 - \gamma^2\|w\|_2^2 \leq -\frac{\varepsilon}{\|\mathcal{T}_{wr}\|_2^2}\|w\|_2^2$$
and the result follows. ■

Let $\mathcal{G}$ be the finite dimensional linear time varying system with state space realization

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_1(t)w(t) + B_2(t)u(t) \qquad x(0) = 0 \\ z(t) &= C_1(t)x(t) + D_{12}(t)u(t) \\ y(t) &= C_2(t)x(t) + D_{21}(t)w(t) \end{aligned} \tag{5.6}$$

defined on $\mathbb{R}_+$, with all matrices bounded. We will make the following assumptions about $\mathcal{G}$.

a) $D_{12}(t)'D_{12}(t) = I$ for all $t \in \mathbb{R}_+$

b) $D_{21}(t)D_{21}(t)' = I$ for all $t \in \mathbb{R}_+$

c) (A, B_2) is stabilizable

d) (A, C_2) is detectable

We would now like to derive a set of sufficient conditions, based on Riccati differential inequalities, for existence of a stabilizing controller such that $\|\mathcal{T}_{zw}\| < \gamma$. In the proof, we will use the separation theory of [22], and essentially follow [43] for the time varying case.

Theorem 5.15. *Let $\mathcal{G}$ be a system of the form* (5.6) *satisfying the above assumptions. Then there exists a controller $\mathcal{K}$ which exponentially stabilizes the closed loop system and results in $\|\mathcal{T}_{zw}\| < \gamma$ if*

a) there exists a bounded non-negative definite solution X to the Riccati differential inequality

$$\begin{aligned} -\dot{X} \geq (A - B_2D_{12}'C_1)'X + X(A - B_2D_{12}'C_1) \\ - X(B_2B_2' - \gamma^{-2}B_1B_1')X + C_1'D_{12}^{\perp}D_{12}^{\perp\prime}C_1 \end{aligned} \tag{5.7}$$

such that $A - B_2D_{12}'C_1 - (B_2B_2' - \gamma^{-2}B_1B_1')X$ is exponentially stable, and

b) there exists a bounded non-negative definite solution Z to the Riccati differential inequality

$$\dot{Z} \geq A_zZ + ZA_z' - Z(C_z'C_z - \gamma^{-2}C_k'C_k)Z + B_1D_{21}^{\perp\prime}D_{21}^{\perp}B_1 \tag{5.8}$$

such that $A_z - Z(C_z'C_z - \gamma^{-2}C_k'C_k)$ is exponentially stable, where

$$\begin{aligned} A_z &= A - B_1D_{21}'C_2 + \gamma^{-2}B_1D_{21}^{\perp\prime}D_{21}^{\perp}B_1'X \\ C_k &= -D_{12}'C_1 - B_2'X \\ C_z &= C_2 + \gamma^{-2}D_{21}B_1'X. \end{aligned}$$

If these conditions are satisfied, then an admissible controller $\mathcal{K}$ is given by

$$\begin{aligned} \dot{\bar{x}} &= A_k\bar{x} + B_k y \qquad \bar{x}(0) = 0 \\ u &= C_k\bar{x} \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} A_k &= A + \gamma^{-2}B_1B_1'X + B_2C_k - (B_1D_{21}' + ZC_z')C_z \\ B_k &= B_1D_{21}' + ZC_z'. \end{aligned}$$

Proof. We use the by now standard change of variables, first introduced in [22], setting $v = u - C_k x$ and $r = w - \gamma^{-2}B_1'Xx$. Then we can rewrite the closed loop in terms of these variables, resulting in a system $\mathcal{P}$ defined by

$$\begin{aligned} \dot{x} &= (A + B_2C_k)x + B_1w + B_2v \\ z &= (C_1 + D_{12}C_k)x + D_{12}v \\ r &= -\gamma^{-2}B_1'Xx + w \end{aligned}$$

and a controller $\mathcal{Q}$ given by

$$\begin{aligned} \dot{e} &= (A_z - ZC_z'C_z)e + (B_1 - B_kD_{21})r \\ v &= -C_ke. \end{aligned}$$

where $e = x - \bar{x}$. We aim to show that these satisfy the assumptions of Lemma 5.14. First, we show that $\mathcal{P}$ is exponentially stable. Rewrite (5.7) as

$$\dot{X} + (A+B_2C_k)'X + X(A+B_2C_k) \leq -XB_2B_2'X - \gamma^{-2}XB_1B_1'X - C_1D_{12}^{\perp}D_{12}^{\perp\prime}C_1. \tag{5.10}$$

By assumption, $A + B_2C_k + \gamma^{-2}B_1B_1'X$ is exponentially stable, and so the pair $(A + B_2C_k, \gamma^{-1}B_1'X)$ is detectable. Then it follows from Lemma 5.1 that $\mathcal{P}$ is exponentially stable. In order to show the norm bound, algebra gives

$$\frac{d}{dt}x'Xx \leq \gamma^2 w(t)'w(t) - z(t)'z(t) + v(t)'v(t) - \gamma^2 r(t)'r(t).$$

Since $\mathcal{P}$ is exponentially stable, and $x(0) = 0$, integrating over $[0, \infty)$ gives

$$\gamma^2\|r\|_2^2 + \|z\|_2^2 \leq \gamma^2\|w\|_2^2 + \|v\|_2^2.$$

Partitioning $\mathcal{P}$ in the obvious way, straightforward manipulations give

$$\mathcal{P}_{21}^{-1} = \left[\begin{array}{c|c} A - B_2D_{12}'C_1 - (B_2B_2' - \gamma^{-2}B_1B_1')X & B_1 \\ \hline \gamma^{-2}B_1'X & I \end{array}\right]$$

which is exponentially stable by assumption.

Next we establish stability of $\mathcal{Q}$. Rewrite (5.8) as

$$\dot{Z} - (A_z - ZC_z'C_z)Z - Z(A_z - ZC_z'C_z)' \geq ZC_z'C_zZ + \gamma^{-2}ZC_k'C_kZ + B_1D_{21}^{\perp\prime}D_{21}^{\perp}B_1'.$$

Since by assumption $A_z - ZC_z'C_z + \gamma^{-2}ZC_k'C_k$ is exponentially stable, the pair $(A_z - ZC_z'C_z, \gamma^{-2}ZC_k')$ is stabilizable, and thus by Lemma 5.2, $A_z - ZC_z'C_z$ is exponentially stable. In order to show that $\|\mathcal{Q}\| < \gamma$, we establish that $\|\mathcal{Q}^*\| < \gamma$, where $\mathcal{Q}^*$ is the dual of $\mathcal{Q}$, defined on $(-\infty, 0]$. The dual system $\mathcal{Q}^*$ has realization

$$\begin{aligned} \dot{e}^* &= (A_z' - C_z'C_zZ)e^* - C_k'r^* \\ v^* &= (B_1' - D_{21}'B_k')e^* \end{aligned}$$

where the time varying system matrices are now functions of $-t$, where Z satisfies the time reversed Riccati inequality

$$\dot{Z} \le -(A_z - ZC_z'C_z)Z - Z(A_z - ZC_z'C_z)' - ZC_z'C_zZ - \gamma^{-2}ZC_k'C_kZ - B_1D_{21}^{\perp\prime}D_{21}^{\perp}B_1'.$$

Then algebra gives

$$\frac{d}{dt}e^{*\prime}Ze^* \le \gamma^2 r^{*\prime}r^* - \gamma^2|r^* + \gamma^{-2}C_kZe^*|^2 - v^{*\prime}v$$

and, since $A_z - ZC_z'C_z$is exponentially stable, integrating over $(-\infty, 0]$ gives

$$\|v^*\|_2^2 - \gamma^2\|r^*\|_2^2 \le -\gamma^2\|r^* + \gamma^{-2}C_kZe^*\|_2^2 - e^*(0)'Z(0)e^*(0).$$

Let L be the system mapping r to $r^* + \gamma^{-2}C_kZe^*$. Then

$$L = \left[\begin{array}{c|c} A_z' - C_z'C_zZ & -C_k' \\ \hline \gamma^{-2}C_kZ & I \end{array}\right]$$

which has inverse

$$L^{-1} = \left[\begin{array}{c|c} A_z' - C_z'C_zZ - \gamma^{-2}C_k'C_kZ & -C_k' \\ \hline -\gamma^{-2}C_kZ & I \end{array}\right].$$

By assumption, $A_z' - C_z'C_zZ$ is exponentially stable, and therefore L^{-1} is a bounded linear operator. So

$$\|v^*\|_2^2 - \gamma^2\|r^*\|_2^2 \le -\frac{\gamma^2}{\|L^{-1}\|^2}\|r\|_2^2 - e^*(0)'Z(0)e^*(0) \qquad \text{for all } r^* \in \mathscr{L}_2(-\infty, 0]$$

and since $Z(0) \ge 0$, we have $\|\mathcal{Q}\| = \|\mathcal{Q}^*\| < \gamma$. Hence we have shown that, with this controller, the constructed systems $\mathcal{P}$ and $\mathcal{Q}$ satisfy the assumptions of Lemma 5.14, and so the closed loop norm $\|\mathcal{T}_{zw}\| < \gamma$ and the system is internally stable. ■

Remark 5.16. Note that the above theorem is in fact necessary also. To show necessity, we need only show that existence of solutions to the Riccati differential *equations* is necessary, since these will also satisfy the inequalities. This is proved in the case when $B_1D_{21}' = 0$ and $C_1'D_{12} = 0$ by Ravi, Nagpal, and Khargonekar in [43], and generalization to the case with cross terms is not difficult. □

We would now like to remove the assumptions in Theorem 5.15 that

$$A - B_2D_{12}'C_1 - (B_2B_2' - \gamma^{-2}B_1B_1')X \qquad A_z - Z(C_z'C_z - \gamma^{-2}C_k'C_k)$$

are exponentially stable. In order to do this, as in the bounded real lemma, we will solve uniformly negative definite Ricatti differential inequalities.

Theorem 5.17. *Suppose in addition to the above assumptions on $\mathcal{G}$ that the pair $(A - B_2 D_{12}' C_1, D_{12}^{\perp\prime} C_1)$ is detectable. Then if*

a) there exists a bounded non-negative definite solution X to the Riccati differential inequality

$$\dot{X} + (A - B_2 D_{12}' C_1)' X + X(A - B_2 D_{12}' C_1) - X(B_2 B_2' - \gamma^{-2} B_1 B_1')X + C_1' D_{12}^{\perp} D_{12}^{\perp\prime} C_1 \leq 0 \quad (5.11)$$

and

b) there exists a constant $\varepsilon > 0$ and a bounded non-negative definite solution Z to the Riccati differential inequality

$$-\dot{Z} + A_z Z + Z A_z' - Z(C_z' C_z - \gamma^{-2} C_k' C_k)Z + B_1 D_{21}^{\perp\prime} D_{21}^{\perp} B_1' \leq -\varepsilon I \quad (5.12)$$

then the controller $\mathcal{K}$ defined by equation (5.9) is stabilizing, and results in a closed loop induced norm which satisfies $\|\mathcal{T}_{zw}\| \leq \gamma$.

Proof. The proof follows the same outline as that of Theorem 5.15, with the following changes in detail.

First, note that to show stability of $\mathcal{P}$, since $(A - B_2 D_{12}' C_1, D_{12}^{\perp\prime} C_1)$ is detectable, so is $(A - B_2 D_{12}' C_1 - B_2 B_2' X, \begin{bmatrix} C_1' D_{12}^{\perp} & X B_2' \end{bmatrix}')$, and therefore equation 5.10 implies that $\mathcal{P}$ is exponentially stable. Stability of $\mathcal{Q}$ is established by observing that $(A_z - Z C_z' C_z, \varepsilon^{1/2} I)$ is stabilizable, and that the Riccati inequality in Z can be written

$$\dot{Z} - (A_z - Z C_z' C_z)Z - Z(A_z - Z C_z' C_z)' \geq Z C_z' C_z Z + \gamma^{-2} Z C_k' C_k Z + B_1 D_{21}^{\perp\prime} D_{21}^{\perp} B_1' + \varepsilon I.$$

Considering the time reversed version of this inequality as before gives

$$\|v^*\|_2^2 + \varepsilon \|e^*\|_2^2 \leq \gamma^2 \|r^*\|_2^2.$$

Since $\|v^*\|_2^2 \leq \delta \|e^*\|_2^2$ for some $\delta > 0$, we arrive at

$$\left(1 + \frac{\varepsilon}{\delta}\right) \|v^*\|_2^2 \leq \gamma^2 \|r^*\|_2^2$$

which implies $\|\mathcal{Q}\| < \gamma$. Then applying Lemma 5.14 gives $\|\mathcal{T}_{zw}\| \leq \gamma$. ■

5.3 Connections to differential games

We also present here, for the finite horizon case, some connections to the game theoretic approach of Chapter 3. We will consider the linear time varying system on the finite horizon $[t_0, t_f]$ described by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_1(t)w(t) + B_2(t)u(t) \qquad x(t_0) = x_0 \\ z(t) &= C_1(t)x(t) + D_{12}(t)u(t) \\ y(t) &= C_2(t)x(t) + D_{21}(t)w(t) \end{aligned}$$

We again make the standard assumptions that

$$D_{21}(t)B_1(t)' = 0 \qquad\qquad D_{12}(t)'C_1(t) = 0$$

for all $t \in [t_0, t_f]$, and use the cost function

$$J(x_0, \mu, w) = -\gamma^2 x_0' Q_0 x_0 + \int_{t_0}^{t_f} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt + x(t_f)'Q_f x(t_f)$$

where $Q_f > 0$ is a symmetric positive definite matrix. In the state feedback case, we will assume the initial state $x_0 = 0$, and in the output feedback case we will require $Q_0 > 0$, and we will consider the initial state x_0 to be part of the unknown disturbance, that is part of the maximizing player's strategy. We would then like to solve the output feedback problem

$$\inf_{\mu \in \mathscr{U}_{of}} \sup_{\substack{w \in \mathscr{L}_2 \\ x_0 \in \mathbb{R}^n}} J(x_0, \mu, w).$$

The solution to this problem is well known, and is derived in Chapter 3. A γ-feasible controller μ exists satisfying

$$\sup_{\substack{w \in \mathscr{L}_2 \\ x_0 \in \mathbb{R}^n}} J(x_0, \mu, w) \le 0$$

if and only if there exist bounded solutions to the two Riccati differential equations

$$\begin{aligned} \dot{Y}_\infty &= AY_\infty + Y_\infty A' - Y_\infty(C_2'C_2 - \gamma^{-2}C_1'C_1)Y_\infty + B_1B_1' \\ Y_\infty(t_0) &= Q_0^{-1} \end{aligned}$$

and

$$\begin{aligned} -\dot{X}_\infty &= A'X_\infty + X_\infty A - X_\infty(B_2B_2' - \gamma^{-2}B_1B_1')X_\infty + C_1'C_1 \\ X_\infty(t_f) &= Q_f. \end{aligned}$$

satisfying the coupling condition

$$X_\infty(t) - \gamma^2 Y_\infty^{-1}(t) < 0.$$

5.3.1 State feedback

The state feedback result is easily derived. We construct a controller for the same system with cost function

$$J(x_0, \mu, w, \Delta_1) = \int_{t_0}^{t_f} z'z + x'\Delta_1 x - \gamma^2 w'w\, dt + x(t_f)'Q_f x(t_f).$$

where $\Delta_1 \ge 0$ is a given bounded positive semidefinite matrix function of time, and the initial state $x_0 = 0$. Clearly, if we can design a controller such that this cost function is negative for all disturbances w, then this same controller will also ensure that the original cost function is negative also. Let X_Δ satisfy

$$\begin{aligned} -\dot{X}_\Delta &= A'X_\Delta + X_\Delta A - X_\Delta(B_2B_2' - \gamma^{-2}B_1B_1')X_\Delta + C_1'C_1 + \Delta_1 \\ X_\Delta(t_f) &= Q_1. \end{aligned} \tag{5.13}$$

A trivial modification of the completion of squares argument of Chapter 3 gives

$$J(x(t_0), \mu, w, \Delta_1) = x(t_0)' X_\Delta(t_0) x(t_0) + \|\mu(\cdot, x(\cdot)) - u^*\|_2^2 \\ - \gamma^2 \|w - w^*\|_2^2 - x(t_f)'(Q_1 - Q_f) x(t_f)$$

for any state feedback controller $\mu \in \mathscr{U}_{sf}$ and any input w, where

$$u^*(t) = -B_2(t)' X_\Delta(t) x(t)$$

and

$$w^*(t) = \gamma^{-2} B_1(t)' X_\Delta(t) x(t).$$

Hence, if $Q_1 > Q_f$, since $J(x_0, \mu, w, 0) \leq J(x_0, \mu, w, \Delta_1)$

$$J(x(0), \mu, w, 0) \leq x(t_0)' X_\Delta(t_0) x(t_0) + \|\mu(\cdot, x(\cdot)) - u^*\|_2^2 - \gamma^2 \|w - w^*\|_2^2.$$

Therefore, when $x(t_0) = 0$, the state feedback strategy $\mu\big(t, x(t)\big) = -B_2(t)' X_\Delta(t) x(t)$ is a γ-feasible state feedback controller for the original problem. The important thing to note here is that the controller depends on X_Δ and not on X_∞. Also, since existence of a bounded solution for X_Δ implies existence of a γ-feasible state feedback controller, existence of a bounded solution for X_Δ implies existence of a bounded solution for X_∞.

Further, the key point is that Δ_1 is *arbitrary*. This means that we do not need to know it in order to synthesize a controller. This also allows us to avoid having to solve exactly the Riccati differential equation, instead we may solve

$$\begin{aligned} -\dot{X}_\Delta &\geq A' X_\Delta + X_\Delta A - X_\Delta (B_2 B_2' - \gamma^{-2} B_1 B_1') X_\Delta + C_1' C_1 \\ X_\Delta(t_f) &\geq Q_f \end{aligned} \tag{5.14}$$

In particular, in the case when the Riccati differential equation is monotonic, it is easy to guarantee that the solution we are constructing has a gradient *below* some prespecified value, whereas previously we could only assume that the constructed solution was γ-feasible provided that the numerical solution to the Riccati differential equation was sufficiently close to the actual solution. Monotonicity is guaranteed for linear time invariant systems for specific choices of the terminal weight Q_f, as we shall show in the next Chapter.

5.3.2 Output feedback

In the output feedback case, we will assume the cost function contains an initial state term, with $Q_0 > 0$. Then Proposition 3.19 implies that both the RDEs for X_∞ and Y_∞ have strictly positive definite solutions, since by assumption they have strictly positive definite boundary conditions. Then we can write

$$-\frac{d}{dt} Y_\infty^{-1} = Y_\infty^{-1} A + A' Y_\infty^{-1} - C_2' C_2 + \gamma^{-2} C_1' C_1 + Y_\infty^{-1} B_1 B_1' Y_\infty^{-1}$$

Clearly we could simply add Δ_1 to the X_∞ equation and $\gamma^{-2}\Delta_1$ to the Y_∞^{-1} equation, since in this case we would be simply increasing the size of $\|z\|_2^2$ by increasing the size of $C_1' C_1$, and if a controller is γ-feasible for this problem then it must be so for the smaller problem also. But by doing so we gain little. In fact, the only difference would be that we have made it harder to synthesis γ-feasible controllers, since we would then need to know Δ_1 in order to construct solutions to both RDEs with the same Δ_1.

So we would like to add *different* positive matrices to each of the Riccati equations, and still guarantee the induced norm bound. In this section we will derive conditions under which we can do this.

Our approach will be to again consider the certainty equivalence formulation as in Section 3.5.2, but again we will not rely on the proof of the general certainty equivalence theorem. In fact, since we would simply like to give sufficient conditions for construction of a γ-feasible controller, we will see that the required result is conceptually very simple.

Our approach will be to use different cost functions for the past and future halves of the cost function. Define

$$V\big(t, x(t)\big) := x(t)' X_\Delta(t) x(t).$$

As we have seen, $V\big(t, x(t)\big)$ is an upper bound for the cost of the game with the minimax state feedback controller in place. Recall the definition of the following disturbance class, which is the set of all possible (w, x_0) disturbances consistent with information available at time τ.

$$\Omega_0(\tau) := \Bigg\{ (x_0, w)\ ; x_0 \in \mathbb{R}^n,\ w \in \mathscr{L}_2[t_0, \tau],\ \text{such that} \\ \begin{array}{l} \dot{x}(t) = A(t)x(t) + B_1(t)w(t) + B_2(t)u^\tau(t) \\ y^\tau(t) = C_2(t)x(t) + D_{21}(t)w(t) \end{array} \qquad \text{for } t \in [t_0, \tau] \Bigg\}.$$

In order to accommodate different initial conditions for the solution to Y_∞, define

$$G(\tau, x_0, u^\tau, w^\tau) := -\gamma^2 x_0' Q_2 x_0 + \int_{t_0}^{\tau} z'z + x'\Delta_2 x - \gamma^2 w'w\, dt + V(\tau, x(\tau)). \qquad (5.15)$$

for some given matrix $Q_2 > 0$, and also define the constrained optimization problem

$$W(\tau, u^\tau, y^\tau) = \sup_{(x_0, w^\tau) \in \Omega_0(\tau)} G(\tau, x_0, u^\tau, w^\tau) \qquad (5.16)$$

Since we are only trying to prove sufficiency, the following result is easily shown.

Theorem 5.18. *For a given controller μ, suppose that, for each $\tau \in [t_0, t_f]$, and for each $y^\tau \in [t_0, \tau]$, the maximization problem* (5.16) *has a bounded solution. Further, suppose that for all $t_1, t_2 \in [t_0, t_f]$ with $t_2 \geq t_1$, and for all y, $W(t_2, u^{t_2}, y^{t_2}) \leq W(t_1, u^{t_1}, y^{t_1})$. That is, W is a decreasing function of τ when the signal u is defined by the feedback law μ. Then, if $Q_2 \leq Q_0$,*

$$\sup_{\substack{x_0 \in \mathbb{R}^n \\ w \in \mathscr{L}_2}} J(x_0, \mu, w, \Delta_2) \leq \sup_{x_0} \big\{ V(t_0, x_0) - \gamma^2 x_0' Q_2 x_0 \big\}.$$

Proof. Clearly, since $Q_2 \leq Q_0$,

$$J(x_0, \mu, w, \Delta_2) \leq W(t_f, u^{t_f}, y^{t_f}).$$

Since W is decreasing,

$$\begin{aligned} J(x_0, \mu, w, \Delta_2) &\le W(t_0, u^{t_0}, y^{t_0}) \\ &= \sup_{x_0 \in \mathbb{R}^n} \left\{ V(t_0, x_0) - \gamma^2 x_0' Q_2 x_0 \right\} \qquad \forall w \in \mathscr{L}_2. \end{aligned}$$

Therefore, with this feedback strategy for the u player, the result holds. ■

Let Y_Δ satisfy the following Riccati differential equation:

$$\begin{aligned} \dot{Y}_\Delta &= AY_\Delta + Y_\Delta A' - Y_\Delta(C_2'C_2 - \gamma^{-2}C_1'C_1 - \gamma^{-2}\Delta_2)Y_\Delta + B_1B_1' \\ Y_\Delta(t_0) &= Q_2^{-1}. \end{aligned} \tag{5.17}$$

Then, it is a trivial modification of Lemma 3.21 to show

Lemma 5.19. *Suppose there exists a matrix $Y_\Delta(t)$ satisfying the Riccati differential equation (5.17) on $[t_0, t_f]$, and such that $X_\Delta(\tau) - \gamma^2 Y_\Delta^{-1}(\tau) < 0$ for $\tau \in [t_0, t_f]$. Then there exists a unique maximum to the constrained maximization problem defined by equation (5.16), and the worst case cost is given by*

$$\begin{aligned} W(\tau, u^\tau, y^\tau) = \int_{t_0}^{\tau} &\left\{ \hat{x}^{\tau\prime}(C_1'C_1 + \Delta_2)\hat{x}^\tau + u'u - \gamma^2 \hat{w}^{\tau\prime}\hat{w}^\tau \right\} dt \\ &+ \hat{x}^\tau(\tau)' X_\Delta(\tau)\hat{x}^\tau(\tau) - \gamma^2 \hat{x}_0^{\tau\prime} Q_2 \hat{x}_0^\tau \end{aligned}$$

where

$$\begin{aligned} \dot{\hat{x}}^\tau &= A\hat{x}^\tau + B_1B_1'Y_\Delta^{-1}\hat{x}^\tau - B_1B_1'Y_\Delta^{-1}\check{x} + B_2u^\tau & \hat{x}^\tau(t_0) &= 0 \\ \dot{\check{x}} &= A\check{x} + \gamma^{-2}Y_\Delta(C_1'C_1 + \Delta_2)\check{x} + Y_\Delta C_2'(y^\tau - C_2\check{x}) + B_2u^\tau & \check{x}(t_0) &= 0 \end{aligned}$$

with boundary conditions

$$\hat{x}^\tau(\tau) = \left(I - \gamma^{-2}Y_\Delta(\tau)X_\Delta(\tau)\right)^{-1}\check{x}(\tau) \tag{5.18}$$

and the worst case disturbance is given by

$$\hat{w}^\tau = B_1'Y_\Delta^{-1}\hat{x}^\tau - B_1'Y_\Delta^{-1}\check{x} + D_{21}'(y^\tau - C_2\hat{x}^\tau). \tag{5.19}$$

The following lemma is straightforward algebra.

Lemma 5.20. *Suppose Y_Δ satisfies equation (5.17), and X_Δ satisfies equation (5.13). Suppose $Y_\Delta(t)(I - \gamma^{-2}X_\Delta(t)Y_\Delta(t))^{-1} > 0$, and define*

$$Z_\Delta(t) = Y_\Delta(t)(I - \gamma^{-2}X_\Delta(t)Y_\Delta(t))^{-1}.$$

Then $Z_\Delta(\tau)$ satisfies

$$\begin{aligned} \dot{Z}_\Delta = (A + \gamma^{-2}B_1B_1'X_\Delta)Z_\Delta &+ Z_\Delta(A + \gamma^{-2}B_1B_1'X_\Delta)' \\ &- Z_\Delta(C_2'C_2 - \gamma^{-2}X_\Delta B_2B_2'X_\Delta + \gamma^{-2}(\Delta_1 - \Delta_2))Z_\Delta + B_1B_1'. \end{aligned}$$

■

Remark 5.21. This result has some important consequences. The solution of the Riccati differential inequalities in X_∞ and Y_∞ together with the coupling condition form a convex problem for each time t. However, we cannot use this to construct solutions to the Riccati differential inequality in Z_∞. □

Let $\bar{x}(\tau) = \hat{x}^\tau(\tau)$ for all τ. The key result of this section is the following one.

Theorem 5.22. *If there exist bounded solutions X_Δ and Y_Δ to equations* (5.13) *and* (5.17), *then the function $W(\tau, u^\tau, y^\tau)$ satisfies*

$$\frac{d}{d\tau}W(\tau,u^\tau,y^\tau) = -\gamma^2|y(\tau) - C_2(\tau)\bar{x}(\tau)|^2 + |u(\tau) + B_2(\tau)'X_\Delta(\tau)\bar{x}(\tau)|^2 \\ - \bar{x}(\tau)'(\Delta_1 - \Delta_2)\bar{x}(\tau)$$

Hence, if $\Delta_1 - \Delta_2 > 0$, then if $u(\tau) = -B_2(\tau)'X_\Delta(\tau)\bar{x}(\tau)$ for all $\tau \in [t_0, t_f]$, then

$$\frac{d}{d\tau}W(\tau,u^\tau,y^\tau) \le 0$$

for all y^τ. Then, with this controller, using Theorem 5.18,

$$\begin{aligned}\sup_{\substack{x_0\in\mathbb{R}^n\\ w\in\mathscr{L}_2}} J(x_0,\hat{\mu},w,\Delta_2) &\le \sup_{x_0}\left\{V(t_0,x_0) - \gamma^2 x_0' Q_2 x_0\right\} \\ &= 0\end{aligned}$$

since by assumption, $Y_\Delta^{-1}(t_0) = Q_2^{-1}$, and $X_\Delta(t_0) - \gamma^2 Y_\Delta^{-1}(t_0) < 0$. Therefore, this controller is γ-feasible for the original problem.

Proof of Theorem 5.22. This proof is identical in method to that of Lemma 3.22. In the same way as that proof, algebra gives

$$\frac{d}{dt}(\hat{x}^{\tau\prime}Y_\Delta^{-1}\check{x}) = u^{\tau\prime}B_2'Y_\Delta^{-1}\check{x} - \check{x}'Y_\Delta^{-1}B_1B_1'Y_\infty^{-1}\check{x} + y^{\tau\prime}C_2\hat{x}^\tau + \hat{x}^{\tau\prime}Y_\Delta^{-1}B_2u^\tau$$

where terms in Δ_2 cancel. We arrive at

$$\begin{aligned}\frac{d}{d\tau}W(\tau,u^\tau,y^\tau) = \frac{d}{d\tau}&\left\{\hat{x}^\tau(\tau)'\big(\gamma^2Y_\Delta^{-1}(\tau) - X_\Delta(\tau)\big)\hat{x}^\tau(\tau)\right\} \\ &+ u(\tau)'u(\tau) - \gamma^2y(\tau)'y(\tau) + \gamma^2\check{x}(\tau)'Y_\Delta^{-1}(\tau)B_1(\tau)B_1(\tau)'Y_\Delta^{-1}(\tau)\check{x}(\tau) \\ &- 2\gamma^2u(\tau)'B_2(\tau)'Y_\Delta^{-1}(\tau)\check{x}(\tau)\end{aligned}$$

and $\bar{x}$ satisfies

$$\begin{aligned}\bar{x} &= (I - \gamma^{-2}Y_\Delta X_\Delta)^{-1}\check{x} \\ &= Z_\Delta Y_\Delta^{-1}\check{x}\end{aligned}$$

and hence

$$\dot{\bar{x}} = A\bar{x} + \gamma^{-2}B_1B_1'X_\Delta\bar{x} + X_\Delta C_2'(y - C_2\bar{x}) \\ + \gamma^{-2}Z_\Delta X_\Delta B_2B_2'X_\Delta\bar{x} + Z_\Delta Y_\Delta^{-1}B_2u - \gamma^{-2}Z_\Delta(\Delta_1 - \Delta_2)\bar{x}.$$

Substituting as in Lemma 3.22 gives the desired result. ■

As it stands, this result is not quite what we would like to achieve, for two reasons. The first is that we need to guarantee that the condition $\Delta_1 \geq \Delta_2$ holds. This means we need to have bounds on Δ_1 and Δ_2. The other reason is that the controller dynamics depend on $\bar{x}$, which itself explicitly is a function of $\Delta_1 - \Delta_2$. We can cure both of these problems. To cure the latter problem, define $\tilde{x}(t) = Y_\Delta^{-1}(t)\check{x}(t)$. Then

$$\dot{\tilde{x}} = -A'\tilde{x} - Y_\Delta^{-1}B_1B_1'\tilde{x} + C_2'y + Y_\Delta^{-1}B_2u$$

with boundary condition $\tilde{x}(t_0) = 0$. The controller formula is then

$$u(t) = -B_2'X_\Delta(t)Z_\Delta(t)\tilde{x}(t)$$

and does not explicitly depend on the Δ_i matrices except through the Riccati solutions.

In order to avoid interdependence of the Δ_i matrices, we make use of the following Lemma.

Lemma 5.23. *Suppose there exists a bounded solution to the the Riccati differential equation*

$$\dot{Y}_\Delta = AY_\Delta + Y_\Delta A' - Y_\Delta(C_2'C_2 - \gamma^{-2}C_1'C_1 - \gamma^{-2}\Delta_2)Y_\Delta + B_1B_1' \tag{5.20}$$

on $[t_0, t_f]$ with boundary condition $Y_\Delta(t_0) = Q_2^{-1}$. If there exists a bounded strictly positive definite solution to the Riccati differential equation

$$\begin{aligned}\dot{Z}_\Delta = {} & (A + (B_1B_1' - \gamma^2B_2B_2')Y_\Delta^{-1})Z_\Delta + Z_\Delta(A + (B_1B_1' - \gamma^2B_2B_2')Y_\Delta^{-1})' \\ & + Z_\Delta(\gamma^2Y_\Delta^{-1}B_2B_2'Y_\Delta^{-1} - \gamma^{-2}\Delta_3 - C_2'C_2)Z_\Delta - B_1B_1' + \gamma^2B_2B_2'\end{aligned} \tag{5.21}$$

with boundary condition

$$Z_\Delta(t_f) = \left(Y_\Delta^{-1}(t_f) - \gamma^{-2}Q_1\right)^{-1} \tag{5.22}$$

then the matrix X_Δ defined by $X_\Delta(t) := \gamma^2Y_\Delta^{-1}(t) - \gamma^2Z_\Delta^{-1}(t)$ satisfies the Riccati differential equation

$$-\dot{X}_\Delta = A'X_\Delta + X_\Delta A - X_\Delta(B_2B_2' - \gamma^{-2}B_1B_1')X_\Delta + C_1'C_1 + \Delta_3 + \Delta_2 \tag{5.23}$$

with boundary condition

$$X_\Delta(t_f) = Q_1 \tag{5.24}$$

and the coupling condition $Y_\Delta^{-1}(t) - \gamma^{-2}X_\Delta(t) > 0$ holds for all $t \in [t_0, t_f]$.

Conversely, assuming there exist a bounded solution Y_Δ to (5.20) with boundary condition $Y_\Delta(t_0) = Q_2^{-1}$, then if there exists a bounded solution X_Δ to equation (5.23) with boundary condition (5.24) satisfying the coupling condition $Y_\Delta^{-1}(t) - \gamma^{-2}X_\Delta(t) > 0$, then the matrix defined by

$$Z_\Delta := \left(Y_\Delta^{-1} - \gamma^{-2}X_\Delta\right)^{-1}$$

satisfies the Riccati differential equation (5.21) with boundary condition (5.22).

Proof. This is straightforward to show by direct differentiation. ■

We now can guarantee that $\Delta_1 \geq \Delta_2$ by choosing arbitrary $\Delta_2 \geq 0$ and $\Delta_3 \geq 0$. Then the matrix function Δ_1 is given by $\Delta_2 + \Delta_3$. As a result, we do not need to know the Δ matrices explicitly; we can simply solve the Riccati differential inequalities.

Hence we can now construct controllers for the finite horizon time varying induced 2-norm problem by solving Riccati differential inequalities. The solution presented here has the advantage over the simpler solution based upon the bounded real lemma by Wu et. al. [79] in that the controllers do not depend on $\dot{X}_\infty$ and $\dot{Y}_\infty$. This in turn implies that, if we solve this problem for parameter varying systems as a special case of the time varying problem, the resulting controller does not depend on the rate of change of the parameter.

However, we lose an important feature of the solutions of [79], and that is that the constructive procedure here is not convex. Since the Riccati differential inequality for Z_Δ depends on the product $Y_\Delta^{-1}Z_\Delta$, we cannot solve both inequalities together and have the problem remain simultaneously convex in Y_Δ and Z_Δ, which we could do were we solving the original inequalities in X_Δ and Y_Δ. Thus we have shown that rate independent γ-suboptimal $\mathscr{H}_\infty$ controllers exists, but the construction we have given here is unfortunately not convex.

Collecting these results gives the following theorem.

Theorem 5.24. *Suppose there exists bounded solutions to the Riccati differential inequalities*

$$\begin{aligned} \dot{\check{Y}} &\leq -\check{Y}A - A'\check{Y} - \check{Y}B_1B_1'\check{Y} + C_2'C_2 - \gamma^{-2}C_1'C_1 \\ \check{Y}(t_0) &\leq Q_0 \end{aligned}$$

and

$$\begin{aligned} \dot{\check{Z}} &\geq -\check{Z}(A + (B_1B_1' - \gamma^2 B_2B_2')\check{Y}) - (A + (B_1B_1' - \gamma^2 B_2B_2')\check{Y})'\check{Z} \\ &\qquad + \check{Z}(B_1B_1' - \gamma^2 B_2B_2')\check{Z} - \gamma^2\check{Y}B_2B_2'\check{Y} + C_2'C_2 \\ \check{Z}(t_f) &\leq \check{Y}^{-1}(t_f) - \gamma^{-2}Q_f. \end{aligned}$$

Then the controller

$$\begin{aligned} \dot{\tilde{x}} &= -A'\tilde{x} - \check{Y}B_1B_1'\tilde{x} + C_2'y + \check{Y}B_2u & \tilde{x}(t_0) = 0 \\ u &= -\gamma^2B_2'(\check{Y}\check{Z}^{-1} - I)\tilde{x} \end{aligned}$$

guarantees

$$\|z\|_2^2 + x(t_f)'Q_fx(t_f) \leq \gamma^2\|w\|_2^2 + \gamma^2x_0'Q_0x_0$$

for all $w \in \mathscr{L}_2[t_0, t_f]$ *and all* $x_0 \in \mathbb{R}^n$.

Although the formulae derived here are similar to those of the previous section, they are not identical. We have now derived independent inequalities for Y and Z, rather than for X and Z. Also, the standard controller state description of Section 5.2 depends on the parameters Δ_1 and Δ_2 in this case.

In this chapter we have extended the solution of [43] to remove the assumptions that the solution to the Riccati differential equations are stabilizing. We have also shown that these equations can be replaced by inequalities. For the dynamic game problem, we have perturbed the separated cost function to derived solutions in terms of Riccati differential inequalities also.

In the next chapter we will use the sufficient conditions for existence and construction of γ-feasible controllers presented in this chapter to show that the moving horizon $\mathscr{H}_\infty$ control problem also results in stabilizing γ-feasible controllers.

6. MOVING HORIZON $\mathscr{H}_\infty$ CONTROL

6.1 Motivation

The moving horizon control technique was developed using the linear-quadratic regulator in the 1970's, with one of the first papers being that by Kleinman [44] in 1970. The techniques in his paper were later reformulated and generalized by various authors, see for example [18, 46, 45]. Kwon and Pearson [46] proved stability for linear time varying systems using a controller which optimized a quadratic cost function integrated over a time interval from the current time t to a fixed distance ahead $t+T$. That is, at each time t, the controller solves

$$\inf_{\mu \in \mathscr{U}_{sf}} \int_t^{t+T} x(s)'R(s)x(s) + u(s)'u(s)\, ds + x(t+T)'Q_f x(t+T).$$

Theoretically, the resulting controller defined on the interval $[t, t+T]$ is only actually used at time t before being recalculated. The solution to this problem was given in terms of a Riccati differential equation, integrated backwards from time $t+T$ at each time t. Kwon et. al. [45] also formulated a general procedure for the forwards integration of such solutions. This allows a continuous update of the time varying controller.

If the terminal weight Q_f is chosen to satisfy certain conditions, or is replaced by the terminal constraint $x(t+T) = 0$, then the closed loop system will be exponentially stable. Indeed, it is intuitively reasonable from the above form of cost function that a large terminal weight would tend to result in a stable closed loop system, and in fact for the quadratic state feedback problem, choosing Q_f sufficiently large will result in stability. We may also regard the constraint $x(t+T) = 0$ as a limiting case, corresponding to $Q_f = \infty$. For time varying systems, this gives a method of stabilizing the system which depends quantitatively on the parametric description of the system for only a finite time ahead, and depends on more qualitative features of the long term plant behaviour.

Further, this method allows stabilization of systems which are known only for the short term future. In dealing with long term variation of systems, there are at present two extreme options. The first is to assume a time invariant system and design a controller robust against time varying perturbations, which usually leads to very conservative controller design. The other is to design fully for time varying systems. In the case of both linear quadratic and $\mathscr{H}_\infty$ controllers, this requires the backwards integration from infinity of a Riccati equation (see for example [69, 43]), and somewhat optimistically assumes knowledge of the system throughout future time. The moving horizon method can be viewed as a compromise between these two methods.

This problem can be viewed in the following way; given the set of stabilizable time varying (A, B) pairs of fixed dimension, does there exist a map to the set of time varying state feedback laws such that for each (A, B) pair the resulting closed loop is exponentially stable, and the corresponding controller depends only *causally* on (A, B).

That is, is it possible, given only information about the past behaviour of the A and B matrices, to decided upon the current input based only on knowledge of the system matrices up until now? The moving horizon strategy does not do quite this, since it uses information about the system matrices up to some fixed time T ahead, and indeed the system has to satisfy other conditions for all time. However, we shall show that not only is the resulting closed loop exponentially stable, but it is also possible to construct such controllers so that the closed loop satisfies a prespecified induced norm bound.

The quadratic moving horizon method described above is motivated as a state feedback strategy, although since it has been shown to be equivalent to an infinite horizon regulator problem for certain choices of terminal weight Q_f, it has also been used as an output feedback strategy with a Kalman filter providing the state estimates.

In this chapter, a receding horizon controller is formulated with each finite horizon optimization based upon an $\mathscr{H}_\infty$ optimization. It is hoped that the advantages of receding horizon control might be combined with the robustness advantages of $\mathscr{H}_\infty$ control. Tadmor [68] proved, under certain assumptions and with a terminal constraint, that a similar controller was stable and satisfied an infinite horizon norm bound. We generalize these results in the state feedback case and construct a new controller for the output feedback case.

In the observation feedback case it is difficult to produce a natural formulation of the $\mathscr{H}_\infty$ receding horizon control problem. For the finite horizon problem, typically the assumptions made are that the initial state at the beginning of each optimization interval is completely known, or is completely unknown. In the latter case it is treated as part of the disturbance and subject to a quadratic weighting [72] [42]. In the receding horizon problem, though, we have observations from *before* the optimization interval.

We first consider the state feedback problem, with the above quadratic cost function replaced by the game theoretic cost function of previous chapters. We will then consider the output feedback case.

6.2 State feedback

We will again consider the continuous time system

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B_1(t)w(t) + B_2(t)u(t)\\ z(t) &= C_1(t)x(t) + D_{12}(t)u(t)\\ y(t) &= C_2(t)x(t) + D_{21}(t)w(t)\end{aligned} \tag{6.1}$$

defined on $\mathbb{R}_+$. Again we will make the nonsingularity assumptions that $D_{12}(t)'D_{12}(t) = I$ and $D_{21}(t)D_{21}(t)' = I$ for all $t \in \mathbb{R}_+$. We will also assume that the system matrices are all bounded functions of time.

6.2.1 The moving horizon differential game

The principle idea behind moving horizon control is that of solving a finite horizon optimization problem at each time $t \geq 0$, over the interval $[t, t+T]$. In this way, at each time t the resulting controller depends only on the plant parameters up to a *finite* time ahead.

We will consider controllers in the space of continuous time state feedback laws given by

$$\mathscr{U}_{sf}^{n,m_2}[t_1,t_2] := \Big\{ v : [t_1,t_2] \times \mathbb{R}^n \to \mathbb{R}^{m_2};\ \forall\, x \in \mathscr{L}_2^n[t_1,t_2],\ v(\cdot, x(\cdot)) \in \mathscr{L}_2^{m_2}[t_1,t_2] \Big\}.$$

The moving horizon problem we will solve, at each time $t > 0$, will be

$$\inf_{\mu \in \mathscr{U}_{sf}[t,t+T]} \sup_{w \in \mathscr{L}_2[t,t+T]} \int_t^{t+T} z(s)'z(s) - \gamma^2 w(s)'w(s)\, ds + x(t+T)'F(t+T)x(t+T) \quad (6.2)$$

where $F(t+T) > 0$ is some given positive definite weighting matrix on the terminal state. At each time t we construct the state feedback law $u(t) = \mu(t, x(t))$. However, ideally this control law is only implemented instantaneously, at time t, although in practice it might be implemented on a short finite interval $[t, t+\delta]$. At time $t+\delta$, a new optimization would be performed over the interval $[t+\delta, t+\delta+T]$. In this chapter we will assume that the controller is updated continuously, and in Chapter 7 we will discuss the case when discrete updates are performed.

For each separate optimization problem, the initial state may not be zero, and hence a direct induced norm interpretation on each finite interval is not possible, since zero input u would give a nonzero output z on the interval $[t, t+T]$, and thus the induced norm on this interval is undefined.

The terminal state penalty F is often incorporated into finite horizon control problems to allow for compromises between the norm of z and the size of the final state $x(t_f)$. In the moving horizon case, we will show that particular choices of F guarantee closed loop stability.

An expression for the moving horizon controller. From the results of Chapter 3, we can solve the game theoretic problem explicitly on each interval $[t, t+T]$. There exists a saddle point solution to the finite horizon state feedback problem (6.2) if and only if there exists a bounded solution to the Riccati differential equation

$$\begin{aligned} -\dot{X}_\infty = (A - B_2 D_{12}' C_1)' X_\infty + X_\infty (A - B_2 D_{12}' C_1) \\ - X_\infty (B_2 B_2' - \gamma^{-2} B_1 B_1') X_\infty + C_1' D_{12}^{\perp} D_{12}^{\perp\prime} C_1 \end{aligned} \quad (6.3)$$

on $[t, t+T]$ with boundary condition $X_\infty(t+T) = F(t+T)$. This result can also be found in [50].

For the moving horizon problem, we will write the solution X_∞ to this Riccati differential equation on the interval $[t_1, t_2]$ with boundary condition $X_\infty(t_2) = F$ as $X_\infty(t_1, t_2, F)$. Then, solving the problem over each interval $[t, t+T]$, it is clear that this must satisfy the matrix partial differential equation

$$\begin{aligned} &-\frac{\partial}{\partial t} X_\infty(t, s, F(s)) \\ &\quad = (A - B_2 D_{12}' C_1)(t)' X_\infty(t, s, F(s)) + X_\infty(t, s, F(s))(A - B_2 D_{12}' C_1)(t) \\ &\qquad - X_\infty(t, s, F(s))(B_2 B_2' - \gamma^{-2} B_1 B_1')(t) X_\infty(t, s, F(s)) + C_1' D_{12}^{\perp} D_{12}^{\perp\prime} C_1(t) \end{aligned} \quad (6.4)$$

for $s \geq t \geq 0$, with boundary condition

$$X_\infty(s, s, F(s)) = F(s) \qquad \text{for all } s \geq T.$$

The state feedback moving horizon controller is then given by

$$u(t) = -\big(B_2(t)'X_\infty(t, t+T, F(t+T)) + D_{12}(t)'C_1(t)\big)x(t).$$

Note that, in the case when the original system is linear time invariant, then the resulting controller is time invariant also. We will use the following lemmas to analyse the moving horizon $\mathscr{H}_\infty$ controller.

Lemma 6.1. *Suppose F is a bounded positive solution of*

$$\begin{aligned}\dot{F} + (A - B_2D_{12}'C_1)'F + F(A - B_2D_{12}'C_1) \\ - F(B_2B_2' - \gamma^{-2}B_1B_1')F + C_1'(I - D_{12}D_{12}')C_1 \leq 0 \quad (6.5)\end{aligned}$$

on the interval $[t_1, t_2]$, and X_∞ is a bounded positive solution of equation (6.3) on the interval $[t_1, t_2]$. Further suppose that $X_\infty(t_2) = F(t_2)$. Then $X_\infty(t) < F(t)$ for $t \in [t_1, t_2]$.

Proof. Using the completion of squares argument of Theorem 3.14, we have

$$x(t)'X_\infty(t)x(t) = \inf_{\mu \in \mathscr{U}_{sf}[t,t_2]} \sup_{w \in \mathscr{L}_2[t,t_2]} \int_t^{t_2} z(s)'z(s) - \gamma^2 w(s)'w(s)\, ds + x(t_2)'F(t_2)x(t_2)$$

and

$$\begin{aligned}x(t)'F(t)x(t) = \inf_{\mu \in \mathscr{U}_{sf}[t,t_2]} \sup_{w \in \mathscr{L}_2[t,t_2]} \bigg\{ \int_t^{t_2} |z(s)|^2 + x(s)'\Delta(s)x(s) - \gamma^2|w(s)|^2\, ds \\ + x(t_2)'F(t_2)x(t_2)\bigg\}\end{aligned}$$

for some matrix function $\Delta \geq 0$. Since

$$f(u, w) \leq g(u, w) \implies \inf_{u \in \mathscr{U}_{sf}} \sup_{w \in \mathscr{L}_2} f(u, w) \leq \inf_{u \in \mathscr{U}_{sf}} \sup_{w \in \mathscr{L}_2} g(u, w)$$

we have immediately the required result. ■

Lemma 6.2. *The solution X_∞ to the Riccati differential equation (6.4) satisfies, for all $t_3 \geq t_2 \geq t_1 \geq 0$ and for all $F \geq 0$,*

a) $X_\infty(t_1, t_2, X_\infty(t_2, t_3, F)) = X_\infty(t_1, t_3, F)$

b) $F > 0 \implies X_\infty(t_1, t_2, F) > 0$

c) $C_1'D_{12}^\perp D_{12}^{\perp\prime}C_1 > 0 \implies X_\infty(t_1, t_2, F) > 0$

d) $F_1 \geq F_2 \geq 0 \implies X_\infty(t_1, t_2, F_1) \geq X_\infty(t_1, t_2, F_2)$

Proof.

a) This is a straightforward consequence of the definition of X_∞.

b) See Proposition 3.19.

c) See Proposition 3.19.

d) This follows, similarly to Lemma 6.1, from

$$f(u,w) \le g(u,w) \implies \inf_{u\in\mathscr{U}_{sf}} \sup_{w\in\mathscr{L}_2} f(u,w) \le \inf_{u\in\mathscr{U}_{sf}} \sup_{w\in\mathscr{L}_2} g(u,w)$$

■

6.2.2 Stability of the receding horizon controller

For the linear quadratic problem with a terminal weight, the closed loop was shown to be stable under certain conditions by Kwon, Bruckstein and Kailath [45]. The problem when $F=\infty$, that is when the terminal state is constrained to be zero was considered by Tadmor [68]. The techniques we will use are similar to those in [45].

We will first need the following Lemma. Related results can be found in [45, 2, 59, 17, 57, 7].

Lemma 6.3. *Suppose F satisfies*

$$\begin{aligned} \dot F + (A - B_2D_{12}'C_1)'F + F(A - B_2D_{12}'C_1) \\ - F(B_2B_2' - \gamma^{-2}B_1B_1')F + C_1'(I - D_{12}D_{12}')C_1 \le 0 \end{aligned} \quad (6.6)$$

and $F(t) \ge 0$ for all $t \ge 0$. Suppose also that for each $t \ge 0$, there exists a bounded nonnegative definite solution to the Riccati differential equation (6.3) on the interval $[t, t+T]$, with boundary condition $X_\infty(t+T) = F(t+T)$. Then

$$X_\infty(t_1, t_2, F(t_2)) \ge X_\infty(t_1, t_3, F(t_3))$$

for all $t_3 \ge t_2 \ge t_1 \ge 0$. That is, $X_\infty(t, s, F(s))$ is non-increasing with respect to s.

Proof. By definition, $X_\infty(t_3, t_3, F(t_3)) = F(t_3)$. Then from Lemma 6.1, we have

$$X_\infty(t, t_3, F(t_3)) \le F(t)$$

for all $t \le t_3$. Therefore, for $t_2 \le t_3$,

$$X_\infty(t_2, t_3, F(t_3)) \le F(t_2).$$

If we consider this equation at time t_2 as a boundary condition, and apply Lemma 6.2(d), we see

$$X_\infty\big(t_1, t_2, X_\infty(t_2, t_3, F(t_3))\big) \le X_\infty\big(t_1, t_2, F(t_2)\big)$$

and hence, applying Lemma 6.2(a),

$$X_\infty(t_1, t_3, F(t_3)) \le X_\infty(t_1, t_2, F(t_2))$$

■

The following theorem is the required stability result.

Theorem 6.4. *Suppose the pair* $(A - B_2 D_{12}' C_1, D_{12}^{\perp\prime} C_1)$ *is detectable, the pair* (A, B_2) *is stabilizable, and let* F *satisfy* (6.6), *with* $F(t) \geq 0$ *for* $t \geq 0$. *Suppose furthermore that for each* $t \geq 0$, *there exists a bounded nonnegative definite solution to the Riccati differential equation* (6.3) *on the interval* $[t, t+T]$, *with boundary condition* $X_\infty(t+T) = F(t+T)$. *Then, if there exists* $\beta > 0$ *such that* $X_\infty(t, t+T, F(t+T)) < \beta I$ *for all* $t \geq 0$, *then the control law*

$$u(t) = -\big(B_2(t)' X_\infty(t, t+T, F(t+T)) + D_{12}(t)' C_1(t)\big) x(t). \tag{6.7}$$

applied to the system (6.1) *results in an exponentially stable closed loop.*

Proof. We will write $X(t) = X_\infty(t, t+T, F(t+T))$. Then, with the controller in place, the homogeneous part of closed loop dynamics is given by $\dot{x}(t) = \Lambda(t) x(t)$, where

$$\Lambda(t) = A(t) - B_2(t) B_2(t)' X(t) - B_2(t) D_{12}(t)' C_1(t).$$

The matrix $X(t)$ satisfies

$$\begin{aligned} \dot{X} + (A - B_2 D_{12}' C_1)' X + X(A - B_2 D_{12}' C_1) - X(B_2 B_2' - \gamma^{-2} B_1 B_1') X \\ + C_1'(I - D_{12} D_{12}') C_1 - \frac{\partial}{\partial s} X_\infty(t, s, F(s))\Big|_{s=t+T} = 0 \end{aligned} \tag{6.8}$$

which we can rewrite as

$$\begin{aligned} \dot{X} + \Lambda' X + X \Lambda + X B_2 B_2' X + \gamma^{-2} X B_1 B_1' X \\ + C_1'(I - D_{12} D_{12}') C_1 - \frac{\partial}{\partial s} X_\infty(t, s, F(s))\Big|_{s=t+T} = 0. \end{aligned}$$

Lemma 6.3 implies

$$\frac{\partial}{\partial s} X_\infty(t, s, F(s))\Big|_{s=t+T} \leq 0$$

and if we write

$$\tilde{C} = \begin{bmatrix} D_{12}^{\perp\prime} C_1 \\ B_2' X \end{bmatrix}$$

then X satisfies

$$\dot{X} + \Lambda' X + X \Lambda \leq -\tilde{C}' \tilde{C}.$$

Since $(A - B_2 D_{12}' C_1, D_{12}^{\perp\prime} C_1)$ is detectable, $(\Lambda, \tilde{C})$ is detectable also, and thus applying Lemma 5.1 gives the desired result. ∎

Remark 6.5. Note that, although in the above proof we have assumed differentiability of $X_\infty(t, s, F(s))$ with respect to s, it is straightforward to extend the result to the case when the differentiability assumption does not hold. In this case, the differential equations for X are replaced by integral equations. This technique is used by Tadmor [69] to solve the general linear $\mathscr{H}_\infty$ problem. □

Remark 6.6. Clearly the assumption that (A, B_2) is stabilizable is necessary for the existence of a stabilizing state feedback linear time varying controller. However, we have not used it in the above argument. This implies that, if $(A - B_2 D_{12}' C_1, D_{12}^{\perp\prime} C_1)$ is detectable, existence of a positive solution X to the Riccati equation (6.8) implies that (A, B_2) is stabilizable. □

Clearly, the condition that F satisfy equation (6.6) is not necessary for exponential stability. Nor indeed is the condition that

$$\frac{\partial}{\partial s} X_\infty(t, s, F(s))\Big|_{s=t+T} \leq 0$$

since this might correspond to a different C_1 matrix, for which

$$C_1'(I - D_{12}D_{12}')C_1 - \frac{\partial}{\partial s} X_\infty(t, s, F(s))\Big|_{s=t+T} > 0.$$

6.2.3 Infinite horizon norm bounds

Theorem 6.7. *If the conditions of Theorem 6.4 hold, and $x_0 = 0$, then the controller (6.7) results in*

$$\|z\|_2^2 - \gamma^2 \|w\|_2^2 \leq 0 \qquad \text{for all } w \in \mathscr{L}_2[0, \infty)$$

where the 2-norms are defined on $\mathscr{L}_2[0, \infty)$. That is, with this controller in place, the closed loop induced norm satisfies $\|\mathcal{T}_{zw}\| \leq \gamma$.

Proof. Completing the square as in Section 3.4.3 gives

$$\int_0^\infty \frac{d}{dt}\{x(t)'X(t)x(t)\} + z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt =$$

$$\int_0^\infty |u + B_2'Xx + D_{12}'C_1 x|^2 - \gamma^2 |w - \gamma^{-2} B_1' X x|^2 + x'\left(\frac{\partial}{\partial s} X_\infty(t, s, F(s))\Big|_{s=t+T}\right) x\, dt$$

Therefore, with $u = -B_2'Xx - D_{12}'C_1 x$, writing $w^* = \gamma^{-2} B_1' X x$, since the closed loop is exponentially stable, and by assumption $X(\cdot)$ is bounded, we have

$$-x_0' X(0) x_0 + \|z\|_2^2 - \gamma^2 \|w\|_2^2 = -\gamma^2 \|w - w^*\|_2^2 + \int_0^\infty x'\left(\frac{\partial}{\partial s} X_\infty(t, s, F(s))\Big|_{s=t+T}\right) x\, dt.$$

Since $x_0 = 0$, using Lemma 6.3, we arrive at

$$\|z\|_2^2 - \gamma^2 \|w\|_2^2 \leq 0 \text{ for all } w \in \mathscr{L}_2[0, \infty)$$

as desired. ■

This is the key result of this section. It shows that, using only the conditions specified in Theorem 6.4 on the terminal state weight $F(t)$, we can achieve an induced norm bound over the whole of future time. There are a number of questions we would like to ask about this scheme. The first is, when does a matrix F satisfying these conditions exist?

Proposition 6.8. *For a given time varying system of the form* (6.1) *such that* (A, B_2) *is stabilizable and* $(A - B_2 D_{12}' C_1, D_{12}^{\perp\prime} C_1)$ *is detectable, if there exists a stabilizing state feedback controller* $\mu \in \mathscr{U}_{sf}[0, \infty)$ *such that the closed loop system has* $\|\mathcal{T}_{zw}\| < \gamma$, *then there exists a solution to the Riccati differential equation*

$$\dot{F} + (A - B_2 D_{12}' C_1)' F + F(A - B_2 D_{12}' C_1) \\ - F(B_2 B_2' - \gamma^{-2} B_1 B_1') F + C_1'(I - D_{12} D_{12}') C_1 = 0$$

This is proved in [43]. This implies that, for any given time varying system, if there exists a γ-feasible stabilizing state feedback controller, then there exists a γ-feasible state feedback moving horizon controller. Of course, this statement is in fact not very useful, since the moving horizon controller would be the infinite horizon controller. In the construction of this controller, it would be necessary to use the exact model of the plant over all future time.

Linear time invariant systems. Similar conditions, which make use of complete knowledge of the future plant to construct moving horizon controllers, are well known for linear time invariant plants for quadratic regulator problems. One in particular is the so-called 'fake' algebraic Riccati equation, introduced by Poubelle et al.[56]. However, Poubelle considered the linear-quadratic moving horizon problem, for which the conditions required for stability are simpler. We derive here a similar result for the $\mathscr{H}_\infty$ problem as follows.

Lemma 6.9. *Consider the LTI system* (6.1) *where all matrices are constant. Suppose that* (A, B_2) *is stabilizable and* $(A - B_2 D_{12}' C_1, D_{12}^{\perp\prime} C_1)$ *is detectable, in the time invariant sense. Let* $F > 0$ *be a fixed symmetric matrix. Suppose for each* $t \geq 0$ *there exists a saddle point for the finite horizon differential game*

$$\inf_{\mu \in \mathscr{U}_{sf}[t, t+T]} \sup_{w \in \mathscr{L}_2[t, t+T]} \int_t^{t+T} z(r)' z(r) - \gamma^2 w(r)' w(r) \, dr + x(t+T)' F x(t+T).$$

and write

$$X := X_\infty(t, t+T, F) \qquad \Delta := \frac{\partial}{\partial t} X_\infty(t, s, F)\Big|_{s=t+T}.$$

Then if $\Delta \geq 0$, *and if* $A - B_2 D_{12}' C_1 - (B_2 B_2' - \gamma^{-2} B_1 B_1') X$ *is stable, then the moving horizon controller*

$$u(t) = -(B_2 X_\infty(t, t+T, F) + D_{12}' C_1) x(t) \tag{6.9}$$

stabilises the closed loop system and results in $\|\mathcal{T}_{zw}\| \leq \gamma$.

Proof. Differentiation implies

$$\Delta = \dot{X} - \frac{\partial}{\partial s} X_\infty(t, s, F)\Big|_{s=t+T}$$

and hence X satisfies

$$(A - B_2 D_{12}' C_1)' X + X(A - B_2 D_{12}' C_1) \\ - X(B_2 B_2' - \gamma^{-2} B_1 B_1') X + C_1'(I - D_{12} D_{12}') C_1 + \Delta = 0.$$

If we now consider the new output

$$\tilde{z} = \tilde{C}_1 x + \tilde{D}_{12} u$$

where

$$\tilde{C}_1 = \begin{bmatrix} C_1 \\ \Delta^{\frac{1}{2}} \end{bmatrix} \qquad \tilde{D}_{12} = \begin{bmatrix} D_{12} \\ 0 \end{bmatrix}.$$

then X satisfies the 'fake' algebraic Riccati equation

$$(A - B_2\tilde{D}_{12}'\tilde{C}_1)'X + X(A - B_2\tilde{D}_{12}'\tilde{C}_1) \\ - X(B_2B_2' - \gamma^{-2}B_1B_1')X + \tilde{C}_1'(I - \tilde{D}_{12}\tilde{D}_{12}')\tilde{C}_1 = 0. \quad (6.10)$$

Then, from [22], since X is a stabilising solution to this equation, with the controller given by equation (6.9) the closed loop norm from w to $\tilde{z}$ must be strictly less than γ. Since $\|\tilde{z}\|_2^2 \geq \|z\|_2^2$ for any w, the original system has an induced norm strictly bounded by γ also. ■

In other words, if the solution to the finite horizon Riccati differential equation is non-decreasing in the above sense, then the moving horizon controller will be γ-feasible and, for the LTI problem, the above result gives an alternative proof to Theorem 6.7. Further, we know how conservative such a controller will be, since not only is it γ-feasible for the original system (6.1), but it is also γ-feasible for the system with the larger output $\tilde{z}$. Hence the *rate* of increase of X_∞, as measured by Δ, is important in assessing the level of conservatism of the performance achieved by the moving horizon controller. The following Lemma gives a condition for Δ to be decreasing.

Lemma 6.10. *Let A, R and Q be constant matrices, and let X satisfy*

$$-\dot{X} = A'X + XA - XRX + Q$$

on the interval $[t_0, t_f]$, with boundary condition $X(t_f) = Q_f$. Then, if $\dot{X}(t_f) \leq 0$, then X is non-increasing for all $t \in [t_0, t_f]$. Similarly, if $\dot{X}(t_f) \geq 0$, then X is non-decreasing for all $t \in [t_0, t_f]$.

Proof. Differentiating gives

$$-\ddot{X} = (A - RX)'\dot{X} + \dot{X}(A - RX)$$

which is a Lyapunov equation, and so applying Proposition 3.19 to this equation, we arrive at

$$\dot{X} = \psi(t_f, t)'\dot{X}(t_f)\psi(t_f, t)$$

and the result follows. ■

We can use this result to immediately give the LTI version of Theorem 6.7. For linear time invariant systems, choosing a fixed F such that

$$(A - B_2D_{12}'C_1)'F + F(A - B_2D_{12}'C_1) \\ - F(B_2B_2' - \gamma^{-2}B_1B_1')F + C_1'(I - D_{12}D_{12}')C_1 \leq 0$$

will ensure that

$$\frac{\partial}{\partial t} X_\infty(t, s, F)\Big|_{s=t+T} \geq 0$$

for $t \geq 0$. Then, provided γ is chosen sufficiently large, the resulting moving horizon controller

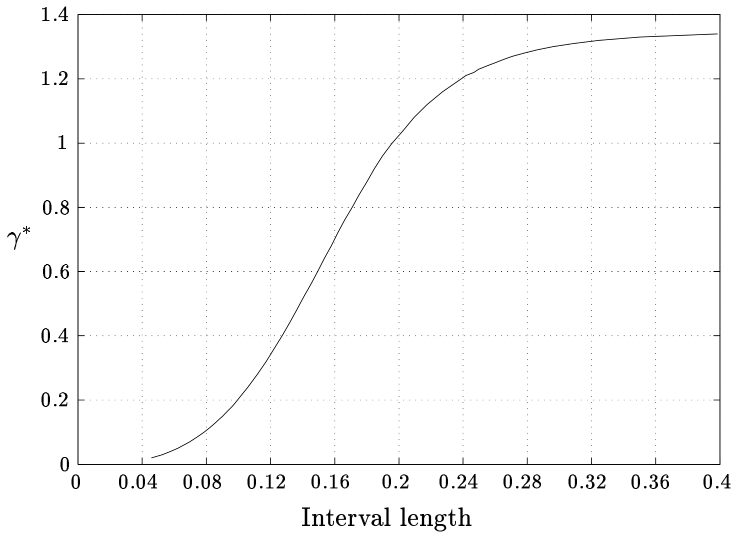
$$u(t) = -(B_2 X_\infty(t, t+T, F) + D_{12}' C_1) x(t)$$

will be a γ-feasible $\mathscr{H}_\infty$ controller for the original problem. Further, Lemma 6.9 tells us exactly how much above the optimal γ must be chosen.

An important point to note is that, if $\bar{X}$ is the solution to the algebraic Riccati equation

$$\begin{aligned}(A - B_2 D_{12}' C_1)' \bar{X} + \bar{X}(A - B_2 D_{12}' C_1)& \\ - \bar{X}(B_2 B_2' - \gamma^{-2} B_1 B_1')\bar{X} + C_1'(I - D_{12} D_{12}') C_1 &= 0\end{aligned}$$

then $F > \bar{X}$ is *not* sufficient for $X_\infty(t, t+T, F)$ to be stabilizing. Counterexamples to this, in the discrete time case for the regulator problem, are given in [13], and it is straightforward to construct similar examples in the continuous time case.



$$A = \begin{bmatrix} -0.0100 & 0 & -30.0000 \\ 0 & -0.0100 & 15.0000 \\ 1.0000 & -1.0000 & 0 \end{bmatrix} \qquad B_1 = \begin{bmatrix} 9.3969 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 6.9500 \\ 0 \\ 0 \end{bmatrix} \qquad C_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -16.3469 & 0 \end{bmatrix} \qquad D_{12} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Figure 6.1: Optimal gamma versus interval length for a finite horizon problem

Figure 6.1 shows the variation in optimal achievable gamma by state feedback against horizon length for a linear time invariant system. The open loop poles of this system are $-0.0050 + 6.7082i$, $-0.0050 - 6.7082i$, and -0.0100, and the open loop frequency response is shown in Figure 6.2.

The system is taken from a spring damper example in [29]. The resonance shown has a period of about 0.93 seconds. However, we can see that, for the state feedback

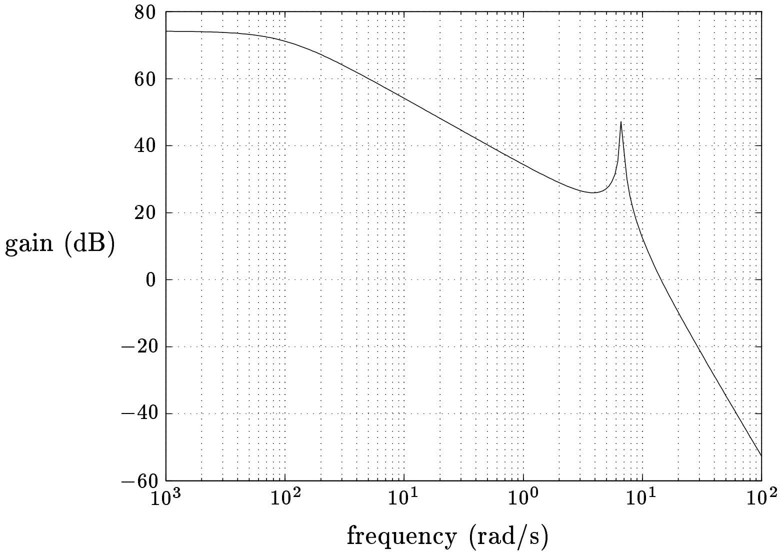


Figure 6.2: Open loop frequency response.

synthesis, the optimal achievable gamma is about constant for intervals greater than 0.4 seconds.

The above analysis for LTI systems is not our main concern, however. It allows construction of an $\mathscr{H}_\infty$ γ-feasible stabilizing controller, but there are already well known techniques for this. It gives two extra 'tuning' parameters, but in practice these are of little value, since we have no idea how to tune them. Further, there are known design techniques using weights as tuning parameters for the $\mathscr{H}_\infty$ problem, and the relationship between the weights and the resulting closed loop properties are far clearer than the relationship F and T have to the closed loop properties. Indeed, by increasing F we may actually cause the closed loop to become unstable. It would appear therefore, that the main utility of these results is for time varying systems. Before investigating further interpretations, we will turn to the output feedback problem.

6.3 Measurement feedback

In this section, we will combine the output feedback results of Chapter 5 with the state feedback results of the previous section. In order to do this, we will derive results for the filtering Riccati equation which are analogous to those of the the previous section for the control Riccati equation.

Let $\mathcal{G}$ be the finite dimensional linear time varying system with state space realization

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_1(t)w(t) + B_2(t)u(t) \qquad x(0) = 0 \\ z(t) &= C_1(t)x(t) + D_{12}(t)u(t) \\ y(t) &= C_2(t)x(t) + D_{21}(t)w(t) \end{aligned} \tag{6.11}$$

defined on $\mathbb{R}_+$, with all matrices bounded. We will make the following assumptions about $\mathcal{G}$.

6.A1) $D_{12}(t)'D_{12}(t) = I$ for all $t \in \mathbb{R}_+$

6.A2) $D_{21}(t)D_{21}(t)' = I$ for all $t \in \mathbb{R}_+$

6.A3) (A, B_2) is stabilizable

6.A4) (A, C_2) is detectable

6.A5) $(A - B_2D_{12}'C_1, D_{12}^{\perp\prime}C_1)$ is detectable

The stabilizability and detectability assumptions are not necessary for the finite horizon problem, but will of course be necessary for the moving horizon problem in order to ensure stability. As before, the control Riccati differential equation is

$$\begin{aligned} -\dot{X}_\infty = (A - B_2D_{12}'C_1)'X_\infty + X_\infty(A - B_2D_{12}'C_1) \\ - X_\infty(B_2B_2' - \gamma^{-2}B_1B_1')X_\infty + C_1'(I - D_{12}D_{12}')C_1 \end{aligned} \tag{6.12}$$

and we restate the definitions needed for the filtering Riccati equation.

$$\begin{aligned} A_z &= A - B_1D_{21}'C_2 + \gamma^{-2}B_1D_{21}^{\perp\prime}D_{21}^{\perp}B_1'X \\ C_k &= -D_{12}'C_1 - B_2'X \\ C_z &= C_2 + \gamma^{-2}D_{21}B_1'X. \end{aligned}$$

and the Riccati differential equation is

$$\dot{Z}_\infty = A_zZ_\infty + Z_\infty A_z' - Z_\infty(C_z'C_z - \gamma^{-2}C_k'C_k)Z_\infty + B_1D_{21}^{\perp\prime}D_{21}^{\perp}B_1'. \tag{6.13}$$

A solution to the output feedback finite horizon $\mathscr{H}_\infty$ problem is given by the following theorem. In this case, $X = X_\infty$.

Theorem 6.11. *For the linear time varying system $\mathcal{G}$ defined by equations* (6.11) *on the finite horizon $[0, T]$, a controller $\mathcal{K}$ exists giving $\|\mathcal{T}_{zw}\| < \gamma$, where the norm is the finite horizon induced $\mathscr{L}_2$ norm, if and only if there exist positive definite bounded solutions to the Riccati differential equations* (6.12) *and* (6.13) *on the interval $[0, T]$. In this case, one such controller is given by*

$$\begin{aligned} \dot{\bar{x}} &= A_k\bar{x} + B_ky \qquad \bar{x}(0) = 0 \\ u &= C_k\bar{x} \end{aligned}$$

where

$$\begin{aligned} A_k &= A + \gamma^{-2}B_1B_1'X + B_2C_k - (B_1D_{21}' + Z_\infty C_z')C_z \\ B_k &= B_1D_{21}' + Z_\infty C_z'. \end{aligned}$$

Proof. A variant on this theorem is proved in Chapter 3. See also Limebeer, Anderson, Khargonekhar and Green [50]. ■

We will write the solution Z_∞ to the Riccati differential equation (6.13) on the interval $[t_1, t_2]$ with boundary condition $Z_\infty(t_1) = G$ as $Z_\infty(t_2, t_1, G)$, which therefore satisfies the matrix partial differential equation

$$\begin{aligned} -\frac{\partial}{\partial t}Z_\infty(t, s, G(s)) = A_z(t)Z_\infty(t, s, G(s)) + Z_\infty(t, s, G(s))A_z(t)' \\ - Z_\infty(t, s, G(s))(C_z'C_z - \gamma^{-2}C_k'C_k)(t)Z_\infty(t, s, G(s)) + B_1D_{21}^{\perp\prime}D_{21}^{\perp}B_1(t)'. \end{aligned} \tag{6.14}$$

for $t \geq s$, with boundary condition

$$Z_\infty(t,t,G(t)) = G(t) \qquad \text{for all } t \geq 0.$$

We now state the following Lemmas, directly analogous to Lemmas 6.1 and 6.2.

Lemma 6.12. *Suppose G is a bounded positive solution of*

$$-\dot{G} + A_z G + G A_z' - G(C_z' C_z - \gamma^{-2} C_k' C_k) G + B_1 D_{21}^{\perp\prime} D_{21}^{\perp} B_1' \leq 0 \tag{6.15}$$

on the interval $[t_1, t_2]$, and Z_∞ is a bounded positive solution of equation (6.13) *on the interval $[t_1, t_2]$. Further suppose that $Z_\infty(t_1) = G(t_1)$. Then $Z_\infty(t) \leq G(t)$ for all $t \in [t_1, t_2]$.*

Proof. By substituting $\hat{t} = t_2 - t + t_1$, we arrive at an equation in the same form as equation (6.3). Hence we can apply Lemma 6.1 and the result follows. ∎

Lemma 6.13. *The solution Z_∞ to the Riccati differential equation* (6.14) *satisfies, for all $t_3 \leq t_2 \leq t_1$ and for all $G \geq 0$,*

a) $Z_\infty(t_1, t_2, Z_\infty(t_2, t_3, G)) = Z_\infty(t_1, t_3, G)$

b) $G > 0 \implies Z_\infty(t_1, t_2, G) > 0$

c) $B_1 D_{21}^{\perp\prime} D_{21}^{\perp} B_1(t)' > 0 \implies X_\infty(t_1, t_2, G) > 0$

d) $G_1 \geq G_2 \geq 0 \implies Z_\infty(t_1, t_2, G_1) \geq Z_\infty(t_1, t_2, G_2)$

Proof. The proof of this follows exactly that of Lemma 6.2, with the only slight difficulty being part d). This can be transformed into the form of Lemma 6.2 part d) by defining $\tilde{Z} = -Z_\infty$. ∎

We can now prove the filtering analogue of Lemma 6.3.

Lemma 6.14. *Suppose G satisfies the Riccati differential inequality*

$$-\dot{G} + A_z G + G A_z' - G(C_z' C_z - \gamma^{-2} C_k' C_k) G + B_1 D_{21}^{\perp\prime} D_{21}^{\perp} B_1' \leq 0 \tag{6.16}$$

and $G(t) \geq 0$ for all $t \geq -T_2$. Suppose also that, for each $t \geq -T_2$, there exists a bounded nonnegative definite solution to the Riccati differential equation (6.14) *on the interval $[t, t - T_2]$, with boundary condition $Z_\infty(t - T_2) = G(t - T_2)$. Then*

$$Z_\infty(t_1, t_2, G(t_2)) \geq Z_\infty(t_1, t_3, G(t_3))$$

for all $-T_2 \leq t_3 \leq t_2 \leq t_1$.

Proof. The proof follows exactly the lines of Lemma 6.3, with a few inequalities reversed. In particular, note that the order $-T_2 \leq t_3 \leq t_2 \leq t_1$ is reversed from that used in the state feedback expressions. The details follow.

By definition, $Z_\infty(t_3, t_3, G(t_3)) = G(t_3)$, and then applying Lemma 6.12 we have

$$Z_\infty(t, t_3, G(t_3)) \leq G(t)$$

for all $t \geq t_3$. Thus, for $t_2 \geq t_3$,

$$Z_\infty(t_2, t_3, G(t_3)) \leq G(t_2).$$

From Lemma 6.13d), we know that $G_1 \leq G_2$ implies that $Z_\infty(t, t_2, G_1) \leq Z_\infty(t, t_2, G_2)$ for $t_2 \leq t$. Hence

$$Z_\infty(t_1, t_2, Z_\infty(t_2, t_3, G(t_3))) \leq Z_\infty(t_1, t_2, G(t_2))$$

and applying Lemma 6.13a), we have

$$Z_\infty(t_1, t_3, G(t_3)) \leq Z_\infty(t_1, t_2, G(t_2)).$$

■

Remark 6.15. Note that this implies that

$$\frac{\partial}{\partial s} Z_\infty(t, s, G(s)) \geq 0$$

for $t \geq s$. □

The following Lemma will allow us to construct output feedback moving horizon $\mathscr{H}_\infty$ controllers.

Lemma 6.16. *Suppose G satisfies* (6.16), *and let $Z(t) = Z_\infty(t, t - T_2, G(t - T_2))$. Then Z satisfies the Riccati differential inequality*

$$\dot{Z} \geq A_z Z + Z A_z' - Z(C_z' C_z - \gamma^{-2} C_k' C_k) Z + B_1 D_{21}^{\perp\prime} D_{21}^{\perp} B_1'. \tag{6.17}$$

Proof. Differentiating Z gives

$$\dot{Z} = A_z Z + Z A_z' - Z(C_z' C_z - \gamma^{-2} C_k' C_k) Z + B_1 D_{21}^{\perp\prime} D_{21}^{\perp} B_1' + \left.\frac{\partial}{\partial s} Z_\infty(t, s, G(s))\right|_{s = t - T_2}$$

Applying Lemma 6.14 gives the desired result. ■

Note that in the coefficients A_k, C_k, and C_z, it does not matter whether X is defined by the moving horizon control Riccati equation or by an infinite horizon solution. Also we can easily ensure that the inequalities are satisfied uniformly, since by adding εI to the term $B_1 D_{21}^{\perp\prime} D_{21}^{\perp} B_1'$, it is clear that if G satisfies the Riccati differential inequality uniformly, and $Z_\infty(t)$ is defined on the finite horizon by

$$\dot{Z}_\infty = A_z Z_\infty + Z_\infty A_z' - Z_\infty(C_z' C_z - \gamma^{-2} C_k' C_k) Z_\infty + B_1 D_{21}^{\perp\prime} D_{21}^{\perp} B_1' + \varepsilon I$$

then $Z_\infty(t, t - T_2, G(t - T_2)$ will satisfy the Riccati differential inequality (6.17) uniformly also.

The output feedback controller. We now construct the output feedback controller. The interpretation here is that we would like to solve the differential game problem

$$\inf_{\mu \in \mathscr{U}_{of}[t,t+T]} \sup_{w \in \mathscr{L}_2[t,t+T]} \int_t^{t+T} z(s)'z(s) - \gamma^2 w(s)'w(s)\, ds + x(t+T)'F(t+T)x(t+T).$$

However, we no longer can solve this problem conditional on the state at time t, since we do not have this information. In the moving horizon problem for the quadratic regulator, this problem is usually solved by assuming a Kalman filter provides the state estimate. However, we would like to guarantee an infinite horizon $\mathscr{H}_\infty$ norm bound, and so the assumption of w being white noise does not fit well with this objective. Therefore we now solve, at time t, the new differential game

$$\inf_{\mu \in \mathscr{U}_{of}[t-T_2,t+T]} \sup_{\substack{w \in \mathscr{L}_2[t-T_2,t+T] \\ x(t-T_2) \in \mathbb{R}^n}} \Big\{ -\gamma^2 x(t-T_2)'H(t-T_2)x(t-T_2) + \int_{t-T_2}^{t+T} z(s)'z(s) - \gamma^2 w(s)'w(s)\, ds + x(t+T)'F(t+T)x(t+T) \Big\}.$$

The matrix $H(t-T_2) > 0$ now represents our assumed uncertainty in the state at time $t - T_2$. By applying Lemma 6.16 and Theorem 5.17, we can now state the output feedback moving horizon result in full.

Theorem 6.17. *Let the system $\mathcal{G}$ be defined by equations* (6.11) *satisfying assumptions* (1)-(5)*. Suppose $F \in \mathscr{L}_\infty^{n\times n}$ satisfies*

$$\dot{F} + (A - B_2D_{12}'C_1)'F + F(A - B_2D_{12}'C_1) - F(B_2B_2' - \gamma^{-2}B_1B_1')F + C_1'(I - D_{12}D_{12}')C_1 \le 0$$

and $G \in \mathscr{L}_\infty^{n\times n}$ satisfies

$$-\dot{G} + A_zG + GA_z' - G(C_z'C_z - \gamma^{-2}C_k'C_k)G + B_1D_{21}^{\perp\prime}D_{21}^{\perp}B_1' \le -\varepsilon I \tag{6.18}$$

with $F \ge 0$ and $G \ge 0$. Suppose also that for each $t \ge 0$, there exists a bounded solution to the Riccati differential equation (6.3) *on the interval $[t, t+T]$ with boundary condition $X_\infty(t+T) = F(t+T)$, and define $X(t) = X_\infty(t, t+T, F(t+T))$ by the PDE* (6.4)*. Suppose further that, for some constant $\varepsilon > 0$, for each $t \ge 0$, there exists a bounded solution to the Riccati differential equation*

$$\dot{Z}_\infty = A_zZ_\infty + Z_\infty A_z' - Z_\infty(C_z'C_z - \gamma^{-2}C_k'C_k)Z_\infty + B_1D_{21}^{\perp\prime}D_{21}^{\perp}B_1' + \varepsilon I$$

on the interval $[t - T_2, t]$, with boundary condition $Z_\infty(t-T_2) = G(t-T_2)$, and define $Z(t) = Z_\infty(t, t-T_2, G(t-T_2))$ as that solution. Then the controller

$$\dot{\bar{x}} = A_k\bar{x} + B_ky \qquad \bar{x}(0) = 0$$
$$u = C_k\bar{x}$$

exponentially stabilizes the closed loop, and results in a closed loop induced norm bound $\|\mathcal{T}_{zw}\| \le \gamma$. In terms of the moving horizon Riccati differential equations, the parameters are

$$
\begin{aligned}
A_z &= A - B_1 D_{21}' C_2 + \gamma^{-2} B_1 D_{21}^{\perp\prime} D_{21}^{\perp} B_1' X \\
C_k &= -D_{12}' C_1 - B_2' X \\
C_z &= C_2 + \gamma^{-2} D_{21} B_1' X \\
A_k &= A + \gamma^{-2} B_1 B_1' X + B_2 C_k - (B_1 D_{21}' + Z C_z') C_z \\
B_k &= B_1 D_{21}' + Z C_z'.
\end{aligned}
$$

Remark 6.18. Note that in the above solution, we require knowledge of the system matrices A, B, C, D from before the time the controller is implemememented, on the interval $[-T_2, 0]$. This is a somewhat artificial restriction of the method. □

In fact, the matrix $H(t - T_2)$ is given by

$$H(t - T_2) = G(t - T_2)^{-1} + \gamma^{-2} X_\infty(t - T_2, t + T_1, F(t + T_1))$$

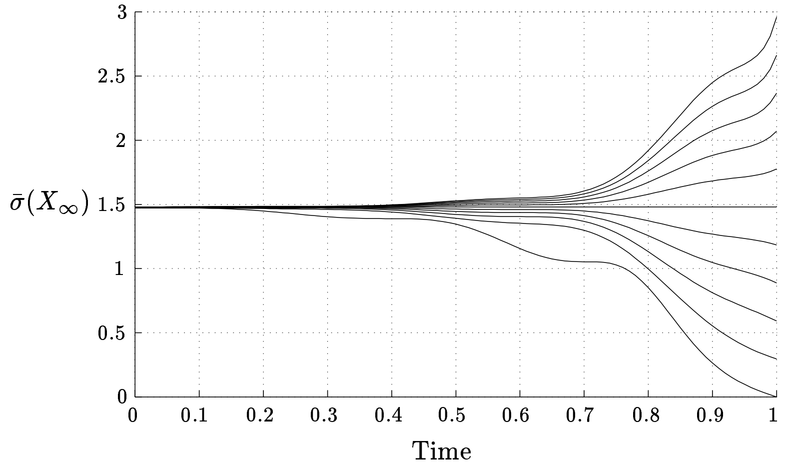
using equations (3.30) and (3.24). This implicit choice of initial uncertainty of the state ensures that the usual coupling condition $X_\infty(t) - \gamma^2 Y_\infty^{-1}(t) < 0$ is satisfied by any positive choice of G satisfying (6.18).

This theorem shows that the moving horizon technique gives a stabilizing γ-feasible controller for a linear time varying system. In some sense, however, the proof illustrates an apparent flaw in this methodology. If we can construct F and G, why not simply use them as X and Z in the controller formulae? In fact, why not simply construct F and G by integrating forwards in time the Riccati equations? One reason is that the finite horizon Riccati differential equations converge to the solutions of the infinite horizon Riccati differential equations, so we do not expect the solution to be sensitive to the choice of F and G. we shall return to consideration of these points in the sequel.

In the meantime, we would expect to gain something by using recent past and future information about the system to construct the matrices F and G. Heuristically, for the moving horizon controller, we expect a large choice of F to give stability, since this then corresponds to a large terminal state weighting matrix.

Indeed, our intuition from the finite horizon state feedback problem is as follows. For the purpose of this discussion, consider a linear time invariant system, and the construction of a state feedback controller with a fixed terminal weight on a finite horizon $[0, T]$. Then we know that we can choose any positive semi-definite terminal state weight, Q_f, and construct a finite horizon γ-feasible $\mathscr{H}_\infty$ controller. However, we also know that we are not restricted to choosing solutions to the *equation*, but we can also choose solutions to the inequality. Further, since we are integrating backwards, given any specific final value Q_f, any trajectory of X which increases sufficiently fast backwards in time will be a solution of the inequality, and hence will provide a γ-feasible $\mathscr{H}_\infty$ controller.

Figure 6.3 shows typical solutions for the control Riccati equation for a 3rd order system. The system here is taken from [29, p. 299]. The finite horizon is $[0, 1]$, and the final conditions are all multiples of the infinite horizon stabilizing solution. In this particular example, it is easy to see that, since the stabilizing solution to the algebraic Riccati equation is positive definite, and in this case the term $B_2 B_2' - \gamma^2 B_1 B_1'$ is sign



$$A = \begin{bmatrix} -0.0100 & 0 & -30.0000 \\ 0 & -0.0100 & 15.0000 \\ 1.0000 & -1.0000 & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} -1.2659 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} -31.6481 \\ 0 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1.3166 & 0 \end{bmatrix} \quad D_{12} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad X_\infty = \begin{bmatrix} 0.0140 & 0.0283 & 0.0948 \\ 0.0283 & 0.3142 & 0.4776 \\ 0.0948 & 0.4776 & 1.2759 \end{bmatrix} \text{ in steady state}$$

Figure 6.3: Solutions to the Riccati differential equation on $[0, 1]$.

definite, applying Lemma 6.10 implies that all solutions with a scalar multiple of that solution as a final boundary condition will be monotonic. The graph shows only the maximum singular value of the solutions.

Figure 6.4 illustrates possible solutions to the Riccati differential inequality on $[0, 1]$. In this figure, at times $0, 0.1, \ldots, 1$, the Riccati differential equation has been perturbed by adding $0.05I$ to $\dot{X}_\infty$. Integrating backwards, we can always add to the solution X_∞, but we cannot subtract from it. A game theoretic interpretation of this statement will also be given in the next chapter.

Figure 6.5 illustrates a solution to the Riccati differential inequality, which satisfies the equation for $t \notin [0.75, 0.8]$. Within this interval, it is perturbed by $0.05I$. Note that, after the perturbation, the solution rapidly converges back to the steady state solution.

For linear time invariant systems, Theorem 6.17 implies that we can choose any of these boundary conditions Q_f which result in monotonic solutions for the Riccati equation as a constant choice for X_∞, with a corresponding result for Z_∞. The resulting controller will be γ-feasible and stabilizing. Lemma 6.10 gives a computable condition for LTI systems which allows us to find monotonic solutions. That is, in order to find monotonic solutions to the Riccati differential equation for X_∞, we need to find Q_f satisfying

$$\begin{aligned}(A - B_2 D_{12}' C_1)' Q_F + Q_f (A - B_2 D_{12}' C_1) \\ - Q_f (B_2 B_2' - \gamma^{-2} B_1 B_1') Q_f + C_1' (I - D_{12} D_{12}') C_1 < 0.\end{aligned}$$

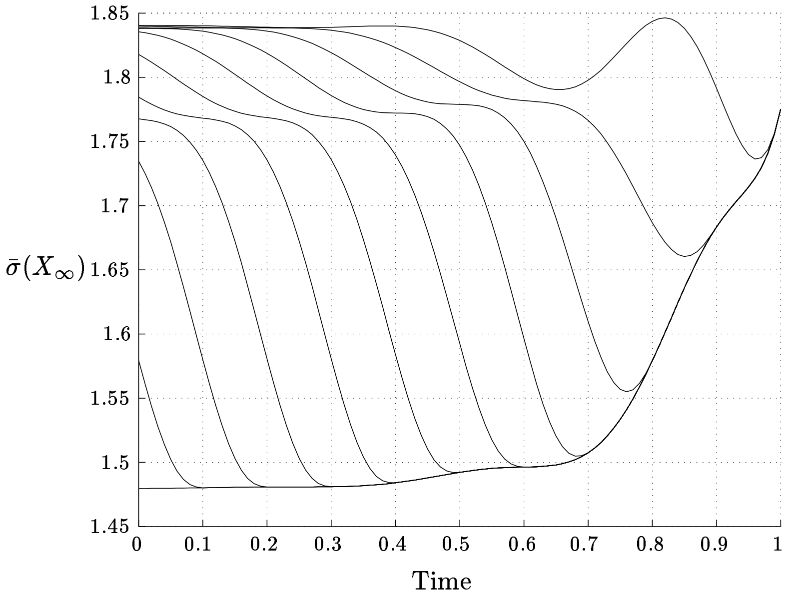


Figure 6.4: Some solutions to the Riccati differential inequality on $[0, 1]$.

The importance of this inequality is well known in the LMI approach to $\mathscr{H}_\infty$ control, and the above inequality can be written

$$\begin{bmatrix} \mathcal{N}_R & 0 \\ 0 & I \end{bmatrix}' \begin{bmatrix} AQ_f^{-1} + Q_f^{-1}A & Q_f^{-1}C_1' & B_1 \\ C_1Q_f^{-1} & -\gamma I & 0 \\ B_1' & 0 & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{N}_R & 0 \\ 0 & I \end{bmatrix} < 0$$

where $\mathcal{N}_R$ denotes a basis for the null space of $\begin{bmatrix} B_2' & D_{12}' \end{bmatrix}$. See [26] for more details on this approach. Here we have given a different intuition to the LMI solution for linear time invariant problems, and extended this approach to cover linear time varying systems also.

6.4 Terminal constraints

We therefore might expect that one possible method for constructing F would be to construct the solution with an infinite terminal state weight. This corresponds to the constraint that, for the finite horizon dynamic game on $[t, t+T]$, the system state is brought to zero at time $t+T$. We can achieve this in the minimax problem by putting an infinite weight on the state at time $t+T$, that is by choosing $Q_f = \infty$. The following Lemma considers the Riccati equation for X^{-1}, to avoid the singularity at the endpoint.

Lemma 6.19. *Suppose for $i = 1, 2$ that R_i is a bounded solution to the Riccati differential equation*

$$\begin{aligned} \dot{R}_i = (A - B_2D_{12}'C_1)R_i + R_i(A - B_2D_{12}'C_1)' \\ + R_iC_1'D_{12}^{\perp}D_{12}^{\perp\prime}C_1R_i - (B_2B_2' - \gamma^{-2}B_1B_1') \end{aligned}$$

on the interval $[t_1, t_2]$, with $R_1(t_2) = Q_1$ and $R_2(t_2) = Q_2$, and $Q_1 \geq Q_2 \geq 0$. Then $R_1(t) \geq R_2(t)$ for all $t \in [t_1, t_2]$.

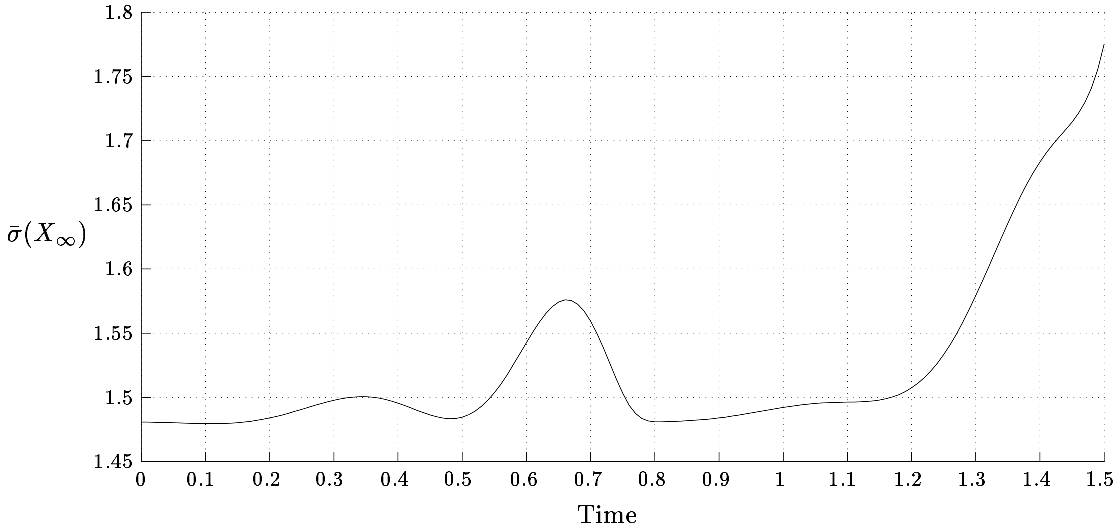


Figure 6.5: A solution to the Riccati differential inequality.

Proof. Note that although $R = X_\infty^{-1}$, we cannot simply apply Lemma 6.2, since we specifically want to apply this to the case $Q_2 \not> 0$. In order to do this, we use the results of Brockett [16, p. 130] on linear quadratic regulation with an indefinite cost on the state. If R_i is a bounded solution to the Riccati differential equation then

$$x(t)'R_i(t)x(t) = \min_{u \in \mathscr{L}_2[t,t_2]} \int_t^{t_2} x(s)'(B_2B_2' - \gamma^{-2}B_1B_1')(s)x(s) + u(s)'u(s)\,ds + x(t_2)'Q_ix(t_2)$$

where x is subject to the dynamics

$$\dot{x}(t) = -(A - B_2D_{12}'C_1)'(t)x(t) + C_1'D_{12}^{\perp}u.$$

Clearly, since this is a minimization problem, and the cost function is greater for $i = 1$ than for $i = 2$, we have $R_1(t) \geq R_2(t)$. ■

Lemma 6.20. *Let $0 \leq t_1 < t_2 \leq t_3$. Suppose that, for $s = t_2$ and $s = t_3$, there exists a bounded solution $R \in \mathscr{L}_\infty^{n\times n}$ to the Riccati differential equation*

$$\dot{R} = (A - B_2D_{12}'C_1)R + R(A - B_2D_{12}'C_1)' + RC_1'D_{12}^{\perp}D_{12}^{\perp\prime}C_1R - (B_2B_2' - \gamma^{-2}B_1B_1') \qquad (6.19)$$

on the interval $[t_1, s]$ such that such that $R(s) = 0$ and $R(t) > 0$ for $t_1 \leq t < s$. Let $R(t,s)$ be the solution at time t with boundary condition at time s, defined on the interval $[t_1, s]$. Then

$$R(t,t_3)^{-1} \geq R(t,t_2)^{-1}$$

for all $t \in [t_1, t_2)$. Further, if for all $t \geq 0$, there exists a solution $R(t, t+T)$ satisfying the above conditions, then $X(t) = R(t, t+T)^{-1}$ satisfies the control Riccati inequality (6.12).

Proof. By assumption, $R(t_2, t_3) > 0$, and $R(t_2, t_2) = 0$. Therefore, in the same way as the proof of Lemma 6.3, we can use time t_3 as a boundary condition. Then, Lemma 6.19 implies that $R(t, t_2) \leq R(t, t_3)$ for all $t_1 \leq t \leq t_2$. The last part of the Lemma then follows by direct differentiation. ∎

Lemma 6.21. *Let $\mathcal{G}$ be an exponentially stable linear time varying system defined on $[0, \infty)$ by*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)w(t) \\ z(t) &= C(t)x(t) + D(t)w(t) \end{aligned}$$

and define $\|\mathcal{G}\|_{[t,s]}$ as the finite horizon induced norm of the system on the interval $[t, s]$ with state $x(t) = 0$. Then there exists $\gamma > 0$ such that $\|\mathcal{G}\|_{[t,s]} < \gamma$ for all $0 \leq t < s$.

Proof. Since $\mathcal{G}$ is exponentially stable, then $\|\mathcal{G}\|$ is bounded. Clearly $\|\mathcal{G}\| \geq \|\mathcal{G}\|_{[t,s]}$ for all $0 \leq t < s$, since the worst case w for the finite horizon problem can be used to generate the same output for the infinite horizon problem. ∎

The above Lemma does not imply that the induced norm is bounded for each finite horizon problem with a terminal constraint. We will use the following sequence of Lemmas to achieve this.

Lemma 6.22. *For given bounded matrix functions A, B, C, D, there exists $\gamma > 0$ such that there exists a bounded solution to the Riccati differential equation* (6.19) *on the finite interval $[t_1, t_2]$ with boundary condition $R(t_2) = 0$.*

Proof. This is proved in [7, Proposition 8.5]. ∎

Lemma 6.23. *If there exists a solution to equation* (6.19) *on the finite interval $[t_1, t_2]$ with boundary condition $R(t_2) = 0$, then a solution exists for all A, B, C, D, and γ in an open neighbourhood of the nominal, and the solution R is uniformly continuous in the state space matrices A, B, C, D and in γ.*

Proof. A proof of continuity for general nonlinear equations satisfying a local Lipschitz condition is in Sontag [61]. For specific details for finite horizon Riccati equations, see also Başar and Bernhard [7, Chapter 8]. ∎

Lemma 6.24. *Consider the set of finite dimensional linear time varying systems $\mathcal{G}$ described by*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned}$$

where $A \in \mathscr{L}_\infty^{n\times n}$, $B \in \mathscr{L}_\infty^{n\times m}$, $C \in \mathscr{L}_\infty^{p\times n}$ and $D \in \mathscr{L}_\infty^{p\times m}$ on the interval $[0, \sigma]$ or $[0, \infty)$. Then, if σ is finite, the induced norm of $\mathcal{G}$ on the finite interval $[0, \sigma]$ is a continuous function of the state space matrices with the uniform metric. On $[0, \infty)$, if A is exponentially stable, then there exists a neighbourhood of the state space matrices in which every system is exponentially stable, and further $\|G\|$ is continuous there.

Proof. Let

$$\mathcal{G}_1 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \qquad \mathcal{G}_2 = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$$

with states x_1 and x_2 respectively. Define $p = x_1 - x_2$ and $q = x_1 + x_2$, and $A_\delta = A_2 - A_1$, $B_\delta = B_2 - B_1$, $C_\delta = C_2 - C_1$, and $D_\delta = D_2 - D_1$. We apply the usual ε, δ arguments for continuity, and show that, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\max(\|A_\delta\|_\infty, \|B_\delta\|_\infty, \|C_\delta\|_\infty, \|D_\delta\|_\infty) < \delta$ implies that $\|\mathcal{G}_1 - \mathcal{G}_2\| \leq \varepsilon$. Then we can rewrite the system equation $z = (\mathcal{G}_1 - \mathcal{G}_2)w$ as

$$\begin{aligned} \begin{pmatrix} \dot p \\ \dot q \end{pmatrix} &= \hat A \begin{pmatrix} p \\ q \end{pmatrix} + \hat B w \\ z &= \hat C \begin{pmatrix} p \\ q \end{pmatrix} + \hat D w \end{aligned}$$

where

$$\begin{aligned} \hat A &= \begin{bmatrix} A_1 + \frac{A_\delta}{2} & -\frac{A_\delta}{2} \\ -\frac{A_\delta}{2} & A_1 + \frac{A_\delta}{2} \end{bmatrix} & \hat B &= \begin{bmatrix} -B_\delta \\ 2B_1 + B_\delta \end{bmatrix} \\ \hat C &= \begin{bmatrix} C + \frac{C_\delta}{2} & -\frac{C_\delta}{2} \end{bmatrix} & \hat D &= D_\delta \end{aligned}$$

Then, from Lemma 5.9 if there exists a bounded matrix X such that

$$\dot X + \hat A' X + X\hat A + \hat C'\hat C + (X\hat B + \hat C'\hat D)(\gamma^2 I - \hat D'\hat D)^{-1}(\hat B' X + \hat D'\hat C) \leq -\varepsilon_1 I$$

for some constant $\varepsilon_1 > 0$, then $\|\mathcal{G}_1 - \mathcal{G}_2\| < \gamma$. Let

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$$

then if $A_\delta = 0$, $B_\delta = 0$, $C_\delta = 0$, and $D_\delta = 0$, we arrive at the equivalent Riccati equations

$$\begin{aligned} -\varepsilon_1 I &\geq A_1' X_1 + X_1 A_1 + \dot X_1 + C_1' C_1 \\ -\varepsilon_1 I &\geq A_1' X_2 + X_2 A_1 + \dot X_2 + 4\gamma^{-2} X_2 B_1 B_1' X_2. \end{aligned}$$

In the finite horizon case, uniformly bounded solutions to these equations always exist, since they correspond to equations for the controllability and inverse of the observability gramians. In the infinite horizon case, solutions will always exist if A_1 is exponentially stable. Since the set of uniformly negative definite matrices is open, Lemma 6.23 implies that choosing A_δ, B_δ, C_δ and D_δ sufficiently small in the uniform norm will ensure that the above $2n \times 2n$ Riccati equation still has a uniformly bounded solution, and that $\gamma^2 I - \hat D'\hat D$ is uniformly invertible, from which the result follows. ■

Corollary 6.25. *Let Ω be a set of matrix functions A, B, C and D defined on $[t_0, t_1]$. Suppose there exists $\delta > 0$ such that $\max(\|A(t)\|, \|B(t)\|, \|C(t)\|, \|D(t)\|) < \delta$ for all $t \geq 0$ and for all $(A, B, C, D) \in \Omega$. Then there exists $\gamma > 0$ such that, for any $(A, B, C, D) \in \Omega$, the induced norm of the corresponding linear time varying system on $[t_0, t_1]$ is less than γ.*

Lemma 6.26. *Define the observability gramians*

$$G_i(t_0, t_1) := \int_{t_0}^{t_1} \phi(t_0, r) B_i(r) B_i(r)' \phi(t_0, r)'\, dr$$

for $i = 1, 2$, where ϕ is the transition matrix for A. Suppose that there exists $\beta > 0$ such that, for all $t \geq 0$,

$$\frac{1}{\beta} I < G_2(t, t+T) < \beta I$$
$$G_1(t, t+T) < \beta I.$$

Then there exists $\gamma > 0$ such that, for each $t \geq 0$, the Riccati differential equation (6.19) *has a bounded solution on each interval $[t, t+T]$ with boundary condition $R(t+T, t+T) = 0$. Further, there exists $\alpha > 0$ such that $R(t, t+T) < \alpha I$ for all $t \geq 0$.*

Proof. Define the operators $\Gamma_i : \mathscr{L}_2[t, t+T] \to \mathbb{R}^n$ by

$$\Gamma_i r := \int_t^{t+T} \phi(t, s) B_i(s) u(s)\, ds$$

for $i = 1, 2$. Then $G_i(t, t+T) = \Gamma_i \Gamma_i^*$. Then, for given inputs $u_{[t,t+T]}$, $w_{[t,t+T]}$, the state $x(t+T)$ is given by

$$x(t+T) = \Gamma_1 w_{[t,t+T]} + \Gamma_2 u_{[t,t+T]}.$$

Therefore, the condition that the state $x(t+T)$ be zero is equivalent to the restriction that $u_{[t,t+T]}$ satisfy

$$\begin{aligned} u_{[t,t+T]} &= -\Gamma_2^* (\Gamma_2 \Gamma_2^*)^{-1} \Gamma_1 w_{[t,t+T]} + v \\ &= \mathcal{L}_t w_{[t,t+T]} + v \end{aligned}$$

for some v in the kernel of Γ_2. From the assumptions in the Lemma, it is easy to see that $\|\mathcal{L}_t\| < \beta^2$ for all $t \geq 0$. At each t, we can write

$$\begin{aligned} z_{[t,t+T]} &= \mathcal{F}_t w_{[t,t+T]} + \mathcal{G}_t u_{[t,t+T]} \\ &= (\mathcal{F}_t + \mathcal{G}_t \mathcal{L}_t) w_{[t,t+T]} + v. \end{aligned}$$

The map $\mathcal{F}_t$ is defined by the state space matrices A, B_1, C_1, and D_{11}, which by assumption are bounded over time, and hence there exists $\alpha_1 > 0$ such that $\|\mathcal{F}_t\| < \alpha_1$ for all $t \geq 0$. Similarly, there exists $\alpha_2 > 0$ such that $\|\mathcal{G}_t\| < \alpha_2$ for all $t \geq 0$. Therefore, for each $t > 0$, the controller can always achieve a finite horizon induced norm on the interval $[t, t+T]$ less than $\alpha_1 + \alpha_2 \beta$, by choosing $v = 0$, and satisfy the terminal constraint. Thus, if $\gamma > \alpha_1 + \alpha_2 \beta$, there exists a bounded solution to each Riccati differential equation (5.1) on every interval of the form $[t, t+T]$.

Finally, we have to show that the solution $R(t, t+T)$ is bounded over all t. This follows since by Lemma 6.23 the solution $R(t, t+T)$ is continuous in the state space matrices of the system. Since these are bounded, $R(t, t+T)$ is bounded also. ■

Remark 6.27. A dual theorem statement holds for the filtering Riccati equation with an infinite initial condition. However, the gramians in this case depend on X_∞, so this does not give rise to a testable condition. We could of course rewrite the problem in terms of the independent X_∞ and Y_∞ equations, with a coupling condition. In this case, we could use this Lemma to guarantee boundedness of both Riccati equations, but not to guarantee that the coupling condition would not fail. □

We can now state the following theorem, which has a terminal constraint for the state feedback problem.

Theorem 6.28. *Let the system $\mathcal{G}$ be defined by equations* (6.11) *satisfying assumptions* (1)-(5)*. Suppose also that $G_2(t, t+T)$ and $G_1(t, t+T)$ satisfy the conditions of Lemma 6.26. Then, let $\gamma > 0$ be such that there exists a bounded solution to the Riccati differential equation*

$$\dot{R} = (A - B_2 D_{12}' C_1) R + R(A - B_2 D_{12}' C_1)' + R C_1' D_{12}^{\perp} D_{12}^{\perp\prime} C_1 R - (B_2 B_2' - \gamma^{-2} B_1 B_1')$$

on the interval $[t, t+T]$ with boundary condition $R(t+T, t+T) = 0$, and with $R(t, t+T)$ bounded over all $t \geq 0$. Then the state feedback controller

$$u = -(B_2(t)' R(t, t+T) + D_{12}(t)' C_1(t)) x(t)$$

exponentially stabilizes the closed loop, and results in a closed loop induced norm which satisfies $\|\mathcal{T}_{zw}\| \leq \gamma$.

6.5 Recursive computation of X_∞

In order to implement the moving horizon controller, we need to calculate the value of $X_\infty(t, t+T, F(t+T))$ for all $t > 0$. This requires the solution of a Riccati differential equation over the interval $[t, t+T]$ for each time t, given boundary conditions $X_\infty(t, t+T, F(t+T)) = F(t+T)$ for all $t > 0$. Kwon, Bruckstein and Kailath [45] applied results from scattering theory in their solution to the quadratic problem, to give a forwards differential equation for $X_\infty(t, t+T, F(t+T))$. Here we state the same solution for the indefinite Riccati equation. These results are derived in [73] and [59] for the case when the Riccati differential equation has a positive definite quadratic term. In this section, $P(\tau, \sigma)$ is used to mean $X_\infty(\tau, \sigma, 0)$.

Define for convenience

$$N(\tau) = B_2(\tau) B_2(\tau)' - \gamma^{-2} B_1(\tau) B_1(\tau)'$$

and let

$$S(\tau, \sigma) = \begin{bmatrix} \Phi(\tau, \sigma) & L(\tau, \sigma) \\ P(\tau, \sigma) & \Psi(\tau, \sigma) \end{bmatrix}.$$

Consider the system of equations

$$\frac{\partial}{\partial \tau} \Phi(\tau, \sigma) = \Phi(\tau, \sigma)[N(\tau) P(\tau, \sigma) - A(\tau)] \tag{6.20}$$

$$\frac{\partial}{\partial \tau} \Psi(\tau, \sigma) = [P(\tau, \sigma) N(\tau) - A'(\tau)] \Psi(\tau, \sigma) \tag{6.21}$$

$$\frac{\partial}{\partial \tau} L(\tau, \sigma) = \Phi(\tau, \sigma) N(\tau) \Psi(\tau, \sigma) \tag{6.22}$$

$$\frac{\partial}{\partial \tau} P(\tau, \sigma) = A'(\tau) P(\tau, \sigma) + P(\tau, \sigma) A(\tau) - P(\tau, \sigma) N(\tau) P(\tau, \sigma) + C_1(\tau)' C_1(\tau) \tag{6.23}$$

with boundary conditions

$$S(\sigma, \sigma) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

The following equations

$$\begin{aligned}
\frac{\partial}{\partial\sigma}\Phi(\tau,\sigma) &= [A(\sigma) + L(\tau,\sigma)C_1(\sigma)'C_1(\sigma)]\Phi(\tau,\sigma)\\
\frac{\partial}{\partial\sigma}\Psi(\tau,\sigma) &= \Psi(\tau,\sigma)[A'(\sigma) + C_1(\sigma)'C_1(\sigma)L(\tau,\sigma)]\\
\frac{\partial}{\partial\sigma}L(\tau,\sigma) &= A(\sigma)L(\tau,\sigma) + L(\tau,\sigma)A'(\sigma) + L(\tau,\sigma)C_1(\sigma)'C_1(\sigma)L(\tau,\sigma) - N(\sigma)\\
\frac{\partial}{\partial\sigma}P(\tau,\sigma) &= \Psi(\tau,\sigma)C_1(\sigma)'C_1(\sigma)\Phi(\tau,\sigma)
\end{aligned}$$

give the partial derivatives with respect to σ. It is straightforward to verify this by taking both sets of second partial derivatives and showing that they are equal, and verifying that along the boundary the total derivative

$$\frac{d}{dt}S(t,t) = \frac{\partial S(\tau,t)}{\partial\tau}\Big|_{\tau=t} + \frac{\partial S(t,\sigma)}{\partial\sigma}\Big|_{\sigma=t}$$

is zero, since Ψ, Φ, L, P are constant along the boundary. We can find a differential equation for $P(t, t+T)$ in terms of t since

$$\frac{d}{dt}S(t,t+T) = \frac{\partial S(\tau,t+T)}{\partial\tau}\Big|_{\tau=t} + \frac{\partial S(t,\sigma)}{\partial\sigma}\Big|_{\sigma=t+T}$$

Hence

$$\frac{d\Phi}{dt} = \Phi[N(t)P - A(t)] + [A(t+T) + LC_1(t+T)'C_1(t+T)]\Phi \tag{6.24}$$

$$\frac{d\Psi}{dt} = [PN(t) - A'(t)]\Psi + \Psi[A'(t+T) + C_1(t+T)'C_1(t+T)L] \tag{6.25}$$

$$\frac{dL}{dt} = \Phi N(t)\Psi + A(t+T)L + LA'(t+T) + LC_1(t+T)'C_1(t+T)L - N(t+T) \tag{6.26}$$

$$\frac{dP}{dt} = A'(t)P + PA(t) - PN(t)P + C_1(t)'C_1(t) + \Psi C_1(t+T)'C_1(t+T)\Phi \tag{6.27}$$

where $\Phi = \Phi(t, t+T)$ etc. This gives a differential equation for $P(t, t+T, 0)$. The boundary conditions for equations (6.24–6.27) are given by solving equations (6.20–6.23) backwards in time from $\tau = \sigma = T$ to $\tau = 0, \sigma = T$. This backwards solution has to be performed only once, when $t = 0$.

We now make use of the following identity.

$$X_\infty(\tau,\sigma,F(\sigma)) = P(\tau,\sigma) + \Psi(\tau,\sigma)F(\sigma)[I - L(\tau,\sigma)F(\sigma)]^{-1}\Phi(\tau,\sigma)$$

This can be verified in a straightforward manner. Then

$$X_\infty(t,t+T,F(t+T)) = P + \Psi F(t+T)[I - LF(t+T)]^{-1}\Phi$$

These equations are simpler than they appear, since $\Phi = \Psi'$. They give us a quadratic differential equation in matrices of size $2n$, which is integrated forwards to give the controller.

6.6 Robust Performance and LMIs

We would now like to discuss the advantages of the moving horizon control synthesis technique presented in this chapter. Indeed, while it is a very interesting theoretical result as it stands, that using only exact knowledge of the plant parameters up to some finite time ahead, we can both stabilize and guarantee an induced norm bound over all time for a linear time varying system, it is not so clear what the merits of this scheme are in a practical sense. The often cited advantages of predictive control, that it gives extra tuning parameters, are not so clear in the $\mathscr{H}_\infty$ context. In particular, almost always the $\mathscr{H}_\infty$ performance index is not really a *performance* index. The system $\mathcal{G}$ does not correspond to the lumped physical plant, and disturbance attenuation is not the goal. The generalized $\mathscr{H}_\infty$ suboptimal synthesis problem is of primary interest because many other problems can be recast into this form. The construction of a controller which is robust to a particular perturbation class, whilst at the same time offering good performance with respect to other, more easily describable engineering specifications is performed via $\mathscr{H}_\infty$ synthesis, but with many design considerations taken into account. These would include, for example, the construction of appropriate weights and scaling matrices. Not only this, but if after the loop has been shifted into the form of the generalized problem it is discovered that the problem is singular or close to singular, then often the design specifications themselves would need to be reconsidered. A good discussion of the techniques associated with this methodology can be found in, for example, Dahleh and Diaz-Bobillo [21].

This stated $\mathscr{H}_\infty$ system methodology, when transferred to the moving horizon context, brings problems. Currently, use of the 'tuning parameters', the horizon length T and the matrices F and G, as extra design 'knobs' does not look hopeful, when the generalized problem is so far removed from the physical problem at hand. Not only this, but one of the key stated advantages was that future plant variation could be dealt with in the future, that is, that the control signal now did not explicitly depend on the time variation of the plant in the future after time $t + T$. For this to be the case, it is necessary that the plant changes are such that redesign of the weights is not necessary, and that the problem does not become singular or poorly posed. However, we hope that these assumptions on the plant are not too stringent.

In the above discussion, we have been rather vague about an important point. This is that, although the controller does not depend explicitly on the system matrices after time $t + T$, we are required to make some choices of F and G in order to construct the controller. The choice of these matrices determines the allowed variation of the plant in the future.

The following two Lemmas express the Riccati differential inequalities as linear matrix inequalities. First define

$$\mathcal{N}_Q = \begin{bmatrix} I & 0 \\ -D_{12}'C_z & D_{21}^{\perp\prime} \end{bmatrix} \qquad \mathcal{N}_R = \begin{bmatrix} I & 0 \\ -D_{12}B_2' & D_{12}^{\perp} \end{bmatrix}.$$

Lemma 6.29. *There exist constants $\varepsilon, \delta > 0$, and a bounded matrix function $X > \delta I$*

such that

$$\dot{X} + (A - B_2 D_{12}' C_1)' X + X(A - B_2 D_{12}' C_1) \\ - X(B_2 B_2' - \gamma^{-2} B_1 B_1') X + C_1' D_{12}^{\perp} D_{12}^{\perp\prime} C_1 < -\varepsilon I \quad (6.28)$$

if and only if there exist constant $\varepsilon, \delta > 0$ and a bounded matrix function $R > \delta I$ such that

$$\begin{bmatrix} \mathcal{N}_R & 0 \\ 0 & I \end{bmatrix}' \begin{bmatrix} -\dot{R} + AR + RA' + \varepsilon I & RC_1' & B_1 \\ C_1 R & -\gamma I & 0 \\ B_1' & 0 & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{N}_R & 0 \\ 0 & I \end{bmatrix} < 0 \qquad (6.29)$$

In this case, one possible choice is $R = \gamma X^{-1}$.

Proof. First, it is easy to see that (6.29) is satisfied if and only if

$$\begin{bmatrix} \dot{R} + (A - B_2 D_{12}' C_1) R + (A - B_2 D_{12}' C_1)' R - \gamma B_2 B_2' + \varepsilon I & RC_1' D_{12}^{\perp} & B_1 \\ D_{12}^{\perp\prime} C_1 R & -\gamma I & 0 \\ B_1' & 0 & -\gamma I \end{bmatrix} < 0$$

Successively applying the Schur complement formula gives the desired result. ■

Lemma 6.30. *There exist constants $\varepsilon, \delta > 0$, and a bounded matrix function $Z > \delta I$ such that*

$$-\dot{Z} + A_z Z + Z A_z' - Z(C_z' C_z - \gamma^{-2} C_k' C_k) Z + B_1 D_{21}^{\perp\prime} D_{21}^{\perp} B_1' < -\varepsilon I$$

if and only if there exist constants $\varepsilon, \delta > 0$ and a bounded matrix function $Q > \delta I$ such that

$$\begin{bmatrix} \mathcal{N}_Q & 0 \\ 0 & I \end{bmatrix}' \begin{bmatrix} \dot{Q} + \tilde{A}' Q + Q\tilde{A} + \varepsilon I & QB_1 & C_k' \\ B_1' Q & -\gamma I & 0 \\ C_k & 0 & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{N}_Q & 0 \\ 0 & I \end{bmatrix} < 0 \qquad (6.30)$$

where $\tilde{A} = A + \gamma^{-2} B_1 B_1' X$. In this case, one possible choice is $Q = \gamma Z^{-1}$.

Remark 6.31. Note that, in both the linear matrix inequalities written above, it is possible to remove the εI term from the $1,1$ block of the matrix, and instead solve the LMI uniformly over time. □

Written in this form, we gain another viewpoint advantage. This shows an interesting property of the moving horizon strategy, which might be regarded as *robust performance*. Replace X and Z in the above LMIs by F and G. Then, for any plant whose system matrices satisfy the inequalities (6.29) and (6.30), the controller defined by F and G will result in a closed loop induced norm less than or equal to γ. Note that inequalities (6.29) and (6.30) are affine, and hence convex, in A and C_1, but not in B_1 for example.

However, it is worth noting that, as described above, this robust performance condition is not very useful. Typically in robust control, the set of plants for which one desires to achieve a given performance level is specified in advance. Then the goal of the robust control strategy is, if a strategy which will achieve the performance level for all plants in the set exists, to give a construction for the controller. The important point, however, is that the robust control strategy must also tell the designer if there does not exist *any* strategy which will achieve the desired performance level. We might also like the design technique to indicate where the trade-offs should be made, etc.

If we first choose F and G, and then discover which set of plants around the nominal we can give robust performance for, then we have not fulfilled this goal. An example of this is the idea that *all* linear time invariant strictly stabilizing controllers are robustly stabilizing, in the sense that there exists some ball of plants around the nominal which is also strictly stabilized. However, the question is not whether some ball exists, but whether the particular ball we need is stabilized

However, this is compensated for in the moving horizon approach by the fact that we do not need to choose F and G in advance. Instead, at time t we can choose F and G according to our knowledge of the system matrices. Therefore, it is possible to have the controller vary according to variations in the plant, rather than design in advance for all possible future variations in the plant.

There is an important difference between this moving horizon approach and the parameter varying approaches of [55, 79]. The parameter varying approach is to consider the system matrices A, B, C, D as functions of a time varying parameter $\rho(t)$. Then $\dot{X}$ can be replaced by $\frac{\partial X}{\partial \rho}\dot{\rho}$. Then, since the LMIs are convex in ρ, we can attempt to find a solution $X(\rho)$ for all parameter values $|\dot{\rho}| < \nu$, for some constant ν. This is usually approached by gridding the parameter space. In this case, the controller achieves an induced 2-norm of γ for all trajectories of the parameter ρ which satisfy the rate limitation. The controller may either be independent of the parameter as in [55] or, less conservatively, may depend on the parameter and its rate of change as in [79]. Note that the parameter varying approach gives only sufficient conditions for existence of a controller.

However, in the moving horizon approach, we do not synthesize for all possible trajectories of a parameter, or equivalently for all possible variations of the system matrices within some set. Instead we use the parameter value in real time to construct a controller for the system. The key difference is that we do not need existence of a parameter dependent X which will satisfies the LMI for *all* parameter variations, but instead simply need to satisfy it for the particular trajectory on which the system is moving. Again, this appears to have moved us away from the robust control paradigm, since we no longer give some specification in advance of the allowable parameter space.

We have so far constructed solutions X_∞ and Z_∞ by integrating over a finite horizon, and choosing F and G appropriately. The finite horizon integrations correspond to well defined moving horizon control and estimation problems. However, we could have chosen the intervals differently, for example so that they overlapped. In the current framework, since the solution to the filtering equation Z_∞ depends on the solution X_∞, we have to integrate the control and filtering equations over separate intervals, so that Z_∞ is integrated over $[t-T_2, t]$ and X_∞ and integrated over $[t, t+T]$.

The LMI approach to controller synthesis produces results in a different form. It results in linear matrix inequalities for $R = \gamma X_\infty^{-1}$ and $S = \gamma Y_\infty^{-1}$, together with a coupling condition $X_\infty - \gamma^2 Y_\infty^{-1} < 0$. These inequalities are independent, so, using similar methods to those in this chapter, we could show that integrating the equations over a finite interval with an appropriate boundary condition will give a solution to the inequality on an infinite interval. We would, of course, still have to satisfy the coupling condition.

This makes it possible for us to choose the finite horizon control and filtering intervals so that they overlap. That is, we can choose to integrate both control and filtering equations either forwards or backwards in time. Unfortunately, the controller formulas in terms of X_∞ and Y_∞ also depend on $\dot{X}_\infty$, see for example Wu. et. al. [79].

Since we are looking for solutions to the Riccati differential equations with no specified boundary conditions, one possible solution might be to pick an initial boundary condition and integrate forwards. However, the problem is how to choose the initial value. We would like to be able to choose a boundary condition which maximizes the set of plants for which our controller is γ-feasible.

Consider the Riccati differential inequality for X

$$\begin{aligned}\dot{X} + (A - B_2 D_{12}' C_1)' X + X(A - B_2 D_{12}' C_1) \\ - X(B_2 B_2' - \gamma^{-2} B_1 B_1') X + C_1' D_{12}^{\perp} D_{12}^{\perp\prime} C_1 < -\varepsilon I.\end{aligned}$$

Since this is an inequality, it is clear that given any $\beta > 0$, we can ensure that $X < \beta I$ for all $t > 0$ by choosing $\dot{X}$ sufficiently negative. However, in order to use this as the quadratic term of a Lyapunov function to guarantee closed loop stability, we need to ensure that $X > 0$ for all $t \in \mathbb{R}_+$. In fact, Proposition 3.19 shows that if X is a solution to the Riccati differential equation that, integrating forwards, becomes negative definite at some time t_1, then for all $t > t_1$, $X(t) \ngeq 0$.

We would like to have a result of the form of the following: if there exists a matrix function X_1 satisfying the above Riccati differential inequality, then for any initial condition $Q > X_1(0)$, we can find a solution to the Riccati differential inequality also, by suitably choosing the degree to which the inequality is satisfied. Unfortunately, it is not clear if this is true or false.

6.7 Summary

In this chapter we have described a generalization of the linear quadratic moving horizon control formulation to the $\mathscr{H}_\infty$ problem. We have shown that, for both state and output feedback problems, with a suitable choice of initial and final conditions it is possible to construct controllers which are both stabilizing and give an induced norm bound over all time. This form of controller uses only the explicit description of the system up to some finite time ahead, and from some finite time in the past. We have shown that for linear time invariant systems the resulting controller is linear time invariant, and that the boundary conditions required for norm boundedness correspond to monotonicity conditions on the solutions of the Riccati equations. We have also given a general recursive technique for the computation of the solution to the Riccati partial differential equations required.

7. DISCRETE INTERVAL MOVING HORIZON CONTROL

7.1 Introduction

In Chapter 6, we considered the moving horizon control problem for continuous time systems. The cost function was defined for each interval $[t, t+T]$, and an attempt was made to optimize this for each $t \geq 0$. The resulting control law was time varying for time varying systems.

Traditionally, we are trying to arrive at an explicit set of differential equations for the controller. In our work on continuously updated moving horizon control laws, we arrived at a solution which, at each time t, required the integration over $[t, t+T]$ of a Riccati differential equation of dimension equal to the state dimension. We then transformed this to a single forwards integration of a Riccati differential equation of dimension twice the state dimension.

In this chapter, we give a new perspective on this, again based on game theory. This will allow us to consider discrete updates of the moving horizon control law explicitly. That is, instead of demanding that we reintegrate the Riccati differential equation on $[t, t+T]$ at every t, and using only the initial value obtained, we will see that it is possible to reintegrate the Riccati differential equation at discrete times only.

The central theme of this chapter is the interpretation of the solution to the Riccati differential equation for X_∞ as a cost-to-go function.

7.2 State feedback

We consider the linear time varying system $\mathcal{G}$ defined by

$$\dot{x}(t) = A(t)x(t) + B_1(t)w(t) + B_2(t)u(t) \qquad x(t_0) = x_0 \tag{7.1}$$

$$z(t) = C_1(t)x(t) + D_{12}(t)w(t). \tag{7.2}$$

Recall the following definition.

Definition 7.1. *Define as before $X_\infty(t, t_2, Q)$ as the solution to the Riccati differential equation*

$$\begin{aligned} -\dot{X}_\infty + (A - B_2 D_{12}' C_1)' X_\infty + X_\infty (A - B_2 D_{12}' C_1) \\ - X_\infty (B_2 B_2' - \gamma^{-2} B_1 B_1') X_\infty + C_1' D_{12}^\perp D_{12}^{\perp\prime} C_1 \end{aligned} \tag{7.3}$$

with boundary condition $X_\infty(t_2) = Q$. □

Denote

$$J(t_1, t_2, Q) = \int_{t_1}^{t_2} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt + x(t_2)'Qx(t_2),$$

where we have omitted the dependence on the initial state $x(t_1)$, the minimizer's strategy μ and the disturbance w for notational convenience. The following lemma shows that, if we solve one particular differential game, we can guarantee a bounded upper value for the same cost function with any smaller terminal state weight.

Lemma 7.2. *Let $Q_1 \geq 0$, $Q_2 \geq 0$ be symmetric positive definite matrices such that $Q_2 \leq Q_1$. Suppose there exists a state feedback saddle point for the differential game*

$$\inf_{\mu \in \mathscr{U}_{sf}[t_0,t_1]} \sup_{w \in \mathscr{L}_2[t_0,t_1]} J(t_0, t_1, Q_1)$$

and let μ_1 be the state feedback saddle point strategy for the minimizing player. With this controller in place, the quadratic optimization problem defined by

$$\sup_{w \in \mathscr{L}_2[t_0,t_1]} J(t_0, t_1, Q_2) \tag{7.4}$$

has a bounded solution, and the optimal value of this problem is $x(t_0)'X_2x(t_0)$, for some positive definite matrix X_2 satisfying $X_2 \leq X_\infty(t_0, t_1, Q_1)$.

Proof. This is almost by definition. Clearly, with the choice of feedback strategy μ_1, the quadratic optimization problem

$$\sup_{w \in \mathscr{L}_2[t_0,t_1]} J(t_0, t_1, Q_1) \tag{7.5}$$

has a bounded supremum, and in fact its value is given by $x(t_0)'X_\infty(t_0, t_1, Q_1)x(t_0)$, from Theorem 3.14. Since $Q_2 \leq Q_1$, the optimal value of the problem defined by equation (7.4) is less than that defined by equation (7.5). Further, since μ_1 is a linear function of the state, the optimal value will be a homogeneous quadratic function of $x(t_0)$. ∎

Hence, if $x_0 = 0$, solving the differential game with terminal weight Q_1 results in a state feedback $\mathscr{H}_\infty$ solution satisfying

$$\sup_{w \in \mathscr{L}_2[t_0,t_1]} \frac{\|z\|_2^2 + x(t_1)'Q_2x(t_1)}{\|w\|_2^2} \leq \gamma$$

for all $Q_2 \leq Q_1$.

Definition 7.3. *Given two state feedback strategies, $\mu_1 \in \mathscr{U}_{sf}[t_0, t_1]$ and $\mu_2 \in \mathscr{U}_{sf}[t_1, t_2]$, define $\mu_3 \in \mathscr{U}_{sf}[t_0, t_2]$, the* ***concatenation*** *of μ_1 and μ_2, in the obvious way, with*

$$\mu_3(t, x(t)) = \begin{cases} \mu_1(t, x(t)) & \text{if } t_0 \leq t < t_1, \\ \mu_2(t, x(t)) & \text{if } t_1 \leq t \leq t_2. \end{cases}$$

□

The next result shows that this gives us freedom in our choice of an $\mathscr{H}_\infty$ suboptimal controller.

Theorem 7.4. *Suppose $t_0 < t_1 < t_2$. Suppose that there exists a saddle point for the differential game*

$$\inf_{\mu \in \mathscr{U}_{sf}[t_0,t_1]} \sup_{w \in \mathscr{L}_2[t_0,t_1]} J(t_0, t_1, Q_1).$$

and let $\mu_1 \in \mathscr{U}_{sf}[t_0, t_1]$ be the infimising state feedback strategy on $[t_0, t_1]$. Further, suppose that there exists a saddle point for the differential game

$$\inf_{\mu \in \mathscr{U}_{sf}[t_1,t_2]} \sup_{w \in \mathscr{L}_2[t_1,t_2]} J(t_1, t_2, Q_2).$$

and let $\mu_2 \in \mathscr{U}_{sf}[t_1, t_2]$ be the infimising state feedback strategy. Let $\mu_3 \in \mathscr{U}_{sf}[t_0, t_2]$ be the concatenation of μ_1 and μ_2. Then, with the controller μ_3 in place, if $Q_1 \geq X_\infty(t_1, t_2, Q_2)$,

$$\sup_{w \in \mathscr{L}_2[t_0,t_2]} J(t_0, t_2, Q_2)$$

is bounded, and its value is given by $x(t_0)'Xx(t_0)$, with $X \leq X_\infty(t_0, t_1, Q_1)$. Hence, when $x_0 = 0$, this controller results in $\|\mathcal{T}_{zw}\| \leq \gamma$, with the induced norm defined over $[t_0, t_2]$.

Proof. By standard dynamic programming arguments, we can see that for any fixed state feedback strategy $\mu \in \mathscr{U}_{sf}[t_0, t_2]$

$$\begin{aligned}
&\sup_{w \in \mathscr{L}_2[t_0,t_2]} \left\{ \int_{t_0}^{t_2} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt + x(t_2)'Q_2 x(t_2) \right\} \\
&\qquad = \sup_{w \in \mathscr{L}_2[t_0,t_1]} \left\{ \int_{t_0}^{t_1} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt \right. \\
&\qquad\quad \left. + \sup_{w \in \mathscr{L}_2[t_1,t_2]} \left\{ \int_{t_1}^{t_2} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt + x(t_2)'Q_2 x(t_2) \right\} \right\}.
\end{aligned}$$

Then, with the choice of μ in the lemma,

$$\sup_{w \in \mathscr{L}_2[t_1,t_2]} \left\{ \int_{t_1}^{t_2} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt + x(t_2)'Q_2 x(t_2) \right\} = x(t_1)'X_\infty(t_1, t_2, Q_2)x(t_1)$$

so

$$\begin{aligned}
&\sup_{w \in \mathscr{L}_2[t_0,t_2]} \int_{t_0}^{t_2} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt + x(t_2)'Q_2 x(t_2) \\
&= \sup_{w \in \mathscr{L}_2[t_0,t_1]} \int_{t_0}^{t_1} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt + x(t_1)'X_\infty(t_1, t_2, Q_2)x(t_1) \\
&\leq \sup_{w \in \mathscr{L}_2[t_0,t_1]} \int_{t_0}^{t_1} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt + x(t_1)'Q_1 x(t_1) \\
&= x(t_0)'X_\infty(t_0, t_1, Q_1)x(t_0)
\end{aligned}$$

■

This result has the following interpretation; suppose we would like to design a control law for the finite interval $[t_0, t_2]$, for a particular system. Suppose we have a linear time varying model for the system. Then, given γ, we could calculate $X_\infty(t_1, t_2, Q_2)$. However, if we expect that the model is very poor on $[t_1, t_2]$, then we can choose Q_1 greater than our approximation of $X_\infty(t_1, t_2, Q_2)$, and design using Q_1 on $[t_0, t_1]$. Then at time t_1, we might expect to have some experience with which to assume a better nominal model on $[t_1, t_2]$. If with this new model $X_\infty(t_1, t_2, Q_2) \le Q_1$, then the resulting control law will still give a bound of γ on the induced 2-norm on the whole interval $[t_0, t_2]$. *All* plants which agree with the model on $[t_0, t_1]$ and satisfy $X_\infty(t_1, t_2, Q_2) \le Q_1$, can be controlled with performance level γ from time t_0. It is not until time t_1 that we need to know the exact model for the plant on $[t_1, t_2]$.

Clearly however, we cannot expect to achieve the same level of optimal γ by increasing Q_1, although we expect the reduction in achievable gamma to be more significant over shorter intervals, since then the controller has to attenuate the size of the state more quickly.

The next result generalizes this idea to multiple intervals.

Theorem 7.5. *Let $t_0 < t_1 < t_2 < \cdots < t_k$, and let $Q_i, 1 \le i \le k$ be a sequence of symmetric positive definite matrices such that $Q_i \ge X_\infty(t_i, t_{i+1}, Q_{i+1})$, for $0 \le i < k$. Suppose there exists a saddle point for each of the differential games*

$$\inf_{\mu \in \mathscr{U}_{sf}[t_i, t_{i+1}]} \sup_{w \in \mathscr{L}_2[t_i, t_{i+1}]} J(t_i, t_{i+1}, Q_{i+1})$$

for each $i, 0 \le i < k$, and let μ_i be the infimising solution on each interval $[t_i, t_{i+1}]$. Let $\mu \in \mathscr{U}_{sf}[t_0, t_k]$ be the concatenation of the $\mu_i, 0 \le i < k$. Then with this state feedback control law, the quadratic optimization problem

$$\sup_{w \in \mathscr{L}_2[t_0, t_k]} J(t_0, t_k, Q_k)$$

has a bounded solution, and its value is $x(t_0)'Xx(t_0)$, with $X \le X_\infty(t_0, t_1, Q_1)$. Hence, when $x(t_0) = 0$, this controller results in $\|\mathcal{T}_{zw}\| \le \gamma$, where the induced norm is defined on the interval $[t_0, t_k]$.

Proof. First, we show that for a fixed controller, for any interval, the cost is bounded by that of a shorter interval with a sufficiently greater terminal state weight.

$$\begin{aligned}
&\sup_{w \in \mathscr{L}_2[t_0, t_k]} J(t_0, t_k, Q_k) \\
&\quad = \sup_{w \in \mathscr{L}_2[t_0, t_{k-1}]} \left\{ \int_{t_0}^{t_{k-1}} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt \right. \\
&\qquad\quad \left. + \sup_{w \in \mathscr{L}_2[t_{k-1}, t_k]} \left\{ \int_{t_{k-1}}^{t_k} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt + x(t_k)'Q_k x(t_k) \right\} \right\} \\
&\quad = \sup_{w \in \mathscr{L}_2[t_0, t_{k-1}]} J(t_0, t_{k-1}, X_\infty(t_{k-1}, t_k, Q_k)) \\
&\quad \le \sup_{w \in \mathscr{L}_2[t_0, t_{k-1}]} J(t_0, t_{k-1}, Q_{k-1})
\end{aligned}$$

So, if $Q_i \geq X_\infty(t_i, t_{i+1}, Q_{i+1})$, then

$$\sup_{w \in \mathscr{L}_2[t_0,t_{i+1}]} J(t_0, t_{i+1}, Q_{i+1}) \leq \sup_{w \in \mathscr{L}_2[t_0,t_i]} J(t_0, t_i, Q_i)$$

Since this holds for all i, $0 \leq i < k$, we have

$$\begin{aligned} J(t_0, t_k, Q_k) &\leq J(t_0, t_1, Q_1) \\ &= x(t_0)' X_\infty(t_0, t_1, Q_1) x(t_0) \end{aligned}$$

Hence result. ■

We can regard the above problem as

$$\inf_{\mu \in \mathscr{U}_{sf}[t_0,t_k]} \sup_{w \in \mathscr{L}_2[t_0,t_k]} \left\{ \int_{t_0}^{t_k} z(t)'z(t) - \gamma^2 w(t)'w(t)\, dt + x(t_k)'Q_k x(t_k) + \sum_{i=1}^{k-1} x(t_i)'\Delta_i x(t_i) \right\}$$

Here, $\Delta_i = Q_i - X_\infty(t_i, t_{i+1}, Q_{i+1})$. Clearly, if each $\Delta_i \geq 0$, by bounding this cost function we have also bounded the cost function when each $\Delta_i = 0$.

The advantage here is that the Δ_i are *arbitary* positive definite matrices. This means that we do not have to choose them in advance. By choosing Δ_1 sufficiently large, we allow ourselves freedom in choosing the subsequent Δ_i. Naturally, we must choose each Q_i such that the differential game on $[t_{i-1}, t_i]$ has a saddle point. That is, we must choose Q_i such that the Riccati differential equation 7.3 with boundary condition $X_\infty(t_i) = Q_i$ has a bounded solution on $[t_{i-1}, t_i]$. Also, at time t_{i-1}, we will have used Q_{i-1} to construct the controller on the interval $[t_{i-2}, t_{i-1}]$. Therefore, the further condition Q_i must satisfy is that $X_\infty(t_{i-1}, t_i, Q_i) \leq Q_{i-1}$.

Clearly, if there exist a state feedback saddle point solution for the differential game on $[t_0, t_k]$, then there exists a sequence of Q_i satisfying the assumptions of Theorem 7.5. Such a sequence is of course given by $Q_i = X_\infty(t_i, t_k, Q_k)$.

We would now like to choose a sequence of Q_i to satisfy the assumptions of Theorem 7.5, and at the same time gain the advantage of having a large Q_i, and hence a larger set of plants for which the controller is γ-feasible. Clearly, working forwards in time, in order to maximize the available choice of Q_2, we must choose Q_1 as large as possible.

Suppose Q_a is positive definite, then if the solution to the Riccati differential equation for $X_\infty(t, t_1, Q_a)$ is bounded for $t \in [t_0, t_1]$, then from Lemma 6.2d, for any Q_b satisfying $0 \leq Q_b \leq Q_a$ the solution $X_\infty(t, t_1, Q_b)$ will also be bounded for $t \in [t_0, t_1]$. As a result, in order to maximize the range of choice for Q_2, we should choose Q_1 as large as possible.

Remark 7.6. A limiting factor here is that the state feedback saddle point controller on $[t_0, t_1]$ is given by

$$u(t) = -B_2(t)' X_\infty(t, t_1, Q_1) x(t).$$

Therefore, the larger we choose Q_1, the larger the resulting control signals will be. □

We can view the above controller synthesis technique quite simply. By choosing Q_1 large, we are adding a large state penalty to the cost function at time t_1. Therefore, we are synthesizing a controller which both satisfies the γ-feasibility requirement, and ensures that the state at time t_1 is small. Intuitively, we would expect this to enlarge the class of plants for which, after time t_1, we can synthesis a γ-feasible controller over the entire interval $[t_0, t_k]$. By reducing the size of the state at time t_1, it becomes far less important to the controller at time t_0 what the plant does after time t_1.

Having chosen Q_1, we now need to choose Q_2 such that $X_\infty(t, t_2, Q_2)$ is bounded for $t \in [t_1, t_2]$, and such that $X_\infty(t_1, t_2, Q_2) < Q_1$. Again, one possible choice is given by $Q_2 = X_\infty(t_2, t_k, Q_k)$, since in this case $X_\infty(t_1, t_2, Q_2) = X_\infty(t_1, t_k, Q_k) \leq Q_1$.

Since we wish to maximize Q_2, we could simply choose to integrate forwards from Q_1. This gives the maximal choice of Q_2, if the solution to the Riccati differential equation is bounded with this initial condition on the interval $[t_1, t_2]$.

In the extreme case, by inverting the Riccati equation we could choose each $Q_i = \infty$, and design a controller which would bring the state to zero at each time t_i. In this way, at any given t_i, as long as there exists a γ-feasible controller for the interval $[t_i, t_k]$, the concatenated controller will satisfy the induced norm bound over the whole interval $[t_0, t_k]$..

This is clearly an extreme control strategy, although it is a natural discretely updated controller for the $\mathscr{H}_\infty$ suboptimal problem. Indeed, in the sense defined above, it maximizes the allowable set of plants of order n for which this controller is γ-feasible, for this particular discrete set of intervals.

However, this successive interval control does not quite fit into the moving horizon framework of the Chapter 6. A more consistent view is given by consideration of overlapping intervals, as in the next result.

Theorem 7.7. *Let the time sequences $s_0 < s_1 < s_2 < \cdots < s_{k+1}$ and $t_0 < t_1 < t_2 < \cdots < t_k$ satisfy $s_{i+1} \leq t_i$ for $0 \leq i \leq k$, with $s_{k+1} = t_k$. That is, $[s_i, t_i]$ form a sequence of overlapping intervals covering $[s_0, t_k]$.*

Suppose Q_i is a sequence of symmetric positive definite matrices such that the differential game

$$\inf_{\mu \in \mathscr{U}_{sf}[s_i, t_i]} \sup_{w \in \mathscr{L}_2[s_i, t_i]} J(s_i, t_i, Q_i)$$

has a saddle point for $0 \leq i < k$, and $Q_i \geq X_\infty(t_i, t_{i+1}, Q_{i+1})$, for $0 \leq i < k$. Let μ_i be the infimising saddle point solution on the interval $[s_i, t_i]$, and let $\hat{\mu}_i$ be the truncations to the intervals $[s_i, s_{i+1}]$. Construct $\mu \in \mathscr{U}_{sf}[s_0, t_k]$ as the concatenation of the $\hat{\mu}_i$. Then with this control law, the quadratic optimization

$$\sup_{w \in \mathscr{L}_2[s_0, t_k]} J(s_0, t_k, Q_k)$$

is bounded, and its value is given by $x(t_0)' X x(t_0)$, where $X \leq X_\infty(s_0, t_0, Q_0)$. Hence, when $x(t_0) = 0$, this controller results in $\|\mathcal{T}_{zw}\| \leq \gamma$, where the induced norm is defined on $[s_0, t_k]$.

Proof. Clearly, each $\hat{\mu}_i$ is the infimising saddle point solution for the differential game

$$\inf_{\mu \in \mathscr{U}_{sf}[s_i, s_{i+1}]} \sup_{w \in \mathscr{L}_2[s_i, s_{i+1}]} J(s_i, s_{i+1}, X_\infty(s_{i+1}, t_i, Q_i)).$$

Therefore, applying the previous theorem, if

$$X_\infty(s_{i+1}, t_i, Q_i) \geq X_\infty(s_{i+1}, s_{i+2}, X_\infty(s_{i+2}, t_{i+1}, Q_{i+1}))$$

then the result holds. This condition holds if and only if

$$X_\infty(s_{i+1}, t_i, Q_i) \geq X_\infty(s_{i+1}, t_{i+1}, Q_{i+1})$$

which holds if

$$Q_i \geq X_\infty(t_i, t_{i+1}, Q_{i+1}) \tag{7.6}$$

■

This result is the discretized analogue of Theorem 6.7. The endpoint condition given by equation (7.6) is the analogue of the Riccati differential inequality for F given by equation (6.6).

7.3 Conclusions

In this chapter we have given the discretely updated analogue of the moving horizon results of Chapter 6. We have considered the state feedback problem, and shown that for a finite interval it is possible to break up the interval into smaller overlapping intervals. Provided the terminal state weights on each subinterval satisfy certain inequalities, the resulting controller will be γ-feasible.

8. CONCLUSIONS

By way of conclusion we summarise the main contributions of this thesis.

- In chapter 3, we have given an explicit derivation of the certainty equivalence principle developed by Whittle [78] and Başar and Bernhard [7] for the finite horizon continuous time linear differential game with a quadratic cost function.

- We have given necessary and sufficient conditions for the existence of state feedback controllers which attenuate the induced $\mathscr{L}_2$ norm of a multi-rate system with jumps to less than some prespecified level γ on a finite horizon. If such a controller exists, we have provided an explicit construction for one such controller. The class of systems for which this formulation holds includes continuous systems connected with multiple generalized hold functions.

- We have also developed necessary and sufficient conditions for the existence of measurement feedback controllers for multi-rate sampled-data systems which cause the closed loop induced norm from $\ell_2 \oplus \mathscr{L}_2$ to $\mathscr{L}_2$ to be less than some prespecified level γ. If such a controller exists, we have provided an explicit construction for one such controller, and shown that this particular construction depends upon the future hold times, but not on the future sample times.

- We have expressed the solution to the $\mathscr{H}_\infty$ synthesis problem for continuous finite dimensional linear time varying systems in terms of two coupled Riccati differential inequalities, and shown that there exists a controller for parameter varying systems which does not explicitly depend on the rate of change of the parameter. Unfortunately, the conditions are not convex and so cannot be used for gain scheduling; however, they can be used for explicit time varying synthesis. We have reduced the assumptions required for this solution.

- We have investigated the moving horizon control formulation with a game-theoretic cost function, and given sufficient conditions on the terminal state weight and initial state uncertainty such that the closed loop is stable. When it is stable, we have shown that, by enforcing a bound on the induced norm up to a finite time ahead, we can achieve the same induced norm over the whole of future time. In this way we have given a synthesis technique which only depends explicitly on the system parameters up to a finite time ahead.

- In the output feedback case, we have given a dual construction for moving horizon controllers, where estimation is based on information from a finite time in the past and control is based on information up to a finite time in the future.

- We have shown that, for finite horizon continuous differential games, it is possible to divide the interval up into possibly overlapping regions, and solve separate differential games on each separate subinterval. Provided that the boundary conditions on each interval satisfy certain inequalities, the resulting controllers will be γ-feasible on the entire interval. This corresponds to the moving horizon problem with the horizon moved discretely rather than continuously.

8.1 Suggestions for future research

- One clear direction for future research is to extend the synthesis conditions of Chapter 4 for systems with jumps to include the output feedback case. This would provide as a by-product a complete solution to the aperiodic $\mathscr{H}_\infty$ multi-rate sampled-data problem on a finite horizon, and provide a unified treatment of discrete, continuous, sampled-data and general hybrid systems.

- It would also be possible to specialize the above results to the periodic case to provide a time domain solution of the $\mathscr{H}_\infty$ synthesis problem for general hybrid systems on the *infinite* horizon. With the sampling and hold operators periodic, we expect that the limiting solution of the Value function as the horizon length tends to infinity would also be periodic, and hence could be evaluated algebraicly for time invariant systems.

- One of the advantages of the game theoretic approach to controller synthesis as we have used it in this thesis is that we have not needed to use explicit far-future information about a time varying system in order to synthesis γ-feasible $\mathscr{H}_\infty$ controllers. However, the multi-rate sampled-data solution of Chapter 4 does depend explicitly on all future hold instants. This is intuitively unreasonable; we would not expect very much difference in the controller's strategy if some of the future hold times were perturbed slightly. It may be possible to combine the moving horizon results with the multi-rate sampled-data results so that infinite horizon synthesis could be performed for sampled-data systems, without complete future knowledge of the hold times.

- Similarly, based on the results in Chapter 4 and 5, we expect that it is possible to characterise the solution to the $\mathscr{H}_\infty$ synthesis problem for sampled-data systems in terms of linear matrix inequalities which are affine in the system matrices. This would allow, for example, construction of scheduled controllers for sampled-data systems. By gridding the time interval between sampled and hold times, it may also be possible to construct controllers valid for a range of inter-sample and hold periods.

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