Local stability analysis of nonlinear systems

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Abstract

This paper considers local stability properties of systems comprising stable linear time-invariant operators in combination with a scalar nonlinearity. We consider those nonlinearities whose gain can be related to the peak value of their input signal. It is assumed that the nonlinearity has some nominal gain for small signals (ie with peak value less than some number), and that the gain then increases for larger inputs.

It is shown that there is a class of exogenous inputs, characterised by their energy, such that all signals in the system are bounded, and the effective gain of the nonlinearity is no greater than the nominal value.

It is further shown that, providing a stated condition is satisfied, there is a larger class of exogenous inputs, again characterised by their energy, such that all signals in the system are bounded. This condition is shown to be an inequality between known parameters of the nonlinearity and the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms of the linear parts of the system.

Key Words: stability, local stability, nonlinear analysis, disturbance rejection

1 Introduction

For many nonlinearities, it is possible to express the $\mathcal{L}_2$-$\mathcal{L}_2$ gain as being bounded by a known function of the peak value ($\mathcal{L}_\infty$-norm) of its input signal u. For example, consider the "ideal deadzone", which satisfies

- if $\|u\|_\infty \leq 1$ then $\|\Delta\| = 0$
- if $\|u\|_\infty < r$ $(r > 1)$ then $\|\Delta\| < 1 - \frac{1}{r}$

Knowing this information, and given some system incorporating this nonlinearity, we may well ask:

- Can we determine a class of exogenous inputs to the system such that all the signals in the system remain bounded?

If we can find such a class, and in practice we nearly always can, then we say that the system is *locally stable* with respect to inputs in this class. Local stability of systems with the ideal deadzone nonlinearity was considered in [1]; this paper can be considered a generalisation of that work.

Such a concept makes intuitive sense if applied to real systems - for example, an aircraft cannot maintain stability in the presense of arbitrarily large wind gusts, but it may well be stable for all gusts with sufficiently small energy.

We see also that, for this particular nonlinearity, the gain has a nominal value (zero) for small u (those with peak value no greater than 1). For larger signals, the gain grows to some upper value (the $\mathcal{L}_2$-$\mathcal{L}_2$ gain of the nonlinearity.) This pattern occurs with a number of common memoryless nonlinearities, and leads us to consider a second question:

- Can we determine a class of exogenous inputs to the system such that the nonlinear gain is no larger than the nominal value?

If we can find such a class, then we say that the system is *nominally stable* with respect to inputs in this class.

The usefulness of this concept is that, if we believe that the disturbances will be in this class (or are *likely* to be in this class), then we can model the nonlinearity as being norm-bounded by the nominal value, and hence use any of the many design techniques for such nonlinearities.

However, if we did this, then we would certainly desire that there is some (strictly) larger class of signals for which the system is locally stable, in order to have some guarantee on what happens if the disturbance is just a little bit too large.

We will see that we can find answers to both of these questions, along with a simple sufficient condition to determine whether the former class is larger than the latter.

It should be noted that for particular nonlinearities, there are numerical methods for calculating such bounds (eg [3]), and due to the fact that these methods use more information about the nonlinearity, they can be less conservative that the method described here. The strength of this work is that it provides insight into the relationship between the properties of the system components (eg $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms of transfer functions) and the overall local stability properties.

2 Notation

Let $\mathcal{L}_2[0,\infty)$ be the set of bounded-energy time-varying scalar signals $v(t), t \geq 0$, with norm $\|v\|_2 := \sqrt{\int_0^\infty v^*(t)v(t)\,dt}$. Similarly, let $\mathcal{L}_\infty[0,\infty)$ be the set of bounded-magnitude time-varying scalar signals $v(t), t \geq 0$, with norm $\|v\|_\infty := \sup_{t\in[0,\infty)} |v(t)|$. Let Π_T be the truncation operator

$$(\Pi_T v)(t) := \begin{cases} v(t) & \text{if } t \in [0,T], \\ 0 & \text{otherwise.} \end{cases}$$

Then let $\mathcal{L}_{2e}$ be the set of bounded-energy time-varying scalar signals $v(t), t \geq 0$ for which $\|\Pi_{T_0} v\|_2$ is finite for any $T < \infty$.

Let $\mathcal{RH}_2$ be the set of real-rational transfer matrices $G(s)$, analytic in the open right half plane and square integrable on the $j\omega$ axis, with norm $\|G\|_2 := \sqrt{\int_{-\infty}^\infty G^*(j\omega)G(j\omega)\,d\omega}$. Similarly, let $\mathcal{RH}_\infty$ be the set of real-rational transfer matrices $G(s)$, analytic in the open right half plane and essentially bounded on the $j\omega$ axis, with norm $\|G\|_\infty := \sup_\omega\{G^*(j\omega)G(j\omega)\}$.

Derivatives of functions will be shown with apostrophes, ie $f'(x_0)$ means the derivative of $f(x)$ evaluated at x_0. A one-sided derivative will be shown as $f'(x_0^+)$.

3 Problem Definition

3.1 System interconnections

We consider the interconnections shown in Figure 1, which can be described by the following equations

$$u = Fx + Gy \tag{1}$$
$$y = \Delta u \tag{2}$$

where $x \in \mathcal{L}_2[0,\infty)$ is a (possibly vector-valued) exogenous input, u and y are real scalar signals, F and G are stable, strictly proper, real-rational transfer functions, and Δ is a causal scalar nonlinearity with finite gain and finite uniform instantaneous gain.

Lemma 1 *The interconnection formed by F, G and Δ is well-posed.*

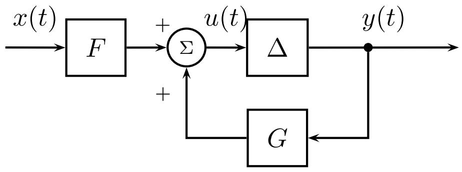

Figure 1: Linear system with scalar nonlinearity

Proof of Lemma 1: G has zero uniform instantaneous gain (since it is strictly proper), so the product of the uniform instantaneous gains of G and Δ is zero. Hence the interconnection is well-posed. (Details may be found in eg. [5]) ∎

Hence unique solutions $y, u \in \mathcal{L}_{2e}$ exist on any finite interval $[0,T]$ for any $x \in \mathcal{L}_{2e}$.

3.2 Properties of $[F\ G]$

Define a function $\gamma_{F,G}(.)$ on $[0,\infty)$, which characterises the gain of $\begin{bmatrix} F & G \end{bmatrix}$ with respect to *individual* norm bounds on x and y

$$\gamma_{F,G}(\phi) := \frac{\phi}{\sup_{\|x\|_2\leq 1, \|y\|_2 \leq \phi} \|Fx + Gy\|_2} \tag{3}$$

which can be calculated [2] as

$$\gamma_{F,G}(\phi) = \frac{\phi}{\inf_{\alpha\in(0,\infty)} \left\{ \left\| \begin{bmatrix} \frac{1}{\alpha}F & G \end{bmatrix} \right\|_\infty \sqrt{\alpha^2 + \phi^2} \right\}}$$

It may be shown that $\gamma_{F,G}(.)$ is continuous and strictly monotonic, with $\gamma_{F,G}(0) = 0$ and $\lim_{\phi\to\infty} \gamma_{F,G}(\phi) = \frac{1}{\|G\|_\infty}$, and that the following inequality holds

$$\frac{\phi}{\|F\|_\infty + \|G\|_\infty\,\phi} \leq \gamma_{F,G}(\phi) \leq \frac{\phi}{\sqrt{\|F\|_\infty^2 + \|G\|_\infty^2\,\phi^2}} \tag{4}$$

3.3 Properties of Δ

We assume a known function $\psi_\Delta(.)$ defined on (r_0,∞) for some $r_0 \geq 0$ such that for any $u \in \mathcal{L}_{2e}$, $T > 0$, and $y = \Delta u$

- if $\|\Pi_T u\|_\infty < r$ for some $r > r_0$, then
$$\|\Pi_T y\|_2 < \psi_\Delta(r)\,\|\Pi_T u\|_2 \tag{5}$$

We define

$$\beta_0 := \lim_{r\to r_0} \psi_\Delta(r)$$
$$\phi_0 := \begin{cases} 0 & \text{if } \beta_0 = 0 \\ \gamma_{F,G}^{-1}(\beta_0) & \text{if } \beta_0 \in (0, \frac{1}{\|G\|_\infty}) \end{cases}$$
$$\beta_1 := \sup_{r\in(r_0,\infty)} \psi_\Delta(r)$$
$$\phi_1 := \begin{cases} \gamma_{F,G}^{-1}(\beta_1) & \text{if } \beta_1 \in (0, \frac{1}{\|G\|_\infty}) \\ \infty & \text{if } \beta_1 \geq \frac{1}{\|G\|_\infty} \end{cases}$$

and assume that $\psi_\Delta(.)$ is differentiable with finite, strictly positive derivative on (r_0, ∞), which implies that $\psi_\Delta^{-1}(.)$ is well-defined, continuous and strictly increasing on (β_0, β_1).

Examples of $\psi_\Delta(.)$ are given in Figure 3 (for the "ideal deadzone" nonlinearity) and Figure 5 (for the "saturated squaring" nonlinearity.)

3.4 Assumptions

We make the following assumptions

- $\beta_1 \|G\|_\infty \geq 1$ (since otherwise the interconnection would be globally stable, by the small gain theorem [6])
- $\beta_0 \|G\|_\infty < 1$

4 Main Results

Definition 1 *An interconnection is said to be* ***locally stable*** *with respect to some restriction(s) on its exogenous inputs if for* ***any*** *inputs satisfying the restriction(s)*

- *all signals are bounded*

Definition 2 *A nonlinearity satisfying Equation 5 is said to be operating in its* ***nominal regime*** *if*

- $\|u\|_\infty \leq r_0$

Definition 3 *The interconnection in Figure 1 is said to be* ***nominally stable*** *with respect to some restriction(s) on the exogenous inputs if for* ***any*** *inputs satisfying the restriction(s)*

- *the interconnection is locally stable,* and
- *the nonlinearity is operating in its nominal regime.*

Theorem 1 (Nominal Stability) *For any Δ satisfying Equation 5, the interconnection in Figure 1 is nominally stable with respect to exogenous inputs x satisfying*

$$\|x\|_2 \leq \frac{r_0}{\|F\|_2 + \|G\|_2 \phi_0}$$

Theorem 2 (Local Stability) *For any Δ satisfying Equation 5, the interconnection in Figure 1 is locally stable with respect to exogenous inputs x satisfying*

$$\|x\|_2 < \sup_{\phi \in (\phi_0, \phi_1)} \frac{\psi_\Delta^{-1}(\gamma_{F,G}(\phi))}{\|F\|_2 + \|G\|_2 \phi}$$

Furthermore, if

$$r_0 \psi'_\Delta(r_0^+) \frac{\|F\|_\infty \|G\|_2}{\|F\|_2} < (1 - \beta_0 \|G\|_\infty)^3$$

then

$$\sup_{\phi \in (\phi_0, \phi_1)} \frac{\psi_\Delta^{-1}(\gamma_{F,G}(\phi))}{\|F\|_2 + \|G\|_2 \phi} > \frac{r_0}{\|F\|_2 + \|G\|_2 \phi_0}$$

Remarks

- It is important to note that β_0, r_0 and $\psi'_\Delta(r_0^+)$ are simply known properties of the nonlinear gain bound function $\psi_\Delta(.)$ (see Figure 3), and that $\phi_0 := \gamma_{F,G}^{-1}(\beta_0)$
- If Δ is such that $r_0 = 0$ then there is effectively no nominal regime (by Theorem 1), but in that case there will always be a non-empty class of admissable x guaranteeing local stability (by Theorem 2)
- For any particular F and G (and Δ) it is possible to calculate the bounds on $\|x\|_2$ quite easily. However it is often more useful to use the inequality in Equation 4 to obtain analytic (but possibly quite conservative) lower bounds, using just the $\mathcal{H}_\infty$- and $\mathcal{H}_2$-norms of F and G. This technique will be demonstrated in the examples.

Proof of these Theorems will require a number of preliminary Lemmas.

Lemma 2 *For any $v \in \mathcal{L}_{2e}$, $\|\Pi_T v\|_2$ is a continuous non-decreasing function of T.*

Proof of Lemma 2: For any $v \in \mathcal{L}_{2e}$, $\|\Pi_T v\|_2$ is non-decreasing in T (this is a basic assumption about Π_T) and $\|\Pi_T v\|_2$ is continuous in T (see eg. [4]). ∎

Lemma 3 *For any x, y, u satisfying Equations 1 and 2, there exists some $T > 0$ such that $\|\Pi_T x\|_2 \neq 0$ and $\|\Pi_T y\|_2 \leq \phi_0 \|\Pi_T x\|_2$.*

Proof of Lemma 3: It follows from well-posedness of the interconnection that for sufficiently small T, $\|\Pi_T y\|_2 \leq \beta_0 \|F\|_\infty \|\Pi_T x\|_2$, and it may be shown (using, for example, the inequality in Equation 4) that $\phi_0 \geq \beta_0 \|F\|_\infty$. ∎

Lemma 4 *For any x, y, u satisfying Equations 1 and 2, any $T > 0$ and any $\phi \in (\phi_0, \phi_1)$*

$$\|\Pi_T x\|_2 < \frac{\psi_\Delta^{-1}(\gamma_{F,G}(\phi))}{\|F\|_2 + \|G\|_2 \phi} \Longrightarrow \|\Pi_T y\|_2 < \phi \|\Pi_T x\|_2$$

Proof of Lemma 4: We prove the converse, ie that $\|\Pi_T y\|_2 \geq \phi \|\Pi_T x\|_2$ implies $\|\Pi_T x\|_2 \geq \frac{\psi_\Delta^{-1}(\gamma_{F,G}(\phi))}{\|F\|_2 + \|G\|_2 \phi}$.

Assume that $\|\Pi_T y\|_2 \geq \phi \|\Pi_T x\|_2$. Then by Lemmas 2 and 3 there exists some $T' \leq T$ such that $\|\Pi_{T'} y\|_2 = \phi \|\Pi_{T'} x\|_2$ and $\|\Pi_{T'} x\|_2 \neq 0$.

By Equation 3 we have that

$$\begin{aligned} \|\Pi_{T'} u\|_2 &\leq \frac{\|\Pi_{T'} y\|_2}{\gamma_{F,G}(\phi)} \quad \text{ie} \\ \|\Pi_{T'} y\|_2 &\geq \gamma_{F,G}(\phi) \|\Pi_{T'} u\|_2 \end{aligned}$$

and by Equation 5, and noting the monotonicity of $\psi_\Delta^{-1}(.)$, we deduce that

$$\|\Pi_{T'} u\|_\infty \geq \psi_\Delta^{-1}(\gamma_{F,G}(\phi))$$

It is then standard that

$$\begin{aligned} \|\Pi_{T'} u\|_\infty &\leq \|F\|_2 \|\Pi_{T'} x\|_2 + \|G\|_2 \|\Pi_{T'} y\|_2 \\ &= (\|F\|_2 + \|G\|_2 \phi) \|\Pi_{T'} x\|_2 \end{aligned}$$

so

$$\|\Pi_T x\|_2 \geq \|\Pi_{T'} x\|_2 \geq \frac{\psi_\Delta^{-1}(\gamma_{F,G}(\phi))}{\|F\|_2 + \|G\|_2 \phi} \quad \blacksquare$$

Proof of Theorem 1: This result follows by letting $T \to \infty$ and $\phi \to \phi_0$ in Lemma 4. $\blacksquare$

Lemma 5 *Given the following function $g(.)$ defined on* $(0, \infty)$

$$\begin{aligned} g(z) &:= \frac{1}{\gamma_{F,G}^2(\sqrt{z})} \\ &= \inf_{\alpha \in (0,\infty)} \left\{ \left\| \begin{bmatrix} F & \alpha G \end{bmatrix} \right\|_\infty^2 \left(\frac{1}{\alpha^2} + \frac{1}{z} \right) \right\} \end{aligned}$$

then

$$g(z_0 + \delta z) \leq g(z_0) - \frac{\|F\|_\infty^2 \delta z}{z_0(z_0 + \delta z)}$$

Proof of Lemma 5: For $\epsilon > 0$ and $z_0 \in (0, \infty)$, define

$$\mathcal{A}_\epsilon(z_0) := \{\alpha : \left(\frac{1}{\alpha^2} + \frac{1}{z_0} \right) \left\| \begin{bmatrix} F & \alpha G \end{bmatrix} \right\|_\infty^2 < g(z_0) + \epsilon\}$$

Then for any $\epsilon > 0$, $z_0 \in (0, \infty)$ and $\alpha \in \mathcal{A}_\epsilon(z_0)$

$$\begin{aligned} g(z_0 + \delta z) &\leq \left\| \begin{bmatrix} F & \alpha G \end{bmatrix} \right\|_\infty^2 \left(\frac{1}{\alpha^2} + \frac{1}{z_0 + \delta z} \right) \\ &< g(z_0) + \epsilon - \frac{\delta z}{z_0(z_0 + \delta z)} \left\| \begin{bmatrix} F & \alpha G \end{bmatrix} \right\|_\infty^2 \\ &\leq g(z_0) + \epsilon - \frac{\|F\|_\infty^2 \delta z}{z_0(z_0 + \delta z)} \end{aligned}$$

and by taking the limit as $\epsilon \to 0$ we get the result stated. $\blacksquare$

Proof of Theorem 2: The first statement follows by letting $T \to \infty$ in Lemma 4.

Proof of the second statement involves finding sufficient conditions for the existence of some $\delta\phi > 0$ such that

$$\frac{\psi_\Delta^{-1}(\gamma_{F,G}(\phi_0 + \delta\phi))}{\|F\|_2 + \|G\|_2 (\phi_0 + \delta\phi)} > \frac{r_0}{\|F\|_2 + \|G\|_2 \phi_0}$$

or equivalently (noting that $\psi_\Delta(.)$ is monotonic)

$$\gamma_{F,G}(\phi_0 + \delta\phi) > \psi_\Delta \left(\frac{\|F\|_2 + \|G\|_2 (\phi_0 + \delta\phi)}{\|F\|_2 + \|G\|_2 \phi_0} r_0 \right)$$

If $\beta_0 \neq 0$ (implies $\phi_0 \neq 0$) then we apply the chain rule to the result of Lemma 5 to show that

$$\gamma_{F,G}(\phi_0 + \delta\phi) \geq \gamma_{F,G}(\phi_0) + \frac{\|F\|_\infty^2 \gamma_{F,G}^3(\phi_0)}{\phi_0^2} \delta\phi - \mathcal{O}(\delta\phi^2)$$

Otherwise, if $\beta_0 = 0$ (implies $\phi_0 = 0$), we use the approximation in Equation 4 to say that

$$\begin{aligned} \gamma_{F,G}(\phi_0 + \delta\phi) &\geq \frac{(\phi_0 + \delta\phi)}{\|F\|_\infty + \|G\|_\infty (\phi_0 + \delta\phi)} \\ &= \frac{\delta\phi}{\|F\|_\infty + \|G\|_\infty \delta\phi} \end{aligned}$$

Noting that

$$\psi_\Delta(r_0 + \delta r) \leq \psi_\Delta(r_0) + \psi'_\Delta(r_0)\delta r + \mathcal{O}(\delta r^2)$$

and using Equation 4, it is simply algebra to show that the condition stated does indeed imply the existence of such a $\delta\phi$. $\blacksquare$

5 Examples

5.1 Example 1

We consider a system with the "ideal deadzone" nonlinearity

$$\begin{aligned} y(t) &= \Delta(u(t)) \\ &:= \begin{cases} u(t) + 1 & \text{if } u(t) < -1, \\ 0 & \text{if } -1 \leq u(t) \leq 1, \\ u(t) - 1 & \text{if } u(t) > 1, \end{cases} \end{aligned}$$

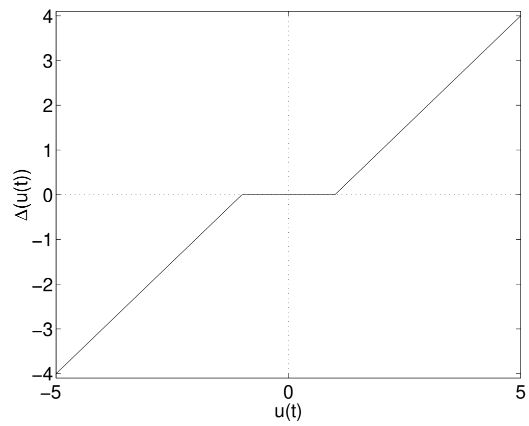

Figure 2: $\Delta(.)$ for the "ideal deadzone" nonlinearity

For this nonlinearity we can use

$$\psi_{\Delta}(r) := 1 - \frac{1}{r} \quad (r > 1)$$

with $\beta_0 = 0$, $\beta_1 = 1$, $r_0 = 1$ and $\psi'_{\Delta}(r_0^+) = 1$. We assume $\|G\|_\infty > 1$ so as not to satisfy the small gain theorem.

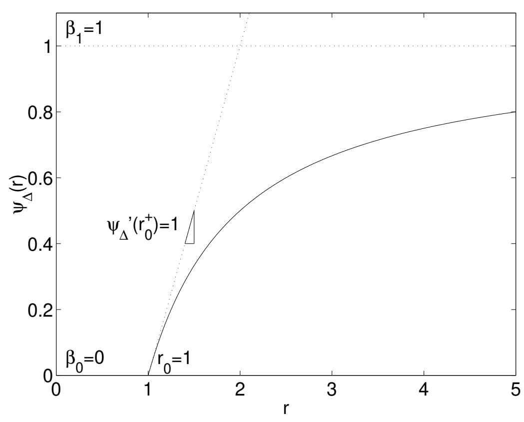

Figure 3: $\psi_{\Delta}(.)$ for the "ideal deadzone" nonlinearity

Applying Theorem 1, we see that this interconnection remains in its nominal regime if $\|x\|_2 \leq \frac{1}{\|F\|_2}$.

Applying Theorem 2, we see that the condition is satisfied if $\frac{\|F\|_\infty \|G\|_2}{\|F\|_2} < 1$.

Assuming this condition is satifsied, we use the inequality in Equation 4 to obtain an analytic lower bound on $\sup_{\phi \in (\phi_0, \phi_1)} \frac{\psi_{\Delta}^{-1}(\gamma_{F,G}(\phi))}{\|F\|_2 + \|G\|_2 \phi}$ which uses just the $\mathcal{H}_\infty$- and $\mathcal{H}_2$-norms of F and G. For convenience, write $\mu := \|G\|_\infty$ and $\rho := \frac{\|F\|_\infty \|G\|_2}{\|F\|_2}$. Then the interconnection is locally stable with respect to x satisfying

$$\|x\|_2 < \frac{1}{\|F\|_2} \left(\frac{\sqrt{\rho(\mu - 1)} + \sqrt{\mu - \rho}}{\sqrt{\rho(\mu - \rho)} + \sqrt{\mu - 1}} \right)^2$$

provided $\frac{\|F\|_\infty \|G\|_2}{\|F\|_2} < 1$.

5.2 Example 2

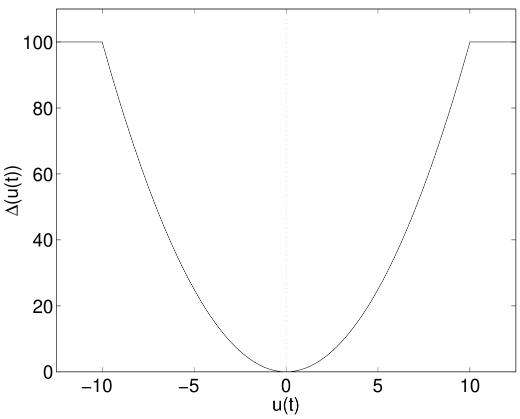

Figure 4: $\Delta(.)$ for the "saturated squaring" nonlinearity (shown for $M = 10$)

Secondly, we consider a system with a "saturated squaring" nonlinearity

$$\begin{aligned} y(t) &= \Delta(u(t)) \\ &:= \begin{cases} |u(t)|^2 & \text{if } |u(t)| \leq M, \\ M^2 & \text{if } |u(t)| \geq M, \end{cases} \end{aligned}$$

for some $M \gg 1$. For this nonlinearity we can use

$$\psi_{\Delta}(r) := \begin{cases} r & \text{if } r \leq M \\ M + \frac{r-M}{r-M+1} & \text{if } r \geq M \end{cases}$$

with $\beta_0 = 0$, $\beta_1 = M + 1$, $r_0 = 0$ and $\psi'_{\Delta}(r_0^+) = 1$. Note the term $M + \frac{r-M}{r-M+1}$, which is intentionally conservative in order to satisfy the condition that the function $\psi_{\Delta}(.)$ be differentiable and strictly increasing.

As noted in the Remarks to Theorems 1 and 2, since $r_0 = 0$, this interconnection does not have a nominal regime, but always has a local stability property.

We again use the inequality in Equation 4 to obtain an analytic lower bound on $\sup_{\phi \in (\phi_0, \phi_1)} \frac{\psi_{\Delta}^{-1}(\gamma_{F,G}(\phi))}{\|F\|_2 + \|G\|_2 \phi}$ which uses just the $\mathcal{H}_\infty$- and $\mathcal{H}_2$-norms of F and G. For convenience, write $\mu := \|G\|_\infty$ and $\rho := \frac{\|F\|_\infty \|G\|_2}{\|F\|_2}$. Then the interconnection is locally stable with respect

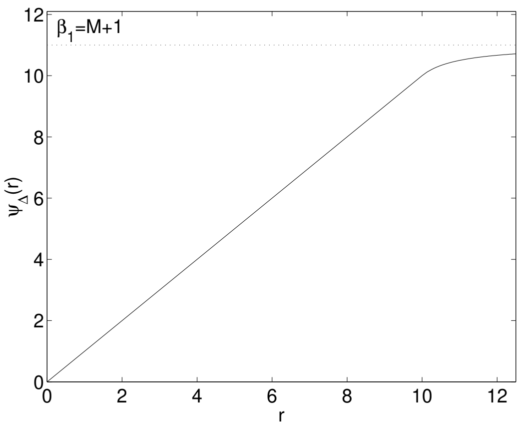

Figure 5: $\psi_\Delta(.)$ for the "saturated squaring" nonlinearity (shown for $M = 10$)

to x satisfying

$$\begin{aligned}\|x\|_2 &< \frac{1}{\|F\|_2}\frac{1}{(\sqrt{\mu}+\sqrt{\rho})^2} \\ &= \frac{1}{(\sqrt{\|F\|_2\,\|G\|_\infty}+\sqrt{\|G\|_2\,\|F\|_\infty})^2}\end{aligned}$$

It is clear that if M is sufficiently large, then with any x in this class, $\|u\|_\infty < M$, and hence this "saturated squaring" nonlinearity is indistinguishable from the "ideal squaring" nonlinearity $y(t) = |u(t)|^2$ (which, having infinite gain, cannot be considered directly in this framework.)

This leads us to conclude that this result holds for the "ideal squaring" nonlinearity.

6 Conclusions

In this paper we have considered the local stability properties of certain nonlinear systems. We have identified a regime of nominal operation, and determined a class of exogenous inputs such that the system remains in that nominal regime.

Furthermore, we have found a sufficient condition on the system components which, if satisfied, guarantees that there is a larger class of exogenous inputs such that the system remains bounded.

The methods have been illustrated with two examples.

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