Some Remarks on Repetitive Control Systems

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Abstract

We investigate some robustness compromises involved in achieving attenuation for periodic disturbances. We consider asymptotic rejection or attenuation of disturbance harmonics and transient response.

1 Introduction

A goal of repetitive control is to ensure tracking/disturbance rejection of periodic references/disturbances. If it is desired to achieve *asymptotic* tracking/rejection, the internal model principle (see [2]) suggests that the controller needs to contain poles at the frequency of the periodic signal and all its non-zero harmonics (possibly infinitely many). This was the original idea of Inoue *et al.* [8] who proposed using a positive feedback around a time delay to generate an internal model of the periodic disturbance. It was observed by a number of authors that such a scheme lacks robustness (see [5], [6]). We demonstrate this fact here in the context of gap metric uncertainty (Proposition 1). On the other hand we show that a reasonable level of asymptotic *attenuation* of periodic disturbances can always be achieved without serious robustness compromises (Proposition 2). In the case where asymptotic rejection of a harmonic does not conflict with robustness, we show that to approach optimal robustness necessitates an arbitrarily poor disturbance transient response (Proposition 3).

2 Discussion

For simplicity we will restrict attention to (rational) scalar plants and controllers in the standard feedback configuration (Figure 1). We will use the gap metric robustness margin as our measure of uncertainty tolerance which is given by [4]:

$$b_{P,C} := \left\| \begin{pmatrix} I \\ P \end{pmatrix} (I - CP)^{-1}(I, C) \right\|_\infty^{-1}.$$

The optimal robustness margin of P is $b_{\mathrm{opt}}(P) := \sup_{C \text{ stblz}} b_{P,C}$. We consider a finite set of frequencies $\Omega_n := \{\pm j\omega_1, ..., \pm j\omega_n\}$. We consider the problem of asymptotically rejecting disturbances u_1 at e_2 which are periodic and have a finite number of harmonics belonging only to Ω_n. This requires $P(I-CP)^{-1} = 0$ for all $s \in \Omega_n$. We define an optimal robustness margin with this additional constraint:

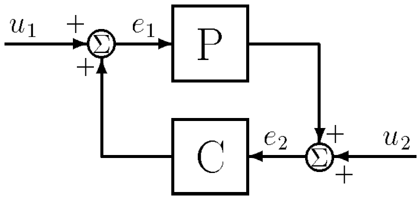

Figure 1: Standard Feedback Configuration.

$$b_{\mathrm{opt}}(P, \Omega_n) := \sup_{C \text{ stblz}} \left\{ b_{P,C} : P(I - CP)^{-1} = 0 \, \forall s \in \Omega_n \right\}.$$

Proposition 1 *Let $P(j\omega_k) \neq 0$ for each $j\omega_k \in \Omega_n$. Then*

$$b_{\mathrm{opt}}(P, \Omega_n) = \min_{j\omega_k \in \Omega_n} \left\{ b_{\mathrm{opt}}(P), \frac{|P(j\omega_k)|}{\sqrt{1 + |P(j\omega_k)|^2}} \right\}.$$

The above proposition illustrates the impossibility of achieving asymptotic disturbance rejection of general periodic inputs. Indeed, if the plant P is strictly proper then the robustness margin will be zero if all harmonics are included. Even if P is invertible at infinity (unlikely in practice) it is easy to check that the control system fails to be w-stable in the sense of [3]. Thus, asymptotic rejection of all harmonics is not a practicable proposition. However, no robustness compromise is required at a given harmonic if $\frac{|P(j\omega_k)|}{\sqrt{1+|P(j\omega_k)|^2}} \geq b_{\mathrm{opt}}(P)$. We mention that the above proposition is closely related to results of [1] on the regulator problem with robust stability. We now turn to the question of the best attenuation which can be achieved subject to a given robustness level.

Proposition 2 *Suppose $0 < \alpha \leq b_{\mathrm{opt}}(P)$. For any $\epsilon > 0$ there exists a C_ϵ such that $b_{P,C_\epsilon} \geq \alpha - \epsilon$ and*

$$|P(I - C_\epsilon P)^{-1}(j\omega_k)| \leq \xi_k$$

for all $j\omega_k \in \Omega_n$, if and only if

$$\begin{array}{lll} \xi_k & \geq & 0 \qquad \textit{when } |P(j\omega_k)| \geq \frac{1}{\sqrt{\alpha^{-2}-1}} \\ \xi_k & \geq & g(\alpha, |P(j\omega_k)|) \qquad \textit{otherwise,} \end{array}$$

where $g(\alpha, x) := \frac{x(1-x\sqrt{\alpha^{-2}-1})}{1+x^2}$.

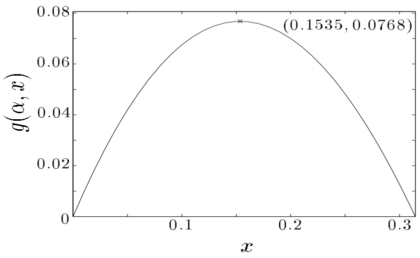

Figure 2: The function $g(\alpha, x)$ for $\alpha = 0.3$.

The above proposition tells us a minimum modulus for the plant which is consistent with a certain robustness margin. For example if we require $\alpha = 0.3$ (a reasonable level in practice) then asymptotic rejection of a harmonic is possible providing $|P(j\omega_k)| > 0.314$, and otherwise, attenuation to less than 7.68% of the disturbance magnitude is always possible as shown by Figure 2. Note that $g(\alpha, x)$ has a maximum of $g_{\max} = \frac{\alpha}{2(1+\sqrt{1-\alpha^2})}$ at $x_{\max} = \frac{\alpha}{1+\sqrt{1-\alpha^2}}$, so even with $\alpha = 1$, a 50% reduction is always possible. We next consider the transient response to a sinusoidal disturbance at a frequency ω_0 as measured by the quantity:

$$\left\| \frac{P(1-CP)^{-1}(s) - P(1-CP)^{-1}(j\omega_0)}{s - j\omega_0} \right\|_2 .$$

Proposition 3 *Suppose $b_{\text{opt}}(P) < \frac{|P(j\omega_0)|}{\sqrt{1+|P(j\omega_0)|^2}}$ for some ω_0. Consider any sequence of stabilizing compensators C_n with $b_{P,C_n} \to b_{\text{opt}}(P)$ as $n \to \infty$ and $P(1-C_nP)^{-1}(j\omega_0) = 0$ for each n. Then the sequence*

$$\left\| \frac{1}{s - j\omega_0} P(1 - C_n P)^{-1}(s) \right\|_2 \to \infty$$

as $n \to \infty$.

The above propositions suggest an approach to repetitive control by adopting specifications at the frequencies of the various disturbance harmonics according to the corresponding open loop plant magnitude. For large plant magnitudes, asymptotic rejection of a harmonic can be sought. For small plant magnitudes there is a natural tendency to attenuate harmonics which can be preserved in the closed loop. For magnitudes in an intermediate range *asymptotic* attenuation can be sought to a level suggested by Proposition 2. For the transient response, it is clear that some departure from optimal robustness is required, at least if asymptotic rejection is required within the plant bandwidth. The extent of such compromises and the interaction of transient requirements at different frequencies requires further investigation.

One possible design approach is to consider the $\mathcal{H}_\infty$ loop-shaping method with a weighting function of the form:

$$W(s) = W_1(s) \frac{s+\gamma_0}{s} \prod_{k=1}^{n} \frac{s^2 + \beta_k s + \gamma_k}{s^2 + \delta_k s + \omega_k^2}.$$

where $W_1(s)$ is an ordinary loop-shaping weight and $\delta_k = 0$ for some lower harmonics. This gives an approximate approach with the potential for a reasonable trade-off between the factors mentioned above.

Proofs of Propositions. Propositions 1 and 2 use some standard facts on $\mathcal{H}_\infty$ boundary interpolation and Proposition 2 uses some bounds of [11, Theorem V.2]. Proposition 3 uses some standard facts on $\mathcal{H}_\infty$ optimal interpolants and a normal families argument.

3 Other work on repetitive control

The reader is referred to [7] for further references as well as for questions of robustness, adaptation to unknown frequency periods and several applications. We mention work on stability conditions and synthesis methods in [5], [6] and [12]. For discrete time approaches see [10] and the references therein. A sampled-data repetitive control design methodology, in which inter-sample behaviour was taken into account, can be found in [9].

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