Local stability properties of systems with saturation and deadzone nonlinearities

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Abstract

This paper considers the local stability of systems comprising linear time-invariant operators in combination with a deadzone nonlinearity. The behaviour of systems which are *not* globally bounded-input bounded-output stable is investigated, and it is shown that under certain conditions, such systems are bounded-output stable for a restricted class of inputs. A sufficient condition for this property is stated as a simple norm inequality, and the restricted input class is shown to be characterised by the energy of the signal.

The applicability of this work to systems with saturation nonlinearity, and in particular the well-known "anti-windup" problem is shown, and an example given.

Key Words: saturation, deadzone, anti-windup, stability

1 Introduction

It is an unavoidable fact that all physical systems exhibit some form of input saturation, which can have a severely detrimental affect on performance, and can sometimes lead to instability. In view of the fact that well-proven strategies have been developed for designing linear controllers for linear plants, many attempts have been made to include (post-design) compensation for input saturation; this problem has been termed the "anti-windup" problem (see for example: [1], [2], [3], [4] and references therein.)

A reasonable definition of the problem, and one which has been the focus of much recent work (eg. [2], [3]), is to firstly design a linear controller which gives acceptable performance in the unsaturated operating regime, and then to modify this controller such that

- control action (and hence performance) in the unsaturated regime is unaltered
- stability is guaranteed for all external inputs in a given class
- some suitable performance measure is minimised for all such inputs

It was shown recently in [3] that for stable plants it is possible to achieve all three of the above for all square-integrable inputs, with the performance measure an induced norm from the saturation error to certain signals in the loop.

Taking advantage of the fact that "saturation" plus "deadzone" equals the identity, this work considers the analysis of a generalised plant perturbed by a deadzone nonlinearity; it is clear then that the "anti-windup" problem given above can be easily formulated in this way. The condition for such a system to be bounded input-bounded output stable for all square-integrable inputs is stated and assumed to be *unsatisfied*.

Then it will be shown that the system *is* bounded output stable for a smaller class of inputs, subject to a single condition being satisfied. This class of inputs will be shown to be those whose energy is less than a certain number. Furthermore, such systems may be said to exhibit "graceful degradation", that is, small excursions into the saturated regime will result in small errors.

2 Notation

Let $\mathcal{L}_2[0,\infty)$ be the set of bounded-energy time-varying scalar signals $v(t), t \geq 0$, with norm $\|v\|_2 := \sqrt{\int_0^\infty v^*(t)v(t)\,dt}$. Also, let $\mathcal{L}_2[0,T]$ be the set of time-varying scalar signals $v(t), t \geq 0$ such that $\|\Pi_{T_0} v\|_2$ is finite for all $T_0 < T$. Similarly, let $\mathcal{L}_\infty[0,\infty)$ be the set of bounded-magnitude time-varying scalar signals $v(t), t \geq 0$, with norm $\|v\|_\infty := \sup_{t\in[0,\infty)} |v(t)|$. Let Π_T be the truncation operator

$$(\Pi_T v)(t) := \begin{cases} v(t) & \text{if } t \in [0,T], \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{RH}_2$ be the set of real-rational transfer matrices $G(s)$, analytic in the open right half plane and square integrable on the $j\omega$ axis, with norm

$\|G\|_2 := \sqrt{\int_{-\infty}^{\infty} G^*(j\omega)G(j\omega)\,d\omega}$. Similarly, let $\mathcal{RH}_\infty$ be the set of real-rational transfer matrices $G(s)$, analytic in the open right half plane and essentially bounded on the $j\omega$ axis, with norm $\|G\|_\infty := \sup_{\omega\in(-\infty,\infty)}\{G^*(j\omega)G(j\omega)\}$.

3 System set-up

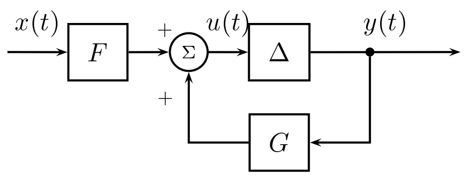

Figure 1: System with deadzone nonlinearity

Figure 1 shows, in block diagram form, the interconnections described by the following equations:

$$u = Fx + Gy \tag{1}$$

$$y(t) = (\Delta u)(t) \tag{2}$$

$$:= \begin{cases} u(t)+1 & \text{if } u(t) < -1, \\ 0 & \text{if } -1 \le u(t) \le 1, \\ u(t)-1 & \text{if } u(t) > 1, \end{cases}$$

where $x(t), y(t), u(t), e(t) \in \mathbb{R}$, F and G are stable, strictly proper transfer functions (ie $F, G \in \mathcal{RH}_2 \cap \mathcal{RH}_\infty$) and Δ is the unity deadzone operator, which is shown in Figure 2. Note that if $|u(t)| \le \gamma$ for all t, then the gain of the deadzone is at most $\frac{\gamma-1}{\gamma} = 1 - \frac{1}{\gamma}$.

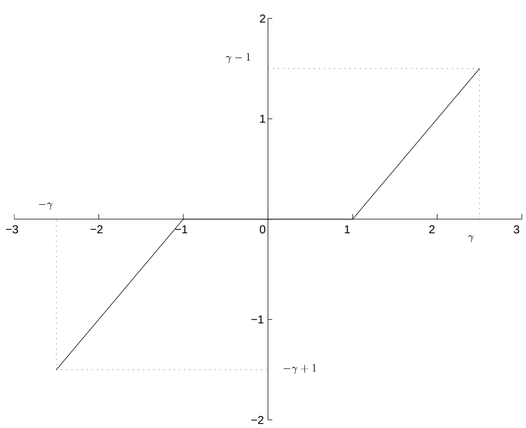

Figure 2: Deadzone nonlinearity

4 Main Result

The main result of this paper is that for $\|G\|_\infty > 1$, and providing the condition

$$\|G\|_2 < \frac{\|F\|_2}{\|F\|_\infty} \tag{3}$$

is satisfied, then there exist $0 < X_0 < X_1$, $0 < Y_1 < \infty$ and $0 < U_1 < \infty$ such that

1. $\|x\|_2 < X_0$ implies $y = 0$ and $\|u\|_2 \le \|F\|_\infty \|x\|_2$,
2. $\|x\|_2 < X_1$ implies $\|y\|_2 < Y_1$ and $\|u\|_2 < U_1$, and
3. for $X \in (X_0, X_1)$, there exist non-decreasing functions $Y(X) \in (0, Y_1)$, $U(X) \in (\|F\|_\infty X_0, U_1)$ such that $\|x\|_2 < X$ implies $\|y\|_2 < Y(X)$ and $\|u\|_2 < U(X)$.

and hence the system in Figure 1 exhibits a form of "graceful degradation". Figure 3 shows a representative plot of this result.

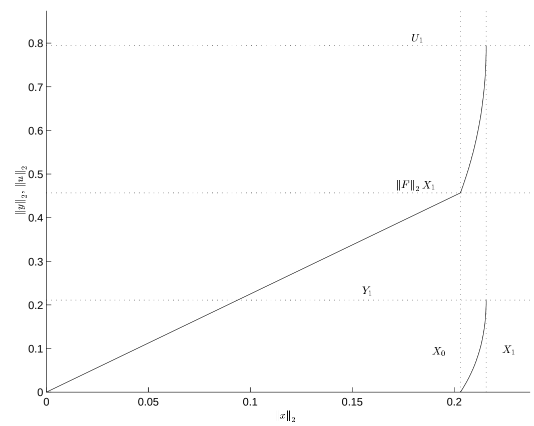

Figure 3: Example of norm bounds

5 Preliminaries

Assumptions

1. $x \in \mathcal{L}_2[0,\infty)$
2. $\|G\|_\infty > 1$

Remarks:

1. If $\|G\|_\infty < 1$ then the feedback loop would satisfy the Small-Gain Theorem ([5]), which would mean that $x \in \mathcal{L}_2[0,\infty)$ implies $y \in \mathcal{L}_2[0,\infty)$.
2. We do not consider the borderline case $\|G\|_\infty = 1$ in this paper, since some systems with $\|G\|_\infty = 1$ have $x \in \mathcal{L}_2[0,\infty)$ implying $y \in \mathcal{L}_2[0,\infty)$, while others do not, and simple norm conditions are not sufficient to determine this.

Definition

$$\theta := \frac{\|G\|_2}{\|F\|_2}$$

Well-posedness of feedback equations

Lemma 1 *The loop formed by G and Δ is well-posed.*

Remark: Hence for any $x \in \mathcal{L}_2[0,\infty)$ and any $T \in (0,\infty)$ there are unique solutions $y, u \in \mathcal{L}_2[0,T]$ to Equations 1 and 2.

Proof of Lemma 1: Firstly, $F \in \mathcal{RH}_\infty$ and $x \in \mathcal{L}_2[0,\infty)$ imply that $Fx \in \mathcal{L}_2[0,\infty)$. Secondly, $G \in \mathcal{H}_2$ implies that the product of the uniform instantaneous gains of G and Δ is zero. Hence the loop is well-posed. (Details may be found in eg. [6]) ∎

6 Main Theorem

Define the following functions of $\phi \in (0,\infty)$:

$$\Gamma(\phi) := \inf_{\alpha \in (0,\infty)} \left\{ \left\| \begin{bmatrix} \frac{1}{\alpha}F & G \end{bmatrix} \right\|_\infty \sqrt{\alpha^2\phi^2 + 1} \right\}$$

$$\hat{x}(\phi) := \frac{1}{\|F\|_2} \frac{\phi}{\phi + \theta} \frac{\Gamma(\phi)}{\Gamma(\phi) - 1}$$

Theorem 1 *For any $T \in (0,\infty)$ such that $\|\Pi_T x\|_2 > 0$, and for any $\phi \in (0,\infty)$:*

$$\|\Pi_T x\|_2 < \hat{x}(\phi) \Longrightarrow \|\Pi_T y\|_2 < \frac{1}{\phi} \|\Pi_T x\|_2$$

Corollary 1 *For any $T \in (0,\infty)$ such that $\|\Pi_T x\|_2 > 0$:*

$$\|\Pi_T y\|_2 < \frac{1}{\sup\{\phi : \|\Pi_T x\|_2 < \hat{x}(\phi)\}} \|\Pi_T x\|_2$$

Corollary 2 *For any $T \in (0,\infty)$:*

$$\|\Pi_T x\|_2 < \frac{1}{\|F\|_2} \Longrightarrow \|\Pi_T y\|_2 = 0$$

Proof of Theorem 1 will require the following three Lemmas:

Lemma 2 *For any $T \in (0,\infty)$:*

$$\|\Pi_T y\|_2 > 0 \Longrightarrow \|\Pi_T x\|_2 \geq \hat{x}\left(\frac{\|\Pi_T x\|_2}{\|\Pi_T y\|_2}\right) \qquad (4)$$

Proof of Lemma 2: We consider the cases $\|\Pi_T u\|_\infty \leq 1$ and $\|\Pi_T u\|_\infty > 1$ separately:

Case 1: if $\|\Pi_T u\|_\infty \leq 1$ then $y(t) = 0$ for all $t \in [0,T]$ and $\|\Pi_T y\|_2 = 0$. Hence this case is not applicable.

Case 2: if $\|\Pi_T u\|_\infty > 1$ then $\|\Pi_T y\|_2 > 0$, and the following inequality holds:

$$\|\Pi_T y\|_2 \leq \left(1 - \frac{1}{\|\Pi_T u\|_\infty}\right) \|\Pi_T u\|_2 \qquad (5)$$

Defining

$$\phi_T := \frac{\|\Pi_T x\|_2}{\|\Pi_T y\|_2}$$

it is then standard that

$$\|\Pi_T u\|_\infty \leq \|F\|_2 \|\Pi_T x\|_2 + \|G\|_2 \|\Pi_T y\|_2 \qquad (6)$$

and also true that

$$\|\Pi_T u\|_2 \leq \Gamma(\phi_T) \|\Pi_T y\|_2 \qquad (7)$$

Substituting Equations 6 and 7 into Equation 5, eliminating $\|\Pi_T y\|_2 \neq 0$, and rearranging, we deduce that

$$\|\Pi_T x\|_2 \geq \frac{1}{\|F\|_2} \frac{\phi_T}{\phi_T + \theta} \frac{\Gamma(\phi_T)}{\Gamma(\phi_T) - 1}$$

Hence the statement is proved. ∎

Remark: The bound in Equation 7 is the smallest lower bound, ie there exist x, y such that $\|\Pi_T(Fx + Gy)\|_2 = \Gamma(\frac{\|\Pi_T x\|_2}{\|\Pi_T y\|_2}) \|\Pi_T y\|_2$ (this was shown in [7].)

Lemma 3 *$\|\Pi_T x\|_2$ and $\|\Pi_T y\|_2$ are continuous non-decreasing functions of T.*

Proof of Lemma 3: For any $v \in \mathcal{L}_2[0,\infty)$, $\|\Pi_T v\|_2$ is non-decreasing in T (this is a basic assumption about Π_T) and $\|\Pi_T v\|_2$ is continuous in T (see [8]). ∎

Lemma 4 *There exists some $T_1 > 0$ such that $\|\Pi_{T_1} x\|_2 > 0$ and $\|\Pi_{T_1} y\|_2 = 0$.*

Proof of Lemma 4: We know that $x \neq 0$ and $\|\Pi_T x\|_2$ is continuous in T, hence for any $\delta \in (0, \|x\|_2$ it is possible to find T_1 such that $\|\Pi_{T_1} x\|_2 = \delta$. Then, if δ is sufficiently small, one solution to the feedback equations in the interval $[0, T_1]$ is $y = 0$, $u = \Pi_{T_1}(Fx)$.

But Lemma 1 states that there are unique solutions in any time interval $[0, T_1]$. Hence this *is* the unique solution, so $\|\Pi_{T_1} x\|_2 = \delta > 0$ and $\|\Pi_{T_1} y\|_2 = 0$. ∎

Proof of Theorem 1: Assume that there exist some $T \in (0,\infty)$ and $\phi \in (0,\infty)$ such that $\|\Pi_T x\|_2 < \hat{x}(\phi)$ and $\|\Pi_T y\|_2 \geq \frac{1}{\phi}\|\Pi_T x\|_2$.

Then by considering Lemmas 3 and 4, there exists some $T_\phi \in (0,T]$ such that $\left\|\Pi_{T_\phi} y\right\|_2 = \frac{1}{\phi}\left\|\Pi_{T_\phi} x\right\|_2$ and $\left\|\Pi_{T_\phi} y\right\|_2 > 0$.

But Lemma 2 states (noting that $\left\|\Pi_{T_\phi} y\right\|_2 > 0$)

$$\begin{aligned}\left\|\Pi_{T_\phi} x\right\|_2 &\geq \hat{x}\left(\frac{\left\|\Pi_{T_\phi} x\right\|_2}{\left\|\Pi_{T_\phi} y\right\|_2}\right)\\ &= \hat{x}(\phi)\end{aligned}$$

which contradicts the assumption above. ∎

Proof of Corollary 1: Immediate from Theorem 1. ∎

Proof of Corollary 2: Noting that $\lim_{\phi\to\infty}\hat{x}(\phi) = \frac{1}{\|F\|_2}$, this result is immediate from Corollary 1. ∎

6.1 Continuation to infinite time

Hence by considering $T \to \infty$, we get the following Corollary:

Corollary 3 *1. If we define*

$$X_0 := \frac{1}{\|F\|_2}$$

then $\|x\|_2 < X_0$ implies

$$\begin{aligned} y &= 0\\ \|u\|_2 &\leq \|F\|_\infty \|x\|_2\end{aligned}$$

2. If $\sup_{\phi\in(0,\infty)}\{\hat{x}(\phi)\} > \frac{1}{\|F\|_2}$, and we define

$$\begin{aligned}\phi_1 &:= \arg\sup_{\phi\in(0,\infty)}\{\hat{x}(\phi)\}\\ X_1 &:= \hat{x}(\phi_1)\\ Y_1 &:= \frac{1}{\phi_1}X_1\\ U_1 &:= \|F\|_\infty X_1 + \|G\|_\infty Y_1\end{aligned}$$

then $\|x\|_2 < X_1$ implies

$$\begin{aligned}\|y\|_2 &< Y_1\\ \|u\|_2 &< U_1\end{aligned}$$

3. For any $X \in (X_0, X_1)$, if we define

$$\begin{aligned}Y(X) &:= \frac{1}{\sup\{\phi : X < \hat{x}(\phi)\}}X\\ U(X) &:= \|F\|_\infty X + \|G\|_\infty Y(X)\end{aligned}$$

then $\|x\|_2 < X$ implies

$$\begin{aligned}\|y\|_2 &< Y(X)\\ \|u\|_2 &< U(X)\end{aligned}$$

Proof of Corollary 3: These results follow immediately from Corollaries 1 and 2, and from the simple norm inequality $\|u\|_2 \leq \|F\|_\infty \|x\|_2 + \|G\|_\infty \|y\|_2$. ∎

It remains only to give conditions such that $\sup_{\phi\in(0,\infty)}\{\hat{x}(\phi)\} > \frac{1}{\|F\|_2}$:

Theorem 2 *Define $\mu := \|G\|_\infty$, $\rho := \frac{\|G\|_2\|F\|_\infty}{\|F\|_2}$ and*

$$\begin{aligned}\Psi(\lambda) :=&(1-\rho^2)\lambda^6 + 4\rho\lambda^5 + \rho^2(4-3\mu^2)\lambda^4\\ &+ 2\rho\mu^2\lambda^3 + \rho^2\mu^2(4-3\mu^2)\lambda^2\\ &- \rho^2\mu^4(\mu^2-1)\end{aligned}$$

Then

1. *$\rho < 1$ implies*

$$\begin{aligned}\sup_{\phi\in(0,\infty)}\{\hat{x}(\phi)\} &> \frac{1}{\|F\|_2}\\ X_1 &\geq \frac{1}{\|F\|_2}\left(\frac{\sqrt{\rho(\mu-1)}+\sqrt{\mu-\rho}}{\sqrt{\rho(\mu-\rho)}+\sqrt{\mu-1}}\right)^2\end{aligned}$$

2. *$\sup_{\phi\in(0,\infty)}\{\hat{x}(\phi)\} > \frac{1}{\|F\|_2}$ implies*

$$\begin{aligned}\rho &\leq 1\\ X_1 &\leq \frac{1}{\|F\|_2}\frac{\lambda_1}{\lambda_1+\rho}\frac{\sqrt{\lambda_1^2+\mu^2}}{\sqrt{\lambda_1^2+\mu^2}-1}\end{aligned}$$

where λ_1 is any positive real solution to $\Psi(\lambda) = 0$.

Remarks:

1. For any $\rho \in (0,1)$ and any $\mu \in (1,\infty)$, there is at least one positive real solution to $\Psi(\lambda) = 0$, since $\Psi(0) < 0$, $\Psi(\lambda) > 0$ for sufficiently large λ, and $\Psi(\lambda)$ is continuous.
2. We believe, based on numerical results, that for any $\rho \in (0,1)$ and any $\mu \in (1,\infty)$ there is precisely one positive real solution to $\Psi(\lambda) = 0$, and hence that λ_1 is uniquely defined.
3. Note that in addition to proving the *sufficient* condition stated in Equation 3, this Theorem also gives a *necessary* condition for $\sup_{\phi\in(0,\infty)}\{\hat{x}(\phi)\} > \frac{1}{\|F\|_2}$.

Proof of Theorem 2:

1. By considering the approximation

$$\left\|\left[\tfrac{1}{\alpha}F \quad G\right]\right\|_\infty \leq \sqrt{\frac{1}{\alpha^2}\|F\|_\infty^2 + \|G\|_\infty^2}$$

it follows that

$$\Gamma(\phi) \leq \|F\|_\infty \phi + \|G\|_\infty$$

and hence, since $\hat{x}(\phi)$ is monotonic in $\Gamma(\phi)$, it is straightforward to verify the stated lower bound on X_1, and the conditions under which this bound is larger than $\frac{1}{\|F\|_2}$.

2. By considering the approximation

$$\left\| \begin{bmatrix} \frac{1}{\alpha}F & G \end{bmatrix} \right\|_\infty \geq \max\left\{ \frac{1}{\alpha} \|F\|_\infty, \|G\|_\infty \right\}$$

it follows that

$$\Gamma(\phi) \geq \sqrt{\|F\|_\infty^2 \phi^2 + \|G\|_\infty^2}$$

and hence, since $\hat{x}(\phi)$ is monotonic in $\Gamma(\phi)$, it is straightforward to verify the stated upper bound on X_1, and the conditions under which this bound is not larger than $\frac{1}{\|F\|_2}$.

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7 Application to SISO systems with saturation nonlinearity

In [3], a system comprising a plant P and controller C with a saturation nonlinearity at the controller output was considered. It was shown that if the plant and controller are given as $P(s) = N(s)M^{-1}(s)$ and $C(s) = \tilde{V}_0^{-1}(s)\tilde{U}_0(s)$ such that $M(s)M^*(s) + N(s)N^*(s) = I$ and $M(s)\tilde{V}_0(s) + N(s)\tilde{U}_0(s) = I$, then all coprime factorisations of that same controller can be parameterised as $C(s) = \tilde{V}^{-1}(s)\tilde{U}(s)$ with $\tilde{V}(s) = Q(s)\tilde{V}_0(s)$ and $\tilde{U}(s) = Q(s)\tilde{U}_0(s)$ where $Q, Q^{-1} \in \mathcal{RH}_\infty$.

Noting that $\text{Sat}(u) \equiv (1 - \Delta)(u)$, and assuming that the saturated signal is available, then this set-up can be implemented as in Figure 4. Note that this implementation will give the same response in unsaturated operation as the original system.

It may then be seen that Figure 4 can be redrawn in the form of Figure 1, in which case F and G are given by

$$F(s) = M(s)\tilde{U}_0(s)$$
$$G(s) = I - M(s)Q^{-1}(s)$$

It was proved in [3] that for unstable plants P there is no $Q, Q^{-1} \in \mathcal{H}_\infty$ such that $\left\|I - MQ^{-1}\right\|_\infty < 1$, and hence that it is not possible to guarantee stability in this case. However, in certain cases, the results of this work can be used to show stability for suitably small inputs x.

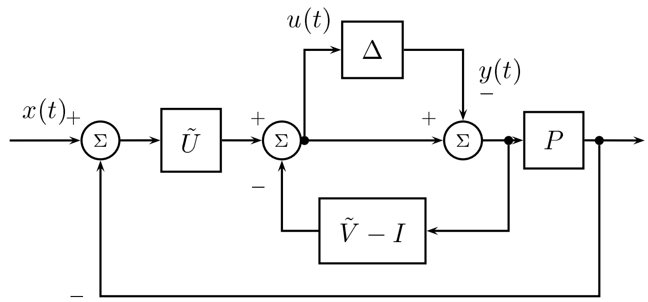

Figure 4: Anti-windup control problem

Example
Let

$$P(s) = \frac{1}{s-1} = \left(\frac{1}{s+\sqrt{2}}\right)\left(\frac{s-1}{s+\sqrt{2}}\right)^{-1}$$

$$C(s) = \frac{21}{s+11} = \left(\frac{(s+11)(s+\sqrt{2})}{s^2+10s+10}\right)^{-1}\left(\frac{21(s+\sqrt{2})}{s^2+10s+10}\right)$$

$$Q(s) = \frac{s+a}{s+b}$$

with $a, b > 0$. For this system, F and G are given by

$$F(s) = \frac{21(s-1)}{s^2+10s+10} \quad \text{independent of } a, b$$

$$G(s) = \frac{(a+\sqrt{2}+1-b)s + (b+a\sqrt{2})}{(s+\sqrt{2})(s+a)}$$

and hence $X_0 = 0.203$, independently of a and b. This means that for any x such that $\|x\|_2 < 0.0203$, $y = 0$.

Then we plot X_1 as a function of a and b: this is shown as a contour plot in Figure 5. Picking the best a and b from this data ($a = 1.8197$, $b = 1.2023$) gives $X_1 = 0.2157$, which means that for any x such that $\|x\|_2 < 0.02157$, $\|y\|_2$ is bounded. Note that the lower bound on X_1 from Theorem 2 is 0.2128, which is not a bad approximation.

Figure 3 shows $Y(X)$ and $U(X)$ for this F and G; the plot also shows X_0, X_1, Y_1 and U_1.

8 Further applications

Many of the results given are not limited to the case studied ($\|G\|_\infty > 1$, $\|G\|_2 < \frac{\|F\|_2}{\|F\|_\infty}$); in particular, Theorem 1 and its Corollaries can be used in the case

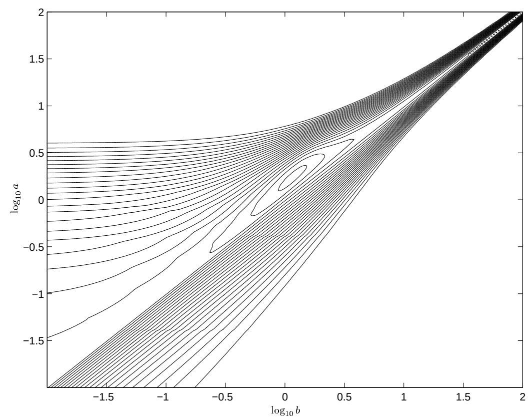

Figure 5: Contour plot of X_1 vs a and b

$\|G\|_\infty < 1$ (and sometimes also in the case $\|G\|_\infty = 1$) to derive bounds from $\|x\|_2$ to $\|y\|_2$.

9 Conclusions

In this work we have shown that systems which fail the Small-Gain test for global stability are nevertheless locally stable to small-energy disturbances, providing the condition in Equation 3 is satisfied. Such systems then exhibit a "graceful degradation" in performance for small excursions into the saturated regime.

The applicability of this analysis to a system comprising an unstable linear plant, stable linear controller and input saturation has been demonstrated.

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