Necessity of Nonlinear Control for Robust Rejection of Periodic Disturbances

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Abstract

Consider a linear plant with a strictly proper rational transfer function and an input which is a known periodic waveform of unknown amplitude and phase. In the case when the periodic waveform has infinitely many non-zero harmonics, it is pointed out that a *linear* controller cannot robustly stabilize the plant and achieve asymptotic rejection of the disturbance. For a specific linear plant and a triangular wave disturbance, it is shown by construction that the problem is soluble with a *nonlinear* controller. Robustness is measured using the gap metric robust stability margin.

Keywords: Repetitive control; Robustness; Asymptotic disturbance rejection; Internal model principle; Nonlinear control; Triangular waveforms.

1 Introduction

The aim of repetitive control is to achieve tracking/rejection of periodic references/disturbances. The Internal Model Principle [2] suggests that the dynamic structure of the reference/disturbance signals has to be incorporated in the linear controller. This means that when a linear controller is used, it needs to have poles (possibly infinitely many) at the harmonics of the periodic signals. Inoue *et al.* [7, 8] implemented this idea using a positive feedback around a time delay as an internal model for a general periodic signal with known fundamental period. It was pointed out by a number of authors that this scheme of complete tracking/rejection of periodic signals lacks robustness [5, 6]. A similar conclusion making use of the gap metric robustness measure for linear systems was drawn in [9, 10], namely, asymptotic rejection of infinitely many harmonics in the case that the plant rolls off at high frequency implies a zero robustness margin. In fact it can be seen that any such control loop would be destabilized by an arbitrarily small time delay.

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The linear control approach to asymptotic rejection of a periodic waveform with multiple (possibly infinitely many) harmonics necessitates the inclusion of poles in the controller at all frequencies which are harmonics of the periodic signal. Such a linear controller would seek asymptotic rejection for *any* periodic signal with the same set of harmonics. Therefore, if it is known that the disturbance is a specific periodic waveform (e.g. a triangular wave) then such a controller would not exploit this fact. The question arises whether robust asymptotic rejection is possible with nonlinear control for a specific type of periodic waveform. The purpose of this paper is to show by construction that the answer is in the affirmative, at least for a specific class of plants and triangular wave disturbances. In effect, we illustrate an extension of the Internal Model Principle to a case which necessitates nonlinear control.

The problem will be stated precisely in Section 2. Some preliminary background on robustness will be mentioned in Section 3. The main result is the construction of a nonlinear controller, with the considerations of asymptotic rejection and robust stability, which will be described in Section 4. A simple numerical example with simulations (Section 5) and conclusions (Section 6) are also given.

2 Problem Statement

We will consider the class of triangular waves of known fundamental period but unknown amplitude and phase. Denote the standard triangular wave of frequency 1 rad/sec with unit amplitude and zero phase and support on $[0, \infty)$:

$$f_0(t) = \begin{cases} \frac{2}{\pi}(t - 2k\pi), & 2k\pi \le t < 2(k + 1/4)\pi, \\ 2 - \frac{2}{\pi}(t - 2k\pi), & 2(k + 1/4)\pi \le t < 2(k + 3/4)\pi, \\ \frac{2}{\pi}(t - 2k\pi) - 4, & 2(k + 3/4)\pi \le t < 2(k + 1)\pi, \end{cases}$$

where k is any non-negative integer. Any triangular wave $f(t)$ of fundamental frequency ω_0 rad/sec, amplitude A and phase ϕ can be written as $f(t) = A\, f_0(\omega_0 t + \phi)$. These waves are piecewise C^1 on $[0, \infty)$ and are equal to their Fourier series (having infinitely many non-zero Fourier coefficients). Denote by $\mathcal{T}(\omega_0)$ the set of such triangular waves, i.e.

$$\mathcal{T}(\omega_0) := \{A\, f_0(\omega_0 t + \phi) : A, \phi > 0\}\,.$$

We will restrict attention to scalar plants and controllers in the standard feedback configuration (Figure 1). We choose to formulate our repetitive control problem as the asymptotic distur-

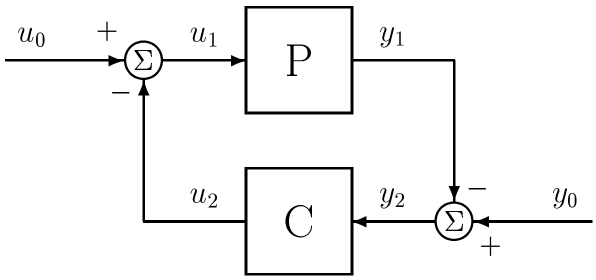

Figure 1: Standard Feedback Configuration.

bance rejection of triangular waves $u_0 \in \mathcal{T}(\omega_0)$ at the plant output y_1, namely $\lim_{t\to\infty} y_1(t) = 0$ when $u_0 \in \mathcal{T}(\omega_0)$ and $y_0 = 0$. We will require that the feedback loop, denoted by $[P, C]$, is robust in the sense of having a non-zero gap metric robustness margin $b_{P,C}$, which is defined as follows for linear P and C [3, 1]:

$$b_{P,C} := \left\| \begin{pmatrix} I \\ P \end{pmatrix} (I - CP)^{-1}(I, -C) \right\|_{\mathcal{H}_\infty}^{-1}. \tag{1}$$

Note that $b_{P,C}$ is the inverse of the $\mathcal{L}_2$-induced norm of the mapping from (u_0, y_0) to (u_1, y_1) in Figure 1. Formally we define our control problem as follows.

Problem 2.1. *Given a fundamental frequency ω_0 and a rational, scalar, strictly proper plant P with $P(j\omega_0) \neq 0$, find a controller C with $b_{P,C} > 0$ to achieve robust asymptotic rejection at y_1 for any $u_0 \in \mathcal{T}(\omega_0)$.*

We are able to show that Problem 2.1 is not soluble with a linear C, assuming that C belongs to a certain class of linear systems which includes all finite dimensional and some infinite dimensional ones. We first note that the conclusion is trivial if C is finite dimensional. This follows since any element of $\mathcal{T}(\omega_0)$ contains infinitely many non-zero harmonics and any rational function has finitely many zeros, so asymptotic disturbance rejection requirement cannot even be achieved. We therefore need to admit controllers which are infinite dimensional. Let us assume that C is chosen so that $G := P(I - CP)^{-1} \in \mathcal{H}_\infty^\alpha$ for some $\alpha < 0$, where $\mathcal{H}_\infty^\alpha$ denotes the space of bounded and analytic functions in the right half plane $\mathrm{Re}(s) > \alpha$. It is possible to show using the techniques of Weiss and Häfele 1997 (see Lemma 9 and Proposition 11 of [11]) that we can decompose the output y_1, where $\hat{y}_1 = G\hat{u}_0$ for $u_0 = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \in \mathcal{T}(\omega_0)$, uniquely as: $y_1 = y_{1,\mathrm{ss}} + y_{1,\mathrm{tr}}$ where $y_{1,\mathrm{ss}} = \sum_{k=-\infty}^{\infty} G(jk\omega_0) a_k e^{jk\omega_0 t}$ and $y_{1,\mathrm{tr}},\ \dot{y}_{1,\mathrm{tr}} \in \mathcal{L}_2[0,\infty)$. The latter implies that $y_{1,\mathrm{tr}} \to 0$ as $t \to \infty$. So the condition that $y_{1,\mathrm{tr}} \to 0$ as $t \to \infty$ requires that $G(jk\omega_0) = 0$ for all (infinitely many) k where $a_k \neq 0$. As shown in [9, 10] this requires that $b_{P,C} < |P(jk\omega_0)|/\sqrt{1 + |P(jk\omega_0)|^2}$ for all k where $a_k \neq 0$. Since P is strictly proper, we cannot have $b_{P,C} > 0$, which shows that Problem 2.1 is not soluble with such a linear C. It seems unlikely that Problem 2.1 is soluble for more general linear time-invariant controllers, so we will turn our attention to nonlinear controllers. In order to show that there is a solution for a nonlinear C, we need to generalize the definition of $b_{P,C}$.

3 Preliminaries

For the notion of robust stability, the nonlinear gap metric robustness margin [4] is adopted. For the sake of completeness, some pertinent definitions are stated in this section.

Plants and controllers will be viewed as causal operators $\mathbf{P} : \mathcal{U} \to \mathcal{Y}$ and $\mathbf{C} : \mathcal{Y} \to \mathcal{U}$ satisfying $\mathbf{P}0 = 0$ and $\mathbf{C}0 = 0$, where $\mathcal{U}$ and $\mathcal{Y}$ are appropriate signal spaces such as a Lebesgue space $\mathcal{L}_\infty[0,\infty)$ or an extended space $\mathcal{L}_{2e}[0,\infty)$. *Graphs* of the plant $\mathbf{P}$ and the controller $\mathbf{C}$ are defined as

$$\mathcal{G}_{\mathbf{P}} := \left\{ \begin{pmatrix} u \\ \mathbf{P}u \end{pmatrix} : u \in \mathcal{U}, \mathbf{P}u \in \mathcal{Y} \right\}$$

and

$$\mathcal{G}_{\mathbf{C}} := \left\{ \begin{pmatrix} \mathbf{C}y \\ y \end{pmatrix} : \mathbf{C}y \in \mathcal{U}, y \in \mathcal{Y} \right\}.$$

Define the mapping $\mathbf{H}_{\mathbf{P},\mathbf{C}}$ as

$$\mathbf{H}_{\mathbf{P},\mathbf{C}} \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} = \left(\begin{pmatrix} u_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \right)$$

and also the *parallel projections*:

$$\Pi_{\mathcal{G}_\mathbf{P}//\mathcal{G}_\mathbf{C}} := \Pi_1 \mathbf{H}_{\mathbf{P},\mathbf{C}} \text{ and } \Pi_{\mathcal{G}_\mathbf{C}//\mathcal{G}_\mathbf{P}} := \Pi_2 \mathbf{H}_{\mathbf{P},\mathbf{C}}$$

where Π_i denotes the natural projection onto the ith component ($i = 1, 2$). These parallel projections are the mappings from the external inputs to the internal signals defining the plant and the controller graph respectively.

Define the *truncation operator* $\mathbf{T}_\tau$ as

$$(\mathbf{T}_\tau f)(t) = f(t) \text{ on } [0, \tau]$$

and zero otherwise. Let $\mathcal{X}$ and $\mathcal{Z}$ be signal spaces. A causal operator $\mathbf{F} : \mathcal{X} \to \mathcal{Z}$ is said to be *stable* if $\mathbf{F}0 = 0$ and

$$\|\mathbf{F}\| := \sup \left\{ \frac{\|\mathbf{T}_\tau(\mathbf{F}(f))\|_\mathcal{Z}}{\|\mathbf{T}_\tau f\|_\mathcal{X}} : f \in \mathcal{X}, \tau > 0, \|\mathbf{T}_\tau f\|_\mathcal{X} \neq 0 \right\} < \infty.$$

Similarly, a causal operator $\mathbf{F} : \mathcal{X} \to \mathcal{Z}$ is said to be *incrementally stable* if $\mathbf{F}0 = 0$ and

$$\begin{aligned} \|\mathbf{F}\|_\triangle \quad := \quad \sup & \left\{ \frac{\|\mathbf{T}_\tau(\mathbf{F}(f_1) - \mathbf{F}(f_2))\|_\mathcal{Z}}{\|\mathbf{T}_\tau(f_1 - f_2)\|_\mathcal{X}} : f_1, f_2 \in \mathcal{X}, \right. \\ & \left. \tau > 0, \|\mathbf{T}_\tau(f_1 - f_2)\|_\mathcal{X} \neq 0 \right\} < \infty. \end{aligned}$$

Let $\mathcal{W}_1$, $\mathcal{W}_2$ be two nonempty subsets of an ambient space. The *directed gap* and the *gap* between $\mathcal{W}_1$ and $\mathcal{W}_2$ are defined as

$$\vec{\delta}(\mathcal{W}_1, \mathcal{W}_2) := \begin{cases} \inf\{ \|(\mathbf{\Phi} - \mathbf{I})\| : \mathbf{\Phi} \text{ is a causal, bijective mapping} \\ \qquad \text{from } \mathcal{W}_1 \text{ onto } \mathcal{W}_2 \text{ with } \mathbf{\Phi}0 = 0\} \\ \infty \text{ if no such operator } \mathbf{\Phi} \text{ exists} \end{cases}$$

and

$$\delta(\mathcal{W}_1, \mathcal{W}_2) := \max \left\{ \vec{\delta}(\mathcal{W}_1, \mathcal{W}_2), \vec{\delta}(\mathcal{W}_2, \mathcal{W}_1) \right\}.$$

Let $\mathbf{P}$, $\mathbf{P}'$ be causal operators from $\mathcal{U}$ to $\mathcal{Y}$ and $\mathbf{C}$, $\mathbf{C}'$ be causal operators from $\mathcal{Y}$ to $\mathcal{U}$. We have the following robustness theorem [4].

Lemma 3.1. *Let $\mathbf{H}_{\mathbf{P}',\mathbf{C}'}$ be stable. If*

$$\vec{\delta}(\mathcal{G}_{\mathbf{P}'}, \mathcal{G}_\mathbf{P}) \|\Pi_{\mathcal{G}_{\mathbf{P}'}//\mathcal{G}_{\mathbf{C}'}}\| + \vec{\delta}(\mathcal{G}_{\mathbf{C}'}, \mathcal{G}_\mathbf{C}) \|\Pi_{\mathcal{G}_{\mathbf{C}'}//\mathcal{G}_{\mathbf{P}'}}\| < 1$$

then $\mathbf{H}_{\mathbf{P},\mathbf{C}}$ is stable and

$$\|\Pi_{\mathcal{G}_\mathbf{P}//\mathcal{G}_\mathbf{C}}\| \leq \frac{\|\Pi_{\mathcal{G}_{\mathbf{P}'}//\mathcal{G}_{\mathbf{C}'}}\| (1 + \vec{\delta}(\mathcal{G}_{\mathbf{P}'}, \mathcal{G}_\mathbf{P}))}{1 - \vec{\delta}(\mathcal{G}_{\mathbf{P}'}, \mathcal{G}_\mathbf{P}) \|\Pi_{\mathcal{G}_{\mathbf{P}'}//\mathcal{G}_{\mathbf{C}'}}\| - \vec{\delta}(\mathcal{G}_{\mathbf{C}'}, \mathcal{G}_\mathbf{C}) \|\Pi_{\mathcal{G}_{\mathbf{C}'}//\mathcal{G}_{\mathbf{P}'}}\|}.$$

The quantity $b_{\mathbf{P}',\mathbf{C}'} := \|\Pi_{\mathcal{G}_{\mathbf{P}'}//\mathcal{G}_{\mathbf{C}'}}\|^{-1}$ is called the *gap metric robustness margin* for the closed-loop system $[\mathbf{P}', \mathbf{C}']$. We remark that this definition of $b_{\mathbf{P}',\mathbf{C}'}$ depends on the choice of the underlying signal spaces. If $\mathcal{U} = \mathcal{Y} = \mathcal{L}_2[0, \infty)$ and the plant and controller are linear time-invariant, then this definition coincides with the definition in (1). In the following, we will make the choice $\mathcal{U} = \mathcal{Y} = \mathcal{L}_\infty[0, \infty)$. It should be noted that for linear time-invariant P and C, the $\mathcal{L}_\infty$-induced version of $b_{\mathbf{P},\mathbf{C}}$ is less than or equal to the $\mathcal{L}_2$-induced version, which is itself equal to $b_{P,C}$. Thus Problem 2.1 still has no solution with such a linear controller.

4 Construction

In this section, a causal, memoryless nonlinear controller will be given to achieve the asymptotic rejection of inputs $u_0 \in \mathcal{T}(\omega_0)$ at the output y_1 for some given frequency ω_0 and to achieve $b_{\mathbf{P},\mathbf{C}} > 0$.

Define a linear controller

$$\tilde{\mathbf{C}} := \begin{pmatrix} \tilde{\mathbf{C}}_1 \\ \tilde{\mathbf{C}}_2 \end{pmatrix}$$

with transfer function

$$\tilde{C}(s) = \begin{pmatrix} \tilde{C}_1(s) \\ \tilde{C}_2(s) \end{pmatrix} := \begin{pmatrix} -(ks+\gamma)/(s^2+\omega_0^2) \\ -(\gamma s - k\omega_0^2)/(\omega_0(s^2+\omega_0^2)) \end{pmatrix},$$

where $k, \gamma > 0$. Define also the nonlinear mapping $N : \mathbb{R}^2 \to \mathbb{R}$ as

$$N\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{cases} 0, & \text{if } x_1 = x_2 = 0, \\ \frac{2\sqrt{x_1^2+x_2^2}}{\pi} \sin^{-1}\left(\frac{x_1}{\sqrt{x_1^2+x_2^2}}\right), & \text{otherwise,} \end{cases}$$

where $\sin^{-1}(\cdot)$ takes its principal value in the range $-\pi/2$ to $\pi/2$. We will use the notation $\mathbf{N}$ to denote the operator acting on $\mathcal{L}_\infty[0,\infty) \times \mathcal{L}_\infty[0,\infty)$ and defined by applying N pointwise in time. Define the (scalar) nonlinear controller $\mathbf{C} := \mathbf{N} \circ \tilde{\mathbf{C}}$ (see Figure 2). The remaining part of

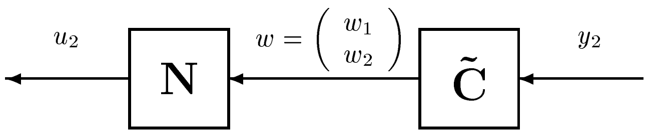

Figure 2: A Nonlinear Controller $\mathbf{C} := \mathbf{N} \circ \tilde{\mathbf{C}}$.

this section will show that the above controller $\mathbf{C}$ can solve Problem 2.1. The intuition behind the controller is as follows. The linear controller $\tilde{\mathbf{C}}$ can produce in the steady state, for an input which is eventually zero, any pair of sinusoids of the same amplitude and with a $\pi/2$ phase shift between them. If such a signal is applied to $\mathbf{N}$, then the output of $\mathbf{N}$ is, in the steady state, a triangular wave of the same frequency, amplitude and phase as the first component of the output of $\tilde{\mathbf{C}}$. Thus $\mathbf{C}$ plays the role of an internal model for the disturbances in $\mathcal{T}(\omega_0)$. In order that $\mathbf{C}$ solves Problem 2.1, it is necessary that $\mathbf{H}_{\mathbf{P},\mathbf{C}}$ is a stable operator. This will be established by showing that the directed gap between $\mathbf{C}$ and some stabilizing linear controller is small enough for Lemma 3.1 to apply. To prove formally that asymptotic disturbance rejection is achieved, the Contraction Mapping Theorem will be used.

4.1 Asymptotic Rejection

Consider the feedback configuration of Figure 1 with $\mathbf{C} = \mathbf{N} \circ \tilde{\mathbf{C}}$. We will assume that $\mathbf{P}$ is stable and that $[\mathbf{P}, \tilde{\mathbf{C}}_1]$ is stable. Let $y_0 = 0$ and consider any $u_0 \in \mathcal{T}(\omega_0)$, say $u_0(t) = A\, f_0(\omega_0 t + \phi)$. Define the corresponding vector signal

$$v_0(t) := \begin{pmatrix} A\sin(\omega_0 t + \phi) \\ A\cos(\omega_0 t + \phi) \end{pmatrix}$$

and observe that $u_0 = \mathbf{N}v_0$. It can easily be verified that the following equations are satisfied

$$w = -\tilde{\mathbf{C}}\mathbf{P}(\mathbf{N}v_0 - \mathbf{N}w), \tag{2}$$

$$y_1 = \mathbf{P}(\mathbf{N}v_0 - \mathbf{N}w). \tag{3}$$

Define the operator

$$\tilde{\mathbf{G}} := -\left(\mathbf{I} - \tilde{\mathbf{C}}\mathbf{P}(1,0)\right)^{-1}\tilde{\mathbf{C}}\mathbf{P}.$$

$\tilde{\mathbf{G}}$ can be shown to have the transfer function

$$\tilde{G} = \frac{-\tilde{C}_1 P}{1 - \tilde{C}_1 P}\begin{pmatrix} 1 \\ \frac{\gamma s - k\omega_0^2}{\omega_0(ks+\gamma)} \end{pmatrix}. \tag{4}$$

By subtracting the term $\tilde{\mathbf{C}}\mathbf{P}(1,0)w$ from both sides of (2) and pre-multiplying by $\left(\mathbf{I} - \tilde{\mathbf{C}}\mathbf{P}(1,0)\right)^{-1}$, we obtain the equation $w = \tilde{\mathbf{G}}(\mathbf{N}v_0 - \mathbf{N}w) + \tilde{\mathbf{G}}(1,0)w$. Defining also

$$f := w - v_0,$$

we therefore have the equation

$$f = \tilde{\mathbf{G}}\mathbf{\Gamma}(f) + \left\{\tilde{\mathbf{G}}(1,0) - \mathbf{I}\right\} v_0 \tag{5}$$

where the operator $\mathbf{\Gamma}$ is defined as

$$\mathbf{\Gamma}(f) := \mathbf{N}v_0 - \mathbf{N}(v_0 + f) + (1,0)f. \tag{6}$$

Since $P(\pm j\omega_0) \neq 0$, we have

$$\tilde{G}(\pm j\omega_0) = \begin{pmatrix} 1 \\ \pm j \end{pmatrix}$$

and therefore $\left\{\tilde{\mathbf{G}}(1,0) - \mathbf{I}\right\} v_0 \in \mathcal{L}_2[0,\infty)$. We will see below that the operator $\mathbf{\Gamma}$ maps the space $\mathcal{L}_2[0,\infty)$ into itself. If we can also prove that the incremental gains of $\tilde{\mathbf{G}}$ and $\mathbf{\Gamma}$ satisfy the inequality

$$\|\tilde{\mathbf{G}}\|_\triangle \cdot \|\mathbf{\Gamma}\|_\triangle < 1 \tag{7}$$

then (5) has a unique solution $\tilde{f} := (\mathbf{I} - \tilde{\mathbf{G}}\mathbf{\Gamma})^{-1}\left\{\tilde{\mathbf{G}}(1,0) - \mathbf{I}\right\} v_0 \in \mathcal{L}_2[0,\infty)$, by the Contraction Mapping Theorem. Under the assumptions that P is strictly proper and stable then $sP \in \mathcal{H}_\infty$, so that $\frac{d}{dt} \circ \mathbf{P}$ is a bounded operator on $\mathcal{L}_2[0,\infty)$. Thus, asymptotic rejection is achieved at the output y_1 since

$$y_1 = \mathbf{P}\left(\mathbf{N}v_0 - \mathbf{N}(v_0 + \tilde{f})\right) = \mathbf{P}\left(\mathbf{\Gamma}(\tilde{f}) - (1,0)\tilde{f}\right) \in \mathcal{L}_2[0,\infty),$$

and

$$\dot{y}_1 = \left(\frac{d}{dt} \circ \mathbf{P}\right)\left(\mathbf{\Gamma}(\tilde{f}) - (1,0)\tilde{f}\right) \in \mathcal{L}_2[0,\infty).$$

We will now show that $\|\mathbf{\Gamma}\|_\triangle = 2/\pi$ regarding $\mathbf{\Gamma}$ as a mapping from $\mathcal{L}_2[0,\infty)$ into $\mathcal{L}_2[0,\infty)$. Note that, from the definition in (6), for any 2×1 vector functions f and g on $[0,\infty)$, we have

$$\mathbf{\Gamma}(f) - \mathbf{\Gamma}(g) = \mathbf{N}(v_0 + g) - \mathbf{N}(v_0 + f) + (1,0)\big((v_0 + f) - (v_0 + g)\big).$$

Write

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = v_0(t) + g(t), \quad \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = v_0(t) + f(t).$$

By noting that

$$N\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} = N\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and that N is differentiable in $\mathbb{R}^2$ and using the mean value theorem, we have

$$\begin{aligned} \left|\big(\mathbf{\Gamma}(f) - \mathbf{\Gamma}(g)\big)(t)\right| &= \left| N\begin{pmatrix} a_1 \\ |b_1| \end{pmatrix} - N\begin{pmatrix} a_2 \\ |b_2| \end{pmatrix} - (a_1 - a_2) \right| \\ &= \left| \left(\frac{\partial N}{\partial x_1}(\underline{\hat{x}}) - 1, \frac{\partial N}{\partial x_2}(\underline{\hat{x}}) \right) \cdot \begin{pmatrix} a_1 - a_2 \\ |b_1| - |b_2| \end{pmatrix} \right| \end{aligned}$$

for some $\underline{\hat{x}} = (\hat{x}_1, \hat{x}_2)^T = \mu(a_1, |b_1|)^T + (1-\mu)(a_2, |b_2|)^T$, $\mu \in (0,1)$. Note that

$$\begin{aligned} \frac{\partial N}{\partial x_1}(\underline{\hat{x}}) &= \frac{2}{\pi}\left(\cos(\hat{\theta}) + \hat{\theta}\sin(\hat{\theta})\right) \\ \frac{\partial N}{\partial x_2}(\underline{\hat{x}}) &= \frac{2}{\pi}\left(\hat{\theta}\cos(\hat{\theta}) - \sin(\hat{\theta})\right) \end{aligned}$$

for some $\hat{\theta} \in [-\pi/2, \pi/2]$ where

$$\sin(\hat{\theta}) := \frac{\hat{x}_1}{\sqrt{\hat{x}_1^2 + \hat{x}_2^2}}, \qquad \cos(\hat{\theta}) := \frac{\hat{x}_2}{\sqrt{\hat{x}_1^2 + \hat{x}_2^2}}.$$

Defining

$$h(\theta) := \sqrt{\left(\frac{2}{\pi}(\cos(\theta) + \theta\sin(\theta)) - 1\right)^2 + \left(\frac{2}{\pi}(\theta\cos(\theta) - \sin(\theta))\right)^2},$$

and noting that

$$\left| \begin{pmatrix} a_1 - a_2 \\ |b_1| - |b_2| \end{pmatrix} \right| \leq |f(t) - g(t)|,$$

we have

$$\left|\big(\mathbf{\Gamma}(f) - \mathbf{\Gamma}(g)\big)(t)\right| \leq h(\hat{\theta}) \cdot |f(t) - g(t)|$$

for some $\hat{\theta} \in [-\pi/2, \pi/2]$. Since $\max_{\theta \in [-\pi/2, \pi/2]} h(\theta) = h(\pm\pi/2) = 2/\pi$, $\mathbf{\Gamma}$ maps $\mathcal{L}_2[0,\infty)$ into itself and its incremental gain is less than or equal to $2/\pi$. In fact, $\|\mathbf{\Gamma}\|_\triangle = 2/\pi$. This can be observed by choosing any $\varepsilon > 0$,

$$g := \begin{pmatrix} 1 \\ 0 \end{pmatrix} - v_0, \qquad f := \begin{pmatrix} 1 - \varepsilon \\ \sqrt{2\varepsilon - \varepsilon^2} \end{pmatrix} - v_0$$

and observing that

$$\left|\big(\mathbf{\Gamma}(f) - \mathbf{\Gamma}(g)\big)(t)\right| = \left| 1 - \varepsilon - \frac{2}{\pi}\sin^{-1}(1 - \varepsilon) \right|$$

and

$$|f(t) - g(t)| = \sqrt{2\varepsilon}.$$

The ratio $\left|\big(\mathbf{\Gamma}(f) - \mathbf{\Gamma}(g)\big)(t)\right| / |f(t) - g(t)|$ has a limit $2/\pi$ when ε tends to zero.

Finally we note that $\tilde{\mathbf{G}}$ is linear and therefore the incremental gain of $\tilde{\mathbf{G}}$ acting on $\mathcal{L}_2[0,\infty)$ is equal to $\|\tilde{G}\|_{\mathcal{H}_\infty}$. We thus obtain a set of sufficient conditions for asymptotic rejection of disturbances in $\mathcal{T}(\omega_0)$ as follows.

Sufficient Conditions for Asymptotic Rejection of Disturbances in $\mathcal{T}(\omega_0)$

1. P is strictly proper, stable and $P(\pm j\omega_0) \neq 0$;
2. $[P, \tilde{C}_1]$ is stable where $\tilde{C}_1 = -(ks+\gamma)/(s^2+\omega_0^2)$ for some $k, \gamma > 0$;
3. $\|\tilde{G}\|_{\mathcal{H}_\infty} < \pi/2$ where $\tilde{G}$ is defined in (4).

4.2 Robust Stability

In this section, we will give conditions for the nonlinear $b_{\mathbf{P},\mathbf{C}}$ with $\mathbf{C} = \mathbf{N} \circ \tilde{\mathbf{C}}$ to be bounded away from zero. We will make use of Lemma 3.1 with $\mathbf{P}' = \mathbf{P}$, $\mathbf{C} = \mathbf{N} \circ \tilde{\mathbf{C}}$ and $\mathbf{C}' = \lambda \tilde{\mathbf{C}}_1$, for some appropriate $\lambda > 0$, to obtain an upper bound on $b^{-1}_{\mathbf{P},\mathbf{C}} = \left\|\Pi_{\mathcal{G}_\mathbf{P}//\mathcal{G}_\mathbf{C}}\right\|$. We will take the signal spaces to be $\mathcal{U} = \mathcal{Y} = \mathcal{L}_\infty[0,\infty)$ We first note that N satisfies the property

$$k_1|x_1| \leq |N(\underline{x})| \leq k_2|x_1| \tag{8}$$

for all $\underline{x} = (x_1, x_2)^T \in \mathbb{R}^2$, where $k_1 := 2/\pi > 0$ and $k_2 := 1 < \infty$. (This follows from the inequalities $1 \leq |(\sin^{-1} x)/x| \leq \pi/2$ for $-1 \leq x \leq 1$.) We therefore see that

$$\operatorname{dom}\left(\lambda \tilde{\mathbf{C}}_1\right) = \operatorname{dom}\left(\mathbf{N} \circ \tilde{\mathbf{C}}\right).$$

For the rest of this section, we use the following (abuse of) notation. Let $F(s)$ be the transfer function of a stable linear system and $y \in \mathcal{L}_\infty[0,\infty)$. Then $F(s)y$ denotes the time domain output in $\mathcal{L}_\infty[0,\infty)$ of the linear system with input y, i.e. we use Fy to mean $\mathbf{F}y$. We also denote the $\|F(s)y\|_\infty$ as the supremum norm of this time domain signal. We now observe that

$$\operatorname{dom}\left(\lambda \tilde{\mathbf{C}}_1\right) = \left\{\frac{s^2+\omega_0^2}{(s+1)^2} y : y \in \mathcal{Y}\right\} =: \frac{s^2+\omega_0^2}{(s+1)^2}\,\mathcal{Y}.$$

Thus, the graphs of the two controllers are given by

$$\mathcal{G}_{\lambda\tilde{\mathbf{C}}_1} = \left\{ \begin{pmatrix} \lambda \frac{ks+\gamma}{(s+1)^2} y \\ -\frac{s^2+\omega_0^2}{(s+1)^2} y \end{pmatrix} : y \in \mathcal{Y} \right\}$$

and

$$\mathcal{G}_{\mathbf{N}\circ\tilde{\mathbf{C}}} = \left\{ \begin{pmatrix} \mathbf{N}\left(\left(\frac{ks+\gamma}{(s+1)^2} y, \frac{\gamma s - k\omega_0^2}{\omega_0 (s+1)^2} y\right)^T\right) \\ -\frac{s^2+\omega_0^2}{(s+1)^2} y \end{pmatrix} : y \in \mathcal{Y} \right\}.$$

Define

$$V(s) := \begin{pmatrix} \lambda \frac{ks+\gamma}{(s+1)^2} \\ -\frac{s^2+\omega_0^2}{(s+1)^2} \end{pmatrix}.$$

Then $\mathcal{G}_{\lambda\tilde{\mathbf{C}}_1} = V\,\mathcal{Y}$ and $\|Vy\|_\infty \geq \lambda \left\| \frac{ks+\gamma}{(s+1)^2} y \right\|_\infty$. Define also the mapping $\mathbf{\Phi_0} : \mathcal{G}_{\lambda\tilde{\mathbf{C}}_1} \to \mathcal{G}_{\mathbf{N}\circ\tilde{\mathbf{C}}}$ by

$$\mathbf{\Phi_0}(Vy) = \begin{pmatrix} \mathbf{N}\left(\left(\frac{ks+\gamma}{(s+1)^2} y, \frac{\gamma s - k\omega_0^2}{\omega_0 (s+1)^2} y\right)^T\right) \\ -\frac{s^2+\omega_0^2}{(s+1)^2} y \end{pmatrix}.$$

Then this $\mathbf{\Phi_0}$ is causal, bijective and $\mathbf{\Phi_0}0 = 0$. We now choose $\lambda = (k_1 + k_2)/2$. Thus from (8),

$$\begin{aligned}\|(\mathbf{\Phi_0} - \mathbf{I})Vy\|_\infty &= \left\| \mathbf{N}\left(\left(\frac{ks+\gamma}{(s+1)^2}y, \frac{\gamma s - k\omega_0^2}{\omega_0(s+1)^2}y\right)^T\right) - \lambda\frac{ks+\gamma}{(s+1)^2}y\right\|_\infty \\ &\le \frac{k_2-k_1}{2}\left\|\frac{ks+\gamma}{(s+1)^2}y\right\|_\infty \le \frac{k_2-k_1}{2\lambda}\|Vy\|_\infty .\end{aligned}$$

Hence,

$$\vec{\delta}\left(\mathcal{G}_{\lambda\tilde{\mathbf{C}}_1}, \mathcal{G}_{\mathbf{N}\circ\tilde{\mathbf{C}}}\right) \le \frac{k_2-k_1}{k_2+k_1} = \frac{1-2/\pi}{1+2/\pi} = 0.2220.$$

On the other hand, the parallel projection operator $\Pi_{\mathcal{G}_{\lambda\tilde{\mathbf{C}}_1}//\mathcal{G}_\mathbf{P}}$ has the transfer function

$$\begin{pmatrix} \lambda\tilde{C}_1 \\ 1 \end{pmatrix}(1-\lambda\tilde{C}_1P)^{-1}(-P,1).$$

It is known [12] that the $\mathcal{L}_\infty[0,\infty)$ induced norm for this parallel projection is equal to

$$\begin{aligned}\left\|\Pi_{\mathcal{G}_{\lambda\tilde{\mathbf{C}}_1}//\mathcal{G}_\mathbf{P}}\right\| = \max\Bigg\{ &\left\|\mathscr{L}^{-1}\left(\frac{\lambda\tilde{C}_1P}{1-\lambda\tilde{C}_1P}\right)\right\|_1 + \left\|\mathscr{L}^{-1}\left(\frac{\lambda\tilde{C}_1}{1-\lambda\tilde{C}_1P}\right)\right\|_1, \\ &\left\|\mathscr{L}^{-1}\left(\frac{P}{1-\lambda\tilde{C}_1P}\right)\right\|_1 + \left\|\mathscr{L}^{-1}\left(\frac{1}{1-\lambda\tilde{C}_1P}\right)\right\|_1 \Bigg\}\end{aligned}$$

where $\mathscr{L}^{-1}$ stands for the inverse Laplace transform and

$$\left\|D + C(sI-A)^{-1}B\right\|_1 := |D| + \int_0^\infty \left|Ce^{At}B\right| dt$$

for SISO systems with stable A. Based on Lemma 3.1, we can establish the set of sufficient conditions for robust stability of the closed loop $[\mathbf{P}, \mathbf{N}\circ\tilde{\mathbf{C}}]$.

Sufficient Conditions for Robust Stability of $[\mathbf{P}, \mathbf{N}\circ\tilde{\mathbf{C}}]$

1. $[P, \lambda\tilde{C}_1]$ is stable, where $\lambda = (k_1+k_2)/2$, $k_1 = 2/\pi$ and $k_2 = 1$;
2. $\frac{k_2-k_1}{k_2+k_1} \cdot \left\|\Pi_{\mathcal{G}_{\lambda\tilde{\mathbf{C}}_1}//\mathcal{G}_\mathbf{P}}\right\| < 1$.

If these sufficient conditions are satisfied, the (nonlinear) gap metric robustness margin has a lower bound

$$b_{\mathbf{P},\mathbf{N}\circ\tilde{\mathbf{C}}} \ge \frac{1 - \frac{k_2-k_1}{k_2+k_1}\left\|\Pi_{\mathcal{G}_{\lambda\tilde{\mathbf{C}}_1}//\mathcal{G}_\mathbf{P}}\right\|}{\left\|\Pi_{\mathcal{G}_\mathbf{P}//\mathcal{G}_{\lambda\tilde{\mathbf{C}}_1}}\right\|}.$$

5 An Example

This section illustrates the applicability of the conditions derived in the last section with some simulation results. Take the first order plant

$$P(s) = \frac{50}{s+50} \tag{9}$$

and

$$\tilde{C}(s) = \begin{pmatrix} \tilde{C}_1(s) \\ \tilde{C}_2(s) \end{pmatrix} := \begin{pmatrix} -(ks+\gamma)/(s^2+\omega_0^2) \\ -(\gamma s - k\omega_0^2)/(\omega_0(s^2+\omega_0^2)) \end{pmatrix}$$

with $k = \gamma = 10$, $\omega_0 = \pi/2$. It is easy to compute that $\|\tilde{G}\|_{\mathcal{H}_\infty} = 1.4936 < \pi/2$ and the 1-norms

$$\left\| \mathscr{L}^{-1}\left(\frac{1}{1-\lambda\tilde{C}_1 P}\right)\right\|_1 = 2.2682, \quad \left\| \mathscr{L}^{-1}\left(\frac{\lambda\tilde{C}_1}{1-\lambda\tilde{C}_1 P}\right)\right\|_1 = 1.2697,$$

$$\left\| \mathscr{L}^{-1}\left(\frac{P}{1-\lambda\tilde{C}_1 P}\right)\right\|_1 = 1.8336, \quad \left\| \mathscr{L}^{-1}\left(\frac{\lambda\tilde{C}_1 P}{1-\lambda\tilde{C}_1 P}\right)\right\|_1 = 1.2682.$$

These 1-norms were computed using MAPLE with reliable accuracy. So,

$$\begin{aligned} & \vec{\delta}\left(\mathcal{G}_{\lambda\tilde{\mathbf{C}}_1}, \mathcal{G}_{\mathbf{N}\circ\tilde{\mathbf{C}}}\right) \left\|\Pi_{\mathcal{G}_{\lambda\tilde{\mathbf{C}}_1}//\mathcal{G}_{\mathbf{P}}}\right\| \\ \leq\ & 0.2220 \times \max\{1.2682 + 1.2697,\ 1.8336 + 2.2682\} = 0.9106 < 1. \end{aligned}$$

Similarly, $\Pi_{\mathcal{G}_{\mathbf{P}}//\mathcal{G}_{\lambda\tilde{\mathbf{C}}_1}}$ has the transfer function

$$\begin{pmatrix} 1 \\ P \end{pmatrix} (1-\lambda\tilde{C}_1 P)^{-1}(1, -\lambda\tilde{C}_1)$$

and its induced norm is equal to $\max\{2.2682 + 1.2697,\ 1.8336 + 1.2682\} = 3.5379$. Therefore, the closed loop $[\mathbf{P}, \mathbf{N}\circ\tilde{\mathbf{C}}]$ is stable and $b_{\mathbf{P},\mathbf{N}\circ\tilde{\mathbf{C}}} \geq (1-0.9106)/3.5379 = 0.0253$, from Lemma 3.1. The plant in (9) and the controller $\mathbf{N}\circ\tilde{\mathbf{C}}$ defined above were simulated using SIMULINK for u_0 equal to a unit amplitude triangular wave of period 4 secs, zero phase shift and $y_1 = 0$. The results of a simulation are shown in Figures 3 and 4, and illustrate the asymptotic rejection property.

6 Conclusion

This paper investigates the possibility of robust asymptotic rejection of a class of periodic waves using a nonlinear controller. The nonlinear controller consists of a linear compensator and an appropriate nonlinear mapping. The paper illustrates a form of the Internal Model Principle for a class of signals which necessitates a nonlinear controller. Robust stability is also discussed using the nonlinear gap metric robustness margin. The requirements of asymptotic disturbance rejection and robust stability are treated separately.

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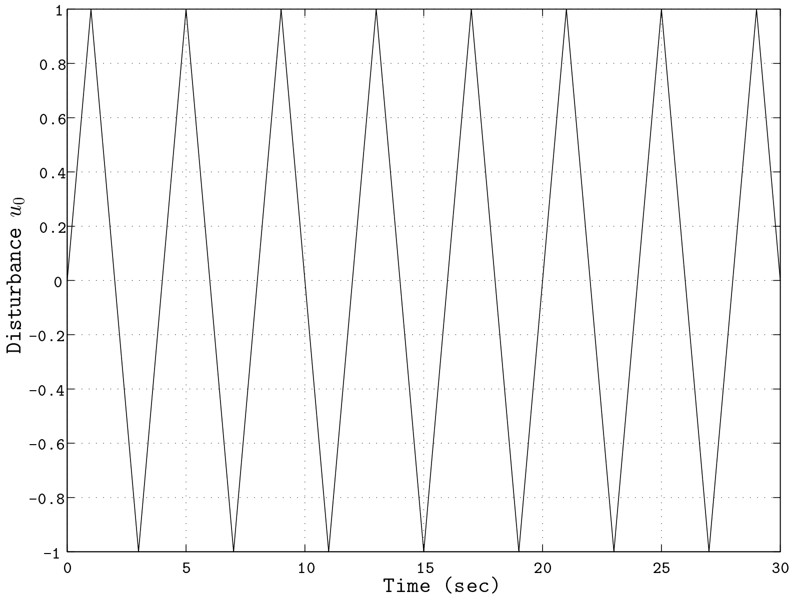

Figure 3: The Triangular Waveform Disturbance u_0 of frequency $\omega_0 = \pi/2$.

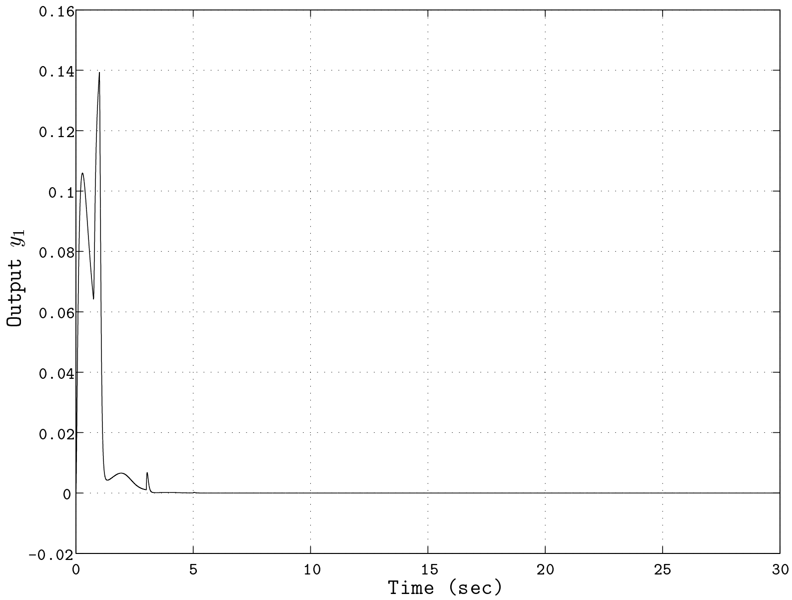

Figure 4: The Plant Output at y_1.