Anti-windup synthesis for guaranteed $\mathcal{L}_2$ performance

S. Crawshaw and G. Vinnicombe
sc10003@eng.cam.ac.uk, gv@eng.cam.ac.uk
Department of Engineering, University of Cambridge
Cambridge, CB2 1PZ, United Kingdom

Abstract

We consider the problem of anti-windup compensation based on coprime factorizations, for asymptotically stable plants with an input saturation nonlinearity. Firstly, we state a stability criterion for the nonlinear system which guarantees finite-gain $\mathcal{L}_2$ stability. Then we propose an $\mathcal{H}_\infty$-based synthesis method which is guaranteed *a priori* to stabilize the system with a guaranteed level of $\mathcal{L}_2$ performance.

1 Introduction

In this paper we extend the coprime factor based anti-windup synthesis method of [2], which was in turn built on the framework of [1]. The plant is assumed to be linear and time-invariant, apart from a saturation at the input, and a linear time-invariant controller is also assumed to be given. Together these define the required closed-loop behaviour in the unsaturated regime. Compensation is then added to the controller. This compensation only acts when the plant input saturates, and its purpose is to lessen the effects of this saturation. The framework is an extension of that of Kothare *et al* [1] which employed a parameterization of all coprime factors of the linear controller which have the same McMillan degree as that controller. This was extended in [2] to a parameterization of *all* coprime factors of the controller and, crucially, the problem of choosing a suitable coprime factorization of the controller was then seen to be equivalent to that of choosing a suitable coprime factorization of the plant. An $\mathcal{H}_\infty$ optimization problem was defined in terms of coprime factors of the plant. This problem is guaranteed to always have a solution (if the plant is stable) and the resulting anti-windup compensation scheme is guaranteed to be globally stabilizing and to have a prescribed level of $\mathcal{L}_2$ performance (in a certain sense). An alternative synthesis technique, which is also guaranteed to always lead to a globally stabilizing solution, was presented by Teel and Kapoor [5]. Their scheme can also be interpreted as a choice of plant coprime factorization, although nonlinear factorizations of a linear plant are also considered. This paper also introduced a performance criterion relating the difference between the real behaviour and the idealized behaviour without saturation to the amount the idealized plant input exceeds the saturation level. The present paper makes two contributions. First the stability condition, which in [2] was based on small gain, is relaxed to one based on the multivariable circle criterion without fundamentally changing the underlying $\mathcal{H}_\infty$ optimization – potentially allowing performance to be further improved. Secondly, we show that the solution to this modified synthesis problem satisfies a bound on the performance index of [5].

From a practical point of view, a major advantage of the proposed synthesis technique is that it is particularly well suited to rapid prototyping. Once weights have been chosen which result in good performance with one controller it is then a simple matter to recalculate the compensation after controller redesign – the reason being that the optimization, and indeed the anti-windup performance, is defined solely in terms of the plant and its coprime factors.

2 Preliminaries

For scalars we define the standard saturation and deadzone functions $\mathrm{Sat}_a : \mathbb{R} \to \mathbb{R}$ and $\mathrm{Dzn}_a : \mathbb{R} \to$

$\mathbb{R}$ $(a > 0)$ as

$$\mathrm{Sat}_a(v) := \begin{cases} a & \text{if } v > a \\ v & \text{if } |v| \le a \\ -a & \text{if } v < -a \end{cases} \tag{1}$$

$$\mathrm{Dzn}_a(v) := v - \mathrm{Sat}_a(v) \tag{2}$$

and if $a = 1$ we shall drop the subscript.

Given a diagonal matrix $A = \mathrm{diag}\{a_1, a_2, \cdots, a_n\}$ with $a_i > 0$ for each $i \in \{1, 2, \cdots, n\}$, we denote by $\mathbf{Sat}_A : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{Dzn}_A : \mathbb{R}^n \to \mathbb{R}^n$ the component-wise multivariable saturation and deadzone functions, i.e.

$$\mathbf{Sat}_A(\boldsymbol{v}) := \begin{bmatrix} \mathrm{Sat}_{a_1}(v_1) \\ \mathrm{Sat}_{a_2}(v_2) \\ \vdots \\ \mathrm{Sat}_{a_n}(v_n) \end{bmatrix} \tag{3}$$

$$\mathbf{Dzn}_A(\boldsymbol{v}) := \begin{bmatrix} \mathrm{Dzn}_{a_1}(v_1) \\ \mathrm{Dzn}_{a_2}(v_2) \\ \vdots \\ \mathrm{Dzn}_{a_n}(v_n) \end{bmatrix} \tag{4}$$

and if $A = I$ we shall drop the subscript. Note that for any $v \in \mathbb{R}$ and $a > 0$

$$\mathrm{Dzn}_a(av) = a\,\mathrm{Dzn}(v) \tag{5}$$

and hence for any $\boldsymbol{v} \in \mathbb{R}^n$ and A as above

$$\mathbf{Dzn}_A(A\boldsymbol{v}) = A\mathbf{Dzn}(\boldsymbol{v}) \tag{6}$$

Lemma 1

For any $a > 0$ and any $x, y \in \mathbb{R}$

$$|\mathrm{Dzn}_a(x + y)| \le |\mathrm{Dzn}_a(x)| + |y|$$

Furthermore, for any $v, w \in \mathcal{L}_2$

$$\|\mathrm{Dzn}_a(v + w)\|_2 \le \|\mathrm{Dzn}_a(v)\|_2 + \|w\|_2$$

and hence for any $\boldsymbol{v}, \boldsymbol{w} \in \mathcal{L}_2^n$

$$\|\mathbf{Dzn}_A(\boldsymbol{v} + \boldsymbol{w})\|_2 \le \|\mathbf{Dzn}_A(\boldsymbol{v})\|_2 + \|\boldsymbol{w}\|_2$$

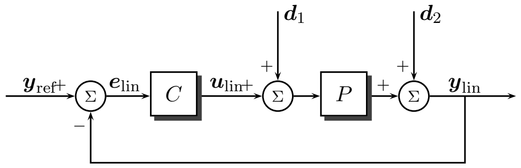

Figure 1: Nominal linear system

3 Problem Definition

Figure 1 shows a nominal tracking problem where the signals $\boldsymbol{y}_{\mathrm{ref}}$, $\boldsymbol{u}_{\mathrm{lin}}$, $\boldsymbol{y}_{\mathrm{lin}}$ and $\boldsymbol{e}_{\mathrm{lin}}$ are the reference signal, controller output, plant output measurement and error signal respectively, and $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are disturbances. The plant and controller are assumed to be finite dimensional, linear, time-invariant systems with transfer functions $P(s)$ and $C(s)$ respectively, and the plant is furthermore assumed to be asymptotically stable and strictly proper.

A significant proportion of the work of the control community over the past decades has been devoted to the analysis and synthesis problems for linear time-invariant systems; we therefore assume that the controller $C(s)$ is specified in advance, having been designed using some (unspecified) method to guarantee some (also unspecified) properties of this linear interconnection. In particular, and as a minimum requirement, we assume that the interconnection is internally stable. If we write the closed-loop relation for this interconnection as

$$\begin{bmatrix} \boldsymbol{e}_{\mathrm{lin}} \\ \boldsymbol{y}_{\mathrm{lin}} \\ \boldsymbol{u}_{\mathrm{lin}} \end{bmatrix} = \begin{bmatrix} S & -SP & -S \\ I - S & SP & S \\ CS & -CSP & -CS \end{bmatrix} \begin{bmatrix} \boldsymbol{y}_{\mathrm{ref}} \\ \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \end{bmatrix} \tag{7}$$

where $S := (I + PC)^{-1}$ is the sensitivity function, then internal stability is equivalent to each element in this transfer matrix being stable.

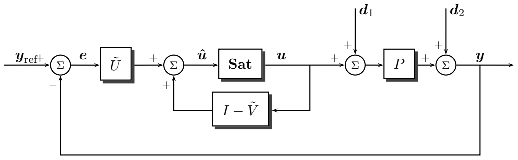

Figure 2: System with saturation and anti-windup compensator

The same basic system with input saturation and a linear anti-windup compensator is shown in Figure 2, where $\hat{\boldsymbol{u}}$ and $\boldsymbol{u}$ are the input and output of the saturation nonlinearity. Identical external signals $\boldsymbol{y}_{\mathrm{ref}}, \boldsymbol{d}_1, \boldsymbol{d}_2$ are assumed, and unity saturation is assumed, without loss of generality.

Remark 1 *Note that this is an idealized picture for the purpose of analysis. In practice, the saturation element shown would be implemented as part of the controller – thus guaranteeing that the saturation at the plant input itself never activates. As a result, the signal d_1 can only represent disturbances at the plant input which enter independently of the saturating actuators; it cannot be used to represent noise on the controller output.*

A general framework, including almost all existing linear anti-windup schemes, has been proposed in [1], where it was suggested that the controller output should be given by

$$\hat{\boldsymbol{u}} = \tilde{U}\boldsymbol{e} + (I - \tilde{V})\boldsymbol{u}$$

where $C = \tilde{V}^{-1}\tilde{U}$ is a left-coprime factorization of C. In [1] the anti-windup controller was parameterized by all *fixed-order* left-coprime factors of C; it was later proposed in [2] to use *all* left-coprime factors of C, which can be parameterized by

$$\tilde{V} = Q\tilde{V}_0$$
$$\tilde{U} = Q\tilde{U}_0$$

with $Q, Q^{-1} \in \mathcal{RH}_\infty$, where $C = \tilde{V}_0^{-1}\tilde{U}_0$ is an arbitrary left-coprime factorization of C. Denote by $\mathcal{Q}$ the set of units in $\mathcal{RH}_\infty$, i.e.

$$\mathcal{Q} := \left\{ Q : Q, Q^{-1} \in \mathcal{RH}_\infty \right\} \tag{8}$$

We choose initial left- and right-coprime factorizations of the controller $C = \tilde{V}_0^{-1}\tilde{U}_0$ and the plant $P = N_0 M_0^{-1}$ such that they satisfy the Bezout identity $\tilde{V}_0 M_0 + \tilde{U}_0 N_0 = I$.

4 Stability analysis

By considering the interconnection of Figure 2 as a LFT on **Dzn** ([3], [7], it may be seen that stability of Figure 2 is equivalent to stability of Figure 3.

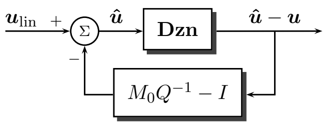

Figure 3: Subsystem for stability analysis

$\boldsymbol{e}, \hat{\boldsymbol{u}}, \boldsymbol{u}$ and $\boldsymbol{y}$ in Figure 2 can then be given in terms of the nominal linear responses of Figure 1 and the output of the deadzone nonlinearity in Figure 3:

$$\boldsymbol{e} - \boldsymbol{e}_{\mathrm{lin}} = N_0 Q^{-1}(\hat{\boldsymbol{u}} - \boldsymbol{u}) \tag{9}$$
$$\hat{\boldsymbol{u}} - \boldsymbol{u}_{\mathrm{lin}} = (I - M_0 Q^{-1})(\hat{\boldsymbol{u}} - \boldsymbol{u}) \tag{10}$$
$$\boldsymbol{u} - \boldsymbol{u}_{\mathrm{lin}} = -M_0 Q^{-1}(\hat{\boldsymbol{u}} - \boldsymbol{u}) \tag{11}$$
$$\boldsymbol{y} - \boldsymbol{y}_{\mathrm{lin}} = -N_0 Q^{-1}(\hat{\boldsymbol{u}} - \boldsymbol{u}) \tag{12}$$

For well-posedness of the interconnection in Figure 3, it is desirable that $M_0 Q^{-1}(\infty) = I$; by a happy coincidence, and under the mild assumption that P is strictly proper, this is equivalent to $Q\tilde{V}_0(\infty) = I$, which guarantees that there is no algebraic loop in the interconnection of Figure 2.

Theorem 1

The interconnection in Figure 2 is stable if $\tilde{U} = Q\tilde{U}_0$ and $\tilde{V} = Q\tilde{V}_0$ for some $Q \in \mathcal{Q}$ such that $M_0 Q^{-1}(\infty) = I$, and there exists a diagonal matrix $A = diag\{a_1, a_2, \cdots, a_n\}$ with $a_i \in (0, 2)$ for each $i \in \{1, 2, \cdots, n\}$, such that

$$\left\| A M_0 Q^{-1} - I \right\|_\infty < 1$$

Furthermore, provided this condition is satisfied, $\|\hat{\boldsymbol{u}} - \boldsymbol{u}\|_2$ is bounded by

$$\|\hat{\boldsymbol{u}} - \boldsymbol{u}\|_2 \leq \frac{\|A\|}{1 - \|A M_0 Q^{-1} - I\|_\infty} \|\mathbf{Dzn}(\boldsymbol{u}_{lin})\|_2$$

PROOF OF THEOREM 1:

Consider the interconnection in Figure 4, which is equivalent to that in Figure 3. The dotted lines outline two new operators; it is simple to see that the linear operator is now given by $AM_0Q^{-1} - I$, and to check that the nonlinear operator is now given by $\mathbf{Dzn}_A$.

The stability condition then follows immediately by the small gain theorem ([8]), noting that the

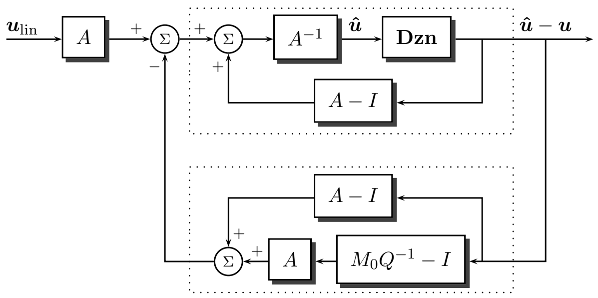

Figure 4: Equivalent representation of Figure 3

new interconnection is well-posed for any such A ($\left\|(AM_0Q^{-1}-I)(\infty)\right\| = \|A-I\| < 1$ and $\mathbf{Dzn}_A$ has uniform instantaneous gain of unity.)

By Lemma 1 we can then deduce that

$$\begin{aligned}\|\hat{\boldsymbol{u}}-\boldsymbol{u}\|_2 &\leq \|\mathbf{Dzn}_A(A\boldsymbol{u}_{\text{lin}})\|_2 \\ &\quad + \left\|AM_0Q^{-1}-I\right\|_\infty \|\hat{\boldsymbol{u}}-\boldsymbol{u}\|_2 \\ &\leq \frac{\|\mathbf{Dzn}_A(A\boldsymbol{u}_{\text{lin}})\|_2}{1-\|AM_0Q^{-1}-I\|_\infty}\end{aligned}$$

provided the stability condition is satisfied. But

$$\mathbf{Dzn}_A(A\boldsymbol{u}_{\text{lin}}) = A\mathbf{Dzn}(\boldsymbol{u}_{\text{lin}})$$

and so the bound follows immediately. ∎

In the scalar case, Theorem 1 can be interpreted in terms of the Nyquist plot of $\left\{M_0Q^{-1}-I\right\}$. For any $a \in (0,2)$ the stated condition is equivalent to requiring that $\left\{M_0Q^{-1}-I\right\}(j\omega)$ remains in the circle with diameter on $(-1, \frac{2}{a}-1)$ for all $\omega \in \mathbb{R} \cup \infty$. It is easily verified that this corresponds to the well-known scalar circle criterion for a nonlinearity in sector $[-\frac{a}{2-a}, 1]$.

Furthermore, we see that in the limit as $a \to 0$, this circle approximates the entire half-plane to the right of $-1+0j$, corresponding to the scalar circle criterion for sector $[0,1]$.

In fact, it may be verified easily that the stability condition of Theorem 1 is applicable when the deadzone nonlinearity of Figure 3 is replaced with *any* diagonal nonlinearity in sector $[0,1]$ (although the performance inequality uses special features of the deadzone).

5 Synthesis for performance

In [2] it is suggested that a suitable performance criterion is to minimize the $\mathcal{L}_2$ gain from the perturbation signal $\hat{\boldsymbol{u}}-\boldsymbol{u}$ to the difference between the actual and nominal outputs $\boldsymbol{y}-\boldsymbol{y}_{\text{lin}}$, weighted by some suitable W.

In [5] it is suggested that a suitable performance criterion is to minimize the $\mathcal{L}_2$ gain from $\mathbf{Dzn}(\boldsymbol{u}_{\text{lin}})$ to the difference between any actual signal and its corresponding nominal signal, such as $\boldsymbol{u}-\boldsymbol{u}_{\text{lin}}$ or $\boldsymbol{y}-\boldsymbol{y}_{\text{lin}}$. A simple upper bound on the $\mathcal{L}_2$ gain from $\mathbf{Dzn}(\boldsymbol{u}_{\text{lin}})$ to $\boldsymbol{u}-\boldsymbol{u}_{\text{lin}}$, for interconnections satisfying the (multivariable) Circle Criterion, was recently given in [4].

In the light of Theorem 1 and Equations 9 to 12, we claim that the synthesis result which was given in [2], and which we state below as Proposition 1, is applicable to both of these performance criteria. First, though, the following remark shows that we must be careful when interpreting these bounds.

Remark 2 *Note that* no *anti-windup scheme can possibly achieve better than*

$$\|\boldsymbol{u}-\boldsymbol{u}_{lin}\|_2 \leq \|\mathbf{Dzn}(\boldsymbol{u}_{lin})\|_2 \tag{13}$$

and

$$\|\boldsymbol{y}-\boldsymbol{y}_{lin}\|_2 \leq \|P\|_\infty \|\mathbf{Dzn}(\boldsymbol{u}_{lin})\|_2 \tag{14}$$

over all $\mathcal{L}_2$ signals $\boldsymbol{u}_{lin}$. (To see this, for example, let $u_o \in \mathcal{L}_2$ be a signal such that $\|Pu_0\|_2 > (\|P\|_\infty - \delta)\|u_0\|_2$ for some suitably small δ and let $\boldsymbol{u}_{lin,k} = (1/\epsilon)\, u_0$ for some suitably small ϵ.) One scheme which achieves these bounds is the unmodified IMC scheme (see [9]), which sets $Q = M_0$; a choice which can be seen to cut the nonlinear loop in Figure 3. However, it has been pointed out by a number of authors (eg [7]) that this choice may lead to poor performance if P has lightly damped or slow modes. The challenge is to to find schemes which provide better performance locally whilst achieving, or coming close to, these lower bounds globally – and to do so robustly. We claim that the following synthesis scheme can be effective in this light; it always gives global bounds, which are often close to (13) and (14), and has a weight which can be chosen to enhance the local performance. A remaining challenge is to find the best way of capturing *this enhanced local performance rigorously. The robustness of the original scheme is demonstrated in [3], and the conclusions remain valid here.*

Proposition 1 ([2],[3])

For an asymptotically stable plant P and any weight W such that $W, W^{-1} \in \mathcal{RH}_\infty$, there always exists $Q \in \mathcal{Q}$ such that

$$\left\| \begin{bmatrix} WN_0Q^{-1} \\ M_0Q^{-1} - I \end{bmatrix} \right\|_\infty < 1 \tag{15}$$

and $M_0Q^{-1}(\infty) = I$.

Furthermore, any $Q^{-1} \in \mathcal{RH}_\infty$ satisfying (15) will also satisfy $Q \in \mathcal{RH}_\infty$.

Note that M_0Q^{-1}, which is not directly optimized in Proposition 1, may be bounded by $\left\|M_0Q^{-1} - I\right\|_\infty + 1$, and hence $\mathcal{L}_2$ performance to *all* of the signals $\boldsymbol{y} - \boldsymbol{y}_{\text{lin}}$ etc is guaranteed by this result.

We extend the synthesis method by utilizing Theorem 1 for the stability test and performance bounds:

Theorem 2

For an asymptotically stable plant P, any weight W such that $W, W^{-1} \in \mathcal{RH}_\infty$, and any diagonal matrix $A = diag\{a_1, a_2, \cdots, a_n\}$ with $a_i \in (0, 2)$ for each $i \in \{1, 2, \cdots, n\}$, there always exists $Q \in \mathcal{Q}$ such that

$$\left\| \begin{bmatrix} WN_0Q^{-1} \\ AM_0Q^{-1} - I \end{bmatrix} \right\|_\infty < 1 \tag{16}$$

and $M_0Q^{-1}(\infty) = I$.

Furthermore, any $Q^{-1} \in \mathcal{RH}_\infty$ satisfying (16) will also satisfy $Q \in \mathcal{RH}_\infty$.

PROOF OF THEOREM 2:

By a change of variable $\hat{Q} = AM_0Q^{-1}$, we consider the unconstrained problem

$$\inf_{\hat{Q} \in \mathcal{Q}} \left\| \begin{bmatrix} WPA^{-1}\hat{Q} \\ \hat{Q} - I \end{bmatrix} \right\|_\infty$$

which has infimum given by ([2])

$$\frac{\left\|WPA^{-1}\right\|_\infty}{\sqrt{1 + \|WPA^{-1}\|_\infty^2}} < 1$$

We then conclude that the full problem has infimum

$$\max \left\{ \frac{\left\|WPA^{-1}\right\|_\infty}{\sqrt{1 + \|WPA^{-1}\|_\infty^2}}, \|A - I\| \right\} < 1$$

where the second term is due to the interpolation constraint $M_0Q^{-1}(\infty) = I$, which does not otherwise put up the achievable norm (see eg [6] Theorem 1.29) ∎

6 Conclusions

We have given a stability criterion, for systems with input saturation and coprime factor anti-windup compensation, which guarantees finite-gain $\mathcal{L}_2$ stability. Secondly, we have given a synthesis method which is guaranteed *a priori* to stabilize the system with a guaranteed level of $\mathcal{L}_2$ performance.

Further investigation is required into the effect of varying A in Theorems 1 and 2, and into the most appropriate choice of both W and A.

A Proof of Lemma 1

PROOF OF LEMMA 1:

The deadzone function has unity Lipschitz constant, ie

$$|\mathrm{Dzn}_a(x + y) - \mathrm{Dzn}_a(x)| \leq |y|$$

from which the first statement follows by application of the triangle inequality:

$$\begin{aligned} |\mathrm{Dzn}_a(x + y)| \leq {} & |\mathrm{Dzn}_a(x + y) - \mathrm{Dzn}_a(x)| \\ & + |\mathrm{Dzn}_a(x)| \end{aligned}$$

The second statement then follows immediately. ∎

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