Performance-Preserving Controller Approximation

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Abstract

This dissertation is concerned with the approximation of high order controllers without degrading the achieved closed loop performance.

We firstly consider the approximation of linear time invariant (LTI) controllers which have been designed for LTI models and satisfy a prespecified level of closed loop $\mathcal{H}_\infty$ performance. We derive sufficient conditions for any other controller to be stabilising and satisfy the same level of $\mathcal{H}_\infty$ performance. Such controllers are said to be (P, γ)-admissible, where P is the model of the plant under consideration and γ is the required level of prespecified $\mathcal{H}_\infty$ performance. The conditions are expressed as norm bounds on particular frequency weighted errors where the weights are selected to make a specific transfer function a contraction. The design of reduced order (P, γ)-admissible controllers is then formulated as a frequency weighted model reduction problem. We demonstrate that when used in conjunction with a combined model reduction/convex optimisation scheme, the proposed design procedures are effective in substantially reducing controller complexity.

It is advantageous for the required weights to be large in some sense. Hence we characterise solutions which maximise either the trace or the determinant of the weights. For scalar controllers a graphical interpretation of the derived sufficient conditions is presented. We show that for scalar weights the procedure for maximising the trace of the weights, and the procedure for maximising the determinant of the weights, both result in the best possible weights. Hence in the scalar case the two procedures are equivalent.

We then consider the approximation of linear parameter varying (LPV) controllers which have been designed for LPV models. Two well known LTI approximation methods – balanced truncation and optimal Hankel norm approximation – are extended to the LPV framework. Given a high order quadratically stabilising controller we then derive a bound on the degradation in quadratic performance when the high order controller is replaced with a low order approximant.

Keywords: Robust Control Theory, H-infinity Control, Controller Approximation, Frequency Weighted Model Reduction, Parameter Dependent Systems, Quadratic Performance.

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As required by University Statute, I hereby declare that this dissertation is not substantially the same as any that I have submitted for a degree at any other University, is the result of my own work, and includes nothing which is the outcome of work done in collaboration.

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Notation

Mathematical Symbols	
$\in$	is an element of
$\iff$	if and only if
$\triangleq$	equals by definition
$\mathrm{Re}(\cdot)$	the real part of $(\cdot)$
$\mathbb{R}$	the real numbers
$\mathbb{C}$	the complex numbers
$\mathbb{R}^{p\times m}$	real valued matrices with p rows and m columns
$\mathbb{C}^{p\times m}$	complex valued matrices with p rows and m columns
M'	the (complex conjugate) transpose of the matrix M
$M^\dagger$	the Moore-Penrose pseudo-inverse of matrix M

For $M \in \mathbb{R}^{n\times n}$	
$\lambda_i(M)$	the i^{th} eigenvalue of M
$\bar{\sigma}(M)$	the maximum singular value of M
$\sigma_i(M)$	the i^{th} singular value of M
$\underline{\sigma}(M)$	the minimum singular value of M
$\rho(M)$	the spectral radius of M
$M > 0$	$M = M'$ is positive definite
$M \geq 0$	$M = M'$ is positive semi-definite
$\mathrm{trace}(M)$	the trace of M
$\det(M)$	the determinant of M

Acronyms	
ARE	Algebraic Riccati Equation
LFT	Linear Fractional Transformation
LMI	Linear Matrix Inequality
LPV	Linear Parameter Varying
LQG	Linear Quadratic Gaussian
LTI	Linear Time Invariant

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Chapter 1

Introduction

1.1 Overview

Approximation plays an important role in all model based control methodologies, from the identification of a mathematical model to describe plant dynamics, through to controller synthesis.

The aim of identification is to derive a set of mathematical equations, the behaviour of which approximates the behaviour of the plant under consideration. The model formed by this set of equations may subsequently be used for synthesis of a controller which meets appropriately defined performance specifications. However, there exists a trade off between accuracy and simplicity – the derived model must be sufficiently accurate to capture the salient features of the plant, yet simple enough to manipulate and understand. In this dissertation we focus on approximation as a controller synthesis tool and assume that a model (of acceptable accuracy) has been identified and is available.

Plants which may be modeled by a single set of finite dimensional, linear time invariant (LTI), differential equations have been studied extensively in terms of both controller synthesis and model approximation. However a problem arises in that the physical system to be controlled almost certainly possesses non-linearities or parameters which vary as a function of time and/or the operating point. One possible method of dealing with these unmodeled variations is to amalgamate them into an uncertainty block (along with uncertainty which may arise from other sources,) and design a controller which is robust to their combined effect. For the chosen class of uncertainty, such techniques result in a figure of merit which indicates how well a controller designed using the model will perform when connected to the actual plant. That is, the robustness of the design to unmodeled dynamics. One such figure of merit is expressed in terms of the $\mathcal{H}_\infty$ norm of the closed loop transfer function. (See Francis [23] or Doyle et al. [14] for an introduction to the $\mathcal{H}_\infty$ control problem.) Given a linear, time invariant, plant model P, the standard suboptimal $\mathcal{H}_\infty$ control problem is to design a stabilising controller such that the $\mathcal{H}_\infty$ norm of the closed loop transfer function is less than some prespecified performance level $\gamma > 0$. Controllers satisfying the $\mathcal{H}_\infty$ suboptimal control problem are said to be (P, γ)-admissible.

A drawback of present $\mathcal{H}_\infty$ design procedures (see for instance [14, 38, 24, 26]) is that the order of resulting (P, γ)-admissible controllers is typically of the same order as the model.

Since models for physical systems are often of high order, controllers designed to satisfy the $\mathcal{H}_\infty$ suboptimal control problem via these procedures will also be of high order. However, in practice low order controllers are preferred to high order controllers. They are less complex, have a shorter processing time and are less likely to run into either software or hardware problems. Hence, before implementation some form of controller approximation is often employed to obtain *simpler* controllers. In this dissertation we are mainly concerned with the issue of reducing the state-dimension of controllers designed in an $\mathcal{H}_\infty$ framework such that there is no degradation in closed loop $\mathcal{H}_\infty$ performance.

Sometimes the plant dynamics may vary to such an extent that a single linear model proves insufficient to capture the essential features of the plant and a series of models which describe the plant behaviour at a number of operating points is required. Controllers may then be designed to meet performance specifications at each of these points and switched between as the operating point varies. Alternatively it may be possible to schedule the controllers by interpolating controller parameters between the frozen point designs. In this way the whole operating range of the plant may be covered by a set of controllers designed to be robust to uncertainties about a fixed point. For these strategies to be successful real time information about the plant operating point is required to enable accurate switching/scheduling to take place. (A comparison of the switching and scheduling strategies for an advanced fighter aircraft is given by Hyde [45].)

Unfortunately, although the switched/scheduled controllers are guaranteed to meet performance specifications at each of the frozen design points, there is no guarantee of meeting performance requirements at intermediate points. However, if heuristic rules such as *the design points should be selected to be close enough together*, and that *the switching/scheduling parameter should vary slowly*, are satisfied, then it is reasonable to assume that satisfactory performance will be achieved across the whole operating range. (See the work of Shamma and Athens [84, 85], Sharuz and Behtash [82] and Meressi and Paden [66] for a detailed discussion of these stability and performance rules.)

Given that it is possible to model a plant across its whole operating range by a linear parameter varying (LPV) model, much work has recently been directed towards exploiting linear matrix inequality (LMI) techniques to design controllers which guarantee performance across the whole operating range. (Here by performance we mean a bound on the induced $\mathcal{L}_2$ gain for arbitrarily varying parameters.) Some success has been achieved in this area by Packard [73] using a small gain framework, and by Apkarian et al. [4], Boyd et al. [10] and Becker [7] when the design points can be interpreted as the vertices of a polytope which completely captures the structure and variations of the plant. A difficulty encountered with this type of formulation is that the computational effort generally increases rapidly as a function of the state dimension [7]. Hence it is desirable to keep the state-dimension as low as possible. This desire motivates the development of approximation techniques for LPV models presented in Chapter 6.

1.2 Synthesis of Reduced Order Controllers

There are three paths which may be followed to synthesize low order controllers for a plant which is modeled by a set (or sets) of high order equations: reduction of the model state-dimension followed by the design of a low order controller using the reduced order model; design of a high order controller from the high order model followed by reduction of the high order controller; and direct design via optimisation of controller parameters. A comprehensive list of references may be found in [3].

Since systematic model based controller design methodologies such as LQG, $\mathcal{H}_\infty$ and μ result in controllers whose state dimension is typically equal to that of the model plus appropriate performance weights, an intuitive procedure is to reduce the state dimension of the model prior to, and without consideration of, the synthesis stage. There are numerous algorithms which may be employed to perform this reduction, including Padè approximation [83, 74], balanced truncation [68, 75], and optimal Hankel norm approximation [34, 36]. However, although the low order controller is designed to perform well with the low order model, there is no guarantee that performance specifications will be met if it is connected to the high order model let alone the actual plant. This suggests that there is a fundamental relationship between the controller synthesis step and the approximation step which the above procedure fails to recognise.

The second approach to the synthesis of low order controllers is to design a high order controller based on a high order model and then reduce the controller. This method is based on the (reasonable) premise that if a reduced order controller is close (in some sense) to a high order controller which is known to provide satisfactory performance then the reduced order controller will also provide satisfactory performance. An advantage of this approach is that a controller satisfying the required performance specifications is known to exist (since one has been calculated), and hence it is then a matter of calculating a low order controller which gives a similar response. A drawback is that there may be satisfactory controllers of the same or lower degree which have a frequency response markedly different to the nominal high order controller.

It is a minimum requirement that any reduced order controller derived in this manner preserve stability of the closed loop. Further, it is desirable to know a priori the likely degradation in performance. There are a number of procedures ([3, 17, 55, 56, 69]) which address this problem, the most general of which incorporate information about the closed loop into frequency weights as a prelude to frequency weighted model reduction. The weights are chosen such that the reduced order controller retains the salient features of the closed loop at the frequencies of greatest interest (e.g. around crossover). The techniques of balanced truncation and optimal Hankel norm approximation have been extended to allow frequency weighting by Enns [17, 18], and Anderson [54] and Glover et al. [39] respectively, although their performance is not guaranteed by tight a priori error bounds.

The third possible path, direct design, is to impose a degree constraint and structure on the controller and use parameter optimisation to obtain an acceptable performance. Extensive work has been carried out in this area by Hyland et al. [46, 47, 48] and Ly et al. [61]. This is particularly convenient if switching/scheduling is required between several controllers since the same structure may be imposed on the controller at each design point. However the techniques are slightly ad-hoc since it is often not clear what constitutes an appropriate structure to impose, nor what constitutes an appropriate degree constraint.

In this dissertation we concentrate on the second of these paths, i.e. the design of a high order controller for a high order plant then reduction of the controller. Given a high order controller which meets satisfactory closed loop performance specifications we propose procedures which result in controllers of low order without incurring any degradation in performance.

1.3 Outline

This dissertation consists of eight chapters. The contents of each chapter will now be outlined.

Chapter 2: Preliminaries

In this chapter we present some background results from linear algebra and $\mathcal{H}_\infty$ control theory. State space representations, function spaces and linear fractional transformation are discussed. We conclude the chapter by presenting the Glover and Doyle [38] parameterisation of all (P,γ)-admissible controllers. This parameterisation is central to the results of Chapter 4.

Chapter 3: Principles of Controller Approximation

This chapter motivates the controller approximation problem by reviewing current model and controller reduction techniques. The techniques of both unweighted and frequency weighted balanced truncation and optimal Hankel norm approximations are considered in some detail.

Techniques which provide a priori conditions for reduced order controllers to preserve closed loop stability and which give an indication of performance degradation are of particular interest. We present a new derivation of a sufficient condition for synthesis of reduced order stabilising controllers. This sufficient condition is a simplified version of the sufficient condition for synthesis of reduced order (P,γ)-admissible controllers which is derived in Chapter 4. The chapter concludes with a review of two alternative approaches which have been proposed for the design of reduced order (P,γ)-admissible controllers.

Chapter 4: Synthesis of Reduced Order (P,γ)-admissible Controllers

Here we propose procedures for the design of reduced order controllers which are (P,γ)-admissible. Given a nominal (P,γ)-admissible controller K^0, we derive weights such that,

$$\|W_2^{-1}(K-K^0)W_1^{-1}\|_\infty < 1 \quad \Rightarrow \quad \|\mathcal{F}_l(P,K)\|_\infty < \gamma$$

or (for different weights),

$$\|W_2^{-1}(\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix} - \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix})W_1^{-1}\|_\infty < 1 \quad \Rightarrow \quad \|\mathcal{F}_l(P,K)\|_\infty < \gamma$$

where $K = \tilde{V}^{-1}\tilde{U}$ and $K^0 = \tilde{V}_0^{-1}\tilde{U}_0$. This is equivalent to finding the set of K or $\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix}$ which lie within a weighted $\mathcal{L}_\infty$ ball centered on K^0 or $\begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}$ respectively.

For (a number of) the proposed procedures the weights are required to be calculated such that a specific transfer function is a contraction. Subject to this constraint, the approximation is made easier if the above weighted $\mathcal{L}_\infty$ balls are made as large as possible. Hence the weights should be made large in some sense. Procedures for maximising the trace or the determinant of the weights are presented.

Chapter 5: A Graphical Interpretation of SISO $\mathcal{H}_\infty$ Controller Approximation

In this chapter we show that for scalar controllers the sufficient conditions of Chapter 4 have a natural frequency by frequency interpretation as either the inside or the outside of a disk in the complex plane. If at every frequency a controller's Nyquist plot lies inside (or where appropriate, outside) the circle which corresponds to the boundary of the disk at that frequency then the controller is guaranteed to be (P,γ)-admissible. This interpretation leads to a graphical design procedure. Further, by comparing the size of the circles derived from each sufficient condition, the relative conservatism of each approach is examined.

We show that given a MIMO controller K^0, maximising the determinant of the weights for the condition $\|W_2^{-1}(K - K^0)W_1^{-1}\|_\infty < 1$ always leads to a *direction* where the weights are as large as possible (subject to keeping a specified transfer function a contraction). When K^0 is scalar, this implies the weights are the best possible. Further, we show that given a scalar controller K^0, maximising the trace of the weights always leads to a *direction* where the weight is the largest possible (subject to keeping a specified transfer function a contraction). Hence, in the scalar case the procedure presented to maximise the determinant of the weights and the procedure presented to maximise the trace of the weights are equivalent.

Chapter 6: Approximation of Linear Parameter-Varying Systems

In this chapter we extend the LTI techniques of balanced truncation and optimal Hankel norm approximation to cope with LPV models. (However, we stress that the optimal Hankel norm type approach no longer has an interpretation of being optimal in any sense. It retains this name since it uses similar manipulations to its LTI counterpart and reduces to the LTI result when the plant under consideration is purely LTI.) Upper bounds for approximation are derived for both techniques.

Given an LPV controller (of high state-dimension,) for an LPV plant we derive a sufficient condition for a reduced order LPV controller to maintain closed loop quadratic-stability and give an a priori bound on closed loop quadratic-performance degradation.

Chapter 7: Design Examples

In this chapter we present three design examples which highlight the effectiveness of the proposed controller approximation procedures. Some computational issues are also discussed.

The first two examples consider the design of reduced-order, LTI, (P,γ)-admissible controllers for LTI models using the procedures proposed in Chapter 4. The first example considers pitch axis control for a model of an experimental highly manoeuverable aeroplane. The second example considers tracking control for a compact disk mechanism.

In the third example we consider pitch axis control for an LPV model of a missile.

Chapter 8: Conclusion

To conclude, we summarise the main results presented in the dissertation and outline future research in the area of controller approximation.

Chapter 2

Preliminaries

2.1 Introduction

In this chapter we present general background material required in future chapters. It is assumed that the reader has a basic understanding of linear systems theory such as controllability and observability (see for instance [51]) and is familiar with $\mathcal{H}_\infty$ control techniques (see [23]).

In Section 2.2 state-space descriptions for linear, time invariant, systems are presented. Particular attention is paid to Hankel singular values and their relevance to the controllability and observability of a system. Definitions related to function spaces are contained in Section 2.3. The interpretation of the $\mathcal{H}_\infty$ norm for stability robustness and disturbance attenuation is also discussed. In Section 2.4 inner functions are introduced. Coprime and spectral factorisations are then discussed in sections 2.5 and 2.6 respectively. In Section 2.7 the use and interconnection of linear fractional transformations in control theory is presented. Finally, in Section 2.8, we give a parameterisation of all stabilising controllers which satisfy a $\mathcal{H}_\infty$ norm bound on a certain linear, time invariant, closed loop transfer function.

2.2 State-Space Systems

This dissertation is mainly concerned with the design of controllers for plants which may be described by a set of linear, time invariant, finite dimensional, differential equations. Such systems may be represented by the following state-space realisation,

$$\dot{x}(t) = A\,x(t) + B\,u(t) \tag{2.1}$$

$$y(t) = C\,x(t) + D\,u(t) \tag{2.2}$$

where $x(t) \in \mathbb{R}^n$ is called the state vector, $u(t) \in \mathbb{R}^m$ is the input and $y(t) \in \mathbb{R}^p$ is the output. State matrices A, B, C, D are real, constant and of the appropriate dimension. If $p = m = 1$ then the system is called scalar or SISO (single input-single output), otherwise it is called multivariable or MIMO (multiple input-multiple output). For convenience, a state-space system given by equations (2.1) and (2.2) will be denoted by the quadruple (A, B, C, D), or (A, B, C) if the system is assumed to be strictly proper.

Given $U(s)$ and $Y(s)$, the Laplace transforms of $u(t)$ and $y(t)$ respectively, and assuming zero initial conditions, the transfer function matrix from input to output is defined as $G(s)$ where

$$Y(s) \;=\; G(s)U(s).$$

From this definition we have

$$G(s) \;=\; C(sI - A)^{-1}B \;+\; D. \tag{2.3}$$

We adopt the convention of writing the transfer function of equation (2.3) as $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. For ease of notation, the Laplace transform variable s and the time variable t are usually omitted. The domain in which we are operating will be clear from the context.

Of all possible realisations (A, B, C, D) of G, those realisations with the smallest state dimension are called *minimal*. This is equivalent to (A, B) being controllable and (C, A) being observable. Given that (A, B, C, D) is minimal, the poles of G are defined as being the eigenvalues of A, and are closely related to the behaviour of the system. The system G is said to be stable if $\mathrm{Re}(\lambda_i(A)) < 0$, and minimum phase if $\mathrm{Re}(z) < 0$ where z is a transmission zero of the system.

We require the following results on transfer function matrices.

Definition 2.2.1 *The conjugate transpose of $G(s)$ is denoted*

$$G^{\sim}(s) \;\triangleq\; \left[\begin{array}{c|c} -A' & C' \\ \hline -B' & D' \end{array}\right].$$

When evaluated on the $j\omega$ axis, $[G(j\omega)]' = G^{\sim}(j\omega)$.

Definition 2.2.2 *$G^{\dagger}$ is a right (left) inverse of G if $GG^{\dagger} = I$ ($G^{\dagger}G = I$). Suppose D has full row (column) rank and $DD^{\dagger} = I$ ($D^{\dagger}D = I$). Then*

$$G^{\dagger}(s) \;=\; \left[\begin{array}{c|c} A - BD^{\dagger}C & -BD^{\dagger} \\ \hline D^{\dagger}C & D^{\dagger} \end{array}\right]$$

is a right (left) inverse of G.

2.2.1 Hankel Singular Values

Consider a stable transfer function G with realisation (A, B, C, D). Then the controllability Gramian, P, and the observability Gramian, Q, of G are defined as

$$P \;\triangleq\; \int_0^{\infty} e^{At}BB'e^{A't} \geq 0 \tag{2.4}$$

$$Q \;\triangleq\; \int_0^{\infty} e^{A't}C'Ce^{At} \geq 0. \tag{2.5}$$

By direct substitution it is easily verified that P and Q satisfy the following Lyapunov equations

$$\begin{aligned} AP + PA' + BB' &= 0 \\ A'Q + QA + C'C &= 0. \end{aligned}$$

Furthermore, (A, B) is controllable if and only if $P > 0$, and (C, A) is observable if and only if $Q > 0$.

In the state-space the ellipsoids $x'P^{-1}x \leq 1$ and $x'Qx \leq 1$ are called the controllability ellipsoid and the observability ellipsoid respectively. The controllability ellipsoid represents the set of states which can be reached from zero initial conditions with a unit energy input. The observability ellipsoid represents the set of states which, as initial conditions, would produce a unit energy output when released from rest. The length and direction of the ellipsoid semiaxes are closely related to the Gramian eigenvalues and eigenvectors. More precisely, the length of the controllability ellipsoid semiaxes are the square root of the controllability Gramian eigenvalues, with direction corresponding to its eigenvectors. Similarly, the length of the observability ellipsoid semiaxes are the inverse of the square root of the observability Gramian eigenvalues, with direction corresponding to its eigenvectors. Consequently, uncontrollable modes (i.e. those possessing zero controllability Gramian eigenvalues) will correspond to zero length semiaxes of the controllability ellipsoid, while unobservable modes (i.e. those possessing zero observability Gramian eigenvalues) will possess infinite length semiaxes on the observability ellipsoid. Obviously eigenvalues can therefore be used to give an indication of the controllability or observability of any particular mode. (For a more detailed discussion of this geometric interpretation see Moore [68] or Enns [17]).

The eigenvalues of the product PQ are invariant under invertible state transformations and hence form a set of input-output invariants.

Definition 2.2.3 *Using the notation introduced above, the Hankel singular values of G are defined as*

$$\sigma_i(G) \triangleq \sqrt{\lambda_i(PQ)}.$$

By convention the σ_i are usually written in descending order of magnitude, $\sigma_1 \geq \ldots \geq \sigma_n$.

We will see in Chapter 3 that a priori bounds on the model approximation error can often be written in terms of Hankel singular values. Since such bounds play a central role in determining reduced stability margins and performance degradation when reduced order controllers are substituted for high order controllers [17, 69, 63, 34], Hankel singular values are integral to the techniques of controller approximation.

2.3 Function Spaces

In $\mathcal{H}_\infty$ control the performance of a system is expressed in terms of the $\mathcal{H}_\infty$ norm of the closed loop transfer function. In this section we define what is meant by the term $\mathcal{H}_\infty$ and give two interpretations of an $\mathcal{H}_\infty$ norm bounded transfer function.

In the context of this dissertation we will use the following definitions. (For a more general discussion of signals, systems, and their norms, see for instance [23, 89, 101]).

Definition 2.3.1 *The Hilbert space $\mathcal{L}_2$ is the space of signals, $u(t)$, with bounded energy, i.e.*

$$\int_{-\infty}^{\infty} u(t)'u(t)dt \quad < \quad \infty.$$

The inner product and norm for $u, v \in \mathcal{L}_2$ are given by,

$$\langle u, v \rangle \quad \triangleq \quad \int_{-\infty}^{\infty} u(t)'v(t)dt$$

and $\|u\|_2 = \sqrt{\langle u, u \rangle}$ respectively. $\mathcal{L}_2$ may be decomposed into $\mathcal{L}_{2+}$ and $\mathcal{L}_{2-}$, where $\mathcal{L}_{2+}$ consists of those signals in $\mathcal{L}_2$ which are zero for negative time, and $\mathcal{L}_{2-}$ consists of those signals in $\mathcal{L}_2$ which are zero for positive time.

$\mathcal{L}_2$ may be defined analogously in the frequency domain by considering the Fourier transform of u (and v). The inner product is then defined as an integration along the length of the imaginary axis. In the time domain, $\mathcal{L}_2$ may be decomposed into $\mathcal{H}_2$ and $\mathcal{H}_2^-$, where $\mathcal{H}_2$ consists of those signals whose inverse Fourier transform is in $\mathcal{L}_{2+}$, and $\mathcal{H}_2^-$ consists of those signals whose inverse Fourier transform is in $\mathcal{L}_{2-}$. Hence by $u \in \mathcal{H}_2$ we will mean that u is a signal with bounded energy which when considered in the time domain is zero for negative time.

For the study of linear, time varying (LPV) systems in Chapter 6 we also require the space of extended $\mathcal{L}_{2e}$ signals. A signal u is said to be in the extended $\mathcal{L}_{2e}$ space if $\Pi_T u \in \mathcal{L}_{2+}$ where Π_T is the truncation operator,

$$\Pi_T u(t) \quad = \quad \begin{cases} u(t) & t \leq T \\ 0 & t > T. \end{cases}$$

We will make extensive use of the $\mathcal{L}_\infty$ and $\mathcal{H}_\infty$ function spaces.

Definition 2.3.2 *The Banach space $\mathcal{L}_\infty$ is the space of matrix valued functions that are (essentially) bounded on the imaginary axis. The $\mathcal{L}_\infty$ norm is defined as,*

$$\|G\|_\infty \triangleq ess \sup_{\omega \in \mathbb{R}} \bar{\sigma}(G(j\omega)).$$

In state-space theory, $\mathcal{L}_\infty$ corresponds to those proper transfer functions which have no poles on the $j\omega$ axis.

Definition 2.3.3 *$\mathcal{H}_\infty$ is the (closed) subspace of functions in $\mathcal{L}_\infty$ that are analytic and bounded in the open right half plane. The $\mathcal{H}_\infty$ norm is defined as*

$$\|G\|_\infty \triangleq ess \sup_{Re(s)>0} \bar{\sigma}(G(s)) = ess \sup_{\omega \in \mathbb{R}} \bar{\sigma}(G(j\omega)).$$

Definition 2.3.4 *$\mathcal{H}_\infty^-$ is the (closed) subspace of functions in $\mathcal{L}_\infty$ that are analytic and bounded in the open left half plane.*

$$\|G\|_\infty \triangleq ess \sup_{Re(s)<0} \bar{\sigma}(G(s)) = ess \sup_{\omega \in \mathbb{R}} \bar{\sigma}(G(j\omega)).$$

In state-space theory, $\mathcal{H}_\infty$ ($\mathcal{H}_\infty^-$) corresponds those transfer functions which have no poles on the imaginary axis or in the open right (left) half plane. Hence $G \in \mathcal{H}_\infty$ implies that G is stable and $G \in \mathcal{H}_\infty^-$ implies that G is antistable. The notation $G \in \mathcal{H}_{\infty,(k)}^-$ will be used to denote functions which have no poles on the imaginary axis and at most k poles in the open left half plane.

We will be considering functions which are real and rational. The prefix $\mathcal{R}$ is used to denote such functions. Note that if G is real and rational then $G \in \mathcal{RH}_\infty$ if and only if G is proper (i.e. finite for all ω), and stable [23].

Note that all $G(s) \in \mathcal{RL}_\infty$ may be decomposed such that

$$G(s) = G_s(s) + G_{un}(s)$$

where $G_s(s) \in \mathcal{RH}_\infty$ and $G_{un}(s) \in \mathcal{RH}_\infty^-$. We will use the notation $G_s(s) = [G(s)]_+$ and $G_{un}(s) = [G(s)]_-$.

A bounded $\mathcal{H}_\infty$ norm has two interpretations of interest in a control framework [69] ; firstly in specifying a level of stability robustness to modeling uncertainty; and secondly an input/output interpretation specifying a level of disturbance attenuation.

2.3.1 Stability Robustness

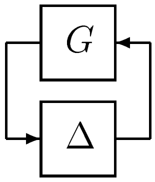


Figure 2.1: Closed Loop for Small Gain Theorem

The stability robustness interpretation of $\|G\|_\infty < \gamma$ relies upon the small gain theorem. First introduced by Zames in [98] it is a simple, but conservative, method for determining closed loop stability.

Theorem 2.3.5 (Small Gain Theorem, [98]) *Consider the block diagram of Figure 2.1 where both G and Δ are in $\mathcal{RH}_\infty$. Then the closed loop is (internally) stable if $\forall\, \omega$*

$$\bar{\sigma}(\Delta)\bar{\sigma}(G) < 1.$$

Alternatively, if $G \in \mathcal{RH}_\infty$ and $\|G\|_\infty < \gamma$ then $\Delta \in \mathcal{RH}_\infty$ and $\|\Delta\|_\infty \leq \gamma^{-1}$ is a sufficient condition to guarantee stability of the closed loop.

Δ is often interpreted as the uncertainty between a model for a system and the actual system. For this interpretation a $\mathcal{H}_\infty$ norm bound of γ implies this level of stability robustness should a controller designed using the model be connected to the actual system. Note that although we have restricted G and Δ to be real rational and stable, the above results hold for infinite dimensional systems and (induced) norm bounded, non-linear, time varying perturbations.

2.3.2 Disturbance Attenuation

From an input/output point of view, the $\mathcal{H}_\infty$ norm represents the maximum energy gain from bounded energy inputs to bounded energy outputs. That is, assume w is an input signal with bounded energy and $G \in \mathcal{RH}_\infty$, then the output Gw will be bounded and

$$\|G\|_\infty = \sup_{\|w\|_2 \le 1} \{\|Gw\|_2\}.$$

Therefore, if we interpret w as a (bounded energy) disturbance signal, $\|G\|_\infty < \gamma$ implies a certain level of disturbance attenuation.

2.4 Inner Functions

Inner transfer functions are used extensively in both $\mathcal{H}_\infty$ synthesis [14] and model approximation [34, 36, 52].

Central to the approximation techniques of Glover [34, 36] is the ability to calculate an approximation which has an all-pass error. This all-pass characteristic leads to approximants which are optimal in the Hankel norm. Similarly, the error bound for balanced truncation approximation may be derived using an all-pass dilation procedure. (Optimal Hankel norm and balanced truncation approximations will be reviewed in more detail in Chapter 3.)

Further, inner functions are essential to the $\mathcal{H}_\infty$ controller design techniques of Doyle et al. [14]. In [14] it is shown, via a change of variable, that the general output feedback problem may be transformed to an equivalent, but simpler problem, which is related to the original by an inner function. Every solution to the simplified problem is also a solution to the original problem.

Definition 2.4.1 *A stable, $p \times m$ transfer function G, where $p \ge m$, is said to be inner if $G^{\sim}G = I$.*

Definition 2.4.2 *A stable, $p \times m$ transfer function G, where $p \le m$, is said to be co-inner if $GG^{\sim} = I$.*

The following state space results are well known [14, 34, 63].

Lemma 2.4.3 *Assume G is a stable, $p \times m$ $(p \ge m)$ transfer function with minimal realisation (A, B, C, D) and observability Gramian $Q = Q' > 0$ satisfying equation (2.5). Then G is inner if and only if* *i)* $D'C + B'Q = 0$

ii) $D'D = I$

Lemma 2.4.4 *Assume G is a stable, $p \times m$ ($p \leq m$) transfer function with minimal realisation (A, B, C, D) and controllability Gramian $P = P' > 0$ satisfying equation (2.5). Then G is co-inner if and only if*

i) $DB' + CP = 0$

ii) $DD' = I$

Notice that if G is partitioned as $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ then G is inner if and only if

$$\begin{aligned} G_{11}^{\sim} G_{11} + G_{21}^{\sim} G_{21} &= I \\ G_{11}^{\sim} G_{12} + G_{21}^{\sim} G_{22} &= 0 \\ G_{12}^{\sim} G_{12} + G_{22}^{\sim} G_{22} &= I \end{aligned} \tag{2.6}$$

and co-inner if and only if

$$\begin{aligned} G_{11} G_{11}^{\sim} + G_{12} G_{12}^{\sim} &= I \\ G_{11} G_{21}^{\sim} + G_{12} G_{22}^{\sim} &= 0 \\ G_{21} G_{21}^{\sim} + G_{22} G_{22}^{\sim} &= I. \end{aligned} \tag{2.7}$$

The importance of inner functions stems from their norm preserving properties. If the input to an inner function has bounded energy then the output will be bounded by the same energy level. The properties of inner functions are discussed further in Section 2.7.1.

2.5 Coprime Factorisations

As with inner functions, coprime factorisations are used extensively in both $\mathcal{H}_\infty$ synthesis [63, 89] and model approximation [3, 32, 58, 60, 64, 65, 67, 72]. A particular advantage of such multiplicative decompositions for model approximation is the stability of the two coprime factors.

In this section we define what is meant by a left coprime factorisation of a transfer function and present some properties of such a fractional representation. Although not stated here, dual results hold for right coprime factorisations. (A detailed discussion of coprime factorisations may be found in [89].)

Definition 2.5.1 ([64], Definition 2.16) *The pair $\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix}$, where $\tilde{U}$, $\tilde{V} \in \mathcal{RH}_\infty$, constitute a left coprime factorisation of K if and only if*

(a) $\tilde{V}$ *is square,* $det(\tilde{V}) \neq 0$

(b) $K = \tilde{V}^{-1}\tilde{U}$

(c) $\tilde{U}$ and $\tilde{V}$ are left coprime, i.e., there exists M, $N \in \mathcal{RH}_\infty$ such that $\tilde{V}M + \tilde{U}N = I$.

Left coprime factorisations are not unique. However they are unique up to left multiplication by a *unit* which we now define.

Definition 2.5.2 ([64], definition 2.17) *A square, invertible, transfer function matrix satisfying U, $U^{-1} \in \mathcal{RH}_\infty$ is called a* unit *in $\mathcal{RH}_\infty$.*

One particular factorisation which is of interest is the *normalised* left coprime factorisation.

Definition 2.5.3 ([64], definition 2.19) *The pair $\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix}$, where $\tilde{U}$, $\tilde{V} \in \mathcal{RH}_\infty$, constitute a normalised left coprime factorisation of K if and only if $\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix}$ is a left coprime factorisation of K and*

$$\tilde{U}\tilde{U}^\sim + \tilde{V}\tilde{V}^\sim = I \quad \forall\, s \in j\omega$$

Normalised coprime factorisations are unique up to left multiplication by a unitary matrix.

2.6 Spectral Factorisations

Spectral factorisations may be viewed as the transfer function matrix equivalent of the square root. Given an Hermitian transfer matrix $\Phi = \Phi^\sim > 0$ which has no poles on the $j\omega$ axis, spectral factorisation allows us to calculate a stable, minimum phase spectral factor W such that $\Phi = W^\sim W$.

To construct such a W, firstly use the decomposition $\Phi = [\Phi]_+ + [\Phi]_- + R$ where $[\Phi]_+$ and $[\Phi]_-$ are strictly proper, and define $(A, B, C) \triangleq [\Phi]_+$. Then

$$W = \left[\begin{array}{c|c} A & B \\ \hline F & L \end{array}\right]$$

where $L'L = R$ may be calculated using a Cholesky factorisation, $F = L'^{-1}(B'X + C)$ and X solves the Riccati equation

$$(A - BR^{-1}C)'X + X(A - BR^{-1}C) - XBR^{-1}B'X - C'R^{-1}C = 0.$$

The dual factorisation, $\Phi = WW^\sim$, is calculated by performing a spectral factorisation on the transpose of Φ then taking the transpose of the result.

2.7 Linear Fractional Transformations

Linear fractional transformations (LFTs) provide a convenient framework in which to formulate most control problems. They are particularly useful in $\mathcal{H}_\infty$ control because of the ease with which controllers and/or uncertainty perturbations may be incorporated into the problem setup.

Figures 2.2 and 2.3 show alternative, but equivalent, forms of LFTs. The closed loop transfer function from w to z is referred to as the LFT of Q with coefficient matrix N and may be defined as follows.

Definition 2.7.1 *A lower linear fractional transformation (see Figure 2.2) is denoted:*

$$T_{zw} = \mathcal{F}_l(\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}, Q) \triangleq N_{11} + N_{12}Q(I - N_{22}Q)^{-1}N_{21} \tag{2.8}$$

whenever $\det(I - N_{22}Q) \neq 0$.

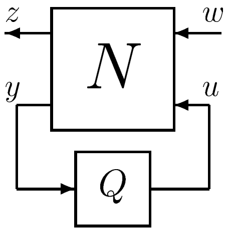


Figure 2.2: Lower Linear Fractional Transformation

Definition 2.7.2 *An upper linear fractional transformation (see Figure 2.3) is denoted:*

$$T_{zw} = \mathcal{F}_u(\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}, Q) \triangleq N_{22} + N_{21}Q(I - N_{11}Q)^{-1}N_{12} \tag{2.9}$$

whenever $\det(I - N_{11}Q) \neq 0$.

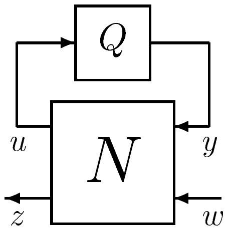


Figure 2.3: Upper Linear Fractional Transformation

Notice that $\mathcal{F}_l(N, Q) = \mathcal{F}_u(\bar{N}, Q)$ where

$$\bar{N} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} N \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

In the sequel we make extensive use of the following lemma on LFT inversion.

Lemma 2.7.3 ([101]) *Using the notation of definition 2.7.1, assume T_{zw} and N are proper,* $\det N(\infty) \neq 0$, $\det(N - \begin{bmatrix} T_{zw} & 0 \\ 0 & 0 \end{bmatrix})(\infty) \neq 0$, *and N_{12} and N_{21} are square and invertible. Then Q is proper and $Q = \mathcal{F}_u(N^{-1}, T_{zw})$.*

An important property of LFTs is that any interconnection of LFTs is also an LFT. A general form of interconnection is called the Redheffer star product [78].

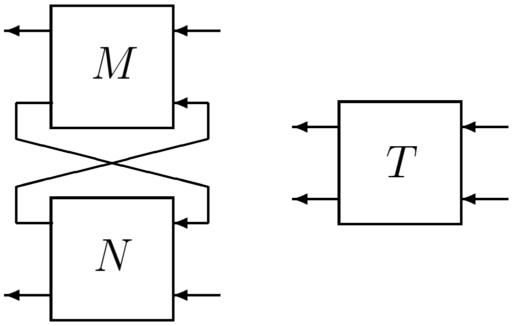


Figure 2.4: Star Product Interconnection

Definition 2.7.4 *Consider the interconnection shown in Figure 2.4. Then the star product of M with N is given by,*

$$\begin{aligned} T &\triangleq \mathcal{S}(M,N) \\ &= \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \\ &= \begin{bmatrix} M_{11}+M_{12}N_{11}(I-M_{22}N_{11})^{-1}M_{21} & M_{12}(I-N_{11}M_{22})^{-1}N_{12} \\ N_{21}(I-M_{22}N_{11})^{-1}M_{21} & N_{22}+N_{21}M_{22}(I-N_{11}M_{22})^{-1}N_{12} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}_l(M,\mathcal{F}_l(N,\bullet)) &= \mathcal{F}_l(T,\bullet) \\ \mathcal{F}_u(N,\mathcal{F}_u(M,\bullet)) &= \mathcal{F}_u(T,\bullet). \end{aligned}$$

For more extensive results on the properties of LFTs and their manipulation see [15, 101].

2.7.1 Inner Functions and LFTs

The following theorem on inner matrices and LFTs is central to our development.

Theorem 2.7.5 ([36],Theorem 2.3 (f) and (g)) *Consider the feedback system of Figure 2.2 where $N_{11} : p_1 \times m_1$, $N_{22} : p_2 \times m_2$ and all other dimensions are compatible. Assume the feedback loop is well posed, i.e.* $\det(I - N_{22}(\infty)Q(\infty)) \neq 0$. *Then*

(i) If rank($N_{21}(j\omega)) = p_2$ $\forall\omega \in \mathbb{R}\cup\infty$ with $N^{\sim}N = I$ then $\|\mathcal{F}_l(N,Q)\|_\infty < 1$ if and only if $\|Q\|_\infty < 1$.

(ii) If rank($N_{12}(j\omega)) = m_2$ $\forall\omega \in \mathbb{R}\cup\infty$ with $NN^{\sim} = I$ then $\|\mathcal{F}_l(N,Q)\|_\infty < 1$ if and only if $\|Q\|_\infty < 1$.

In subsequent chapters we'll encounter the case where N is stable. Assuming the well-posedness condition is satisfied, N_{21} has full row rank (N_{12} has full column rank) at all frequencies and N is inner (co-inner) then Theorem 2.7.5 implies that

$$\{\mathcal{F}_l(N,Q) \in \mathcal{RH}_\infty,\ \|\mathcal{F}_l(N,Q)\|_\infty < 1\} \iff \{Q \in \mathcal{RH}_\infty,\ \|Q\|_\infty < 1\}.$$

2.8 The $\mathcal{H}_\infty$ Sub-optimal Control Problem

In this dissertation we are concerned with the design of (reduced order) controllers which are stabilising and satisfy a pre-specified $\mathcal{H}_\infty$-norm performance bound on the closed loop transfer function. Here we define the problem to be solved and review a widely applied state-space solution.

Consider the following minimal state-space realisation of a finite dimensional, linear, time-invariant system:

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{2.10}$$

where $B_1 \in \mathbb{R}^{n\times m_1}$, $B_2 \in \mathbb{R}^{n\times m_2}$, $C_1 \in \mathbb{R}^{p_1\times n}$ and $C_2 \in \mathbb{R}^{p_2\times n}$. The signals are as follows: $w \in \mathbb{R}^{m_1}$ is the vector of disturbances (e.g. sensor noise and reference signals); $u \in \mathbb{R}^{m_2}$ is the vector of control inputs; $z \in \mathbb{R}^{p_1}$ is the vector of error signals (e.g. tracking errors and penalties on control effort); $y \in \mathbb{R}^{p_2}$ is the vector of control measurements; and $x \in \mathbb{R}^n$ is the state vector. In addition, make the standard assumption that $p_1 \geq m_2$ and $m_1 \geq p_2$.

$P(s)$ is called the *generalised plant*. It includes both the actual plant to be controlled and any frequency dependent weights used in the controller synthesis process [45, 63]. The standard suboptimal $\mathcal{H}_\infty$ control problem is to find all stabilising controllers for the generalised plant, P, such that the $\mathcal{H}_\infty$ norm of the closed loop transfer function, $T_{zw} = \mathcal{F}_l(P, K)$, is less than some prespecified performance level $\gamma > 0$. We shall call this the γ performance problem [7].

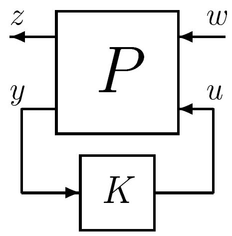


Figure 2.5: General Plant and Controller Interconnection

As in [72], we shall use the following definition for controllers which satisfy the γ performance problem.

Definition 2.8.1 *A controller K is said to be (P, γ)-admissible if K stabilises P and $\|T_{zw}\|_\infty = \|\mathcal{F}_l(P, K)\|_\infty < \gamma$.*

We are concerned here with the problem of finding reduced order (P, γ)-admissible controllers. The starting point for the techniques to be developed in subsequent chapters is the existence of a nominal (high order) (P, γ)-admissible controller. The following section gives necessary and sufficient conditions for the existence of such controllers, and if any exists, a parameterisation of all (P, γ)-admissible controllers. These are expressed in terms of the solution to two Riccati equalities. The parameterisation of all (P, γ)-admissible controllers will be exploited in Chapter 4 to derive sufficient conditions for reduced order controllers to be (P, γ)-admissible. A linear matrix inequality (LMI) based approach, giving necessary and sufficient conditions for the existence of a solution to the γ performance problem, will be presented in Section 3.4.2.

2.8.1 All (P, γ)-admissible Controllers

Provided certain conditions are satisfied, a parameterisation of all (P, γ)-admissible controllers is given by Glover and Doyle in [38]. This involves the solution of two algebraic Riccati equations and is expressed as an LFT on a stable, norm bounded, but otherwise free parameter. The following assumptions are required.

(A1) (A, B_2, C_2, D_{22}) is stabilisable and detectable.

(A2) $D_{22} = 0$.

(A3) $D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $D_{21} = \begin{bmatrix} 0 & I \end{bmatrix}$ and $D_{11} = \begin{bmatrix} D_{1111} & D_{1112} \\ D_{1121} & D_{1122} \end{bmatrix}$,
where $D_{1111} \in \mathbb{R}^{(p_1-m_2)\times(m_1-p2)}$, $D_{1112} \in \mathbb{R}^{(p_1-m_2)\times p_2}$, $D_{1121} \in \mathbb{R}^{m_2\times(m_1-p2)}$ and $D_{1122} \in \mathbb{R}^{m_2\times p_2}$.

(A4) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω.

(A5) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω.

Assumption (A1) is required for the existence of a stabilising controller. Assumption (A2) is made to simplify the following formulae. It may be removed by noting that,

$$\mathcal{F}_l(P, K(I + D_{22}K)^{-1}) \quad = \quad \mathcal{F}_l(P - \begin{bmatrix} 0 & 0 \\ 0 & D_{22} \end{bmatrix}, K)$$

where K is designed with $D_{22} = 0$. The formulae are also considerable simpler if $D_{11} = 0$. A derivation for this case may be found in [14].

Relaxing the rank assumptions on D_{12} and D_{21} in (A3) leads to singular control problems. The LMI based solution reviewed in Section 3.4.2 may be employed to solve such singular problems. In [101] a procedure is given which transforms any plant with D_{12} full column rank and D_{21} full row rank into one which satisfies (A2). Assumptions (A4) and (A5) are required for the corresponding LQG problem to be closed loop asymptotically stable. They are made here for technical reasons.

Before stating the appropriate theorem it is convenient to define the following matrices.

$$R \triangleq D_{1.}'D_{1.} - \begin{pmatrix} \gamma^2 I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{where} \quad D_{1.} \triangleq \begin{bmatrix} D_{11} & D_{12} \end{bmatrix}$$

and

$$\tilde{R} \triangleq D_{.1}D'_{.1} - \begin{pmatrix} \gamma^2 I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{where} \quad D_{.1} \triangleq \begin{bmatrix} D_{11} \\ D_{12} \end{bmatrix}.$$

Also, assuming solutions exist, define X_∞ and Y_∞ as stabilising solutions to the following algebraic Riccati equations,

$$\begin{aligned} &(A - BR^{-1}D'_{1.}C_1)'X_\infty + X_\infty(A - BR^{-1}D'_{1.}C_1) \\ &\qquad -X_\infty BR^{-1}B'X_\infty + C'_1(I - D_{1.}R^{-1}D'_{1.})C_1 = 0 \quad (2.11) \\ &(A - B_1D'_{.1}\tilde{R}^{-1}C)Y_\infty + Y_\infty(A - B_1D'_{.1}\tilde{R}^{-1}C)' \\ &\qquad -Y_\infty C'\tilde{R}^{-1}CY_\infty + B_1(I - D'_{.1}\tilde{R}^{-1}D_{.1})B'_1 = 0 \quad (2.12) \end{aligned}$$

and F and H as state-feedback and output-injection matrices respectively,

$$\begin{aligned} F &= \begin{bmatrix} F_{11} \\ F_{12} \\ F_2 \end{bmatrix} \triangleq -R^{-1}\left[D'_{1.}C_1 + B'X_\infty\right], \\ H &= \begin{bmatrix} H_{11} & H_{12} & H_2 \end{bmatrix} \\ &\triangleq -\left[B_1D'_{.1} + Y_\infty C'\right]\tilde{R}^{-1}. \end{aligned}$$

We are now in a position to state necessary and sufficient conditions for the existence of (P, γ)-admissible controllers and a parameterisation of all such controllers if any exist.

Theorem 2.8.2 ([38], Theorem 1) *For a plant described by equation (2.10) and satisfying assumptions (A1)-(A5):*

(a) There exists an internally stabilising controller K such that $\|\mathcal{F}_l(P, K)\|_\infty < \gamma$ if, and only if,
(i) $\gamma > max(\bar{\sigma}[D_{1111} D_{1112}], \bar{\sigma}[D'_{1111} D'_{1121}])$
and
(ii) there exists $X_\infty \geq 0$ and $Y_\infty \geq 0$ satisfying equations (2.11) and (2.12) respectively and such that $\rho(X_\infty Y_\infty) < \gamma^2$.

(b) Given that the conditions of part (a) are satisfied, then all rational internally stabilising controllers K satisfying $\|\mathcal{F}_l(P, K)\|_\infty < \gamma$ are given by the lower LFT $K = \mathcal{F}_l(K_a, \Phi)$, where $\Phi \in \mathcal{RH}_\infty$, such that $\|\Phi\|_\infty < \gamma$ and K_a is given by

$$\begin{aligned} K_a &\triangleq \begin{bmatrix} K_{a11} & K_{a12} \\ K_{a21} & K_{a22} \end{bmatrix} = \left[\begin{array}{c|cc} A_a & B_{a1} & B_{a2} \\ \hline C_{a1} & D_{a11} & D_{a12} \\ C_{a2} & D_{a21} & 0 \end{array}\right] \\ D_{a11} &= -D_{1121}D'_{1111}(\gamma^2 I - D_{1111}D'_{1111})^{-1}D_{1112} - D_{1122} \end{aligned}$$

$D_{a12} \in \mathbb{R}^{m_2 \times m_2}$ and $D_{a21} \in \mathbb{R}^{p_2 \times p_2}$ are any matrices satisfying

$$\begin{aligned} D_{a12}D'_{a12} &= I - D_{1121}(\gamma^2 I - D'_{1111}D_{1111})^{-1}D'_{1121} \\ D'_{a21}D_{a21} &= I - D'_{1112}(\gamma^2 I - D_{1111}D'_{1111})^{-1}D_{1112} \end{aligned}$$

and

$$\begin{aligned}
B_{a2} &= (B_2 + H_{12})D_{a12} \\
C_{a2} &= -D_{a21}(C_2 + F_{12})Z \\
B_{a1} &= -H_2 + B_{a2}D_{a12}^{-1}D_{a11} \\
C_{a1} &= F_2 Z + D_{a11}D_{a21}^{-1}C_{a2} \\
A_a &= A + HC + B_{a2}D_{a12}^{-1}C_{a1}
\end{aligned}$$

where

$$Z = (I - \gamma^{-2}Y_\infty X_\infty)^{-1}.$$

Remark 2.8.3 *D_{12a}, D_{21a}, and hence K_a, are invertible.*

Remark 2.8.4 *$A_a - B_{2a}D_{12a}^{-1}C_{1a}$ and $A_a - B_{1a}D_{21a}^{-1}C_{2a}$ are both stable, hence $K_{a12}^{-1}, K_{a21}^{-1} \in \mathcal{RH}_\infty$.*

Definition 2.8.5 *Assume the conditions of Theorem 2.8.2 are satisfied, then K_{a11} is called the central controller.*

Chapter 3

Principles of Controller Approximation

3.1 Introduction

For practical reasons it is desirable in most control situations to design simple linear controllers. Not only are the dynamics of such controllers easy to understand and manipulate, but they require less powerful hardware and simpler software to implement. This leads to (possibly) lighter, smaller and cheaper control systems with quicker processing times. Further, since there are likely to be fewer hardware or software bugs, higher integrity can be expected.

As argued by Hyde [45], controller complexity may be measured in terms of the number of calculations required per unit time, the number of controller parameters required to be stored and updated, and by the size of the controller state-dimension. Hyde shows that when there is a requirement to switch between several controllers as the plant operating point changes, a modal realisation (see Definition 3.2.1) for the controllers is particularly convenient. This form of realisation greatly reduces both computational and memory requirements.

The problem of reducing complexity via reducing the state-dimension has received much attention over the last 30 years. See [3, 29, 94] for a comprehensive list of references. However, until recently, few of these approaches considered the fundamental relationship between the model reduction problem and the controller synthesis problem. In this dissertation we investigate this relationship and extend current procedures by deriving sufficient conditions for reduced order controllers to satisfy a prespecified closed loop performance.

The design of reduced order controllers for high order plants may be divided into three categories [3]: direct design via optimisation of controller parameters; model reduction followed by the design of a low order controller using the reduced order model; and design of a high order controller from the high order model, followed by reduction of the high order controller. These three paths are shown in Figure 3.1.

Each of the three paths may be split into two categories: firstly those dealing with strictly open loop considerations – making no reference to how stability or other appropriate closed loop objectives may be effected by the use of such reduced order approximations; and secondly those techniques which look at controller approximation in a closed loop framework. Here

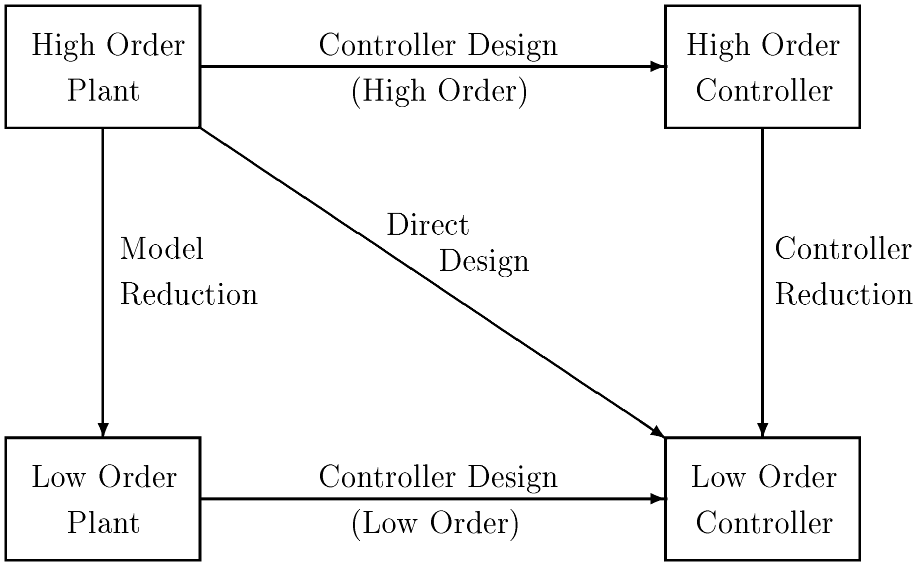


Figure 3.1: Three Avenues for Controller Reduction

we are particularly interested in the techniques which fall into the latter category. Our goal is to derive conditions which guarantee (a priori) that reduced order controllers will stabilise the closed loop and preserve an $\mathcal{H}_\infty$ norm bound on the closed loop transfer function. We achieve this goal by incorporating information about the closed loop into frequency weights and then undertaking frequency weighted model reduction.

Direct design procedures [28, 46, 47, 48, 61] are important since unlike the other two paths they allow the designer to impose not only a degree constraint on the solution but also a specific structure. This is particularly useful if switching is required between several controllers [45]. However, the techniques employed are slightly ad-hoc since it is often not clear what constitutes an appropriate structure to impose, nor what constitutes an appropriate degree constraint. Further, the selection of a suitable initial guess for any parameter optimisation is not often straightforward. Since any parameter optimisation tends to be non-convex there is no guarantee that a global optimal solution will be attained.

There is a fundamental relationship between the model reduction problem and the controller design problem [3, 17, 94, 96]. Since model reduction followed by the design of a low order controller using the reduced order model neglects to take this relationship into account it is generally considered an unsatisfactory approach. In addition, since approximation is undertaken at an earlier stage in the design process, propagation of errors into later stages is more likely to result in a controller giving poor closed loop performance. Iterative schemes, which alternate between model and controller reduction, such as that proposed by Wortelboer et al. [94, 96], may help overcome such drawbacks.

The approach to be taken in this dissertation is that of designing a high order controller from a high order plant, followed by reduction of the high order controller. Information about the closed loop is incorporated into frequency weights and corresponds to requiring the modeling error to be smaller in selected frequency bands, e.g. around crossover. Hence the salient features of the high order controller are retained at the frequencies of most interest.

Before discussing controller approximation in a closed loop framework a review of the most significant open loop model reduction techniques and their frequency weighted extensions is warranted. These techniques are the basis of modern model approximation. In Section 3.2 we present results on open loop model approximation. These generally fall into three categories [94]: parameter matching, projection of dynamics and minimum distance approximation. Those open loop techniques which are readily extended to include frequency weights are discussed in Section 3.3. Included is a new derivation of a sufficient condition for reduced order controllers to maintain stability. To conclude, in Section 3.4 we review controller approximation, with particular reference to closed loop performance degradation when a reduced order controller is substituted for a high order controller.

3.2 Open Loop Approximation

Given a stable model $G(s) \in \mathcal{RH}_\infty$ of McMillan degree n, the open loop approximation problem is to calculate $\hat{G}(s) \in \mathcal{RH}_\infty$ of McMillan degree k $(k < n)$ achieving

$$\min_{\hat{G}(s)\in\mathcal{RH}_\infty} \|G(s) - \hat{G}(s)\|_\infty.$$

For a unit energy input, this is equivalent to finding $\hat{G}(s)$ such that the worst case error has finite energy and is minimised.

Note that there exists a lower bound for this type of approximation problem,

$$\sigma_{k+1}(G) \leq \inf_{\hat{G}(s)\in\mathcal{RH}_\infty} \|G(s) - \hat{G}(s)\|_\infty. \tag{3.1}$$

No reduction technique can give an approximation better than this bound. Many reduction techniques also satisfy an upper bound on the approximation error which may be calculated a priori. The tightness between the upper bound for a particular reduction technique and the general lower bound gives an indication of how well the technique may perform. Intuitively, good approximation techniques are those which are computationally simple, are systematic, and provide a low-order model whose response accurately matches that of the high order model.

Parameter matching methods attempt to identify and retain dominant model parameters. The assumption being that by neglecting parameters which contribute little to the fundamental behaviour of the system a suitable reduced order model may be obtained. Such essential parameters may include time moments, Markov parameters and eigenvalues. The methods of Padé [83] and Routh [44] fall into this category. Although these techniques tend to be computationally very simple, approximation by such methods tends to be somewhat ad hoc. The main criticisms being that there is no clear choice of which parameters are the best to retain, nor is there an a priori guarantee of how good the approximation will be. Given a stable model, classical Padé approximation doesn't even guarantee to calculate a stable approximant [83]. The choice of which parameters are essential, and hence should be retained, depends heavily on the specific problem formulation and designer preference, while the lack of performance guarantee leads to a (possibly) time consuming trial and error approach.

The second category of approximation techniques covers such methods as modal reduction and balanced truncation [68]. The common idea is to transform any given realisation of $G(s)$ into an appropriate, essentially unique, realisation, then truncate those parts of the new realisation considered to contribute least to system behaviour. This successive application of a transformation followed by a truncation is referred to as projection of dynamics [94]. Although these methods do not attempt to optimise with respect to any criterion they are computationally simple and generally work well. In addition, balanced truncation has been shown [17, 34] to satisfy an upper bound on the $\mathcal{H}_\infty$ norm of the error (see Section 3.2.2). Hence they are used widely in practice.

Techniques which attempt to directly minimise the distance between $G(s)$ and $\hat{G}(s)$ comprise the third major category of model reduction procedures. Kavranoğlu and Bettayeb [52] present an implicit solution to the optimal $\mathcal{H}_\infty$ approximation problem, however there are no algorithms for explicitly computing an optimal minimising solution. A closed form solution to the optimal Hankel norm approximation problem, which satisfies a priori upper and lower bounds on the $\mathcal{H}_\infty$ norm of the error, is available and works well.

3.2.1 Modal Reduction

The idea behind modal approximation is to partition a given state-space realisation into two subsystems, one containing dominant (slow) modes, the other non-dominant (fast) modes, followed by the elimination of the non-dominant subsystem. This elimination may be undertaken by a number of techniques [8, 12, 29].

Modal reduction is closely related to truncation of a partial fraction expansion of a given transfer function. This may be represented in state space form as a truncation of a Jordan form realisation [94].

Definition 3.2.1 ([51] Modal Realisation) *The realisation (A, B, C, D) of a transfer function G is said to be in Jordon form if its A matrix is of the form $A = diag(J_1, \ldots, J_N)$ where,*

$$J_i = \begin{bmatrix} \lambda_i & * & & \\ & \lambda_i & \ddots & \\ & & \ddots & * \\ & & & \lambda_i \end{bmatrix} \in \mathbb{C}^{n_i \times n_i},$$

each λ_i is an eigenvalues of A, n_i is the number of times λ_i occurs, and $*$ is either 1 or 0 depending on other properties of A.

Any realisation can be transformed such that it's in a Jordon form realisation. In general there is no particular significance attached to the ordering of the Jordan blocks. However for the purposes of approximation they are required to be ordered such that the norm of the partial fractions associated with each block are in descending order.

Assume $G(s)$ has been transformed to the Jordon form realisation,

$$\begin{aligned} G &= \left[\begin{array}{ccc|c} J_1 & & & B_1 \\ & \ddots & & \vdots \\ & & J_n & B_n \\ \hline C_1 & \ldots & C_n & 0 \end{array}\right] \\ &= G_1 + \ldots + G_k + G_{k+1} + \ldots + G_n \end{aligned}$$

where $G_i \triangleq \left[\begin{array}{c|c} J_i & B_i \\ \hline C_i & 0 \end{array}\right]$ and the J_i are ordered such that $\|G_i\|_\infty$ are in descending order. Then $\hat{G} = G_1 + \ldots + G_k$ is an approximation to G such that

$$\|G - \hat{G}\|_\infty \leq \sum_{i=k+1}^{n} \|G_i\|_\infty.$$

A drawback of modal approximation is that the algorithms to transform G into Jordan form are often numerically unstable [51]. For a more general discussion of modal reduction see [94].

3.2.2 Balanced Truncation

The basis of balanced truncation is the elimination of those states which are least controllable and least observable. The contribution of such states to the overall system input-output response is considered to be negligible and hence they may be ignored without undue loss of accuracy. However, in general, a strongly controllable state may be weakly observable or vice versa. Moore [68] showed how this problem may be overcome by deriving a similarity transformation for stable systems such that the controllability and observability Gramians of the transformed co-ordinates are equal (or *balanced*). Taking the geometric view of Section 2.2.1, the semiaxes of the controllability and observability ellipsoids will be lined up. Since the length of the semiaxes represent the controllability and observability of any particular state, truncation of those states which are the most weakly controllable and observable leads to an acceptable reduced order model.

Definition 3.2.2 ([68]) *A transfer function $G \in \mathcal{RH}_\infty$ with minimal realisation (A, B, C, D) is balanced if its controllability and observability Gramians, P and Q respectively, satisfy $P = Q = \Sigma \triangleq diag(\sigma_1, \ldots, \sigma_n)$. (See [34, 62, 68] for a transformation to take a general realisation to a balanced realisation.)*

We now present some results on balanced realisations. Consider the following conformal partitioning of a balanced realisation.

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \tag{3.2}$$

where $\Sigma_1 \in \mathbb{R}^{k \times k}$ and $\Sigma_2 \in \mathbb{R}^{(n-k)\times(n-k)}$.

The following results, stated without proof, apply.

Lemma 3.2.3 ([75]) *Provided $G(s) \in \mathcal{RH}_\infty$ then subsystems (A_{ii}, B_i, C_i) are balanced with Gramian Σ_i $(i = 1, 2)$.*

Lemma 3.2.4 ([75]) *Provided $G(s) \in \mathcal{RH}_\infty$ then subsystems (A_{ii}, B_i, C_i) are asymptotically stable if Σ_i $(i = 1, 2)$ have no common entries.*

Balanced truncation requires truncation of those states which are least controllable and observable. For a balanced realisation this corresponds to truncating those states corresponding to small σ_i. Therefore, for a k^{th} order approximation we would ideally like $\sigma_k \gg \sigma_{k+1}$. Provided $\sigma_k > \sigma_{k+1}$ (and $G(s)$ is stable), then by Lemmas 3.2.3 and 3.2.4, the subsystem (A_{11}, B_1, C_1) is guaranteed to be balanced and asymptotically stable. $\hat{G}(s)$ with realisation (A_{11}, B_1, C_1) is therefore one possible approximation to $G(s)$.

Although not attempting to optimise any particular cost function, Mustafa and Davidson [71] show that for the specific class of stable models with $A = A'$ and $BB' = C'C = I$ and $n \leq \min\{p, m\}$, balanced truncation is optimal in the Hankel, Hilbert-Schmidt, nuclear and $\mathcal{L}_2$ norms. For general $G(s) \in \mathcal{RH}_\infty$, balanced truncation produces approximants which satisfy the following error bound.

Lemma 3.2.5 ([17, 34]) *For $G(s) \in \mathcal{RH}_\infty$ with balanced realisation (A, B, C) and $\hat{G}(s)$ with realisation (A_{11}, B_1, C_1) an upper bound on the modeling error is given by* twice the sum of the tail *of truncated Hankel singular values.*

$$\|G(s) - \hat{G}(s)\|_\infty \leq 2 \sum_{i=k+1}^{n} \sigma_i = 2tr(\Sigma_2).$$

In fact, more can be said about the error incurred by use of the balance and truncate technique. Such reduced order models possess a good match of the impulse response, however, the step response tends to give steady state errors close to the above upper bound [88]. (For $k = n - 1$ the upper bound is achieved at DC [17].) The technique is therefore better for approximation at high frequency than at low frequency, which is not necessarily desirable as most control systems operate at low or middle frequencies.

Consequently a number of alternative reduction techniques, all employing balanced realisations (but aiming for good low frequency approximation) have been researched since Moore's initial work of 1981. Perhaps the best known of these are those employing a reciprocal transformation [20, 21, 22, 59, 88]. Davidson [9] and Kabamba [50] also proposed alternative techniques in which those states contributing least to the impulse response norm are truncated rather than those states which are least controllable and observable.

3.2.3 Optimal Hankel Norm Approximation

Although approximations obtained using balanced truncation possess many desirable properties they aren't generally optimal in any sense. Ideally we would like an approximation which minimises the infinity norm of the error between $G(s)$ and $\hat{G}(s)$. In [52], Kavranoğlu and Bettayeb give an implicit solution to this problem using an allpass embedding procedure. Unfortunately, except for models with a very specific diagonal structure, as yet there is no closed form solution. However, a closed form solution to the optimal Hankel norm approximation problem does exist [34, 36] and provides a satisfactory solution to the infinity norm approximation problem. A characterisation of all optimal Hankel norm approximations may be found in [34, 36, 62].

The existence of an upper bound on the $\mathcal{L}_\infty$ norm of the error is a desirable property of any model reduction technique. For balanced truncation this upper bound was seen to be given by *twice the sum of the tail* of truncated Hankel singular values (see Lemma 3.2.5), the next lemma implies a similar result for optimal Hankel norm approximations.

Lemma 3.2.6 ([34]) *Let $G(s) \in \mathcal{RH}_\infty$ be strictly proper with Hankel singular values $\sigma_1 > \sigma_2 \ldots > \sigma_N$ where each σ_i has multiplicity r_i, and $\hat{G}(s) \in \mathcal{RH}_\infty$ be a strictly proper k^{th} order optimal Hankel norm approximation to $G(s)$. Then*

$$\|G - \hat{G}\|_\infty \leq 2(\sigma_{k+1} + \sigma_{k+2} + \ldots + \sigma_N)$$

and $\exists\ \hat{D}$ such that

$$\|G - (\hat{G} + \hat{D})\|_\infty \leq (\sigma_{k+1} + \sigma_{k+2} + \ldots + \sigma_N).$$

The above result shows that for optimal Hankel norm approximations the upper bound can be tightened (compared to the balanced truncation approach) in two respects: firstly, by a suitable choice of a $\hat{D}$ matrix the upper bound can be reduced to *the sum of the tail* of the truncated Hankel singular values; and secondly that repeated singular values need only be counted once.

Given the additional degree of freedom here (in the choice of the $\hat{D}$ matrix) it is not surprising that a reduced error bound is derived. However care must be taken as the introduction of this feed through term implies the approximation is no longer strictly proper.

3.2.4 Fractional Approaches

A drawback of the reduction techniques discussed so far is that they assume the model to be approximated is stable. However, it is possible to successfully approximate unstable systems. A (simple) method is to decompose $G \in \mathcal{RL}_\infty$ into a stable and unstable part, $G_s \in \mathcal{RH}_\infty$ and $G_{un} \in \mathcal{RH}_\infty^-$ respectively, such that $G = G_s + G_{un}$, then approximate the stable part by $\hat{G}_s$, before combining $\hat{G}_s$ and G_{un} to obtain an approximation to the original model, $\hat{G} = \hat{G}_s + G_{un}$. If G has $j\omega$-axis poles then they may also be copied directly into $\hat{G}$ but must have precisely the same residues.

The implication here is that no approximation can be undertaken on the unstable part of the model. The approximant is therefore constrained to possess the same unstable poles as the original model. However, there is no reason why good approximations should possess this property.

An alternative to the above procedure is to consider a fractional description, the basis of which is multiplicative decomposition rather than additive decomposition. Fraction approaches to model reduction have received much attention in the literature. The procedure presented next is variously called *fractional balanced reduction* by Meyer [67] and *coprime* *factor model reduction* by Liu and Anderson [58, 3] and McFarlane et al. [63, 64, 65]. It is closely related to the *Riccati balanced model reduction* of Jonckheere and Silverman [49].

Since the factors in a fractional representation are stable by construction, they allow use of the previous reduction techniques without the additive decomposition suggested above. This means that the number of both stable and unstable poles can be reduced. In addition, for a specific configuration, the use of a fractional approach allows derivation of a priori conditions for reduced order controllers to stabilise the closed loop. An indication of the degradation in closed loop performance is also obtainable (see Section 3.4).

The coprime factor model reduction procedure is as follows.

Procedure 3.2.7

a) Write $G = \tilde{M}^{-1}\tilde{N}$ of degree n where $\begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix}$ constitute a left coprime factorisation of G.

b) Approximate $\begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix}$ by $\begin{bmatrix} \tilde{N}_r & \tilde{M}_r \end{bmatrix}$ of degree k ($k < n$) using an appropriate model reduction technique.

c) Form the reduced order approximation $\hat{G} = \tilde{M}_r^{-1}\tilde{N}_r$

Given that a normalised coprime factorisation is chosen in step a), and balanced truncation is chosen in step b), the above procedure produces results which are equivalent to the *Riccati balanced model reduction* method of Jonckheere and Silverman [49]. For such a setup the reduced order $\begin{bmatrix} \tilde{N}_r & \tilde{M}_r \end{bmatrix}$ is guaranteed to be normalised and possess Hankel singular values identical to the first k Hankel singular values of $\begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix}$ [67]. This property may be exploited to derive stability preservation results in terms of the Hankel singular values of $\begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix}$.

3.3 Frequency Weighted Approximation

Controller reduction is a distinct problem from standard open loop model approximation as discussed in the preceding section. As argued in [3], if rationally based, controller reduction schemes must take account of the plant to which the controller is ultimately to be connected. The final objective is, after all, to control the closed loop characteristics such that they meet some performance criteria.

This difference in approach may be accounted for by the incorporation of frequency weights. Controller reduction may therefore be considered as a frequency weighted model reduction problem and corresponds to requiring the modeling error to be smaller in selected frequency bands. Generally the weights are chosen such that the salient features of the controller are retained at the frequencies of most interest.

Whereas the aim of the open loop model reduction schemes discussed in the previous section was to obtain reduced order approximants such that $\|G - \hat{G}\|_\infty$ is small, in this section we consider the more general frequency weighted extension

$$\min_{\substack{\hat{K} \in \mathcal{RH}_\infty \\ \deg(\hat{K}) \leq \deg(K)}} \|W_o(K - \hat{K})W_i\|_\infty \tag{3.3}$$

where $K \in \mathcal{RH}_\infty$, W_o is called an the output weight and W_i is called the input weight. Assume the weights are of McMillan degree n_o and n_i respectively. (Note that from now on we will be considering controller approximation, hence the reason for changing from G to K.)

We now present frequency weighted extensions of the balanced truncation and optimal Hankel norm approximation techniques.

3.3.1 Frequency Weighted Balanced Truncation

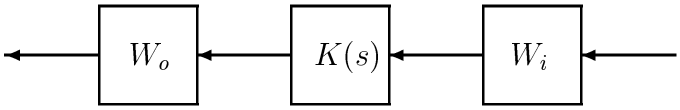


Figure 3.2: Plant with Input and Output Weights

Consider the cascaded interconnection of Figure 3.2. Assume the controller $K \in \mathcal{RH}_\infty$ has the realisation (A, B, C), and the input weight $W_i \in \mathcal{RH}_\infty$ and the output weight $W_o \in \mathcal{RH}_\infty$ have the realisations (A_i, B_i, C_i, D_i) and (A_o, B_o, C_o, D_o) respectively.

A naive approach to take account of the weights would be to form the cascade system $K_c = W_o K W_i$, then balance and truncate K_c to obtain $\hat{K}_c$. An approximation to K may then be calculated as $\hat{K} = W_o^{-1}\hat{K}_c W_i^{-1}$. However, although $\|W_o(K - \hat{K})W_i\|_\infty$ is guaranteed to satisfy an error bound similar to that given in Lemma 3.2.5 the resulting order of $\hat{K}$ will generally be inflated compared to that of K.

The basis of weighted balanced truncation is to truncate a realisation of K whose weighted Gramians are balanced. The procedure to calculate such a balancing transformation is now given.

Procedure 3.3.1 ([17, 18])

a) Given K, W_i and $W_o \in \mathcal{RH}_\infty$ solve the following Lyapunov equation for P_w and Q_w, the weighted input controllability and weighted output observability Gramians respectively.

$$\begin{bmatrix} A & BC_i \\ 0 & A_i \end{bmatrix} \begin{bmatrix} P_w & P_{21}' \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_w & P_{21}' \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} A' & 0 \\ C_i'B' & A_i' \end{bmatrix} + \begin{bmatrix} BD_i \\ B_i \end{bmatrix} \begin{bmatrix} D_i'B' & B_i' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} A' & C'B_o' \\ 0 & A_o' \end{bmatrix} \begin{bmatrix} Q_w & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix} + \begin{bmatrix} Q_w & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix} \begin{bmatrix} A & 0 \\ B_oC & A_o \end{bmatrix} + \begin{bmatrix} C'D_o' \\ C_o' \end{bmatrix} \begin{bmatrix} D_oC & C_o \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

b) Form the frequency weighted balanced realisation

$$A_w = T^{-1}AT \quad B_w = T^{-1}B \quad C_w = CT$$

where $P_w Q_w = T\Sigma_w^2 T^{-1}$ and $\Sigma_w^2 = diag\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2\}$.

c) Conformably partition A_w, B_w and C_w in accordance with Σ_w as in equation (3.2), forming the k^{th} order reduced model $\hat{K}$ using the realisation $(A_{w11}, B_{w1}, C_{w1})$.

Remark 3.3.2 *For one sided weighted approximation, i.e. either $W_i = I$ or $W_o = I$, reduced order controllers calculated using Procedure 3.3.1 satisfy the truncated stability result of Lemma 3.2.4 [17].*

Remark 3.3.3 *Unlike the unweighted case, for general two sided weighted balanced truncation, there is no a priori upper bound on the achievable frequency weighted error. However, assuming the calculated $\hat{K}$ is stable, an upper bound is presented in [53].*

3.3.2 Frequency Weighted Optimal Hankel Norm Approximation

In a similar vein to the unweighted case, one possible approach to the weighted infinity norm approximation problem may be derived using the solution to a related weighted optimal Hankel norm approximation problem.

For two sided weighted optimal Hankel norm approximation both the input and output weights must be invertible. In [100], Zhou presents a procedure to solve the one sided approximation problem with noninvertible weights. Since the design techniques to be presented in Chapter 4 lead to a two sided frequency weighted approximation problem with invertible weights, it is this framework which is now discussed. The following definitions are used throughout: $W_2 \triangleq W_o^{-1}$ and $W_1 \triangleq W_i^{-1}$.

To calculate a solution via the Nehari Extension Theorem [1, 23, 101] we follow the procedure of [54, 39] and solve the equivalent problem

$$\inf_{\mathcal{Q} \in \mathcal{RH}^-_{\infty,(k)}} \|M_o^{\sim -1}(K - \mathcal{Q})M_i^{\sim -1}\|_\infty \tag{3.4}$$

where M_o, M_o^{-1}, M_i, $M_i^{-1} \in \mathcal{RH}_\infty$ such that $M_o^\sim M_o = W_2 W_2^\sim$ and $M_i M_i^\sim = W_1^\sim W_1$, and show that such a solution satisfies bounds on equation (3.3). Note that pre- and post-multiplying by the all-pass factors $M_o^{\sim -1} W_2$ and $W_1 M_i^{\sim -1}$ respectively does not change the value of the infinity norm but will change the value of the Hankel norm of the error [42].

Define $H = [M_o^{\sim -1} K M_i^{\sim -1}]_+ \in \mathcal{RH}_\infty$. Then calculate an optimal Hankel norm approximation $\hat{H}$ to H,

$$\inf_{\hat{H} \in \mathcal{RH}^-_{\infty,(k)}} \|H - \hat{H}\|_\infty = \|H - \hat{H}\|_H = \sigma_{k+1}(H). \tag{3.5}$$

To calculate the required approximation to K notice that

$$\mathcal{Q} = M_o^\sim(\hat{H} + [M_o^{\sim -1} K M_i^{\sim -1}]_-)M_i^\sim \in \mathcal{RH}^-_{\infty,(k)}.$$

Therefore setting $\hat{K} = [\mathcal{Q}]_+$ gives a stable k^{th} order frequency weighted approximation to K.

The following lower bound for frequency weighted approximation is derived from equation (3.5) and the lower bound of equation (3.1),

$$\sigma_{k+1}([M_o^{\sim -1} K M_i^{\sim -1}]_+) = \inf_{\mathcal{Q}\in\mathcal{RH}^-_{\infty,(k)}} \|W_o(K-\mathcal{Q})W_i\|_\infty \tag{3.6}$$

$$\leq \inf_{deg(\hat{K})<deg(K)} \|W_o(K-\hat{K})W_i\|_\infty \tag{3.7}$$

where $K \in \mathcal{RH}_\infty$ has McMillan degree n and $\hat{K} \in \mathcal{RH}_\infty$ has McMillan degree k ($k < n$). Ideally we would like reduction techniques with an upper bound close to this lower bound.

As seen in the previous section, an upper bound on the error for weighted balanced truncation is much more difficult to obtain than for the unweighted case. The same is true for weighted and unweighted Hankel norm approximation. Anderson [2] derives a bound for weighted Hankel norm approximations which is proportional to the unweighted error. The value of the factor of proportionality is dependent solely on the weighting functions. Glover et al. [39] present the following error bound,

$$\|W_o(K-\hat{K})W_i\|_\infty \leq \sigma_{k+1}(H) + \|[\hat{H}]_- - [M_o^{\sim -1}(K-\hat{K})M_i^{\sim -1}]_-\|_\infty.$$

However, if either of the weights have zeros close to the imaginary axis then the second term can become arbitrarily large. Since this bound cannot be calculated a priori, (nor is it generally tight,) it is not as useful as the error bound in the unweighted case.

Remark 3.3.4 *It is also possible to formulate the frequency weighted Hankel norm approximation problem as a four block Nehari extension [39]. Consider calculating $\mathcal{Q} \in \mathcal{RH}^-_{\infty,(k)}$ such that*

$$\inf_{\mathcal{Q}\in\mathcal{RH}^-_{\infty,(k)}} \|W_2^{-1}(K-\mathcal{Q})W_1^{-1}\|_\infty < \gamma.$$

Letting

$$\begin{aligned} R_{21}R_{21}^\sim &= I - \alpha_1^{-2}W_2W_2^\sim \quad R_{21}, R_{21}^{-1} \in \mathcal{RH}_\infty \\ R_{12}^\sim R_{12} &= I - \alpha_2^{-2}W_1^\sim W_1 \quad R_{12}, R_{12}^{-1} \in \mathcal{RH}_\infty \end{aligned}$$

where $\alpha_1 > \|W_2\|_\infty$ and $\alpha_2 > \|W_1\|_\infty$, and defining $E \triangleq (K-\mathcal{Q})$, then leads to the relationships,

$$\begin{aligned} & \|W_2^{-1}EW_1^{-1}\|_\infty \leq \gamma \\ \Leftrightarrow\ & E^\sim W_2^{\sim -1}W_2^{-1}E \leq \gamma^2 W_1^\sim W_1 = \gamma^2\alpha_2^2(I - R_{12}^\sim R_{12}) \\ \Leftrightarrow\ & \left\| \begin{bmatrix} R_{12} \\ W_2^{-1}E/(\gamma\alpha_2) \end{bmatrix} \right\|_\infty \leq 1 \\ \Leftrightarrow\ & \begin{bmatrix} R_{12} \\ E/(\gamma\alpha_1\alpha_2) \end{bmatrix} \begin{bmatrix} R_{12} \\ E/(\gamma\alpha_1\alpha_2) \end{bmatrix}^\sim \leq \begin{bmatrix} I & 0 \\ 0 & \alpha_1^{-2}W_2W_2^\sim \end{bmatrix} = I - \begin{bmatrix} 0 \\ R_{21} \end{bmatrix} \begin{bmatrix} 0 & R_{21}^\sim \end{bmatrix} \\ \Leftrightarrow\ & \left\| \begin{bmatrix} 0 & R_{12} \\ R_{21} & (K-\mathcal{Q})/(\gamma\alpha_1\alpha_2) \end{bmatrix} \right\|_\infty \leq 1 \end{aligned} \tag{3.8}$$

A solution to equation (3.8) may be found in [39] along with the error bound

$$\|W_o(K-\hat{K})W_i\|_\infty \leq \sigma_i(H) + 2\sum_{i=1}^{n+n_i+n_o-k-1} \sigma_i(M_o^{\sim -1}[\mathcal{Q}]_- M_i^{\sim -1}).$$

Experience indicates that none of the frequency weighted approximation techniques discussed generally gives an approximant which comes even close to attaining the lower bound. However, a combined model-reduction/parameter-optimisation approach using the results of frequency weighted model reduction as the starting point for the optimisation seems to work well. Although the lower bound is not necessarily attainable, this combined approach leads to approximants with smaller errors than using frequency weighted approximation alone. This will be borne out in Chapter 7 where we use the following combined Hankel norm/convex optimisation approach [100] to calculate reduced order controllers. Define

$$\hat{H} \triangleq \left[\begin{array}{c|c} A_{\hat{H}} & B_{\hat{H}} \\ \hline C_{\hat{H}} & D_{\hat{H}} \end{array}\right]$$

to be an optimal Hankel norm approximation of $H = [M_o^{\sim -1} K M_i^{\sim -1}]_+$, and define

$$M_i^{\sim -1} \triangleq \left[\begin{array}{c|c} A_W & B_W \\ \hline C_W & D_W \end{array}\right].$$

Then choose

$$\hat{K} = \left[\begin{array}{c|c} A_{\hat{H}} & B_{\hat{H}} \\ \hline Y & D_{\hat{H}} \end{array}\right]$$

where X and Y satisfy the equations

$$\begin{aligned} A_W X - X A_{\hat{H}} + B_W Y &= 0 \\ C_W X + D_W Y &= C_{\hat{H}} \end{aligned}$$

as the starting point for optimisation over the B and D matrices of $\hat{K}$. A similar procedure is available for optimisation over the C and D matrices of $\hat{K}$. Note that both of these optimisations are convex.

3.3.3 Weights for Stability Preservation

To motivate the interpretation of controller approximation as a frequency weighted model reduction problem we now present a new derivation of a sufficient condition for controllers to be stabilising. The condition is expressed as a norm bound on the weighted error between a nominal stabilising controller and any other given controller, where the weight is chosen according to a specific scheme. The technique developed here will be extended in Chapter 4 to derive conditions for controllers to be (P,γ)-admissible. (In Chapter 4 the nominal controller must be (P,γ)-admissible.)

We require the following lemma.

Lemma 3.3.5 *Assume* $R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \in \mathcal{RH}_\infty^{(p_1+p_2)\times(m_1+m_2)}$ *and* $\Delta \in \mathcal{RH}_\infty^{(m_1\times p_1)}$*, then*

$$\|R_{11}\Delta\|_\infty < 1 \quad \Rightarrow \quad \mathcal{F}_u(R,\Delta) \in \mathcal{RH}_\infty.$$

Proof: See [37] ■

We begin by presenting a parameterisation of all stabilising controllers. Using the notation of Section 2.8, making assumptions (A1) and (A2) of Section 2.8.1 and choosing any appropriately dimensioned matrices F and L such that $\mathrm{Re}(\lambda_i(A + B_2F)) < 0$ and $\mathrm{Re}(\lambda_i(A + LC_2)) < 0$, then all controllers which stabilise $\mathcal{F}_l(P,K)$ are parameterised as $K = \mathcal{F}_l(J,Q)$ where

$$J = \left[\begin{array}{c|cc} A + B_2F + LC_2 & -L & B_2 \\ \hline F & 0 & I \\ -C_2 & I & 0 \end{array}\right]$$

and $Q \in \mathcal{RH}_\infty$ [101]. There are two important observations to be made about this parameterisation: firstly that $K = \mathcal{F}_l(J,Q)$ satisfies the assumptions of Lemma 2.7.3, which implies Q may be written as $Q = \mathcal{F}_u(J^{-1},K)$; and secondly that Q is any free parameter satisfying $Q \in \mathcal{RH}_\infty$. Combining these two observations implies that K stabilises P if and only if $\mathcal{F}_u(J^{-1},K) \in \mathcal{RH}_\infty$.

Now if we assume that K may be represented by a nominal stabilising controller, K^0, subject to a stable perturbation, Δ, i.e. $K = K^0 + \Delta$, and define $R \triangleq \mathcal{S}(\begin{bmatrix} 0 & I \\ I & K^0 \end{bmatrix}, J^{-1})$ then we have that

$$Q \in \mathcal{RH}_\infty \quad \Longleftrightarrow \quad \mathcal{F}_u(R,\Delta) \in \mathcal{RH}_\infty.$$

This result is shown pictorially in Figure 3.3.

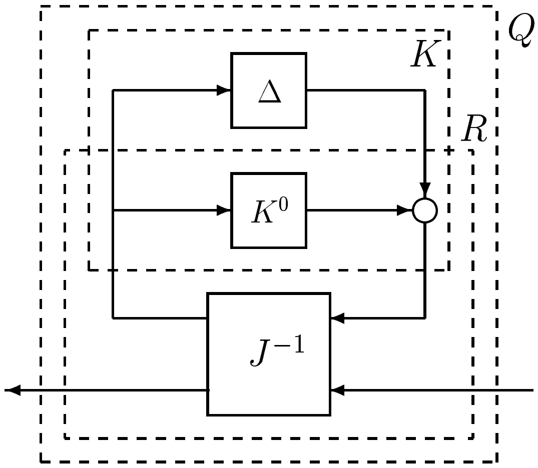


Figure 3.3: Calculation of Weights to Preserve Stability

Therefore, since R is stable for all stabilising K^0, and Δ is stable by design, from Lemma 3.3.5 we have that

$$\begin{aligned} \|R_{11}\Delta\|_\infty < 1 \quad &\Rightarrow \quad \mathcal{F}_u(R,\Delta) \in \mathcal{RH}_\infty \\ &\Longleftrightarrow \quad \mathcal{F}_l(P,K) \in \mathcal{RH}_\infty. \end{aligned}$$

This reduces to the well known condition [3, 17],

$$\|[I - P_{22}K^0]^{-1}P_{22}[K - K^0]\|_\infty < 1 \quad \Rightarrow \quad \mathcal{F}_l(P, K) \in \mathcal{RH}_\infty.$$

Therefore, letting $W_i = I$ and $W_o = [I - P_{22}K^0]^{-1}P_{22}$ in either the weighted balanced truncation, or weighted optimal Hankel norm approximation schemes, gives a tractable frequency weighted approximation problem for the design of reduced order stabilising controllers. Alternatively, this may be written with $W_i = P_{22}[I - K^0P_{22}]^{-1}$ and $W_o = I$ by using the condition $\|\Delta R_{11}\|_\infty < 1$.

3.4 Performance Considerations

The objective of all designs is to control the closed loop transfer function characteristics such that they meet some (pre-specified) performance criteria. Hence, we are particularly interested in those techniques which give an (a priori) indication of how well a reduced order controller will perform when connected to a high order plant. That is, given a nominal controller K^0 with satisfactory performance, we wish to obtain a low order approximant K which gives little or no degradation in closed loop characteristics.

3.4.1 Bounding Performance Degradation

In this section we review design techniques which, although not guaranteed to lead to reduced order (P, γ)-admissible controllers, give an upper bound on the performance degradation. If the degradation is small then it may be considered acceptable to use the derived controller.

Approximation in the Gap Metric

For the specific closed loop feedback structure shown in Figure 3.4 it is possible to determine conditions for the closed loop to remain stable when a controller K is substituted for K^0. Further, given that K is stabilising, it is possible to bound, a priori, the degradation in closed loop performance.

This problem has been tackled by, amongst others, McFarlane et al. [63, 64, 65] using coprime factor approximation, by Mustafa [69, 70] using $\mathcal{H}_\infty$-balanced truncation, and by Georgiou and Smith [32] and Qui and Davidson [77] for approximation in the *gap* metric. A priori bounds on the degradation in closed loop performance are computable for each of these approaches. However, the bound derived by Qui and Davidson is tighter than that of Georgiou and Smith, which is in turn tighter than that of McFarlane et al. [32].

The feedback interconnection of figure 3.4 is internally stable if the transfer function

$$\begin{aligned} \mathcal{H}(G, K) &\triangleq \begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} &\quad (3.9) \\ &= \begin{bmatrix} (I - KG)^{-1} & K(I - GK)^{-1} \\ G(I - KG)^{-1} & (I - GK)^{-1} \end{bmatrix} &\quad (3.10) \end{aligned}$$

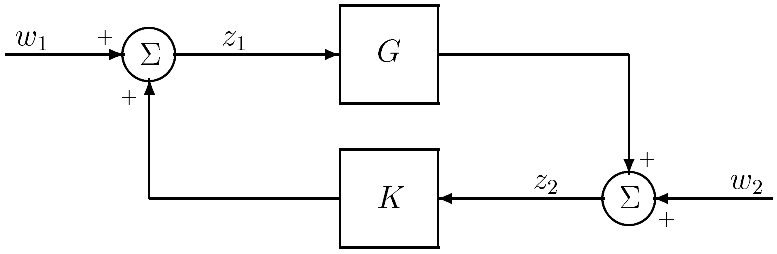


Figure 3.4: Closed Loop for Performance Considerations

from $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ is stable.

The results derived from approximation in the *gap* are motivated by the desire to stabilise the closed loop in the face of simultaneous uncertainty in both the plant and the controller. Nevertheless, they may equally be interpreted in a controller reduction framework. We concentrate on this interpretation.

First we define what is meant by the *graph* of a transfer matrix and the *gap* between two transfer matrices. (For a more detailed discussion see [31, 77, 90].)

Definition 3.4.1 *The graph of a transfer matrix K is the set of all possible bounded input/output pairs,*

$$\mathcal{G}(K) \triangleq \left\{ \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} V \\ U \end{bmatrix} v \; : \; v \in \mathcal{H}_2 \right\}$$

where $K = UV^{-1}$ is a right coprime factorisation of K and $y = Ku$.

Definition 3.4.2 *The gap between two transfer matrices is defined as the gap between their graphs,*

$$\delta(K, \hat{K}) \quad \triangleq \quad \|\Pi_{\mathcal{G}(K)} - \Pi_{\mathcal{G}(\hat{K})}\|$$

where $\Pi_{\mathcal{G}(\bullet)}$ denotes the orthogonal projection onto $\mathcal{G}(\bullet)$.

Note that the gap is the maximum of two directed gaps which are computable if a normalised coprime factorisation is used to define the graph (see [30, 31]).

Assume that K^0 is a given controller such that $\mathcal{H}(G, K^0)$ is stable and K is a (reduced order) controller to be designed. Then the next theorem gives a necessary and sufficient condition for $\mathcal{H}(G, K)$ to be stable. Firstly define the following performance indices,

$$\rho^0 \quad \triangleq \quad \left\| \mathcal{H}(G, K^0) - \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \right\|_\infty^{-1}$$

and

$$\rho \quad \triangleq \quad \left\| \mathcal{H}(G, K) - \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \right\|_\infty^{-1}.$$

Theorem 3.4.3 ([77]) *Assume $\mathcal{H}(G, K^0)$ is stable. Then $\mathcal{H}(G, K)$ is stable for all K such that $\delta(K^0, K) < \epsilon$ if and only if,*

$$\arcsin(\epsilon) + \arccos(\rho^0) \quad \leq \quad \frac{\pi}{2}. \tag{3.11}$$

Theorem 3.4.4 ([77]) *Assume $\mathcal{H}(G, K^0)$ is stable and that ϵ satisfies equation (3.11). Then*

$$\inf_{K:\delta(K^0,K)<\epsilon} (\rho) \quad = \quad \cos(\arcsin(\epsilon) + \arccos(\rho^0))$$

Theorem 3.4.4 gives a bound on the degradation in performance when K is substituted for K^0. To calculate this degradation a priori, we must be able to bound $\delta(K^0, K)$. Appropriate bounds were originally derived by Georgiou and Smith [32]. However, the following bounds, derived by Vinnicombe in his work on the ν-gap [90] are tighter.

Theorem 3.4.5 ([90]) *Let $\begin{bmatrix} \tilde{V}_0 & \tilde{U}_0 \end{bmatrix}$ be a normalised left coprime factorisation for $K^0 = \tilde{V}_0^{-1}\tilde{U}_0$ with Hankel singular values $\sigma_1 \geq \ldots \geq \sigma_{n-r} > \sigma_{n-r+1} = \ldots = \sigma_n$, and define*

$$s_k \quad \triangleq \quad \sum_{i=k+1}^{n-r+1} \arcsin(\sigma_i)$$

and

$$u_k \quad \triangleq \quad \begin{cases} \sin(s_k) & if \quad s_k < \pi/2 \\ 1 & otherwise. \end{cases}$$

Then

$$\sigma_{k+1} \leq \inf_{K:deg(K)\leq k} \delta(K^0, K) \leq u_k.$$

The Lenz Approach

The aim of controller approximation within a $\mathcal{H}_\infty$ framework is to calculate reduced order controllers which are (P, γ)-admissible. This problem has been studied by Lenz et al. [55, 56].

Consider the interconnection of Figure 3.5 where P is the generalised plant and controller $K = K^0 + W_2 \Delta W_1$ is a nominal (P, γ)-admissible controller subject to a weighted perturbation.

Make the assumption that the plant has been scaled such that $\gamma = 1$ and observe that $\mathcal{F}_l(P, K) = \mathcal{F}_l(\tilde{R}, \Delta)$. Hence

$$\|\mathcal{F}_l(P, K)\|_\infty < 1 \quad \Longleftrightarrow \quad \|\mathcal{F}_l(\tilde{R}, \Delta)\|_\infty < 1$$

where

$$\tilde{R} = \begin{bmatrix} I & 0 \\ 0 & W_1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W_2 \end{bmatrix} \tag{3.12}$$

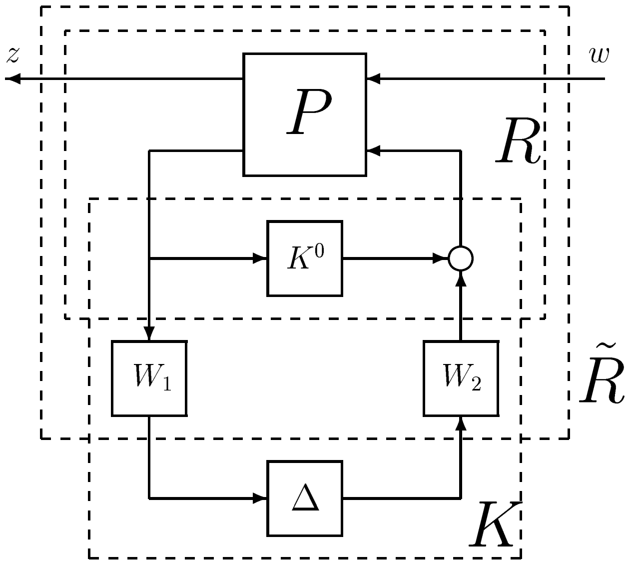


Figure 3.5: Plant and Controller Interconnection for the Lenz Procedure

and R is given by the Redheffer star product [15, 78],

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \mathcal{S}(P, \begin{bmatrix} K^0 & I \\ I & 0 \end{bmatrix}).$$

Then, by Theorem 2.7.5, if weights can be calculated such that $\tilde{R}$ is inner

$$\begin{aligned} \|\Delta\|_\infty < 1 \quad & \Longleftrightarrow \quad \|W_2^{-1}(K - K^0)W_1^{-1}\|_\infty < 1 \\ & \Longleftrightarrow \quad \|\mathcal{F}_l(\tilde{R}, \Delta)\|_\infty < 1 \\ & \Longleftrightarrow \quad \|\mathcal{F}_l(P, K)\|_\infty < 1. \end{aligned}$$

Lenz shows that in general it is not possible to calculate weights such that $\tilde{R}$ is inner. However, weights satisfying

$$R_{21}^\sim W_1^\sim W_1 R_{21} = I - R_{11}^\sim R_{11}$$

and

$$W_2 W_2^\sim = (R_{12}^\sim R_{12} + R_{22}^\sim W_1^\sim W_1 R_{22})^{-1}$$

make the columns of $\tilde{R}$, ($\begin{bmatrix} \tilde{R}_{11} \\ \tilde{R}_{21} \end{bmatrix}$ and $\begin{bmatrix} \tilde{R}_{12} \\ \tilde{R}_{22} \end{bmatrix}$ respectively,) inner and lead to the result that

$$\|W_2^{-1}(K - K^0)W_1^{-1}\|_\infty < \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \|\mathcal{F}_l(P, K)\|_\infty < \sqrt{2}. \tag{3.13}$$

Equation (3.13) represents a sufficient condition for controller K to stabilise the closed loop and give a performance bound of $\|\mathcal{F}_l(P, K)\|_\infty < \sqrt{2}$. This represents a (possible) degradation in performance of $\sqrt{2}$. A similar result may be derived by assuming K is a nominal controller subjected to coprime factor perturbations.

3.4.2 Performance Preserving Techniques

Recall from Section 2.8 that all (P, γ)-admissible controllers may be parameterised as $K = \mathcal{F}_l(K_a, \Phi)$ where K_a is derived in Theorem 2.8.2 and Φ is any appropriately dimensioned, stable transfer matrix such that $\|\Phi\|_\infty < \gamma$. Often Φ is chosen to be zero, which is equivalent to selecting the central controller. However, although this choice minimises the closed loop entropy (for a given γ [69]), in certain two block designs the central controller has the undesirable property of attempting to cancel the zeros of the plant [81]. Hence it may be desirable to use the freedom allowed in selecting Φ to choose another (P, γ)-admissible controller from the possible set of all (P, γ)-admissible controllers.

Here we review two techniques which exploit the freedom in Φ to obtain reduced order (P, γ)-admissible controllers. The first uses the state space matrices of Φ to introduce state cancellations into the controller state-space which may be truncated to give a reduced order controller. The second method uses a constrained linear matrix inequality (LMI) optimisation approach.

Introducing State Cancellations

In [41], Gu et al. propose a procedure to select Φ so as to introduce unobservable modes into $K = \mathcal{F}_l(K_a, \Phi)$. These unobservable modes may then be truncated to obtain a reduced order (P, γ)-admissible controller. Similar state cancellation approaches have been studied by Li and Chang [57] and Davidson [11]. Let Φ be given by

$$\Phi = \left[\begin{array}{c|c} A_\Phi & B_\Phi \\ \hline C_\Phi & D_\Phi \end{array}\right] \in \mathcal{RH}_\infty^{m_2 \times p_2}$$

with state-dimension n_Φ. Then $K = \mathcal{F}_l(K_a, \Phi)$ may be written

$$K = \left[\begin{array}{cc|c} A_a + B_{a2}D_\Phi C_{a2} & B_{a2}C_\Phi & B_{a1} + B_{a2}D_\Phi D_{a21} \\ B_\Phi C_{a2} & A_\Phi & B_\Phi D_{a21} \\ \hline C_{a1} + D_{a12}D_\Phi C_{a2} & D_{a12}C_\Phi & D_{a11} + D_{a12}D_\Phi D_{a21} \end{array}\right]$$

which has state-dimension $n + n_\Phi$. However, if there are cancellations between K and Φ then the McMillan degree of K may be less that $n + n_\Phi$.

Applying the similarity transformation

$$T = \begin{bmatrix} I_n & 0 \\ X & I_{n_\Phi} \end{bmatrix}$$

where X satisfies the two equations

$$\begin{aligned} A_\Phi X - X(A_a - B_{a2}D_{a12}^{-1}C_{a1}) &= B_\Phi C_{a2} && (3.14) \\ C_\Phi X - D_\Phi C_{a2} &= D_{a12}^{-1}C_{a1} && (3.15) \end{aligned}$$

gives the realisation

$$K = \left[\begin{array}{cc|c} \hat{A}_{a11} & \hat{A}_{a12} & \hat{B}_{a1} \\ 0 & \hat{A}_a & \hat{B}_a \\ \hline 0 & \hat{C}_a & \hat{D}_a \end{array}\right]$$

where

$$\begin{aligned}\hat{A}_a &= XB_{a2}C_\Phi + A_\Phi\\ \hat{B}_a &= X(B_{a1} + B_{a1}D_\Phi D_{a21}) + B_\Phi D_{a21}\\ \hat{C}_a &= D_{a12}C_\Phi\\ \hat{D}_a &= D_{a11} + D_{a12}D_\Phi D_{a21}\end{aligned}$$

and

$$\begin{aligned}\hat{A}_{a11} &= A_a + B_{a2}D_\Phi C_{a2} - B_{a2}C_\Phi X\\ \hat{A}_{a12} &= B_{a2}C_\Phi\\ \hat{B}_{a1} &= B_{a1} + B_{a2}D_\Phi D_{a21}.\end{aligned}$$

The problem is now one of choosing the matrices A_Φ, B_Φ, C_Φ, D_Φ and X which satisfy both equations (3.14) and (3.15) and the constraint $\|\Phi\|_\infty < \gamma$. If this is possible, then the state-dimension of K is n_Φ which will typically be chosen to be less that n. K is therefore a reduced order (P,γ)-admissible controller.

There are two general drawbacks to the state cancellation approach. Firstly, there is no systematic method of choosing matrices which a priori satisfy all of the required conditions. For $n_\Phi = n - p_2$, Gu et al. [41] propose a method of characterising matrices B_Φ, C_Φ, D_Φ and X which satisfy equations (3.14) and (3.15) in terms of an arbitrary A_Φ. A search must then be performed to find an A_Φ which is stable and satisfies the norm constraint on Φ. However, there is no guarantee that such an A_Φ exists, nor that it may be calculated if one does exist. Furthermore, since p_2 will generally be small compared with n, the degree of reduction attainable with this method will generally be small.

An LMI-based Approach

Alternative to the Riccati equation based solution to the sub-optimal $\mathcal{H}_\infty$ control problem presented in Section 2.8, necessary and sufficient conditions for the existence of (P,γ)-admissible controllers may be stated in terms of linear matrix inequalities (LMIs). Here the particular (P,γ)-admissible controller chosen depends on which solutions to two LMIs are selected. An advantage of this framework is that no rank assumptions are required on D_{12} and D_{21} and hence *singular* problems may be solved using the same machinery.

Theorem 3.4.6 ([24, 26]) *Consider the plant of equation (2.10), make assumptions (A1) and (A2) (of Section 2.8.1), and let* $\mathcal{N}_R$ *and* $\mathcal{N}_S$ *denote orthonormal bases of the null spaces of* $\begin{bmatrix} B_2' & D_{12}' \end{bmatrix}'$ *and* $\begin{bmatrix} C_2 & D_{21} \end{bmatrix}$ *respectively.*

There exists a (P,γ)-admissible controller if and only if there exist symmetric matrices R and S satisfying the following system of LMI's.

$$\left[\begin{array}{c|c} \mathcal{N}_R' & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{cc|c} AR + RA' & RC_1' & B_1 \\ C_1R & -\gamma I & D_{11} \\ \hline B_1' & D_{11}' & -\gamma I \end{array}\right] \left[\begin{array}{c|c} \mathcal{N}_R & 0 \\ \hline 0 & I \end{array}\right] < 0 \qquad (3.16)$$

$$\left[\begin{array}{c:c} \mathcal{N}_S' & 0 \\ \hdashline 0 & I \end{array}\right] \left[\begin{array}{cc:c} A'S+SA & SB_1 & C_1' \\ B_1'S & -\gamma I & D_{11}' \\ \hdashline C_1 & D_{11} & -\gamma I \end{array}\right] \left[\begin{array}{c:c} \mathcal{N}_S & 0 \\ \hdashline 0 & I \end{array}\right] < 0 \tag{3.17}$$

$$\left[\begin{array}{cc} R & I \\ I & S \end{array}\right] \geq 0. \tag{3.18}$$

Moreover, there exist (P,γ)-admissible controllers of order $k < n$ if and only if equations (3.16) to (3.18) hold for some R and S which further satisfy

$$\mathit{Rank}(I - RS) \leq k. \tag{3.19}$$

Equation (3.19) is always satisfied if $k \geq n$. In this case, equations (3.16) to (3.18) represent necessary and sufficient conditions for the existence of a (P,γ)-admissible controller of the same order as the generalised plant. For a given γ, equations (3.16) to (3.18) are affine in R and S and the set of pairs, R and S, is convex. Hence, given that (P,γ)-admissible controllers exists, it is always possible to find one of order n. Methods to construct a (P,γ)-admissible controller given feasible R and S may be found in [24, 26]. (See [10], and the references therein, for algorithms to perform the required convex optimisation.)

In contrast, due to the inclusion of the rank constraint of equation (3.19), the design of (P,γ)-admissible controllers of order $k < n$ is no longer convex. In [24, 25], Gahinet et al. propose an approach to solving this problem. The aim is to calculate R and S which are solutions of equations (3.16) to (3.18) while maximising the rank deficiency of $(I-RS)$. This is equivalent to minimising the function

$$\Psi(R,S) \triangleq \sum_{i=1}^{n-k} \lambda_i \begin{pmatrix} R & I \\ I & S \end{pmatrix}$$

where $\lambda_1 \leq \cdots \leq \lambda_{n-k}$ denote the $n-k$ smallest eigenvalues of $\begin{pmatrix} R & I \\ I & S \end{pmatrix}$, subject to R and S satisfying equations (3.16) to (3.18).

Therefore, (P,γ)-admissible controllers of order $k < n$ will exist if and only if the global minimum of $\Psi(R,S)$ is zero. Unfortunately the function $\Psi(R,S)$ is non-convex (it is in fact concave) and there is no guarantee of attaining the global minimum. Hence there is no guarantee of being able to calculate reduced order (P,γ)-admissible controllers via this technique. (Given that feasible R and S can be found, a procedure to construct a (P,γ)-admissible controller of order k may be found in [26].)

3.5 Summary

In this chapter the controller approximation problem has been motivated and current techniques to undertake such approximation has been reviewed. Of particular importance are those techniques which give conditions for reduced order controllers to be stabilising, and if so, the associated degradation of closed loop performance. In Chapter 4 the ideas presented in this chapter will be extended to derive sufficient conditions for reduced order controllers to be (P,γ)-admissible.

Chapter 4

Synthesis of Reduced Order (P, γ)-admissible Controllers

4.1 Introduction

Given a (P, γ)-admissible controller K^0, in this chapter we present sufficient conditions for another controller K to be (P, γ)-admissible. The conditions are presented in terms of a norm bound on a particular frequency weighted error, which, for the purpose of reduced order controller synthesis, may be considered as a frequency weighted model reduction problem.

Five approaches, to be denoted $\mathcal{A}(P)$, $\mathcal{CF}(P)$, $\mathcal{A}(\Phi)$, $\mathcal{CF}(\Phi)$ and $\mathcal{CF}(\Theta)$, are studied. The reason for this choice of notation will become clear as we progress. A sufficient condition for reduced order controllers to be (P, γ)-admissible, expressed as a norm bound on a particular frequency weighted error, is derived for each approach. For the $\mathcal{A}(P/\Phi)$[1] and $\mathcal{CF}(P/\Phi)$[2] approaches, weights must be calculated so that a specific transfer function is a contraction – the transfer function being different for each of the four approaches. Two procedures are presented to calculate appropriate weights. For the $\mathcal{CF}(\Theta)$ approach, there is no such freedom in the selection of weights.

Approaches $\mathcal{A}(P)$ and $\mathcal{CF}(P)$ are reminiscent of that taken by Lenz [55] and outlined in Section 3.4.1. However, whereas the Lenz approach led to a sufficient condition for reduced order controllers to be stabilising and guarantee performance degradation of at most a factor of $\sqrt{2}$, the sufficient conditions derived here eliminate the $\sqrt{2}$ factor, thus leading to reduced order (P, γ)-admissible controllers. In both cases we represent all controllers as a nominal (P, γ)-admissible controller subjected to a weighted perturbation. We study additive ($\mathcal{A}(P)$) and coprime factor ($\mathcal{CF}(P)$) perturbations since they encompass the difficulties which may be encountered with other perturbation classes.

For approaches $\mathcal{A}(\Phi)$ and $\mathcal{CF}(\Phi)$ we show that the constraint $\Phi \in \mathcal{RH}_\infty$, $\|\Phi\|_\infty < \gamma$ on the otherwise free parameter in the parameterisation of all (P, γ)-admissible controllers (given in Section 2.8.1) is equivalent to a norm constraint on a specific weighted error. Controllers satisfying this norm constraint are guaranteed to be (P, γ)-admissible. The difference between

[1] $\mathcal{A}(P/\Phi)$ is used to signify either approach $\mathcal{A}(P)$ or $\mathcal{A}(\Phi)$

[2] $\mathcal{CF}(P/\Phi)$ is used to signify either approach $\mathcal{CF}(P)$ or $\mathcal{CF}(\Phi)$

$\mathcal{A}(\Phi)$ and $\mathcal{CF}(\Phi)$ is the same as that for $\mathcal{A}(P)$ and $\mathcal{CF}(P)$, i.e. the particular weighted perturbation chosen – additive for $\mathcal{A}(\Phi)$, coprime factor for $\mathcal{CF}(\Phi)$.

For the additive perturbation approaches, the sufficient conditions are expressed in terms of the weighted error between a (reduced order) controller K and a nominal (P,γ)-admissible controller K^0, e.g.

$$\|W_2^{-1}(K-K^0)W_1^{-1}\|_\infty < 1 \quad \Rightarrow \quad \|\mathcal{F}_l(P,K)\|_\infty < \gamma. \tag{4.1}$$

For the coprime factor perturbation approaches, the sufficient conditions are expressed in terms of the weighted error between left coprime factors of a (reduced order) controller and a nominal (P,γ)-admissible controller, e.g.

$$\|W_2^{-1}(\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix} - \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix})W_1^{-1}\|_\infty < 1 \quad \Rightarrow \quad \|\mathcal{F}_l(P,K)\|_\infty < \gamma \tag{4.2}$$

where $K = \tilde{V}^{-1}\tilde{U}$ and $K^0 = \tilde{V}_0^{-1}\tilde{U}_0$. Note that dual results for right coprime factor perturbations are possible although not discussed here.

Equations (4.1) and (4.2) have a natural interpretation as frequency weighted model reduction problems. Given appropriate weights, the goal is to calculate reduced order K or $\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix}$ which lie within an $\mathcal{L}_\infty$ ball (of right radius W_1 and left radius W_2,) of K^0 or $\begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}$ respectively. $K = \tilde{V}^{-1}\tilde{U}$ is then a reduced order (P,γ)-admissible controller. The larger the $\mathcal{L}_\infty$ ball, the easier the approximation. We therefore wish to make W_1 and W_2 large in some sense.

In each of the $\mathcal{A}(P/\Phi)$ and $\mathcal{CF}(P/\Phi)$ approaches there is freedom to select weights subject to a specific transfer function being contractive. Subject to this constraint, two appropriate methods for maximising the size of the weights are discussed. (Maximising the product of their traces and maximising the product of their determinants.) There is no such freedom for selection of weights in the $\mathcal{CF}(\Theta)$ approach.

This chapter is arranged as follows. In Section 4.2 we modify the procedure of Lenz [55], firstly for the additive perturbation case and then for the coprime factor case. These constitute approaches $\mathcal{A}(P)$ and $\mathcal{CF}(P)$. Approaches $\mathcal{A}(\Phi)$ and $\mathcal{CF}(\Phi)$ are then presented in Section 4.3, and approach $\mathcal{CF}(\Theta)$ is presented in Section 4.4.

The sufficient conditions for approaches $\mathcal{A}(P/\Phi)$ and $\mathcal{CF}(P/\Phi)$, presented in Sections 4.2 and 4.3, are expressed in terms of either $\mathcal{F}_l(\tilde{R},\Delta)$ or $\mathcal{F}_u(\tilde{R},\Delta)$, where $\tilde{R}$ and Δ are different for each approach. The reason for using similar notation will become clear in Section 4.5 where we present a procedure for calculating weights such that $\tilde{R}$ is a contraction. This procedure is valid irrespective of which of the four approaches is being considered. In Section 4.6 we consider maximising the weights of Section 4.5. This is done by maximising the product of the traces of the weights or the product of the determinants of the weights. We show that maximising the product of the determinants may not be appropriate if coprime factor perturbations are being considered.

In Section 4.7 we discuss the relationship between the approaches. We show an equivalence between approaches $\mathcal{A}(P)$ and $\mathcal{A}(\Phi)$, and between approaches $\mathcal{CF}(P)$ and $\mathcal{CF}(\Phi)$. To conclude, we present a condition for which the weights derived for approach $\mathcal{CF}(\Theta)$ satisfy the contractive condition on $\tilde{R}$ of the $\mathcal{CF}(P/\Phi)$ approaches.

4.2 Modified Lenz Approach

In Section 3.4.1 we reviewed a procedure (by Lenz [55]) which led to a sufficient condition for reduced order controllers to guarantee closed loop performance degradation of a factor of at most $\sqrt{2}$. Here we use similar techniques to obtain a sufficient condition for reduced order controllers to be (P, γ)-admissible. Consider the generalised plant and controller interconnection of Figure 4.1.

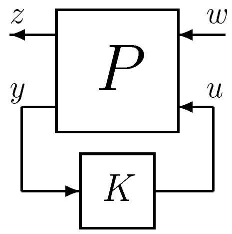


Figure 4.1: Generalised Plant and Controller Interconnection

In the next sections we show how, by considering controller K to be represented by a nominal (P, γ)-admissible controller K^0 subjected to a weighted perturbation, a sufficient condition for any other controller to be (P, γ)-admissible can be obtained. Section 4.2.1 considers the additive perturbation approach, denoted $\mathcal{A}(P)$. Section 4.2.2 considers the coprime factor perturbation approach, denoted $\mathcal{CF}(P)$.

4.2.1 $\mathcal{A}(P)$ – The Additive Perturbation Case

The setup here is analogous to that discussed in Section 3.4.1. Assume that any controller K may be represented by a nominal (P, γ)-admissible controller K^0, subjected to a weighted additive perturbation, i.e. $K = K^0 + W_2 \Delta W_1$, where W_1 and W_2 are stable, minimum phase and invertible and Δ is stable. This assumption imposes the restriction that K and K^0 possess the same right half plane poles, thus to some degree limiting the set of reduced order controllers attainable by our method. However, since we rely on balanced truncation or optimal Hankel norm approximation to calculate K as a weighted approximation to K^0 - where the unstable part of K^0 is copied directly into K - the assumption doesn't impose an unacceptable constraint.

Consider Figure 4.2. Observe that $\mathcal{F}_l(P, K) = \gamma \mathcal{F}_l(\tilde{R}, \Delta)$ and hence

$$\|\mathcal{F}_l(P, K)\|_\infty < \gamma \iff \|\mathcal{F}_l(\tilde{R}, \Delta)\|_\infty < 1 \tag{4.3}$$

where

$$\tilde{R} = \begin{bmatrix} \gamma^{-1/2} I & 0 \\ 0 & W_1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} \gamma^{-1/2} I & 0 \\ 0 & W_2 \end{bmatrix} \tag{4.4}$$

and R is given by the Redheffer star product [15, 78],

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \mathcal{S}(P, \begin{bmatrix} K^0 & I \\ I & 0 \end{bmatrix}). \tag{4.5}$$

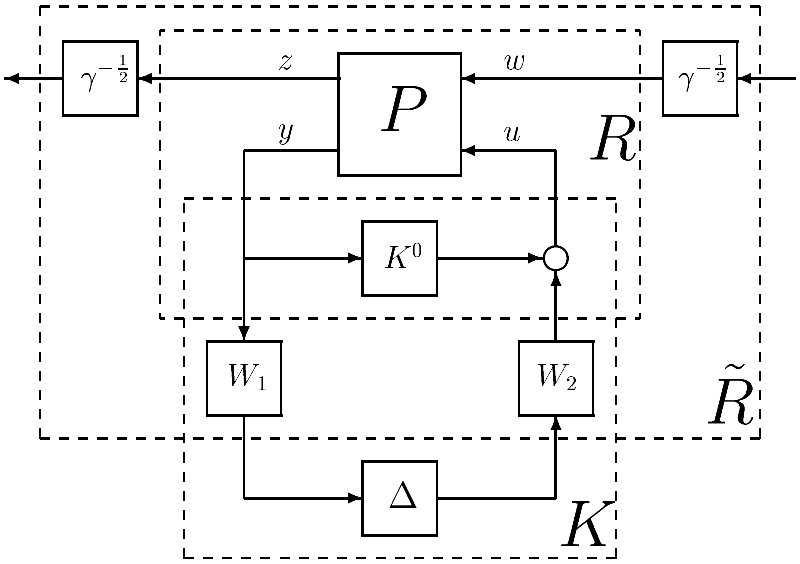


Figure 4.2: Plant and Controller Interconnection for $\mathcal{A}(P)$

The results of Theorem 2.7.5 can now be applied to equation (4.3). Suppose K^0 is some nominal (P, γ)-admissible controller. If stable, minimum phase and invertible weights W_1 and W_2 can be calculated such that $\tilde{R}$ is inner and $\tilde{R}_{21}$ has full row rank (note that this is not generally possible – see Lenz [55]) then

$$\begin{aligned} \|\Delta\|_\infty < 1 \quad &\iff \quad \|W_2^{-1}(K - K^0)W_1^{-1}\|_\infty < 1 \\ &\iff \quad \|\mathcal{F}_l(\tilde{R}, \Delta)\|_\infty < 1 \\ &\iff \quad \|\mathcal{F}_l(P, K)\|_\infty < \gamma. \end{aligned}$$

Hence (with weights such that $\tilde{R}$ is inner and $\tilde{R}_{21}$ has full row rank) we obtain the condition that any controller, $K = K^0 + W_2\Delta W_1$ will also be (P, γ)-admissible if and only if $\|W_2^{-1}(K - K^0)W_1^{-1}\|_\infty < 1$.

Although it is not possible to calculate weights such that the above necessary and sufficient condition is satisfied, it is possible to calculate weights which make $\tilde{R}$ a contraction. Using such weights gives the sufficient condition,

$$\|W_2^{-1}(K - K^0)W_1^{-1}\|_\infty < 1 \;\Rightarrow\; \|\mathcal{F}_l(P, K)\|_\infty < \gamma \tag{4.6}$$

Calculation of appropriate weights is discussed in Section 4.5.

4.2.2 $\mathcal{CF}(P)$ – The Coprime Factor Perturbation Case

The development here is similar to that of the previous section, however, whereas for the $\mathcal{A}(P)$ approach all controllers K were represented by a stable weighted additive perturbation to a nominal (P, γ)-admissible controller, here we express all controllers K as a stable weighted perturbation to the coprime factors of a nominal (P, γ)-admissible.

The advantage of such coprime factor perturbations is that they do not restrict K and K^0 to possess the same number of unstable poles. This gives freedom to the chosen model

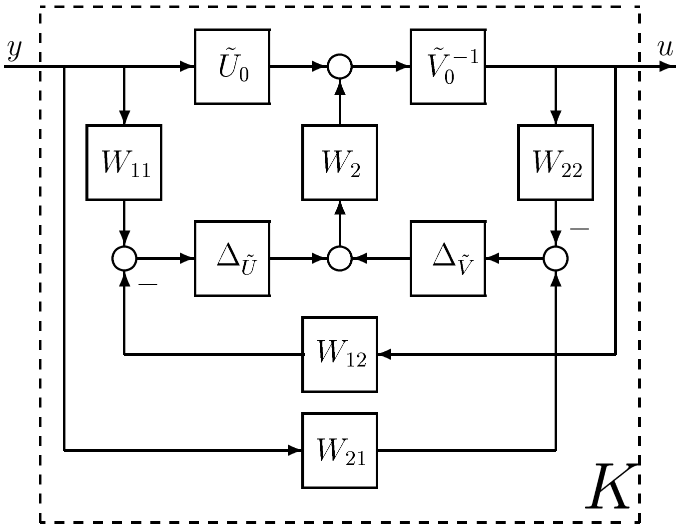


Figure 4.3: Weighted Controller for Approaches $\mathcal{CF}(P)$ and $\mathcal{CF}(\Phi)$

reduction procedure to allow poles to move across the $j\omega$ axis. However there is a disadvantage. Since coprime factorisations are only unique to within multiplication by a *unit* in $\mathcal{RH}_\infty$ it is not clear if any particular factorisation will give better results than any other choice. This makes the selection of an initial factorisation of K^0 somewhat arbitrary. In fact, each choice would lead to a different set of possible (P, γ)-admissible controllers.

To begin, assume that the left coprime factors of $K = \tilde{V}^{-1}\tilde{U}$ may be represented by a weighted perturbation to the left coprime factors of a nominal controller $K^0 = \tilde{V}_0^{-1}\tilde{U}_0$, i.e.

$$\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix} = \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix} + W_2 \begin{bmatrix} \Delta_{\tilde{U}} & \Delta_{\tilde{V}} \end{bmatrix} W_1 \tag{4.7}$$

where W_1 and W_2 are stable, minimum phase and invertible and $\Delta \triangleq \begin{bmatrix} \Delta_{\tilde{U}} & \Delta_{\tilde{V}} \end{bmatrix}$ is stable. This is shown in Figure 4.3 where

$$W_1 = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}.$$

Substituting the $\tilde{V}^{-1}\tilde{U}$ from Figure 4.3 (equation (4.7)) for K in Figure 4.2 leads to the condition

$$\|\mathcal{F}_l(P, K)\|_\infty < \gamma \iff \|\mathcal{F}_l(\tilde{R}, \Delta)\|_\infty < 1 \tag{4.8}$$

where

$$\tilde{R} = \begin{bmatrix} \gamma^{-1/2} I & 0 \\ 0 & W_1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} \gamma^{-1/2} I & 0 \\ 0 & W_2 \end{bmatrix} \tag{4.9}$$

and R is given by the Redheffer star product [15, 78],

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \mathcal{S}(P, \left[\begin{array}{c:c} K^0 & \tilde{V}_0^{-1} \\ \hdashline I & 0 \\ -K^0 & -\tilde{V}_0^{-1} \end{array}\right]). \tag{4.10}$$

Equation (4.9) shows that $\tilde{R}_{21}$ can never have full row rank. Hence the necessity part of Theorem 2.7.5 can never be satisfied. However, we can always calculate weights which make $\tilde{R}$ a contraction. The sufficiency part of Theorem 2.7.5 then leads to the condition,

$$\begin{aligned} \|\Delta\|_\infty < 1 \quad &\iff \quad \|W_2^{-1}\left\{\left[\begin{array}{cc} \tilde{U} & \tilde{V} \end{array}\right] - \left[\begin{array}{cc} \tilde{U}_0 & \tilde{V}_0 \end{array}\right]\right\} W_1^{-1}\|_\infty < 1 \\ &\Rightarrow \quad \|\mathcal{F}_l(\tilde{R}, \Delta)\|_\infty < 1 \\ &\iff \quad \|\mathcal{F}_l(P, K)\|_\infty < \gamma. \end{aligned}$$

A procedure to calculate appropriate weights is given in Section 4.5.

Remark 4.2.1 *For both the $\mathcal{A}(P)$ and $\mathcal{CF}(P)$ approaches, R and $\tilde{R}$ are stable. This is obvious since $\mathcal{F}_l(P, K^0) \in \mathcal{RH}_\infty$ and R have the same A-matrix.*

Remark 4.2.2 *The nominal controller K^0 may be designed using any methodology provided it is (P, γ)-admissible. Further, for the coprime factor case, any left coprime factorisation of K^0 may be chosen.*

4.3 Free Parameter Approach

In Section 4.2 the sufficient conditions for the design of reduced order (P, γ)-admissible controllers were obtained by directly studying the interconnection of the generalised plant P and the controller K. In this section we take an alternative approach which exploits the properties of the parameterisation of all (P, γ)-admissible controllers reviewed in Section 2.8. We begin by using simple LFT manipulation to show how the condition $\Phi \in \mathcal{RH}_\infty$, $\|\Phi\|_\infty < \gamma$ on the otherwise free parameter in Theorem 2.8.2, is equivalent to a norm constraint on a specific weighted error. Remember that $K = \mathcal{F}_l(K_a, \Phi)$ is a (P, γ)-admissible controller if and only if $\Phi \in \mathcal{RH}_\infty$ and $\|\Phi\|_\infty < \gamma$.

Consider the parameterisation of all (P, γ)-admissible controllers as given in Theorem 2.8.2 and represented in Figure 4.4. Since K_a is invertible (see Remark 2.8.3), $K = \mathcal{F}_l(K_a, \Phi)$ may be inverted according to Lemma 2.7.3 to give the mapping from r to v as $\Phi = \mathcal{F}_u(K_a^{-1}, K)$. This is shown in Figure 4.5.

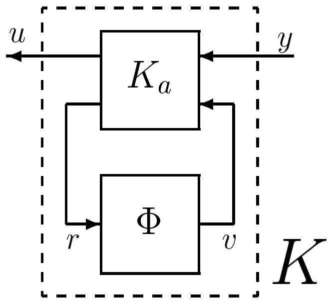


Figure 4.4: LFT Configuration for all (P, γ)-admissible Controllers

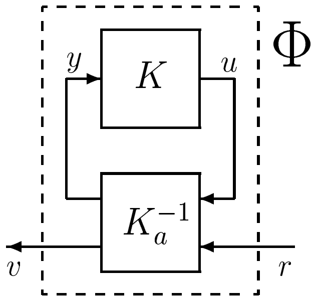


Figure 4.5: LFT Configuration for Φ

Therefore the stability and norm bound constraint on Φ is equivalent to a stability and norm bound constraint on $\mathcal{F}_u(K_a^{-1}, K)$, i.e. K stabilising and

$$\begin{aligned} \|\mathcal{F}_l(P,K)\|_\infty < \gamma \quad &\Longleftrightarrow \quad \|\Phi\|_\infty < \gamma && (4.11)\\ &\Longleftrightarrow \quad \|\mathcal{F}_u(K_a^{-1},K)\|_\infty < \gamma. && (4.12)\end{aligned}$$

In the next two sections we again show how, by considering controllers to be representable by a nominal (P,γ)-admissible controller subjected to a weighted perturbation, sufficient conditions for any other controller to be (P,γ)-admissible may be derived. Section 4.3.1 considers the additive perturbation approach, denoted $\mathcal{A}(\Phi)$. Section 4.3.2 considers the coprime factor perturbation approach, denoted $\mathcal{CF}(\Phi)$.

4.3.1 $\mathcal{A}(\Phi)$ – The Additive Perturbation Case

Assume that the controller to be designed K, may be represented by a nominal (P,γ)-admissible controller K^0, subjected to a weighted additive perturbation, $K = K^0 + W_2\Delta W_1$, where W_1 and W_2 are stable, minimum phase and invertible and Δ is stable.

Comparing Figure 4.6 with Figure 4.5 we see that

$$\begin{aligned} \|\Phi\|_\infty < \gamma \quad &\Longleftrightarrow \quad \|\mathcal{F}_u(K_a^{-1},K)\|_\infty < \gamma \\ &\Longleftrightarrow \quad \|\mathcal{F}_u(K_a^{-1},K^0 + W_2\Delta W_1)\|_\infty < \gamma \\ &\Longleftrightarrow \quad \|\mathcal{F}_u(\tilde{R},\Delta)\|_\infty < 1 && (4.13)\end{aligned}$$

where

$$\tilde{R} = \begin{bmatrix} W_1 & 0 \\ 0 & \gamma^{-1/2} I \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} W_2 & 0 \\ 0 & \gamma^{-1/2} I \end{bmatrix} \tag{4.14}$$

and R is given by the Redheffer star product [15, 78],

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \mathcal{S}\left(\begin{bmatrix} 0 & I \\ I & K^0 \end{bmatrix}, K_a^{-1}\right). \tag{4.15}$$

Comparing equation (4.13) with Theorem 2.7.5 leads to the condition that if K^0 is a nominal (P,γ)-admissible controller and stable, minimum phase and invertible weights W_1

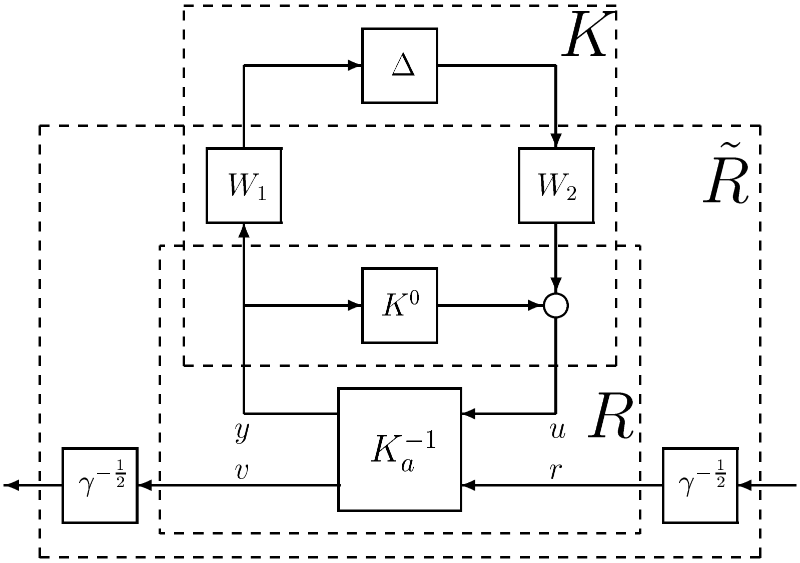


Figure 4.6: Interconnection for Approach $\mathcal{A}(\Phi)$

and W_2 can be calculated such that $\tilde{R}$ is inner and $\tilde{R}_{12}$ has full row rank (note again that this is not generally possible) then

$$\|\Delta\|_\infty < 1 \iff \|\mathcal{F}_u(\tilde{R}, \Delta)\|_\infty < 1.$$

Hence we obtain the condition that any controller, $K = K^0 + W_2 \Delta W_1$ for some $\Delta \in \mathcal{RH}_\infty$ and weights which make $\tilde{R}$ inner with $\tilde{R}_{12}$ full row rank will also be (P, γ)-admissible if and only if $\|W_2^{-1}(K - K^0)W_1^{-1}\|_\infty < 1$.

Again we stress that it is not generally possible to calculate weights such that the above necessary and sufficient condition is satisfied. However it is always possible to calculate weights which make $\tilde{R}$ a contraction. Such weights give the sufficient condition,

$$\|W_2^{-1}(K - K^0)W_1^{-1}\|_\infty < 1 \quad \Rightarrow \quad \|\Phi\|_\infty < \gamma \tag{4.16}$$

$$\iff \quad \|\mathcal{F}_l(P, K)\|_\infty < \gamma. \tag{4.17}$$

Calculation of appropriate weights is the subject of Section 4.5.

4.3.2 $\mathcal{CF}(\Phi)$ – The Coprime Factor Perturbation Case

Analogous to approach $\mathcal{CF}(P)$, in this section we consider the representation of all K as a stable weighted perturbation to the coprime factors of a nominal (P, γ)-admissible controller (see Figure 4.3). Substituting $K = \tilde{V}^{-1}\tilde{U}$ from equation (4.7) for K in equation (4.12) leads to the relationship

$$\|\Phi\|_\infty < \gamma \iff \|\mathcal{F}_u(\tilde{R}, \Delta)\|_\infty < 1 \tag{4.18}$$

where

$$\tilde{R} = \begin{bmatrix} W_1 & 0 \\ 0 & \gamma^{-1/2} I \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} W_2 & 0 \\ 0 & \gamma^{-1/2} I \end{bmatrix} \tag{4.19}$$

and R is given by the Redheffer star product [15, 78],

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \mathcal{S}(\left[\begin{array}{c:c} 0 & I \\ -\tilde{V}_0^{-1} & -K^0 \\ \hdashline \tilde{V}_0^{-1} & K^0 \end{array}\right], K_a^{-1}). \tag{4.20}$$

Along similar lines to previous sections we now compare the right hand side of equation (4.18) with Theorem 2.7.5. The necessity part of Theorem 2.7.5 requires that $\tilde{R}_{12}$ have full row rank. Equation (4.19) shows that this can never be the case. However, we can choose weights W_1 and W_2 such that $\tilde{R}$ is a contraction. These weights give the sufficient condition,

$$\begin{aligned} \|\Delta\|_\infty < 1 \quad & \Longleftrightarrow \quad \|W_2^{-1}\left\{\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix} - \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}\right\}W_1^{-1}\|_\infty < 1 \\ & \Rightarrow \quad \|\Phi\|_\infty < \gamma \\ & \Longleftrightarrow \quad \|\mathcal{F}_l(P,K)\|_\infty < \gamma. \end{aligned} \tag{4.21}$$

Calculation of appropriate weights is the subject of Section 4.5.

Remark 4.3.1 *As with the $\mathcal{A}(P)$ and $\mathcal{CF}(P)$ formulations, for both $\mathcal{A}(\Phi)$ and $\mathcal{CF}(\Phi)$, R and $\tilde{R}$ are stable. To see this, firstly substitute $K^0 = \mathcal{F}_l(K_a, \Phi^0)$ for some $\Phi^0 \in \mathcal{RH}_\infty$, $\|\Phi^0\|_\infty < \gamma$ into equation (4.20), then use the properties of K_a.*

Remark 4.3.2 *The nominal controller K^0 may be designed using any methodology provided it is (P,γ)-admissible. Further, for the coprime factor case any left coprime factorisation of K^0 may be chosen.*

4.4 $\mathcal{CF}(\Theta)$ – An Alternative LFT Approach

In the previous sections we employed standard LFT manipulations to derive sufficient conditions for a given controller to be (P,γ)-admissible. These conditions were expressed in terms of a frequency weighted error where the weights were required to be selected to make a certain transfer function, $\tilde{R}$, a contraction.

Here we take an alternative approach which also leads to a sufficient condition for a given controller to be (P,γ)-admissible, expressed as a frequency weighted error, however, in this approach there is no freedom in the selection of weights. They are completely specified by the coefficient matrix K_a.

Firstly, we require the following alternative form of LFT. It results from a trivial modification to Lemma 2.14 of [64].

Lemma 4.4.1 *Any LFT given by equation (2.8) such that* $\det(N_{12}) \neq 0$ *can also be written*

$$T_{zw} = (M_{22} + QM_{12})^{-1}(M_{21} + QM_{11})$$

where $\det(M_{22} + QM_{12}) \neq 0$,

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \mathcal{F}_l(\left[\begin{array}{cc:c} N_{21} & 0 & N_{22} \\ 0 & 0 & -I \\ \hdashline N_{11} & I & N_{12} - D \end{array}\right], -D^{-1})$$

and D is any appropriately dimensioned invertible matrix.

Since K_{a12} is invertible, the following form of LFT may be used to parameterise all (P,γ)-admissible controllers.

$$K = (\theta_{22} + \Phi\theta_{12})^{-1}(\theta_{21} + \Phi\theta_{11}) = \tilde{V}^{-1}\tilde{U} \tag{4.22}$$

where $\|\Phi\|_\infty < \gamma$ and

$$\begin{aligned}\Theta &\triangleq \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} \\ &= \mathcal{F}_l(\left[\begin{array}{cc:c} K_{a21} & 0 & K_{a22} \\ 0 & 0 & -I \\ \hdashline K_{a11} & I & K_{a12} - D_{a12} \end{array}\right], -D_{a12}^{-1}).\end{aligned} \tag{4.23}$$

Remark 4.4.2 *From equation (4.23) it is simple to show that the A-matrix of Θ is given by $A_a - B_{a2}D_{a12}^{-1}C_{a1}$ and the A-matrix of Θ^{-1} is given by $A_a - B_{a1}D_{a21}^{-1}C_{a2}$. Hence by Remark 2.8.4 Θ is stable and minimum phase.*

Proposition 4.4.3 *Assume there exists a (P,γ)-admissible controller parameterised by equation (4.22), and define*

$$\Delta = \left\{ \begin{bmatrix} \hat{U} & \hat{V} \end{bmatrix} - \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix} \right\} \Theta^{-1} \begin{bmatrix} \gamma^{-1}I & 0 \\ 0 & I \end{bmatrix} \in \mathcal{RH}_\infty$$

where $\begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix} = \begin{bmatrix} \theta_{21} & \theta_{22} \end{bmatrix}$. Then the controller $\hat{K} = \hat{V}^{-1}\hat{U}$ will be (P,γ)-admissible if $\|\Delta\|_\infty < 1/\sqrt{2}$. That is,

$$\begin{aligned}\| \left\{ \begin{bmatrix} \hat{U} & \hat{V} \end{bmatrix} - \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix} \right\} \Theta^{-1} \begin{bmatrix} \gamma^{-1}I & 0 \\ 0 & I \end{bmatrix} \|_\infty < 1/\sqrt{2} \quad &\Rightarrow \quad \|\Phi\|_\infty < \gamma \\ &\iff \quad \mathcal{F}_l(P,\hat{K}) < \gamma.\end{aligned}$$

Proof: By assumption $\Delta \in \mathcal{RH}_\infty$ with $\|\Delta\|_\infty < 1/\sqrt{2}$. Now

$$\begin{bmatrix} \hat{U} & \hat{V} \end{bmatrix} = \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix} + \Delta \begin{bmatrix} \gamma I & 0 \\ 0 & I \end{bmatrix} \Theta$$

which implies

$$\begin{bmatrix} \hat{U} & \hat{V} \end{bmatrix} = \begin{bmatrix} \gamma\Delta_1 & I + \Delta_2 \end{bmatrix} \Theta$$

where $\begin{bmatrix} \Delta_1 & \Delta_2 \end{bmatrix} \triangleq \Delta$. Hence

$$\begin{aligned}\begin{bmatrix} (I+\Delta_2)^{-1}\hat{U} & (I+\Delta_2)^{-1}\hat{V} \end{bmatrix} &= \begin{bmatrix} \gamma(I+\Delta_2)^{-1}\Delta_1 & I \end{bmatrix} \Theta \\ &\triangleq \begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix}\end{aligned}$$

is a coprime factorisation of $\hat{K}$ such that

$$\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix} = \begin{bmatrix} \Phi & I \end{bmatrix} \Theta. \tag{4.24}$$

Now

$$\begin{aligned}\Phi &= \gamma(I+\Delta_2)^{-1}\Delta_1 \\ &= \gamma\mathcal{F}_l(\left[\begin{array}{c:c} 0 & I/\sqrt{2} \\ \hdashline I & 0 \\ 0 & -I/\sqrt{2}\end{array}\right], \sqrt{2}\Delta).\end{aligned} \tag{4.25}$$

and by Theorem 2.7.5,

$$\|\Delta\|_\infty < 1/\sqrt{2} \;\Rightarrow\; \|\Phi\|_\infty < \gamma.$$

■

Remark 4.4.4 *Unlike the previous approaches, $\mathcal{CF}(\Theta)$ requires $\begin{bmatrix}\tilde{U}_0 & \tilde{V}_0\end{bmatrix}$ to be a specific coprime factorisation of the central controller.*

4.5 Weights for Sufficiency

For approaches $\mathcal{A}(P/\Phi)$ and $\mathcal{CF}(P/\Phi)$, weights do not generally exist which satisfy the inner requirement on $\tilde{R}$. However it is always possible to find weights W_1 and W_2 which make $\tilde{R}$ a contraction and thus satisfy the sufficiency part of Theorem 2.7.5. (This gives the sufficient conditions derived earlier in the chapter.) In this section we will show how to calculate these weights. Firstly consider calculating weights such that $\tilde{R}$ of the form

$$\tilde{R} = \begin{bmatrix}\gamma^{-1/2}I & 0 \\ 0 & W_1\end{bmatrix}\begin{bmatrix}R_{11} & R_{12} \\ R_{21} & R_{22}\end{bmatrix}\begin{bmatrix}\gamma^{-1/2}I & 0 \\ 0 & W_2\end{bmatrix}$$

is a contraction. This is appropriate for approaches $\mathcal{A}(P)$ and $\mathcal{CF}(P)$.

Proposition 4.5.1 *Assume $\|R_{11}\|_\infty < \gamma$ and define*

$$L = \begin{bmatrix}L_1 & L_2 \\ L_2^\sim & L_3\end{bmatrix} \triangleq \mathcal{F}_l(\left[\begin{array}{cc:cc} 0 & -R_{22} & 0 & R_{21} \\ -R_{22}^\sim & 0 & R_{12}^\sim & 0 \\ \hdashline 0 & R_{12} & 0 & -R_{11} \\ R_{21}^\sim & 0 & -R_{11}^\sim & 0\end{array}\right], \gamma^{-1}I). \tag{4.26}$$

Define $X \triangleq (W_1^\sim W_1)^{-1}$ and $Y \triangleq (W_2 W_2^\sim)^{-1}$. Then $\tilde{R}$ is a contraction if and only if

$$\begin{bmatrix}X & 0 \\ 0 & Y\end{bmatrix} \geq \begin{bmatrix}L_1 & L_2 \\ L_2^\sim & L_3\end{bmatrix}. \tag{4.27}$$

Proof of Proposition 4.5.1 requires the following theorem on positive semi-definite matrices.

Lemma 4.5.2 ([43], Theorem 7.7.6) *Suppose that an Hermitian matrix is partitioned as $\begin{bmatrix}A & B \\ B^* & C\end{bmatrix}$ where $A = A^*$ and $C = C^* > 0$. This matrix is positive semi-definite if and only if $A - BC^{-1}B^* \geq 0$.*

Proof (of proposition 4.5.1)**:** We require $\|\tilde{R}\|_\infty \leq 1$. Now

$$
\begin{aligned}
&\|\tilde{R}\|_\infty \leq 1 \\
\iff &\begin{bmatrix} I & \tilde{R} \\ \tilde{R}^\sim & I \end{bmatrix} \geq 0 \\
\iff &\left[\begin{array}{cc:cc} \gamma^{1/2}I & 0 & 0 & 0 \\ 0 & W_1^{-1} & 0 & 0 \\ \hdashline 0 & 0 & \gamma^{1/2}I & 0 \\ 0 & 0 & 0 & W_2^{\sim -1} \end{array}\right] \begin{bmatrix} I & \tilde{R} \\ \tilde{R}^\sim & I \end{bmatrix} \left[\begin{array}{cc:cc} \gamma^{1/2}I & 0 & 0 & 0 \\ 0 & W_1^{\sim -1} & 0 & 0 \\ \hdashline 0 & 0 & \gamma^{1/2}I & 0 \\ 0 & 0 & 0 & W_2^{-1} \end{array}\right] \geq 0 \\
\iff &\left[\begin{array}{cc:cc} \gamma I & 0 & R_{11} & R_{12} \\ 0 & (W_1^\sim W_1)^{-1} & R_{21} & R_{22} \\ \hdashline R_{11}^\sim & R_{21}^\sim & \gamma I & 0 \\ R_{12}^\sim & R_{22}^\sim & 0 & (W_2 W_2^\sim)^{-1} \end{array}\right] \geq 0
\end{aligned}
$$

and by interchanging the second and third rows and columns

$$
\iff \left[\begin{array}{cc:cc} \gamma I & R_{11} & 0 & R_{12} \\ R_{11}^\sim & \gamma I & R_{21}^\sim & 0 \\ \hdashline 0 & R_{21} & X & R_{22} \\ R_{12}^\sim & 0 & R_{22}^\sim & Y \end{array}\right] \geq 0
$$

where $X \triangleq (W_1^\sim W_1)^{-1}$ and $Y \triangleq (W_2 W_2^\sim)^{-1}$. Note that

$$
\|R_{11}\|_\infty < \gamma \quad \iff \quad \begin{bmatrix} \gamma I & R_{11} \\ R_{11}^\sim & \gamma I \end{bmatrix} > 0.
$$

Therefore Lemma 4.5.2 implies

$$
\begin{aligned}
&\|\tilde{R}\|_\infty \leq 1 \\
\iff &\begin{bmatrix} X & R_{22} \\ R_{22}^\sim & Y \end{bmatrix} \geq \begin{bmatrix} 0 & R_{21} \\ R_{12}^\sim & 0 \end{bmatrix} \begin{bmatrix} \gamma I & R_{11} \\ R_{11}^\sim & \gamma I \end{bmatrix}^{-1} \begin{bmatrix} 0 & R_{12} \\ R_{21}^\sim & 0 \end{bmatrix} \\
\iff &\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \geq \begin{bmatrix} L_1 & L_2 \\ L_2^\sim & L_3 \end{bmatrix}.
\end{aligned}
$$

■

Remark 4.5.3 *Since $R_{11} = \mathcal{F}_l(P, K^0)$ and K^0 is a (P,γ)-admissible controller, it is guaranteed that $\|R_{11}\|_\infty < \gamma$.*

Remark 4.5.4 *For approaches $\mathcal{A}(\Phi)$ and $\mathcal{CF}(\Phi)$, $\tilde{R}$ is of the form*

$$
\tilde{R} = \begin{bmatrix} W_1 & 0 \\ 0 & \gamma^{-1/2}I \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} W_2 & 0 \\ 0 & \gamma^{-1/2}I \end{bmatrix} \tag{4.28}
$$

with $\|R_{22}\|_\infty < \gamma$. Using Proposition 4.5.1 in conjunction with the transformed system

$$
\bar{R} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tilde{R} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}
$$

gives weights W_1 and W_2 for $\tilde{R}$ of the form of equation (4.28). Note that for the conditions to be necessary the rank requirement on $\tilde{R}_{12}$ in the $\mathcal{A}(\Phi)$ and $\mathcal{CF}(\Phi)$ approaches then becomes a rank requirement on $\bar{R}_{21}$.

Remark 4.5.5 *Note from equation (4.26) that*

$$L = \begin{bmatrix} \gamma R_{21}(\gamma^2 I - R_{11}^{\sim}R_{11})^{-1}R_{21}^{\sim} & -(R_{22}+R_{21}R_{11}^{\sim}(\gamma^2 I - R_{11}R_{11}^{\sim})^{-1}R_{12}) \\ -(R_{22}^{\sim}+R_{12}^{\sim}R_{11}(\gamma^2 I - R_{11}^{\sim}R_{11})^{-1}R_{21}^{\sim}) & \gamma R_{12}^{\sim}(\gamma^2 I - R_{11}R_{11}^{\sim})^{-1}R_{12} \end{bmatrix}.$$

Hence L has the following dimensions and sign (semi-)definiteness.
Approaches $\mathcal{A}(P)$ and $\mathcal{A}(\Phi)$: $L_1 \in \mathcal{RL}_\infty^{p_2\times p_2} > 0$, $L_3 \in \mathcal{RL}_\infty^{m_2\times m_2} > 0$. To see this for the $\mathcal{A}(P)$ approach, notice that from equation (4.5),

$$R = \begin{bmatrix} P_{11}+P_{12}K^0(I-P_{22}K^0)^{-1}P_{21} & P_{12}(I-K^0P_{22})^{-1} \\ (I-P_{22}K^0)^{-1}P_{21} & P_{22}(I-K^0P_{22})^{-1} \end{bmatrix}.$$

The dimensions of P and assumption (A3) of Section 2.8.1 on the rank of D_{12} and D_{21} then give the required result. The $\mathcal{A}(P/\Phi)$ approach follows similarly.
Approaches $\mathcal{CF}(P)$ and $\mathcal{CF}(\Phi)$: $L_1 \in \mathcal{RL}_\infty^{(p_2+m_2)\times(p_2+m_2)} \geq 0$, $L_3 \in \mathcal{RL}_\infty^{m_2\times m_2} > 0$. To see this for the $\mathcal{CF}(P)$ approach, notice that from equation (4.10),

$$R = \begin{bmatrix} P_{11}+P_{12}K^0(I-P_{22}K^0)^{-1}P_{21} & P_{12}(I-K^0P_{22})^{-1}\tilde{V}_0^{-1} \\ \begin{bmatrix} I \\ -K^0 \end{bmatrix}(I-P_{22}K^0)^{-1}P_{21} & \begin{bmatrix} 0 \\ -\tilde{V}_0^{-1} \end{bmatrix} + \begin{bmatrix} I \\ -K^0 \end{bmatrix} P_{22}(I-K^0P_{22})^{-1}\tilde{V}_0^{-1} \end{bmatrix}.$$

As above, the result follows from the dimensions of P and the rank assumptions of D_{12} and D_{21}. From this we can also determine that the rank of L_1 is p_2. Dimensions for the $\mathcal{CF}(\Phi)$ approach follow similarly.

4.6 Weight Maximisation

In preceding sections, provided there exists a nominal (P,γ)-admissible controller, we derived new sufficient conditions for any other controller to be (P,γ)-admissible. These conditions are expressed as a norm bound on a stable transfer function Δ, where, depending on the problem formulation

$$\Delta = \begin{cases} W_2^{-1}(K-K^0)W_1^{-1} & \mathcal{A}(P/\Phi) \text{ approaches} \\ W_2^{-1}\left(\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix} - \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}\right)W_1^{-1} & \mathcal{CF}(P/\Phi/\Theta) \text{ approaches.} \end{cases}$$

To obtain the set of controllers K such that $\|\Delta\|_\infty < 1$ is equivalent to finding the set of controllers K that lie within a weighted $\mathcal{L}_\infty$ ball of left radius W_2 and right radius W_1 centred on K^0 or $\begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}$ respectively. Hence we require both W_1 and W_2 to be large in some sense to make any subsequent approximation as easy as possible. For the $\mathcal{CF}(\Theta)$ approach the weights are completely specified in the problem formulation. However, for the $\mathcal{A}(P/\Phi)$ and $\mathcal{CF}(P/\Phi)$ approaches there is freedom to select the weights subject to the transfer function $\tilde{R}$ being contractive.

In the scalar case maximising $|W_1W_2|$ is appropriate, however in the matrix case the optimal choice is not clear. Here we investigate two satisfactory approaches: firstly to maximise the product $\mathrm{trace}(W_1^\sim W_1)\mathrm{trace}(W_2W_2^\sim)$; and secondly to maximise the product $\det(W_1^\sim W_1)\det(W_2W_2^\sim)$.

However, we will show that for the coprime factor case, maximising the product $\det(W_1^\sim W_1)\det(W_2W_2^\sim)$ may not be a sensible choice as it can lead to *directions* where the modeling error must be zero.

Lemma 4.6.1 (Maximising trace$(W_1^\sim W_1)$trace$(W_2W_2^\sim)$) *Define L as in Proposition 4.5.1 and assume $L_1 \geq 0$, with trace$(L_1) \neq 0$, and $L_3 > 0$. Also define*

$$\alpha_{opt} \triangleq \sqrt{(trace(L_3)/trace(L_1))} \tag{4.29}$$

$$Z_1^2 \triangleq L_2^\sim L_2 \tag{4.30}$$

$$Z_2 \triangleq L_2 Z_1^{-1} L_2^\sim \tag{4.31}$$

where $Z_1 = Z_1^\sim > 0$ and $Z_2 = Z_2^\sim \geq 0$. Then weights W_1 and W_2 which maximise the product trace$(W_1^\sim W_1)$trace$(W_2W_2^\sim)$ subject to equation (4.27) satisfy

$$(W_1^\sim W_1)^{-1} = L_1 + \alpha_{opt}^{-1} Z_2 \tag{4.32}$$

$$(W_2W_2^\sim)^{-1} = L_3 + \alpha_{opt} Z_1 \tag{4.33}$$

Proof: Define $X \triangleq (W_1^\sim W_1)^{-1}$ and $Y \triangleq (W_2W_2^\sim)^{-1}$. We wish to minimise trace(X)trace(Y) subject to equation (4.27).

Given any $Y > L_3$ the unique minimising X is given by

$$X = L_1 + L_2(Y - L_3)^{-1} L_2^\sim.$$

Now define

$$f(Y) \triangleq \mathrm{trace}(X)\mathrm{trace}(Y)$$

and hence, for $\Psi = \Psi^\sim$

$$f(Y+\Psi) = \mathrm{trace}(Y+\Psi)\mathrm{trace}(L_1 + L_2(Y - L_3 + \Psi)^{-1} L_2^\sim)$$

which, making use of the series expansion $(I + M\Psi)^{-1} = I - M\Psi + o(\Psi)$,

$$\begin{aligned} &= (\mathrm{trace}(Y) + \mathrm{trace}(\Psi))\Big(\mathrm{trace}(L_1 + L_2(Y-L_3)^{-1}L_2^\sim) \\ &\qquad - \mathrm{trace}(L_2(Y-L_3)^{-1}\Psi(Y-L_3)^{-1}L_2^\sim)\Big) + o(\Psi) \\ &= f(Y) + \mathrm{trace}(\Psi)\mathrm{trace}(L_1 + L_2(Y-L_3)^{-1}L_2^\sim) \\ &\qquad - \mathrm{trace}(Y)\mathrm{trace}((Y-L_3)^{-1}L_2^\sim L_2(Y-L_3)^{-1}\Psi) + o(\Psi). \end{aligned}$$

$f(Y)$ is therefore at a stationary point if $\forall \Psi = \Psi^\sim$,

$$\mathrm{trace}(\Psi)\mathrm{trace}(L_1 + L_2(Y-L_3)^{-1}L_2^\sim) = \mathrm{trace}(Y)\mathrm{trace}((Y-L_3)^{-1}L_2^\sim L_2(Y-L_3)^{-1}\Psi).$$

By successively considering $\Psi = e_i e_i'$, $\Psi = (e_i e_j' + e_j e_i')$ then $\Psi = \sqrt{-1}(e_i e_j' + e_j e_i')$ where e_i are the standard unit vectors with a 1 as the i^{th} element and zeros elsewhere, gives that

$$(Y - L_3)^{-1} L_2^{\sim} L_2 (Y - L_3)^{-1} = \alpha^{-2} I$$

for some scalar $\alpha > 0$. Therefore noting that $L_2(L_2^{\sim} L_2)^{-1/2} L_2^{\sim} = (L_2 L_2^{\sim})^{1/2}$ for L_2 with full column rank, we have

$$\begin{aligned} Y &= L_3 + \alpha Z_1 \\ X &= L_1 + \alpha^{-1} Z_2 \end{aligned}$$

and

$$\begin{aligned} \mathrm{trace}(Y) &= \mathrm{trace}(L_3) + \alpha \mathrm{trace}(Z_1) \\ \mathrm{trace}(X) &= \mathrm{trace}(L_1) + \alpha^{-1} \mathrm{trace}(Z_1). \end{aligned}$$

Hence

$$f(Y) = \mathrm{trace}(L_1)\mathrm{trace}(L_3) + (\mathrm{trace}(Z_1))^2 + \mathrm{trace}(Z_1)\left(\alpha \mathrm{trace}(L_1) + \alpha^{-1}\mathrm{trace}(L_3)\right)$$

and is minimised when $\alpha = \alpha_{opt}$. ■

Lemma 4.6.2 (Maximising $\det(W_1^{\sim} W_1)\det(W_2 W_2^{\sim})$) *Define L as in Proposition 4.5.1 and assume $L_1 > 0$ and $L_3 > 0$. Also define $SS^{\sim} \triangleq L_1 > 0$, $T^{\sim} T \triangleq L_3 > 0$,*

$$Z_1^2 \triangleq S^{-1} L_2 L_3^{-1} L_2^{\sim} S^{\sim -1} \tag{4.34}$$

and

$$Z_2^2 \triangleq T^{\sim -1} L_2^{\sim} L_1^{-1} L_2 T^{-1}, \tag{4.35}$$

where $Z_1 = Z_1^{\sim} \geq 0$ and $Z_2 = Z_2^{\sim} \geq 0$. Then weights W_1 and W_2 which maximise $\det(W_1^{\sim} W_1)\det(W_2 W_2^{\sim})$ satisfy

$$(W_1^{\sim} W_1)^{-1} = S(I + Z_1)S^{\sim} = L_1 + S Z_1 S^{\sim} \tag{4.36}$$

$$(W_2 W_2^{\sim})^{-1} = T^{\sim}(I + Z_2)T = L_3 + T^{\sim} Z_2 T \tag{4.37}$$

Proof: Define $X = (W_1^{\sim} W_1)^{-1}$ and $Y = (W_2 W_2^{\sim})^{-1}$. We wish to minimise $\det(X)\det(Y)$ subject to equation (4.27).

(a) Firstly assume L_2 has full column rank. Therefore $Y > L_3$ and given a Y the unique minimising X is given by

$$X = L_1 + L_2(Y - L_3)^{-1} L_2^{\sim}.$$

Now define

$$\begin{aligned} f(Y) &\triangleq \frac{\det(X)\det(Y)}{\det(L_1)} \\ &= \frac{\det(Y - L_3 + L_2^{\sim} L_1^{-1} L_2)\det(Y)}{\det(Y - L_3)}. \end{aligned}$$

Therefore, for $\Psi = \Psi^{\sim}$

$$\begin{aligned} f(Y+\Psi) &= \det(Y)\det(I+Y^{-1}\Psi) \\ &\quad \times\frac{\det(Y-L_3+L_2^{\sim}L_1^{-1}L_2)}{\det(Y-L_3)} \\ &\quad \times\frac{\det(I+(Y-L_3+L_2^{\sim}L_1^{-1}L_2)^{-1}\Psi)}{\det(I+(Y-L_3)^{-1}\Psi)}. \end{aligned}$$

In order to find a stationary point observe that

$$(I+M\Psi)^{-1} = I - M\Psi + o(\Psi)$$

and

$$\det(I+M\Psi) = 1+\mathrm{trace}(M\Psi)+o(\Psi)$$

Hence

$$\begin{aligned} f(Y+\Psi) &= f(Y)\,\{\,1+\mathrm{trace}([\,Y^{-1}+(Y-L_3+L_2^{\sim}L_1^{-1}L_2)^{-1} \\ &\quad -(Y-L_3)^{-1}\,]\,\Psi)+o(\Psi)\,\}\,, \end{aligned}$$

giving a stationary point if

$$\mathrm{trace}([\,Y^{-1}+(Y-L_3+L_2^{\sim}L_1^{-1}L_2)^{-1} \;-\; (Y-L_3)^{-1}\,]\,\Psi)=0 \quad \forall\,\Psi=\Psi^{\sim}.$$

However for $M = M^{\sim}$,

$$\mathrm{trace}(M\Psi)=0\ \ \forall\Psi=\Psi^{\sim} \Rightarrow\ M=M^{\sim}=0.$$

Hence $f(Y)$ is at a stationary point if

$$\begin{aligned} && (Y-L_3+L_2^{\sim}L_1^{-1}L_2)^{-1} &= (Y-L_3)^{-1}L_3Y^{-1} \\ &\Rightarrow & Y-L_3+L_2^{\sim}L_1^{-1}L_2 &= YL_3^{-1}(Y-L_3) \\ &\Rightarrow & YL_3^{-1}Y-2Y+L_3-L_2^{\sim}L_1^{-1}L_2 &= 0 \\ &\Rightarrow & (T^{\sim-1}YT^{-1}-I)^2 &= Z_2^2. \end{aligned}$$

Noting that we require $Y-L_3>0$, $\det(X)\det(Y)$ is therefore minimised for

$$Y = T^{\sim}(I+Z_2)T = L_3+T^{\sim}Z_2T.$$

Now note that $X-L_1 = X-SS^{\sim}$ and therefore for the above Y

$$\begin{aligned} (S^{-1}XS^{\sim-1}-I)^2 &= (S^{-1}(X-SS^{\sim})S^{\sim-1})^2 \\ &= (S^{-1}(X-L_1)S^{\sim-1})^2 \\ &= S^{-1}L_2T^{-1}Z_2^{-1}Z_2^2Z_2^{-1}T^{\sim-1}L_2^{\sim}S^{\sim-1} \\ &= S^{-1}L_2L_3^{-1}L_2^{\sim}S^{\sim-1} \\ &= Z_1^2 \end{aligned}$$

or equivalently

$$X = S(I + Z_1)S^{\sim} = L_1 + SZ_1S^{\sim}.$$

(b) If L_2 doesn't have full column rank but does have full row rank then $X > L_1$ and appropriate weights can be calculated by working with the function

$$f(X) \triangleq \frac{\det(X)\det(Y)}{\det(L_3)}$$

instead of $f(Y)$.

(c) If L_2 has neither full column rank nor full row rank then consider the associated augmented problem of minimising $\det(X_a)\det(Y)$ such that

$$\begin{bmatrix} X_{a11} & X_{a12} & 0 \\ X_{a12}^{\sim} & X_{a22} & 0 \\ 0 & 0 & Y \end{bmatrix} \geq \begin{bmatrix} L_1 & 0 & L_2 \\ 0 & I & -\varepsilon I \\ L_2^{\sim} & -\varepsilon I & L_3 \end{bmatrix}$$

where $X_a = X_a^{\sim} > 0$, $L_1 = L_1^{\sim} > 0$ and $L_3 = L_3^{\sim} > 0$. This problem satisfies the full column rank assumption of (a) and hence the previous result applies. Then via a limiting argument it is straightforward to show that $\det(X)\det(Y)$ is minimised by X_{a11} and Y as $\varepsilon \to 0$ and that the limits are as in equations (4.34-4.37). ∎

Remark 4.6.3 *In general it is not possible to exactly solve transfer function factorisations of the form required to calculate Z_1 and Z_2. However a satisfactory technique is presented in Chapter 7.*

4.6.1 The determinant can be a bad choice.

We will now show that for approaches $\mathcal{CF}(P)$ and $\mathcal{CF}(\Phi)$, where coprime factor perturbations are employed, maximising $\det(W_1^{\sim}W_1)\det(W_2W_2^{\sim})$ is not a good choice. We will show that since L_1 is generically rank deficient, the error $(\tilde{V} - \tilde{V}_0)$ must tend to zero for controller $K = \tilde{V}^{-1}\tilde{U}$ to satisfy the sufficient condition for (P, γ) admissibility.

To see this, define $X \triangleq (W_1^{\sim}W_1)^{-1}$, $Y \triangleq (W_2W_2^{\sim})^{-1}$ and notice that L can, after suitable congruence transformation, typically be rewritten as

$$L = \left[\begin{array}{cc:c} 0 & 0 & L_{21} \\ 0 & L_{13} & 0 \\ \hdashline L_{21}^{\sim} & 0 & L_3 \end{array}\right]$$

where $L_{13} = L_{13}^{\sim} > 0$, $L_3 = L_3^{\sim} > 0$ and L_{21} is square and invertible. We wish to maximise the product $\det(W_1^{\sim}W_1)\det(W_2W_2^{\sim})$, or equivalently to minimise $\det(X)\det(Y)$, subject to equation (4.27).

Given any $Y > L_3$ the unique minimising X is

$$\begin{aligned} X &= \begin{bmatrix} 0 & 0 \\ 0 & L_{13} \end{bmatrix} + \begin{bmatrix} L_{21} \\ 0 \end{bmatrix} (Y - L_3)^{-1} \begin{bmatrix} L_{21}^{\sim} & 0 \end{bmatrix} \\ &= \begin{bmatrix} L_{21}(Y - L_3)^{-1}L_{21}^{\sim} & 0 \\ 0 & L_{13} \end{bmatrix}. \end{aligned}$$

Therefore define

$$\begin{aligned} f(Y) &\triangleq \det(Y)\det(L_{21}(Y-L_3)^{-1}L_{21}^{\sim}) \\ &= \frac{\det(L_{21}L_{21}^{\sim})}{\det(I-L_3Y^{-1})} \end{aligned}$$

which reaches a minimum when $\det(X)\det(Y)$ is minimised, or equivalently when $\det(I - L_3Y^{-1})$ is maximised.

Now choose $Y = \mu L_3$ where $\mu \to \infty$. This choice maximises $\det(I - L_3Y^{-1})$ and gives $X_{11} = L_{21}(\mu-1)^{-1}L_3^{-1}L_{21}$. Therefore

$$\det(X)\det(Y) = \det(L_{13})\det(L_{21}L_{21}^{\sim})(\mu-1)^{-m_2}\mu^{m_2} \to \det(L_{13})\det(L_{21}L_{21}^{\sim})$$

where m_2 is the number of controller outputs.

Using the above weights, our sufficient condition becomes

$$\begin{aligned} &\|W_2^{-1}\left\{\begin{bmatrix} \tilde{U} & \tilde{V}\end{bmatrix} - \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0\end{bmatrix}\right\}W_1^{-1}\|_\infty = \\ &\|\mu^{1/2}T\left\{\begin{bmatrix} \tilde{U} & \tilde{V}\end{bmatrix} - \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0\end{bmatrix}\right\}\begin{bmatrix} (\mu-1)^{-1/2}S_1 & 0 \\ 0 & S_2\end{bmatrix}\|_\infty < 1. \end{aligned} \tag{4.38}$$

where $T^{\sim}T = L_3$, $S_1S_1^{\sim} = (L_{21}L_3^{-1}L_{21}^{\sim})$ and $S_2S_2^{\sim} = L_{13}$.

Equation (4.38) shows that as $\mu \to \infty$ the error $(\tilde{V} - \tilde{V}_0)$ must tend to zero for controller $K = \tilde{V}^{-1}\tilde{U}$ to satisfy our sufficient condition and therefore be (P,γ)-admissible.

4.7 Equivalence of Approaches

In this section we establish relationships between approaches $\mathcal{A}(P)$ and $\mathcal{A}(\Phi)$, between approaches $\mathcal{CF}(P)$ and $\mathcal{CF}(\Phi)$, and between approaches $\mathcal{CF}(\Phi)$ and $\mathcal{CF}(\Theta)$.

4.7.1 Equivalence between Approaches $\mathcal{A}(P)$ and $\mathcal{A}(\Phi)$, and between Approaches $\mathcal{CF}(P)$ and $\mathcal{CF}(\Phi)$

We now show that the weights derived via approaches $\mathcal{A}(P)$ and $\mathcal{A}(\Phi)$ are identical, and that the weights derived via approaches $\mathcal{CF}(P)$ and $\mathcal{CF}(\Phi)$ are identical.

To simplify the exposition, as in [14] throughout this section we make the following additional assumptions on the generalised plant P.

(A6) $D_{11} = 0$.

(A7) $D_{12}'C_1 = 0$ and $B_1D_{21}' = 0$.

(A8) $\gamma = 1$.

Assumptions (A6)-(A8) greatly simplify the controller formulae and allow K_a, the co-efficient matrix in the parameterisation of all (P,γ)-admissible controllers, to be written

$$K_a = \left[\begin{array}{c|cc} \hat{A}_\infty & -Z_\infty L_\infty & Z_\infty B_2 \\ \hline F_\infty & 0 & I \\ -C_2 & I & 0 \end{array}\right] \tag{4.39}$$

where

$$\begin{aligned} \hat{A}_\infty &\triangleq A + B_1 B_1' X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2 \\ F_\infty &\triangleq -B_2' X_\infty \\ L_\infty &\triangleq -Y_\infty C_2' \\ Z_\infty &\triangleq (I - Y_\infty X_\infty)^{-1}. \end{aligned}$$

We shall use this parameterisation to prove the following lemma. The subscript notation is taken from [14].

Lemma 4.7.1 *Define*

$$N \triangleq \left[\begin{array}{c|cc} A_{F_\infty} & B_1 & B_2 \\ \hline C_{1F_\infty} & 0 & D_{12} \\ -B_1' X_\infty & I & 0 \end{array}\right]$$

and

$$M \triangleq \left[\begin{array}{c|cc} A_{tmp} - Y_{tmp} C_2' C_2 & B_1 - Y_{tmp} C_2' D_{21} & Y_{tmp} F_\infty' \\ \hline -F_\infty & 0 & I \\ C_2 & D_{21} & 0 \end{array}\right]$$

where

$$\begin{aligned} A_{F_\infty} &\triangleq A + B_2 F_\infty \\ C_{1F_\infty} &\triangleq C_1 + D_{12} F_\infty \\ A_{tmp} &\triangleq A + B_1 B_1' X_\infty \\ Y_{tmp} &= Z_\infty Y_\infty. \end{aligned}$$

Then

(a) N is inner.

(b) M is co-inner.

(c) $P = \mathcal{S}(\mathcal{S}(N, M), \bar{K}_a^{-1})$ where

$$\bar{K}_a^{-1} \triangleq \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} K_a^{-1} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Proof:

(a) A detailed proof may be found in [14]. X_∞ is the positive definite stabilising solution to the ARE of equation (2.11) and guarantees that A_{F_∞} is stable. That N is inner follows by noting that X_∞ is the observability Grammian for N and that the conditions of Lemma 2.4.3 are satisfied.

(b) In [14], Y_{tmp} is defined as the positive definite stabilising solution to an appropriate ARE which guarantees that $A_{tmp} - Y_{tmp}C_2'C_2$ is stable. That M is co-inner follows by noting that Y_{tmp} is the controllability Grammian for M and that the conditions of Lemma 2.4.4 are satisfied.

(c) That $P = \mathcal{S}(\mathcal{S}(N,M), \bar{K}_a^{-1})$ follows from explicitly evaluating the right hand side. ■

We are now in a position to state the main result of this section.

Proposition 4.7.2 *Define L_P as the L derived from the $\mathcal{A}(P)$ approach, and define L_Φ as the L derived from the $\mathcal{A}(\Phi)$ approach (after employing the transformation of Remark 4.5.4). Then $L_P = L_\Phi$, and hence the weights derived for approaches $\mathcal{A}(P)$ and $\mathcal{A}(\Phi)$ are identical.*

For the proof of Proposition 4.7.2 we require the following two results.

Lemma 4.7.3 *Define R_P and R_Φ as the R's derived from approaches $\mathcal{A}(P)$ and $\mathcal{A}(\Phi)$ respectively. Let $\mathcal{N} = \mathcal{S}(N,M)$ where N and M are defined in Lemma 4.7.1. Then*

$$R_P = \mathcal{S}(\mathcal{N}, \bar{R}_\Phi)$$

where

$$\bar{R}_\Phi = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} R_\Phi \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Proof: Recall that

$$R_P = \mathcal{S}(P, \begin{bmatrix} K^0 & I \\ I & 0 \end{bmatrix})$$

and

$$R_\Phi = \mathcal{S}(\begin{bmatrix} 0 & I \\ I & K^0 \end{bmatrix}, K_a^{-1}).$$

We therefore need to show that $P = \mathcal{S}(\mathcal{S}(N,M), \bar{K}_a^{-1})$ where $\bar{K}_a^{-1}$ is given by equation (4.39). This follows from Lemma 4.7.1. ■

Lemma 4.7.4 *Assume N is inner and M is co-inner with N_{21}, M_{12}, $(I - N_{22}M_{11})$ and $(I - M_{11}^\sim M_{11})$ all invertible. Define*

$$\mathcal{M} = \left[\begin{array}{cc:cc} 0 & -I & 0 & 0 \\ 0 & 0 & 0 & I \\ \hdashline I & 0 & 0 & 0 \\ 0 & 0 & -I & 0 \end{array}\right] \left[\begin{array}{cc:cc} 0 & 0 & \multicolumn{2}{c}{\mathcal{S}(N,M)} \\ 0 & 0 & & \\ \hdashline \multicolumn{2}{c:}{\mathcal{S}(N^\sim, M^\sim)} & 0 & 0 \\ & & 0 & 0 \end{array}\right] \left[\begin{array}{cc:cc} 0 & 0 & I & 0 \\ -I & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & -I \\ 0 & I & 0 & 0 \end{array}\right].$$

Then $\mathcal{F}_l(\mathcal{M}, I) = I$.

Proof: The proof involves many pages of matrix manipulations. Here we present only a sketch. Define the signals

$$\begin{bmatrix} s_1 \\ s_2 \\ t_1 \\ t_2 \end{bmatrix} = \mathcal{M} \begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix}.$$

Since we are interested in $\mathcal{F}_l(\mathcal{M}, I)$ it is obvious that $t_1 = v_1$ and $t_2 = v_2$. Hence

$$s_1 = (M_{21}(I - N_{22}M_{11})^{-1}N_{21})v_2 - (M_{22} + M_{21}N_{22}(I - M_{11}N_{22})^{-1}M_{12})u_2 \quad (4.40)$$
$$s_2 = (M_{12}^{\sim}(I - N_{22}^{\sim}M_{11}^{\sim})^{-1}N_{12}^{\sim})v_1 - (M_{22}^{\sim} + M_{12}^{\sim}N_{22}^{\sim}(I - M_{11}^{\sim}N_{22}^{\sim})^{-1}M_{21}^{\sim})u_1 \quad (4.41)$$
$$v_1 = -(N_{11} + N_{12}M_{11}(I - N_{22}M_{11})^{-1}N_{21})v_2 + (N_{12}(I - M_{11}N_{22})^{-1}M_{12})u_2 \quad (4.42)$$
$$v_2 = -(N_{11}^{\sim} + N_{21}^{\sim}M_{11}^{\sim}(I - N_{22}^{\sim}M_{11}^{\sim})^{-1}N_{12}^{\sim})v_1 + (N_{21}^{\sim}(I - M_{11}^{\sim}N_{22}^{\sim})^{-1}M_{21}^{\sim})u_1 \quad (4.43)$$

Now, using the properties of inner and co-inner matrices (equations (2.6) and (2.7)), equations (4.42) and (4.43) imply

$$\begin{aligned}(I - M_{11}^{\sim}M_{11})(I - N_{22}M_{11})^{-1}N_{21}v_2 \\ = M_{21}^{\sim}u_1 + (N_{22} - M_{11}^{\sim})(I - M_{11}N_{22})^{-1}M_{12}u_2. \end{aligned} \quad (4.44)$$

On noting that

$$(I - M_{11}^{\sim}M_{11})^{-1}(N_{22} - M_{11}^{\sim}) = N_{22} - M_{11}^{\sim}(M_{12}M_{12}^{\sim})^{-1}(I - M_{11}N_{22}) \quad (4.45)$$

substituting equation (4.44) into equation (4.40) then gives that $s_1 = u_1$. That $s_2 = u_2$ is shown by substituting equations (4.42) and (4.44) into equation (4.41) and using equation (4.45). ■

Proof (of Proposition 4.7.2)**:** Observe that

$$\left[\begin{array}{cc:cc} 0 & -R_{22} & 0 & R_{21} \\ -R_{22}^{\sim} & 0 & R_{12}^{\sim} & 0 \\ \hdashline 0 & R_{12} & 0 & -R_{11} \\ R_{21}^{\sim} & 0 & -R_{11}^{\sim} & 0 \end{array}\right]$$
$$= \left[\begin{array}{cc:cc} 0 & -I & 0 & 0 \\ 0 & 0 & 0 & I \\ \hdashline I & 0 & 0 & 0 \\ 0 & 0 & -I & 0 \end{array}\right] \left[\begin{array}{cc:cc} 0 & 0 & R_{11} & R_{12} \\ 0 & 0 & R_{21} & R_{22} \\ \hdashline R_{11}^{\sim} & R_{21}^{\sim} & 0 & 0 \\ R_{12}^{\sim} & R_{22}^{\sim} & 0 & 0 \end{array}\right] \left[\begin{array}{cc:cc} 0 & 0 & I & 0 \\ -I & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & -I \\ 0 & I & 0 & 0 \end{array}\right].$$

Hence, we may write

$$L_P = \mathcal{F}_l(\begin{bmatrix} -\mathcal{I}_{12} & \mathcal{I}_{22} \\ \mathcal{I}_{11} & -\mathcal{I}_{21} \end{bmatrix} \begin{bmatrix} 0 & R_P \\ R_P^{\sim} & 0 \end{bmatrix} \begin{bmatrix} -\mathcal{I}_{21} & \mathcal{I}_{11} \\ \mathcal{I}_{22} & -\mathcal{I}_{21} \end{bmatrix}, I)$$

where $\mathcal{I}_{ij}$ $(i, j = 1, 2)$ is defined to be a (four) block matrix whose ij^{th} block is an identity matrix and all other blocks are zero.

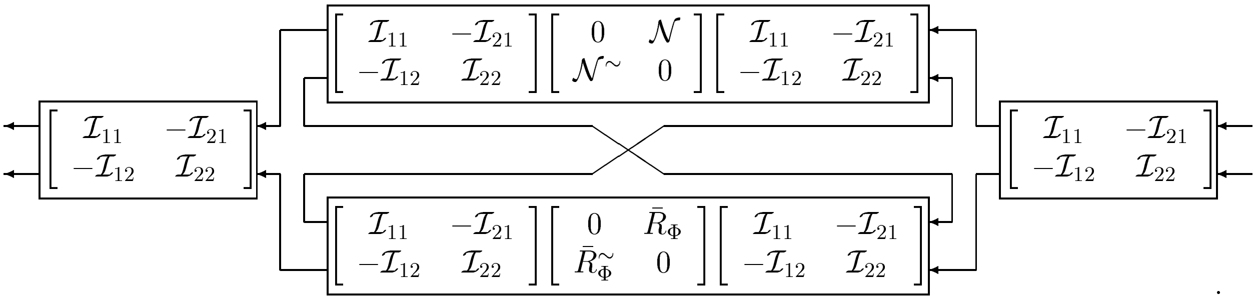
From Lemma 4.7.3 we know R_P may be written $R_P = \mathcal{S}(\mathcal{N}, \bar{R}_\Phi)$ (where $\mathcal{N}$ is defined in Lemma 4.7.3). Hence

$$\begin{bmatrix} 0 & R_P \\ R_P^\sim & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{S}(\mathcal{N}, \bar{R}_\Phi) \\ \mathcal{S}(\mathcal{N}^\sim, \bar{R}_\Phi^\sim) & 0 \end{bmatrix}. \tag{4.46}$$

From equation (4.46) we need to extract the quantity

$$\begin{bmatrix} -\mathcal{I}_{12} & \mathcal{I}_{22} \\ \mathcal{I}_{11} & -\mathcal{I}_{21} \end{bmatrix} \begin{bmatrix} 0 & \bar{R}_\Phi \\ \bar{R}_\Phi^\sim & 0 \end{bmatrix} \begin{bmatrix} -\mathcal{I}_{21} & \mathcal{I}_{11} \\ \mathcal{I}_{22} & -\mathcal{I}_{21} \end{bmatrix}. \tag{4.47}$$

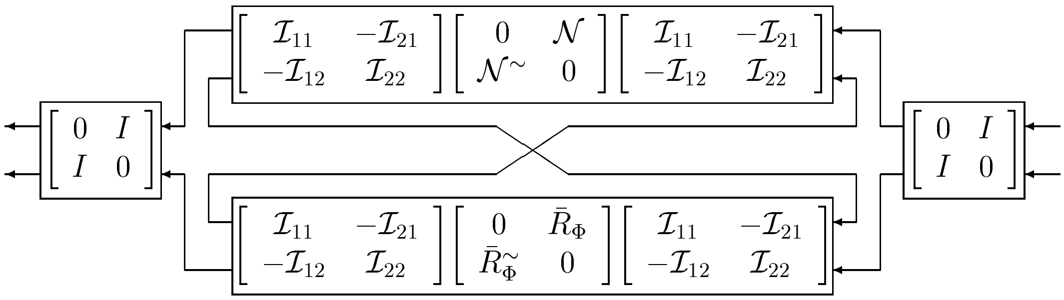
To do this, notice that the right hand side of equation (4.46) may be represented diagrammatically as



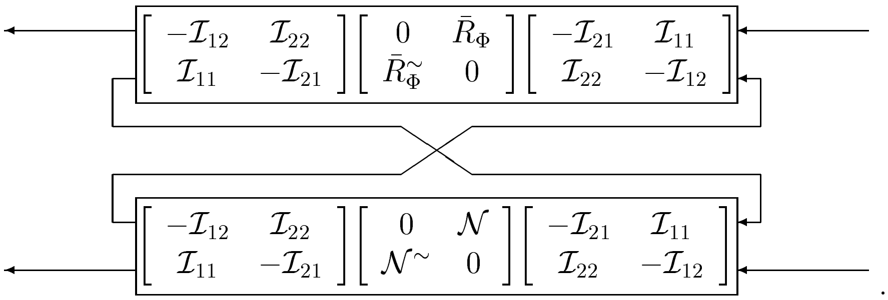
Hence

$$\begin{bmatrix} -\mathcal{I}_{12} & \mathcal{I}_{22} \\ \mathcal{I}_{11} & -\mathcal{I}_{21} \end{bmatrix} \begin{bmatrix} 0 & R_P \\ R_P^\sim & 0 \end{bmatrix} \begin{bmatrix} -\mathcal{I}_{21} & \mathcal{I}_{11} \\ \mathcal{I}_{22} & -\mathcal{I}_{21} \end{bmatrix}$$

may be written



which is equivalent to



Now (taking into account the transformation given in Remark 4.5.4)

$$L_\Phi = \mathcal{F}_l(\begin{bmatrix} -\mathcal{I}_{12} & \mathcal{I}_{22} \\ \mathcal{I}_{11} & -\mathcal{I}_{21} \end{bmatrix}\begin{bmatrix} 0 & \bar{R}_\Phi \\ \bar{R}_\Phi^\sim & 0 \end{bmatrix}\begin{bmatrix} -\mathcal{I}_{21} & \mathcal{I}_{11} \\ \mathcal{I}_{22} & -\mathcal{I}_{12} \end{bmatrix}, I).$$

All that remains to be shown now is that

$$\mathcal{F}_l(\begin{bmatrix} -\mathcal{I}_{12} & \mathcal{I}_{22} \\ \mathcal{I}_{11} & -\mathcal{I}_{21} \end{bmatrix}\begin{bmatrix} 0 & \mathcal{N} \\ \mathcal{N}^\sim & 0 \end{bmatrix}\begin{bmatrix} -\mathcal{I}_{21} & \mathcal{I}_{11} \\ \mathcal{I}_{22} & -\mathcal{I}_{12} \end{bmatrix}, I) = I$$

where $\mathcal{N} = \mathcal{S}(N, M)$ and N and M are defined in Lemma 4.7.1. Noting that N and M satisfy the required assumptions, this is merely the result of Lemma 4.7.4. Hence $L_P = L_\Phi$. ■

Remark 4.7.5 *Lemma 4.7.2 implies that the weights derived from the $\mathcal{A}(P)$ and $\mathcal{A}(\Phi)$ approaches are identical and hence the two approaches are equivalent. Noting that $R_P \in \mathbb{C}^{(p_1+p_2)\times(m_1+m_2)}$, that $R_\Phi \in \mathbb{C}^{(m_2+p_2)\times(p_2+m_2)}$, and that $p_1 \geq m_2$ and $m_1 \geq p_2$, the dimensions involved with $\mathcal{A}(\Phi)$ are generally smaller than with $\mathcal{A}(P)$ and hence it is numerically faster. However, if K^0 has been derived via an LMI technique, and K_a isn't available, then it is appropriate to use the $\mathcal{A}(P)$ approach.*

Using similar techniques to those above it is simple to prove the following proposition.

Proposition 4.7.6 *Define L_P as the L derived from the $\mathcal{CF}(P)$ approach, and define L_Φ as the L derived from the $\mathcal{CF}(\Phi)$ approach (after employing the transform of remark 4.5.4). Then $L_P = L_\Phi$, and hence the weights derived for approaches $\mathcal{CF}(P)$ and $\mathcal{CF}(\Phi)$ are identical.*

Remark 4.7.7 *The notation $\mathcal{A}(P/\Phi)$ is used to signify either approach $\mathcal{A}(P)$ or $\mathcal{A}(\Phi)$. Similarly, the notation $\mathcal{CF}(P/\Phi)$ is used to signify either approach $\mathcal{CF}(P)$ or $\mathcal{CF}(\Phi)$. Further, $\mathcal{CF}(P/\Phi/\Theta)$ signifies either approach $\mathcal{CF}(P)$ or $\mathcal{CF}(\Phi)$ or $\mathcal{CF}(\Theta)$.*

4.7.2 A Relationship between $\mathcal{CF}(P)$ and $\mathcal{CF}(\Theta)$

We now establish a relationship between the weights of approach $\mathcal{CF}(\Theta)$ and weights which make $\tilde{R}$ of approach $\mathcal{CF}(P)$ (equation (4.19)) a contraction. The weights we derive turn out to be optimal in the sense that the choice of W_2 uniquely specifies W_1.

Lemma 4.7.8 *Define $W_1^{-1} = \alpha\Theta^{-1}\begin{bmatrix} \gamma^{-1}I & 0 \\ 0 & I \end{bmatrix}$ and $W_2^{-1} = (\sqrt{2}/\alpha)I$ for Θ as given by equation (4.23) and some scalar α. W_1 and W_2 correspond to the weights required in proposition 4.4.3. They make $\tilde{R}$, given by equation (4.19), a contraction if and only if $\gamma = \alpha^2$.*

Proof: Firstly observe from equation (4.23) that

$$\begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix} = \begin{bmatrix} K_{a12}^{-1}K_{a11} & K_{a12}^{-1} \end{bmatrix}.$$

Substitution into equation (4.20) then gives,

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -K_{a21}^{-1}K_{a22} \\ K_{a11}K_{a21}^{-1}K_{a22} - K_{a12} \end{bmatrix} & \begin{bmatrix} K_{a21}^{-1} \\ -K_{a11}K_{a21}^{-1} \end{bmatrix} \\ I & 0 \end{bmatrix}.$$

Hence

$$\begin{bmatrix} L_1 & L_2 \\ L_2^{\sim} & L_3 \end{bmatrix} = \begin{bmatrix} R_{12}R_{12}^{\sim}/\gamma & -R_{11} \\ -R_{11}^{\sim} & I/\gamma \end{bmatrix}$$

and

$$\Theta^{-1} = \begin{bmatrix} R_{12} & -R_{11} \end{bmatrix}.$$

Also, since $W_2^{-1} = (\sqrt{2}/\alpha)I$, we have $Y \triangleq (W_2W_2^{\sim})^{-1} = (2/\alpha^2)I$, and since

$$W_1^{-1} = \alpha\Theta^{-1}\begin{bmatrix} \gamma^{-1}I & 0 \\ 0 & I \end{bmatrix}$$

we have

$$X \triangleq (W_1^{\sim}W_1)^{-1} = \alpha^2\Theta^{-1}\begin{bmatrix} \gamma^{-2}I & 0 \\ 0 & I \end{bmatrix}\Theta^{\sim -1}.$$

We wish to calculate α such that

$$\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \geq \begin{bmatrix} L_1 & L_2 \\ L_2^{\sim} & L_3 \end{bmatrix}. \tag{4.48}$$

Substituting for L, X and Y, equation (4.48) holds if and only if

$$\begin{aligned} & \begin{bmatrix} R_{12} & -R_{11} \end{bmatrix}\left\{\begin{bmatrix} \frac{\alpha^2}{\gamma^2}I & 0 \\ 0 & \alpha^2 I \end{bmatrix} - \begin{bmatrix} \frac{1}{\gamma}I & 0 \\ 0 & \frac{\alpha^2\gamma}{2\gamma-\alpha^2}I \end{bmatrix}\right\}\begin{bmatrix} R_{12}^{\sim} \\ -R_{11}^{\sim} \end{bmatrix} \geq 0 \\ \Longleftrightarrow \quad & \begin{bmatrix} \frac{\alpha^2-\gamma}{\alpha^2}I & 0 \\ 0 & \frac{\alpha^2(\gamma-\alpha^2)}{2\gamma-\alpha^2}I \end{bmatrix} \geq 0 \qquad (4.49) \\ \Longleftrightarrow \quad & \gamma = \alpha^2. \end{aligned}$$

Note that although $2\gamma < \alpha^2$ is a satisfactory solution to equation (4.49), it violates the constraint $Y - L_3 > 0$. ∎

4.8 Summary

Given a (P, γ)-admissible controller K^0, we have derived sufficient conditions for any other controller K, to be (P, γ)-admissible. The conditions are presented in terms of a norm bound on a particular frequency weighted error, which, for the purpose of reduced order controller synthesis, may be considered as a frequency weighted model reduction problem. If this norm bound is satisfied then it is possible to reduce the state-dimension of $\mathcal{H}_\infty$ controllers without degrading performance. However, the approaches are conservative; only searching for (P, γ)-admissible controllers which lie within a weighted $\mathcal{L}_\infty$ ball of the chosen K^0. Different K^0 lead to different K, and there may be (P, γ)-admissible controllers which lie arbitrarily far from K^0.

Four of the proposed approaches require weights to be calculated which make a specific transfer function a contraction. Any weights which meet this constraint may be selected. However, it is advantageous to select weights which are in some sense *large*. Hence procedures to maximise the product of the traces, or the product of the determinants of the weights have been presented. A major drawback of both these procedures is that a non-standard transfer function factorization is required (to calculate Z_1 and Z_2). At present this factorisation can't be performed exactly, however a satisfactory solution, which can be made arbitrarily close to the exact solution, is presented in Chapter 7.

Chapter 5

A Graphical Interpretation of SISO $\mathcal{H}_\infty$ Controller Approximation

5.1 Introduction

Provided there exists a nominal (P,γ)-admissible controller it was shown in Chapter 4 that there exist sufficient conditions for any other controller to be (P,γ)-admissible and how, for controller synthesis, this may be considered as a frequency weighted model reduction problem. Here we show that for scalar controllers, these sufficient conditions have a natural frequency by frequency interpretation as either the inside or outside of a disk in the complex plane. If at every frequency a controller's Nyquist plot lies inside (or where appropriate, outside) the circle which corresponds to the boundary of the disk at that frequency then the controller is guaranteed to be (P,γ)-admissible. Further, by comparing the size of the circles derived from each approach, the relative conservatism of each approach may be examined.

For a given $K^0 = \tilde{V}_0^{-1}\tilde{U}_0$ and appropriately selected weights, the condition for K to be (P,γ)-admissible given by methods $\mathcal{A}(P/\Phi)$ is

$$\|W_2^{-1}(K - K^0)W_1^{-1}\|_\infty < 1 \tag{5.1}$$

while for methods $\mathcal{CF}(P/\Phi/\Theta)$ it is shown that (for different weights)

$$\|W_2^{-1}(\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix} - \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix})W_1^{-1}\|_\infty < 1 \tag{5.2}$$

guarantees that $K = \tilde{V}^{-1}\tilde{U}$ is (P,γ)-admissible.

Equations (5.1) and (5.2) have a straightforward interpretation as standard frequency weighted model reduction problems. The design of reduced order (P,γ)-admissible controllers by the approaches presented in Chapter 4 is therefore equivalent to finding the set of reduced order K or $\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix}$ that lie within a weighted $\mathcal{L}_\infty$ ball of left radius W_2 and right radius W_1 centred on K^0 or $\begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}$.

To make approximation easier, these weights should be maximised in some sense. A procedure to maximise the product $\mathrm{trace}(W_1^\sim W_1)\mathrm{trace}(W_2 W_2^\sim)$ was presented in Lemma 4.6.1 while a procedure to maximise the product $\det(W_1^\sim W_1)\det(W_2 W_2^\sim)$ was presented in Lemma 4.6.2. Here we investigate possible conservatism in these size criteria.

For scalar controllers the present graphical approach can help to overcome the computational difficulties which arise from the proposed approximation procedures. These are discussed in greater detail in Section 7.2. Since we rely on graphical methods there is no frequency weighted model reduction to perform and hence the non-convexity of these methods becomes a non-issue. Results may be read directly from the graphs. Further, since a parametric description of the weights is not required, there is no need to calculate a parametric approximation of $Z_i^2 = (Z_i^2)^\sim > 0$, $i = 1, 2$ (equations (4.30) and (4.31) and equations (4.34) and (4.35)). Rather, the weights are evaluated on a frequency list, which enables the factorisation to be computed exactly (at each frequency).

In Section 5.2 we show that taken frequency by frequency, for scalar controllers, equations (5.1) and (5.2) represent disks in the complex plane. If at every frequency the point on the controller's Nyquist plot corresponding to that frequency lies inside, or if appropriate outside, the circle corresponding to the boundary of the disk at that frequency, then the controller is (P, γ)-admissible. In Section 5.3 we show that for the $\mathcal{A}(P/\Phi)$ methods and a given K^0, maximising the determinant of the weights (in the matrix case) always leads to a *direction* where the weights are the best possible. (See Definition 5.3.1 for what we mean by best possible.) We also show that for scalar weights, maximising the trace of the weights always leads to a *direction* where the weights are the best possible. Hence in the scalar case Lemma 4.6.1 and Lemma 4.6.2 are equivalent. In Section 5.4 we demonstrate the proposed graphical approach by comparing the results of the graphical approach to those which result from standard frequency weighted model reduction.

The discussion which follows is valid on a frequency by frequency basis. Each frequency corresponds to a particular point on a controller's Nyquist plot and to a particular circle. When we say controllers lying inside/outside of a circle are (P, γ)-admissible, we mean that at every frequency the point on the controller's Nyquist plot corresponding to that frequency lies inside/outside of the circle which corresponds to that frequency.

5.2 A Graphical Approach

In this section we show, in the scalar case, the equivalence between the sufficient conditions of methods $\mathcal{A}(P/\Phi)$ and $\mathcal{CF}(P/\Phi/\Theta)$ and disks in the complex plane. This requires showing the equivalence between equations (5.1) and (5.2) respectively and the equation,

$$K = U + V\Upsilon W \tag{5.3}$$

where $\|\Upsilon\|_\infty < 1$. Notice that for constant U, V, $W \in \mathbb{C}^{1\times 1}$, and $\Upsilon \in \mathbb{C}^{1\times 1}$ varying such that $|\Upsilon| = 1$, equation (5.3) represents a circle in the complex plane with centre U and radius $|VW|$. Further, $|\Upsilon| < 1$ represents the inside of the circle and $|\Upsilon| > 1$ represents the outside of the circle. This allows us to use a graphical procedure to design (P, γ)-admissible controllers.

In the matrix case, to see that $\Delta = W_2^{-1}(K - K^0)W_1^{-1}$ may be written in the form of equation (5.3), simply define $U \triangleq K^0$, $V \triangleq W_2$, $W \triangleq W_1$ and $\Upsilon \triangleq \Delta$. $\|\Delta\|_\infty < 1$ implies K is a (P, γ)-admissible controller. Therefore, in the scalar case, (P, γ)-admissible controllers which satisfy the sufficient condition of equation (5.1) may be interpreted graphically as those controllers whose Nyquist plot lies within a circle of radius $|W_2 W_1|$ centred on K^0.

To show that equation (5.2) may be written in the form of equation (5.3) requires slightly more work. Firstly notice that, in the matrix case, equation (5.2) is equivalent to,

$$\begin{aligned} K &= \mathcal{F}_l(\begin{bmatrix} I & 0 \\ 0 & W_1 \end{bmatrix}\begin{bmatrix} \begin{matrix} K^0 \\ \begin{bmatrix} I \\ -K^0 \end{bmatrix} \end{matrix} & \begin{matrix} \tilde{V}_0^{-1} \\ \begin{bmatrix} 0 \\ -\tilde{V}_0^{-1} \end{bmatrix} \end{matrix} \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & W_2 \end{bmatrix}, \Delta) \qquad (5.4) \\ &\triangleq \mathcal{F}_l(\begin{bmatrix} E & F \\ S & T \end{bmatrix}, \Delta) \end{aligned}$$

where $\Delta = \begin{bmatrix} \Delta_{\tilde{U}} & \Delta_{\tilde{V}} \end{bmatrix} = W_2^{-1}\left\{\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix} - \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}\right\} W_1^{-1}$ and $\|\Delta\|_\infty < 1$.

In the scalar case we have the following proposition.

Proposition 5.2.1 *Consider the scalar valued LFT*

$$K \triangleq \mathcal{F}_l(\begin{bmatrix} E & F \\ S & T \end{bmatrix}, \Delta) \in \mathbb{C}^{1\times 1} \qquad (5.5)$$

where $E,\ F \in \mathbb{C}^{1\times 1}$, $S,\ T \in \mathbb{C}^{2\times 1}$, $\Delta \in \mathbb{C}^{1\times 2}$ and Δ varies such that $\sigma(\Delta) < 1$[1].

i) If $\sigma(T) < 1$ then K describes the inside of a circle in the complex plane.

ii) If $\sigma(T) > 1$, F is invertible and $S'(TT' - I)^{-1}S > 0$ then K describes the outside of a circle in the complex plane.

iii) If $\sigma(T) = 1$ and $S'(TT' - I)^{\dagger}S > 0$ then K describes a half plane.

Before proving Proposition 5.2.1 we derive general matrix valued results then specialise to the scalar case. When $\bar{\sigma}(T) < 1$ we have the following Lemma.

Lemma 5.2.2 *Define*

$$K \triangleq \mathcal{F}_l(\begin{bmatrix} E & F \\ S & T \end{bmatrix}, \Delta) \in \mathbb{C}^{m\times p}$$

where E, F, S, T and Δ are appropriately dimensioned complex valued matrices, $\bar{\sigma}(\Delta) < 1$ and $\bar{\sigma}(T) < 1$. Then K may be written

$$K = U + V\Upsilon W$$

where $\bar{\sigma}(\Upsilon) < 1$ and

$$\begin{aligned} U &= E + F\left\{(I - T'T)^{-1/2}T'(I - TT')^{-1/2}\right\}S \qquad (5.6) \\ V &= F(I - T'T)^{-1/2} \qquad (5.7) \\ W &= (I - TT')^{-1/2}S. \qquad (5.8) \end{aligned}$$

[1]Note that in the scalar case Δ and T each have only one singular value.

Proof: $\bar{\sigma}(\Delta) < 1$ and $\bar{\sigma}(T) < 1$, hence $\bar{\sigma}(T\Delta) < 1$ and $(I - T\Delta)^{-1}$ exists.

Then

$$\begin{aligned} K &\triangleq E + F\Delta(I - T\Delta)^{-1}S \\ &= E + F(I - T'T)^{-1/2}\left\{T' + \mathcal{F}_l(J, \Delta)\right\}(I - TT')^{-1/2}S \\ &= E + F(I - T'T)^{-1/2}T'(I - TT')^{-1/2}S + F(I - T'T)^{-1/2}\Upsilon(I - TT')^{-1/2}S \end{aligned}$$

where $\Upsilon \triangleq \mathcal{F}_l(J, \Delta)$ and

$$J \triangleq \begin{bmatrix} -T' & (I - T'T)^{1/2} \\ (I - TT')^{1/2} & T \end{bmatrix}.$$

Since $J'J = I$ and $\bar{\sigma}(\Delta) < 1$, Theorem 2.7.5 implies $\bar{\sigma}(\Upsilon) < 1$. Hence defining U, V and W as in equations (5.6), (5.7) and (5.8) respectively gives the required result. ■

When $\underline{\sigma}(T) > 1$ we have the following Lemma.

Lemma 5.2.3 *Define*

$$K \triangleq \mathcal{F}_l\left(\begin{bmatrix} E & F \\ S & T \end{bmatrix}, \Delta\right) \in \mathbb{C}^{p\times p}$$

where E, $F \in \mathbb{C}^{p\times p}$ with F invertible, S, T and Δ are appropriately dimensioned complex valued matrices, $\bar{\sigma}(\Delta) < 1$ and $\underline{\sigma}(T) > 1$. Assume $F(T'T - I)^{-1}F' > 0$ and $S'(TT' - I)^{-1}S > 0$. Then K may be written

$$K = U + V\Upsilon W$$

where $\underline{\sigma}(\Upsilon) > 1$ and

$$\begin{aligned} U &= E - F(T'T - I)^{-1}T'S \\ VV' &= F(T'T - I)^{-1}F' \\ W'W &= S'(TT' - I)^{-1}S \end{aligned}$$

Proof: Firstly notice that this is equivalent to finding $\hat{E}$, V and W such that $\bar{\sigma}(W\hat{K}^{-1}V) < 1$ where $\hat{K} = K + (\hat{E} - E)$. $\hat{E}$ represents a translation and V and W represent scalings to normalise the problem.

Then

$$\begin{aligned} \hat{K}^{-1} &= \left(\hat{E} + F(I - \Delta T)^{-1}\Delta S\right)^{-1} \\ &= \left[F(I - \Delta T)^{-1}\left\{(I - \Delta T)F^{-1}\hat{E} + \Delta S\right\}\right]^{-1} \\ &= \left[F^{-1}\hat{E} + \Delta(S - TF^{-1}\hat{E})\right]^{-1}(I - \Delta T)F^{-1} \\ &= \hat{E}^{-1}F\left[I + \Delta(S\hat{E}^{-1}F - T)\right]^{-1}(I - \Delta T)F^{-1} \\ &= \hat{E}^{-1}F\left[I + \Delta(S\hat{E}^{-1}F - T)\right]^{-1}\left[I + \Delta(S\hat{E}^{-1}F - T) - \Delta S\hat{E}^{-1}F\right]F^{-1} \\ &= \hat{E}^{-1} - \hat{E}^{-1}F\left[I + \Delta(S\hat{E}^{-1}F - T)\right]^{-1}\Delta S\hat{E}^{-1} \\ &= \mathcal{F}_l\left(\begin{bmatrix} \hat{E}^{-1} & -\hat{E}^{-1}F \\ S\hat{E}^{-1} & T - S\hat{E}^{-1}F \end{bmatrix}, \Delta\right). \end{aligned}$$

Therefore

$$W\hat{K}^{-1}V = \mathcal{F}_l(\begin{bmatrix} W\hat{E}^{-1}V & -W\hat{E}^{-1}F \\ S\hat{E}^{-1}V & T - S\hat{E}^{-1}F \end{bmatrix}, \Delta).$$

If the co-efficient matrix in the above LFT is unitary then by Theorem 2.3 of [36], if $\bar{\sigma}(\Delta) < 1$ then $\bar{\sigma}(W\hat{K}^{-1}V) < 1$. The co-efficient matrix is unitary if

$$(W\hat{E}^{-1}V)'(W\hat{E}^{-1}V) + (S\hat{E}^{-1}V)'(S\hat{E}^{-1}V) = I \tag{5.9}$$
$$(W\hat{E}^{-1}V)'(-W\hat{E}^{-1}F) + (S\hat{E}^{-1}V)'(T - S\hat{E}^{-1}F) = 0 \tag{5.10}$$
$$(W\hat{E}^{-1}F)'(W\hat{E}^{-1}F) + (T - S\hat{E}^{-1}F)'(T - S\hat{E}^{-1}F) = I. \tag{5.11}$$

Now, equation (5.10) implies

$$\begin{aligned} -W'W\hat{E}^{-1}F - S'S\hat{E}^{-1}F + S'T &= 0 \\ \Rightarrow \qquad W'W + S'S &= S'TF^{-1}\hat{E} \\ \Rightarrow \qquad W'W &= S'(TF^{-1}\hat{E} - S) \end{aligned} \tag{5.12}$$

and equation (5.11) implies

$$W'W + (TF^{-1}\hat{E} - S)'(TF^{-1}\hat{E} - S) = (F^{-1}\hat{E})'(F^{-1}\hat{E}). \tag{5.13}$$

Substituting equation (5.12) into equation (5.13) then gives

$$\begin{aligned} T'TF^{-1} - T'S\hat{E}^{-1} &= F^{-1} \\ \Rightarrow \qquad \hat{E} &= F(T'T - I)^{-1}T'S \end{aligned} \tag{5.14}$$

and substituting equation (5.14) into equation (5.12) gives

$$W'W = S'(TT' - I)^{-1}S. \tag{5.15}$$

Also, equation (5.9) implies

$$W'W + S'S = (V^{-1}\hat{E})'(V^{-1}\hat{E})$$

which when combined with equations (5.14) and (5.15) gives

$$VV' = F(T'T - I)^{-1}F'. \tag{5.16}$$

Hence defining $U = E - \hat{E}$ and calculating V and W by performing Cholesky factorisations on equation (5.15) and (5.16) respectively gives the required result. Note that a solution only exists if the right hand side of equations (5.15) and (5.16) are positive definite. ■

We are now in a position to specialise the general matrix valued results of Lemmas 5.2.2 and 5.2.3 to prove Proposition 5.2.1.

Proof (of Proposition 5.2.1):

i) $\sigma(T) < 1$: Consider Lemma 5.2.2 and notice that by performing a QR factorisation on W and defining $\hat{\Upsilon} \triangleq \Upsilon Q$ and $\hat{W} \triangleq R$, K may be written

$$K = U + V\hat{\Upsilon}\hat{W} \in \mathbb{C}^{1\times 1}$$

where $\hat{\Upsilon} \in \mathbb{C}^{1\times 1}$ such that $|\hat{\Upsilon}| < 1$ and $\hat{W} = |W|$. Since $|\hat{\Upsilon}| < 1$, equation (5.5) maps the inside of a circle in the complex plane.

ii) $\sigma(T) > 1$: This is covered by Lemma 5.2.3. Since $|\Upsilon| > 1$, equation (5.5) maps the outside of a circle in the complex plane.

iii) $\sigma(T) = 1$: This represents the boundary between $\bar{\sigma}(T) < 1$ and $\bar{\sigma}(T) > 1$. K therefore maps into a half plane. ■

Proposition 5.2.1 shows that in the scalar case, equation (5.4), and hence equation (5.2), represents either the inside or outside of a circle in the complex plane. Experience indicates that $\sigma(T) > 1$ for low frequencies and $\sigma(T) < 1$ for medium and high frequencies. Hence in the coprime factor case (P,γ)-admissible controllers may have to lie outside circles in one frequency range and inside circles in another. These ideas will become more concrete when we consider an example (Section 5.4).

In the example we will observe that at every frequency the circle which results from the $\mathcal{CF}(P/\Phi)$ approaches corresponds exactly with the boundary of the region of the complex plane where it is necessary for $K(j\omega)$ to lie for $\bar{\sigma}(\mathcal{F}_l(P(j\omega), K(j\omega))) < \gamma$. Note that there is no guarantee that a controller whose Nyquist plot satisfies $\bar{\sigma}(\mathcal{F}_l(P(j\omega), K(j\omega))) < \gamma\ \forall\omega$ will stabilise the closed loop. From this it would be tempting to conclude that the conditions of the $\mathcal{CF}(P/\Phi)$ approaches are necessary in addition to being sufficient. However, although $\tilde{R}$ can be made to be inner, $\tilde{R}_{21}$ fails the full rank requirement of Theorem 2.7.5.

5.3 Best Possible Weights for the $\mathcal{A}(P/\Phi)$ Approaches

In this section we show that for the $\mathcal{A}(P/\Phi)$ approaches and a given K^0, (in the matrix case) maximising the determinant of the weights always leads to a *direction* where the weights are the best possible. Further, in the scalar case, we show that maximising the trace of the weights also always leads to a *direction* where the weights are the best possible. This confirms that in the scalar case, Lemma 4.6.1 (maximising the trace of the weights) is equivalent to Lemma 4.6.2 (maximising the determinant of the weights). We now define what we mean by the term *best possible* weights.

Definition 5.3.1 *W_1 and W_2 are the best possible weights if $\exists\Delta \in \mathcal{RH}_\infty$ such that $\|\Delta\|_\infty = 1$ and $K = K^0 + W_2\Delta W_1$ such that $\|\mathcal{F}_l(P,K)\|_\infty = \gamma$. This implies there exists a direction where our $\mathcal{L}_\infty$ ball cannot be made any larger without K losing (P,γ)-admissibility.*

Proposition 5.3.2 *Consider $\mathcal{F}_l(\tilde{R},\Delta)$ as given in Figure 5.1. Define X, Y and L as in Proposition 4.5.1. If there exists a vector $v \neq 0$ such that*

$$v'\left\{L_1 - L_2(Y - L_3)^{-1}L_3(Y - L_3)^{-1}L_2^{\sim}\right\}v = 0,$$

then there exists a direction *where W_1 and W_2 are the best possible weights.*

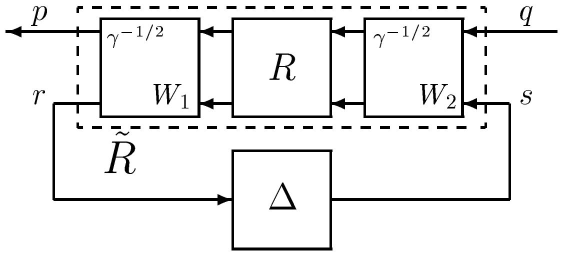


Figure 5.1: $\mathcal{F}_l(\tilde{R}, \Delta)$

Proof: This proof is valid on a frequency by frequency basis. We need to show that $\forall \omega$, $\bar{\sigma}(\tilde{R}) = 1$ and $\exists\ \Delta$ and q such that $|r|^2 = |s|^2$ and $|p|^2 = |q|^2$.

That $\bar{\sigma}(\tilde{R}) = 1$ is guaranteed by selecting weights according to either Proposition 4.6.1 or Proposition 4.6.2. Hence we need only show that $\exists\ \Delta$ and q such that $|r|^2 = |s|^2$ (since $|p|^2 = |q|^2$ follows immediately).

Choose q such that $\begin{bmatrix} q \\ s \end{bmatrix}$ is an eigenvector of $\tilde{R}'\tilde{R}$ with unit eigenvalue. Then

$$\begin{bmatrix} q \\ s \end{bmatrix} = \tilde{R}' \begin{bmatrix} p \\ r \end{bmatrix}$$

and

$$\begin{bmatrix} p \\ r \end{bmatrix} = \tilde{R} \begin{bmatrix} q \\ s \end{bmatrix},$$

or equivalently,

$$\begin{bmatrix} I & \tilde{R} \\ \tilde{R}' & I \end{bmatrix} \left[\begin{array}{c} p \\ r \\ \hdashline -q \\ -s \end{array}\right] = 0. \tag{5.17}$$

Now, using manipulations similar to the proof of Proposition 4.5.1, equation (5.17) implies,

$$\left[\begin{array}{cc:cc} \gamma I & R_{11} & 0 & R_{12} \\ R_{11}' & \gamma I & R_{21}' & 0 \\ \hdashline 0 & R_{21} & X & R_{22} \\ R_{12}' & 0 & R_{22}' & Y \end{array}\right] \left[\begin{array}{c} \bar{p} \\ \bar{r} \\ \hdashline -\bar{q} \\ -\bar{s} \end{array}\right] = 0 \tag{5.18}$$

where $\begin{bmatrix} \bar{p}' & \bar{r}' & -\bar{q}' & -\bar{s}' \end{bmatrix}' = \begin{bmatrix} \gamma^{1/2}p' & W_1'r' & -\gamma^{-1/2}q' & -W_2 s' \end{bmatrix}'$. Therefore,

$$|r|^2 = |s|^2 \quad \Longleftrightarrow \quad \bar{r}'X\bar{r} = \bar{s}'Y\bar{s}.$$

Now, premultiplying equation (5.18) by

$$\begin{bmatrix} I & 0 \\ \begin{bmatrix} 0 & R_{21} \\ R_{12}' & 0 \end{bmatrix} \begin{bmatrix} \gamma & R_{11} \\ R_{11}' & \gamma \end{bmatrix}^{-1} & I \end{bmatrix}$$

implies

$$\left[\begin{array}{cc:cc} \gamma I & R_{11} & 0 & R_{12} \\ R'_{11} & \gamma I & R'_{21} & 0 \\ \hdashline 0 & 0 & X - L_1 & -L_2 \\ 0 & 0 & -L'_2 & Y - L_3 \end{array}\right] \left[\begin{array}{c} \bar{p} \\ -\bar{q} \\ \hdashline \bar{r} \\ -\bar{s} \end{array}\right] = 0.$$

Hence, for $v \neq 0$, choose $X = L_1 + L_2(Y - L_3)^{-1}L'_2$ and

$$\left[\begin{array}{c} \bar{r} \\ -\bar{s} \end{array}\right] = \left[\begin{array}{c} I \\ (Y - L_3)^{-1}L'_2 \end{array}\right] v.$$

Now

$$\begin{aligned} \bar{r}'X\bar{r} - \bar{s}'X\bar{s} &= v'\left\{X - L_2(Y - L_3)^{-1}Y(Y - L_3)^{-1}L'_2\right\} v \\ &= v'\left\{L_1 - L_2(Y - L_3)^{-1}L_3(Y - L_3)^{-1}L'_2\right\} v \end{aligned}$$

But by assumption $\forall\omega$ there exists a $v \neq 0$ such that

$$v'\left\{L_1 - L_2(Y - L_3)^{-1}L_3(Y - L_3)^{-1}L'_2\right\} v = 0.$$

Hence the result follows. ■

Lemma 5.3.3 *Assume weights for the $\mathcal{A}(P/\Phi)$ approaches are calculated according to Lemma 4.6.2 (maximising the product* $\det(W_1^{\sim}W_1)\det(W_2^{\sim}W_2)$*), and let $v \triangleq L_1^{-1}L_2u$. Then $\forall u$,*

$$v'\left\{L_1 - L_2(Y - L_3)^{-1}L_3(Y - L_3)^{-1}L_2^{\sim}\right\} v = 0.$$

Hence, there always exists a direction *where W_1 and W_2 are the best possible weights.*

Proof: Assume L_2 has full column rank. (If not, then perform an augmentation similar to that used in the proof of Lemma 4.6.2.) Using the notation of Lemma 4.6.2 we have,

$$\begin{aligned} L_1 - L_2(Y - L_3)^{-1}L_3(Y - L_3)^{-1}L_2^{\sim} &= L_1 - L_2(T^{\sim}Z_2T)^{-1}L_3(T^{\sim}Z_2T)^{-1}L_2^{\sim} \\ &= L_1 - L_2T^{-1}Z_2^{-1}T^{\sim-1}L_3T^{-1}Z_2^{-1}T^{\sim-1}L_2^{\sim} \\ &= L_1 - L_2T^{-1}Z_2^{-2}T^{\sim-1}L_2^{\sim} \\ &= L_1 - L_2(L_2^{\sim}L_1^{-1}L_2)^{-1}L_2^{\sim} \\ &= S\left\{I - S^{-1}L_2(L_2^{\sim}S^{\sim-1}S^{-1}L_2)^{-1}L_2^{\sim}S^{\sim-1}\right\}S^{\sim} \\ &= S\left\{I - F(F^{\sim}F)^{-1}F^{\sim}\right\}S^{\sim} \end{aligned}$$

where $F = S^{-1}L_2$. Now, since $(I - F(F^{\sim}F)^{-1}F^{\sim})F = 0$, letting $v = L_1^{-1}L_2u$, implies

$$v'\left\{L_1 - L_2(Y - L_3)^{-1}L_3(Y - L_3)^{-1}L_2^{\sim}\right\} v = 0$$

for all vectors u. Hence, by Proposition 5.3.2, there always exists a *direction* where W_1 and W_2 are the best possible weights. ■

Remark 5.3.4 *Note that Lemma 5.3.3 holds for general matrix valued weights. Hence there always exists a* direction *where the $\mathcal{L}_\infty$ ball centred on K^0 with left radius W_2 and right radius W_1 is as large as possible such that $\tilde{R}$ remains a contraction.*

Lemma 5.3.5 *Assume K^0 is a scalar controller and weights for the $\mathcal{A}(P/\Phi)$ approaches are calculated according to Lemma 4.6.1 (maximising the product trace$(W_1^{\sim}W_1)$trace$(W_2^{\sim}W_2)$). Then $\forall v$,*

$$v'\left\{L_1 - L_2(Y-L_3)^{-1}L_3(Y-L_3)^{-1}L_2^{\sim}\right\}v \;=\; 0.$$

Hence, there always exists a direction *where W_1 and W_2 are the best possible weights.*

Proof: Again we prove the result on a frequency by frequency basis. Define $U\Sigma V' \triangleq L_2$. Hence $L_2'L_2 = V\Sigma^2V'$, $Z_1 = V\Sigma V'$ and $L_2Z_1^{-1} = UV'$. Therefore,

$$\begin{aligned}
&L_1 - L_2(Y-L_3)^{-1}L_3(Y-L_3)^{-1}L_2' \\
&\quad = L_1 - L_2\alpha_{opt}^{-1}Z_1^{-1}L_3\alpha_{opt}^{-1}Z_1^{-1}L_2' \\
&\quad = \text{trace}(L_1)\left\{\frac{L_1}{\text{trace}(L_1)} - \frac{L_2Z_1^{-1}L_3Z_1^{-1}L_2'}{\text{trace}(L_3)}\right\} \\
&\quad = \text{trace}(L_1)\left\{\frac{L_1}{\text{trace}(L_1)} - UV'\frac{L_3}{\text{trace}(L_3)}VU'\right\} \\
&\quad = \text{trace}(L_1)U\left\{\frac{U'L_1U}{\text{trace}(L_1)} - \frac{V'L_3V}{\text{trace}(L_3)}\right\}U'.
\end{aligned}$$

Now, $\forall\omega$, since $L_1, L_2, L_3 \in \mathbb{C}^{1\times 1}$

$$L_1 - L_2(Y-L_3)^{-1}L_3(Y-L_3)^{-1}L_2' \;=\; 0.$$

■

Lemma 5.3.3 and Lemma 5.3.5 imply the following result.

Corollary 5.3.6 *For scalar weights, Lemma 4.6.1 and Lemma 4.6.2 are equivalent.*

Graphically, we will see that the disk corresponding to the best possible weights using the $\mathcal{A}(P/\Phi)$ approaches (for a given K^0) just touches the boundary of the region of the complex plane where it is necessary for $K(j\omega)$ to lie for $\bar{\sigma}(\mathcal{F}_l(P(j\omega), K(j\omega))) < \gamma$. Note again that there is no guarantee that a controller whose Nyquist plot satisfies $\bar{\sigma}(\mathcal{F}_l(P(j\omega), K(j\omega))) < \gamma$ $\forall\omega$ will stabilise the closed loop.

5.4 Example

Consider the interconnection shown in Figure 5.2 where G is the SISO pitch loop (including performance weights) of the Generic VSTOL Aircraft Model (GVAM) as studied in [45]. Here we will compare the performance of standard frequency weighted model reduction to the proposed graphical approach.

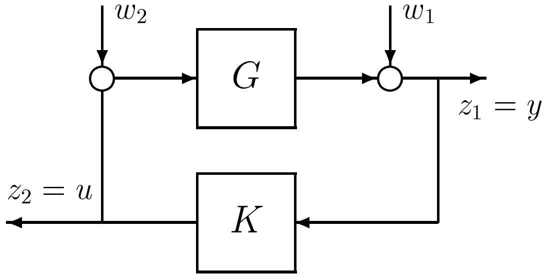


Figure 5.2: GVAM Four Block Interconnection

The model describes the GVAM in hover mode with reaction jets controlling the pitch attitude. Converting this to the general $\mathcal{H}_\infty$ plant and controller LFT interconnection (Figure 2.5) gives a 17 state, 3 input, 3 output generalised plant. The optimal $\gamma_{opt} = 2.12$. Here we use the suboptimal value $\gamma = 4$. Controllers are therefore (P, γ)-admissible if the transfer function T_{zw} from $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ is stable and $\|T_{zw}\|_\infty < 4$.

The results of standard frequency weighted model reduction (without any subsequent parameter optimisation) are presented in Table 5.1. The controller of least degree which satisfies the appropriate sufficient condition, for the frequency weighted methods, or $\|\mathcal{F}_l(P, K)\|_\infty < \gamma = 4$, for the unweighted methods, is presented. The following abbreviations are used:

$\mathcal{A}(\Phi)$	–	Freq. weighted reduction using the $\mathcal{A}(\Phi)$ procedure and maximising the determinant of the weights.
$\mathcal{CF}(\Phi)$	–	Freq. weighted reduction using the $\mathcal{CF}(\Phi)$ procedure and maximising the trace of the weights.
$\mathcal{CF}(\Theta)$	–	Freq. weighted reduction using the $\mathcal{CF}(\Theta)$ procedure.
UOHNA	–	Unweighted optimal Hankel norm approximation.
UBT	–	Unweighted balance and truncate.
LCFR	–	Unweighted optimal Hankel norm reduction of controller normalised left coprime factors.

For all bar $\mathcal{CF}(\Theta)$, K^0 is chosen to be a near optimal 17 state controller, for $\mathcal{CF}(\Theta)$ K^0 must be the central controller. The $\|(\text{weighted error})\|_\infty$ corresponds to the norm of equation (5.1) or (5.2) with the appropriate weights.

The results of Table 5.1 show that for the example considered and using frequency weighted model reduction without parameter optimisation, the smallest order (P, γ)-admissible controller calculated is second order.

However, Table 5.2 shows that the lower bound on the approximation error for a zeroth order controller (i.e. constant feedback) is less than the appropriate sufficient condition. Although the lower bound may not be attainable, these results suggest the possible existence of a zeroth order (P, γ)-admissible controller. Unfortunately the zeroth order controllers obtained by frequency weighted model reduction without optimisation don't even stabilise the closed loop.

Employing parameter optimisation – using the controller obtained from frequency weighted optimal Hankel norm model reduction as a starting point [100] – a zeroth order controller

Reduction Approach	Controller Order	$\lVert(\text{weighted error})\rVert_\infty$	$\lVert\mathcal{F}_l(P,K)\rVert_\infty$
	17 (full)	–	2.12
$\mathcal{A}(\Phi)$	2	0.84	3.44
$\mathcal{CF}(\Phi)$	4	0.67	2.36
$\mathcal{CF}(\Theta)$	2	0.61	3.50
UOHNA	3	–	2.9
UBT	3	–	2.6
LCFR	3	–	2.7

Table 5.1: Frequency Weighted Model Reduction (without Optimisation)

Reduction Approach	Controller Order	Lower bound for $\lVert(\text{weighted error})\rVert_\infty$	Actual $\lVert(\text{weighted error})\rVert_\infty$	$\lVert\mathcal{F}_l(P,K)\rVert_\infty$
$\mathcal{A}(\Phi)$	0	0.68	11.67	Not Stab.
$\mathcal{CF}(\Phi)$	0	0.82	8.84	Not Stab.
$\mathcal{CF}(\Theta)$	0	0.52	1.45	Not Stab.

Table 5.2: Lower Bound for Zeroth Order Approximation

satisfying equation (5.1) for $\mathcal{A}(\Phi)$ was found. Unfortunately, for $\mathcal{CF}(\Phi)$ and $\mathcal{CF}(\Theta)$, no constant $\left[\begin{array}{cc}\tilde{U} & \tilde{V}\end{array}\right]$ satisfying equation (5.2) could be found.

Reduction Approach	Controller Order	$\lVert(\text{weighted error})\rVert_\infty$	$\lVert\mathcal{F}_l(P,K)\rVert_\infty$
$\mathcal{A}(\Phi)$	0	0.956	2.89

Table 5.3: Zeroth Order (P,γ)-admissible Controllers

The results thus far have served to reinforce the belief that frequency weighted model reduction without optimisation may perform poorly and can be improved upon by including a parameter optimisation step. However, even with optimisation, no coprime factorisation of a zeroth order (P,γ)-admissible controller could be found which satisfied equation (5.2). We will now show that by using the graphical approach of Section 5.2 a zeroth order controller may be calculated using either equation (5.1) or (5.2).

Using an appropriate frequency list, calculate U, V, W and Υ as required to write equations (5.1) and (5.2) in the form of equation (5.3). Plots of the calculated circles for the frequencies

$$\begin{aligned}\omega &= 0.001, 0.01, 0.1, 1, 1.5, 2, 3, 4, 6\\ &\quad 10, 13, 18, 20, 30, 50, 100, 1000, \text{ and } 10000\,\text{rad/s}\end{aligned}$$

are shown in Figures 5.3, 5.4 and 5.5. (Note that in practice many more frequency points than this must be examined). Each plot contains two circles and the Nyquist plot for the nominal, near optimal, controller K^0 which has a '+' corresponding to the relevant frequency for that plot. We will now describe each of the circles.

The dashed circles are for the $\mathcal{A}(\Phi)$ approach. Note that they are centered on the '+'. Since the '+' is inside the circle for each of these plots K^0 is (P,γ)-admissible – as it was designed to be. Observe that the region of the real axis between 0.6 and 0.78 (approximately) lies within the circles for every frequency. Constant feedback controllers with a value in this region therefore satisfy equation (5.1) and are guaranteed to be (P,γ)-admissible. This is easily confirmed by calculating the weighted error for such controllers and seeing it is less than unity.

The dotted circles are for the $\mathcal{CF}(\Phi)$ approach. For many of the plots a scale has been chosen such that only a (small) portion of the circle can be seen. However from careful examination of the plots it is a simple task to determine which part of the plane is enclosed. $\sigma(T) > 1$ for the plots up to and including $\omega = 13$rad/s, hence the Nyquist plots at these frequencies must lie outside the dotted circle. It is easy to verify from these plots that K^0 is (P,γ)-admissible.

For $\mathcal{CF}(\Phi)$ the region of the real axis between 0.62 and 1.1 (approximately) always lies in the appropriate region of the plane. Hence there exists a coprime factorisation $\left[\begin{array}{cc}\tilde{U} & \tilde{V}\end{array}\right]$, such that $\tilde{V}^{-1}\tilde{U}$ takes a constant value between 0.62 and 1.1, which satisfies equation (5.2). However, we may not be able to explicitly calculate such a factorisation. Importantly, $K = \tilde{V}^{-1}\tilde{U}$ is known, and is guaranteed to be (P,γ)-admissible.

To obtain an indication of the possible conservatism in our weights we then calculated the region of the plane where it is necessary for $K(j\omega)$ to lie for $\bar{\sigma}(\mathcal{F}_l(P(j\omega), K(j\omega))) < 4$. We stress again that there is no guarantee that a controller whose Nyquist plot satisfies $K(j\omega)$ satisfying $\bar{\sigma}(\mathcal{F}_l(P(j\omega), K(j\omega))) < 4\ \forall\omega$ will stabilise the closed loop.

We have a priori information about this necessary region. The region must always contain the sufficient regions which result from any of the $\mathcal{A}(P/\Phi)$ or $\mathcal{CF}(P/\Phi/\Theta)$ approaches. Hence, when $\sigma(T) > 1$ the boundary of the necessary region must be contained within the dotted circles, and when $\sigma(T) < 1$ the necessary region must contain the dotted circles (and the dashed circles). Further, since we know that the dashed circles are as large as possible, they must always touch the necessary region at at least one point.

Surprisingly, the boundary of this region corresponds exactly with the circle derived from the $\mathcal{CF}(\Phi)$ approach. Further investigations showed that for randomly generated generalised plants, using the $\mathcal{CF}(\Phi)$ approach and maximising the product of the traces of the weights always led to an $\tilde{R}$ such that $\tilde{R}'\tilde{R} = I$. It would be tempting to conclude from this that the conditions of the $\mathcal{CF}(\Phi)$ approach are necessary in addition to being sufficient. We have the following conjecture.

Conjecture 5.4.1 *Assume scalar controllers and use the $\mathcal{CF}(P/\Phi)$ approaches (combined with maximising the product of the traces of the weights). At each frequency $\exists\ K(j\omega)$ such that $\bar{\sigma}(\mathcal{F}_l(P(j\omega), K(j\omega))) < \gamma$ if and only if $\exists\ \Delta(j\omega)$ such that $\bar{\sigma}(\Delta(j\omega)) < 1$ and $K(j\omega) = (\tilde{V}_0(j\omega) + \hat{\Delta}_{\tilde{V}}(j\omega))^{-1}(\tilde{U}_0(j\omega) + \hat{\Delta}_{\tilde{U}}(j\omega))$ where $[\hat{\Delta}_{\tilde{U}}(j\omega)\ \hat{\Delta}_{\tilde{V}}(j\omega)] = W_2(j\omega)\Delta(j\omega)W_1(j\omega)$.*

This seems an unlikely result since although $\tilde{R}'\tilde{R} = I$, $\tilde{R}_{12}$ fails the rank assumption of Theorem 2.7.5. Unfortunately we have been unable to prove (or disprove) this conjecture.

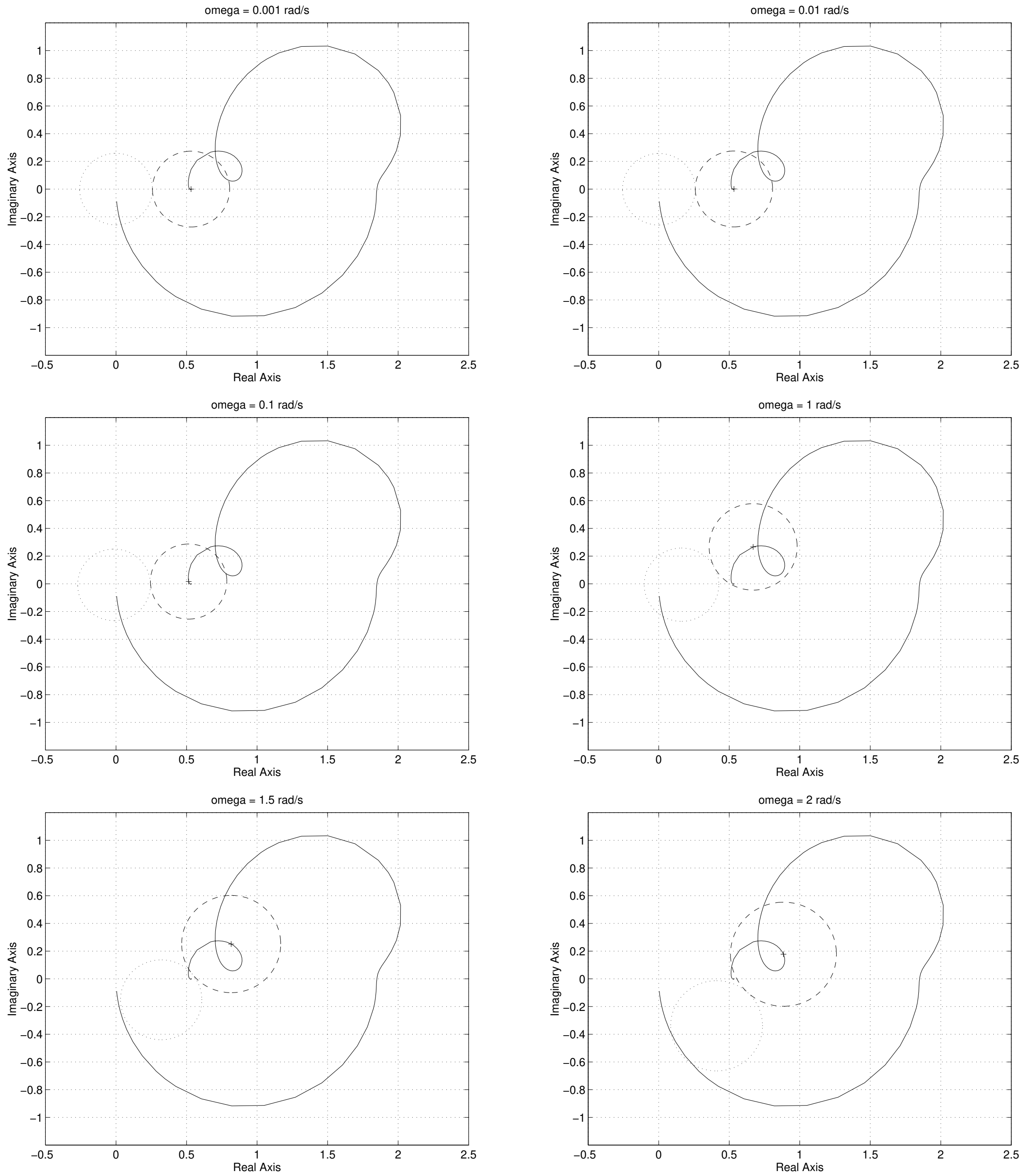


Figure 5.3: Plots for Graphical Design Procedure

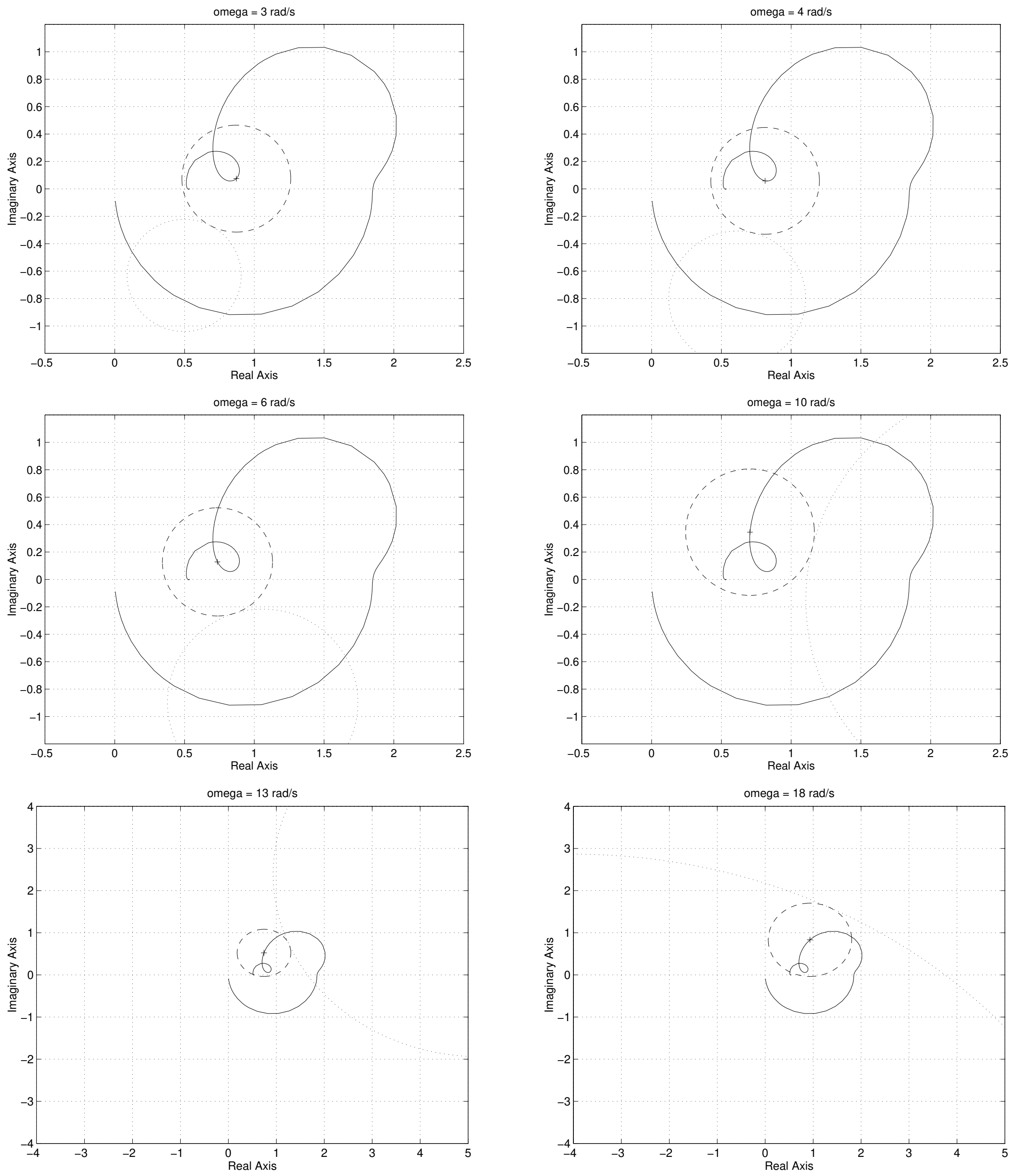


Figure 5.4: Plots for Graphical Design Procedure

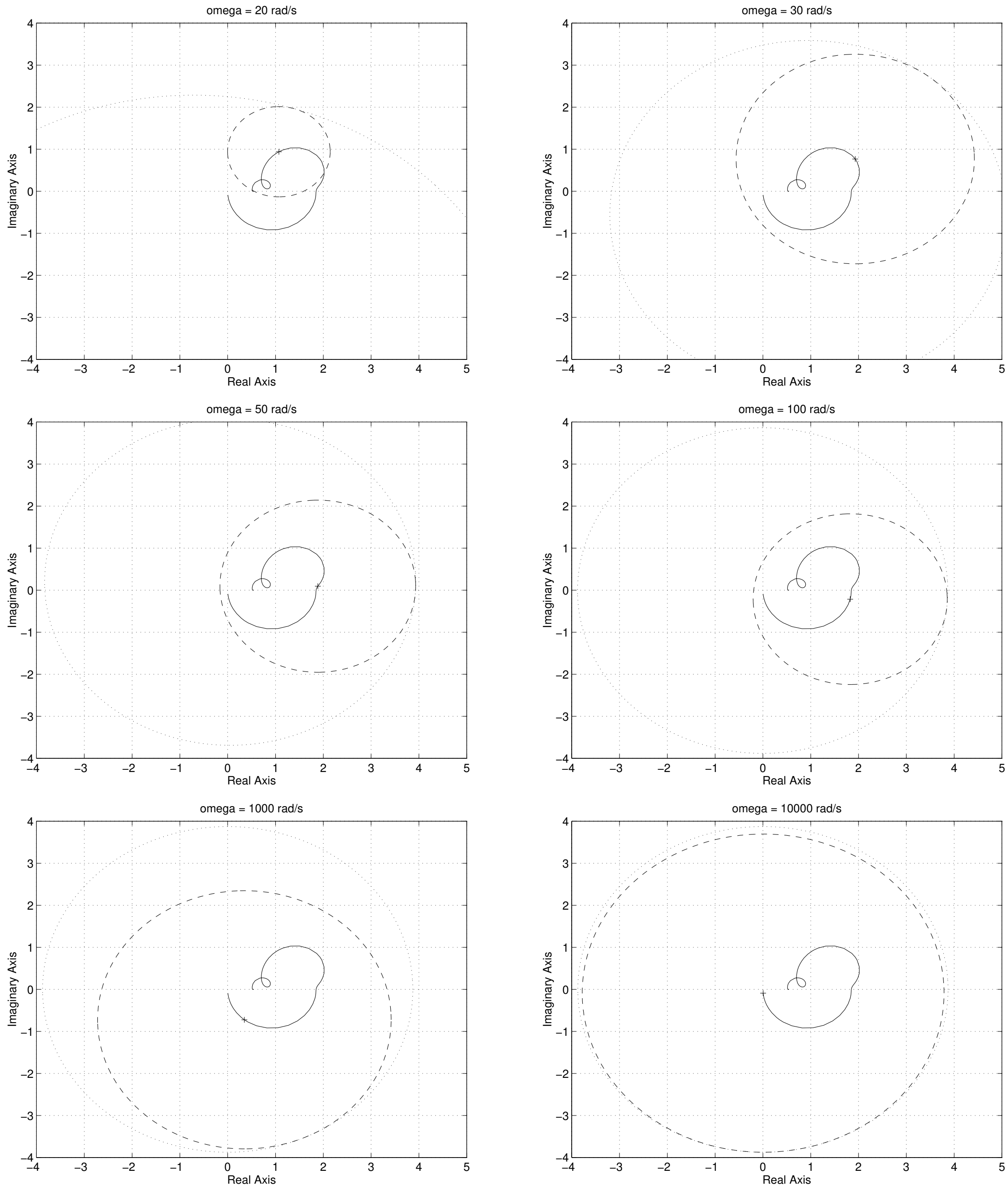


Figure 5.5: Plots for Graphical Design Procedure

5.5 Summary

We have shown that for scalar controllers, equations (5.1) and (5.2) may be interpreted frequency by frequency as disks in the complex plane. Hence the sufficient conditions for the design of reduced order (P, γ)-admissible controllers by the techniques of Chapter 4 have a graphical interpretation. If at every frequency the appropriate point on a controller's Nyquist plot lies inside/outside of the circle which corresponds to the boundary of the disk at that frequency then the controller is (P, γ)-admissible.

We have proved that for the $\mathcal{A}(\Phi)$ approaches and a given K^0, the procedure presented in Lemma 4.6.2 to maximise the product of the determinants of the weights results (in the general matrix case,) in a *direction* where the weights are the best possible. No other maximising criteria can do better in this direction. In the scalar case this implies that the best possible weights have been selected. We also proved that in the scalar case, maximising the product of the traces of the weights results in the best possible weights (when used with $\mathcal{A}(\Phi)$). Hence for scalar weights, Lemma 4.6.1 and Lemma 4.6.2 are equivalent.

In the design example we saw that the proposed graphical approach can perform better than standard frequency weighted model reduction. However, this is not a general rule. It is not difficult to find examples where the graphical technique performs poorly. This is usually the case when the weights, and hence the circles, are small.

That the circles for the $\mathcal{CF}(P/\Phi)$ approaches are larger than those for the $\mathcal{A}(\Phi)$ approaches over the entire frequency range is consistent with the extra degrees of freedom afforded to the fractional representations by allowing unstable poles to move across the $j\omega$ axis. However, we have been unable to theoretically justify why these circles coincide with the circles which correspond to the boundary of the region of necessity. For although $\tilde{R}$ for this approach is inner, $\tilde{R}_{12}$ isn't full row rank and hence we cannot easily conclude that the $\mathcal{CF}(P/\Phi)$ approaches are necessary and sufficient.

Chapter 6

Approximation of Linear Parameter-Varying Systems

6.1 Introduction

In certain applications a single linear, time invariant (LTI) model may be insufficient to capture the salient features of a plant to be controlled. Rather, more sophisticated models (and controller synthesis techniques) are required. One such technique which has received much attention in recent years is that of *gain scheduling* [45, 66, 84, 85, 82, 86]. The idea is to derive LTI models at each of several operating points which cover the range of operation of the plant. An LTI controller may then be designed to meet LTI performance specifications at each of these points. Finally, by interpolating the controller parameters at intermediate points, a single controller which covers the whole range of operation may be derived. (For this strategy to be successful a priori information on the plant operating range, and real time information about the plant operating point is required.)

A drawback of the gain scheduling approach is that although stability and performance are guaranteed to be achieved at the frozen design points, there is in general no guarantee of even stability at intermediate points[84]. Only via simulation can stability and performance for the overall system be checked.

Given that it is possible to model a plant across its whole operating range by a linear, parameter varying (LPV) model, much work has recently been directed towards exploiting linear matrix inequality (LMI) techniques to design controllers which guarantee performance across the whole operating range. (Here by performance we mean a bound on the induced $\mathcal{L}_2$ gain for arbitrarily varying parameters.) Some success has been achieved in this area by Packard [73] using a small gain framework, and by Apkarian et al. [4], Boyd et al. [10] and Becker [7] when the design points can be interpreted as the vertices of a polytope which completely captures the structure and variations of the plant. A difficulty encountered with this type of formulation is that the computational requirements generally increase rapidly as a function of the state dimension [7]. Hence it is desirable to keep the state-dimension as low as possible. This desire motivates the development of approximation techniques for LPV models.

We begin this chapter by defining LPV systems in Section 6.2. In Section 6.3 we extend the standard LTI techniques of balanced truncation and optimal Hankel norm approximation to the LPV framework. Then in Section 6.4 we present a result on bounding the performance degradation when a reduced order LPV controller is substituted for a high order LPV controller.

6.2 Properties of LPV Systems

In this chapter we consider model approximation for the class of linear systems whose state-space matrices depend continuously on one or more exogenous time-varying parameter (e.g. speed or angle of attack in missile control [86]). This class of systems has been studied by Becker [7] for the purpose of controller synthesis. We adopt a similar notation to Becker, and begin by stipulating conditions which guarantee the existence and uniqueness of the class of LPV models to be considered.

Firstly we define $\mathcal{F}_{\mathcal{P}}$, the set of all admissible parameter trajectories. Hence $\rho(t) \in \mathcal{F}_{\mathcal{P}}$ denotes an admissible parameter trajectory for our system.

Definition 6.2.1 *Given a compact subset $\mathcal{P} \subset \mathbb{R}^s$, the set of all piecewise continuous functions mapping $\mathbb{R}^+$ (time) into $\mathcal{P}$ with a finite number of discontinuities in any interval will be denoted $\mathcal{F}_{\mathcal{P}}$.*

Given a compact subset $\mathcal{P} \subset \mathbb{R}^s$ and the continuous functions

$$\begin{array}{ll} A{:}\mathbb{R}^s \to \mathbb{R}^{n\times n} & B{:}\mathbb{R}^s \to \mathbb{R}^{n\times m} \\ C{:}\mathbb{R}^s \to \mathbb{R}^{p\times n} & D{:}\mathbb{R}^s \to \mathbb{R}^{p\times m} \end{array}$$

then the corresponding n^{th} order LPV system may be represented by the equations

$$\dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t) \tag{6.1}$$

$$y(t) = C(\rho(t))x(t) + D(\rho(t))u(t) \tag{6.2}$$

where $\rho(t) \in \mathcal{F}_{\mathcal{P}}$ is a vector of time-varying parameters which are unknown a priori, but may be measured in real time. Note that this characteristic distinguishes LPV systems from linear time-varying (LTV) systems for which the time variation of $\rho(t)$ is known a priori. For convenience, a realisation given by the equations (6.1) and (6.2) will be denoted by the quintuple $(\mathcal{P}, A_\rho, B_\rho, C_\rho, D_\rho)$.

We are now in a position to define the input-output map for an LPV system.

Definition 6.2.2 *Assume that $\Phi(t, t_0)$ is the state transition matrix for the system $(\mathcal{P}, A_\rho, B_\rho, C_\rho, D_\rho)$. Then for $t_0 = 0$, zero initial conditions and any $\rho \in \mathcal{P}$, the causal linear operator $\mathcal{G}_\rho : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ is given by*

$$y(t) = \int_0^t C(\rho(t))\Phi(t,\tau)B(\rho(\tau))u(\tau)d\tau + D(\rho(t))u(t).$$

Analogous to the LTI case, we adopt the convention of writing this operator

$$\mathcal{G}_\rho = \left[\begin{array}{c|c} A_\rho & B_\rho \\ \hline C_\rho & D_\rho \end{array}\right].$$

We now define stability and performance for LPV systems in terms of a quadratic Lyapunov function [7, 10, 91].

Definition 6.2.3 (Quadratic Stability) *The LPV system $\mathcal{G}_\rho$ with realisation $(\mathcal{P}, A_\rho, B_\rho, C_\rho, D_\rho)$ is $\mathcal{Q}$-stable if there exists a $P \in \mathbb{R}^{n\times n}$, $P = P' > 0$ (which is not $\rho(t)$ dependent) such that for all $\rho \in \mathcal{P}$,*

$$A_\rho P + P A_\rho{}' \;<\; 0 \tag{6.3}$$

It is immediate from Definition 6.2.3 that $\mathcal{G}_\rho$ is $\mathcal{Q}$-stable if and only if there exist $P > 0$ and $Q > 0$ (neither of which is $\rho(t)$ dependent) such that the following inequalities hold for all $\rho \in \mathcal{P}$.

$$A_\rho P + P A_\rho{}' + B_\rho B_\rho{}' \;<\; 0 \tag{6.4}$$
$$A_\rho{}' Q + Q A_\rho + C_\rho{}' C_\rho \;<\; 0 \tag{6.5}$$

Remark 6.2.4 *We could use non-strict inequalities in equations (6.4) and (6.5) and allow $P \geq 0$ and $Q \geq 0$. However, as shown by Beck et al. [6], such a system may be reduced (by truncation of states) to one satisfying equations (6.4) and (6.5) without any reduction error. This may be thought of along similar lines to the LTI situation of obtaining a minimal realisation from one which is non-minimal by eliminating any uncontrollable or unobservable states.*

We also require the notions of $\mathcal{Q}$-stabilisability and $\mathcal{Q}$-detectability .

Definition 6.2.5 (Quadratic Stabilisability [7]) *The pair (A_ρ, B_ρ) are $\mathcal{Q}$-stabilisable if there exists a $P_F \in \mathbb{R}^{n\times n}$ and a continuous function $F_\rho : \mathbb{R}^s \to \mathbb{R}^{m\times n}$ such that*

$$P_F(A_\rho + B_\rho F_\rho) + (A_\rho + B_\rho F_\rho)' P_F \;<\; 0$$

for all $\rho \in \mathcal{P}$.

Definition 6.2.6 (Quadratic Detectability [7]) *The pair (A_ρ, C_ρ) are $\mathcal{Q}$-detectable if there exists a $P_L \in \mathbb{R}^{n\times n}$ and a continuous function $L_\rho : \mathbb{R}^s \to \mathbb{R}^{p\times n}$ such that*

$$(A_\rho + L_\rho C_\rho)' P_L + P_L(A_\rho + L_\rho C_\rho) \;<\; 0$$

for all $\rho \in \mathcal{P}$.

We are now ready to define performance for an LPV system.

Definition 6.2.7 (Induced L_2 Gain) *Given a $\mathcal{Q}$-stable LPV system with zero initial conditions, the induced $\mathcal{L}_2$ gain is defined as*

$$\|\mathcal{G}_\rho\|_{i,2} \quad \triangleq \quad \sup_{\rho(t)\in\mathcal{F}_\mathcal{P}} \sup_{\substack{\|u\|_2\neq 0 \\ u\in\mathcal{L}_{2e}}} \frac{\|y\|_2}{\|u\|_2}.$$

The induced $\mathcal{L}_2$ norm represents the largest induced gain from inputs (in $\mathcal{L}_{2e}$) to outputs (in $\mathcal{L}_{2e}$) over the set of all causal linear operators described by the LPV system.

Lemma 6.2.8 (Quadratic Performance [7, 10, 27]) *Given the LPV system $\mathcal{G}_\rho$ with realisation $(\mathcal{P}, A_\rho, B_\rho, C_\rho, D_\rho)$ and a scalar $\gamma > 0$, if there exists an $X \in \mathbb{R}^{n\times n}$, $X = X' > 0$ (which is not $\rho(t)$ dependent) such that for all $\rho \in \mathcal{P}$*

$$\begin{bmatrix} {A_\rho}'X + XA_\rho & XB_\rho & \gamma^{-1}{C_\rho}' \\ {B_\rho}'X & -I & \gamma^{-1}{D_\rho}' \\ \gamma^{-1}C_\rho & \gamma^{-1}D_\rho & -I \end{bmatrix} \; < \; 0 \tag{6.6}$$

then

1. *$\mathcal{G}_\rho$ is $\mathcal{Q}$-stable.*
2. *There exists a $\beta < \gamma$ such that $\|\mathcal{G}_\rho\|_{i,2} \leq \beta$.*

Note that bounding the quadratic performance in this way is inherently conservative. Lemma 6.2.8 gives only a sufficient condition for the induced $\mathcal{L}_2$ gain to be less than γ. However, constrained optimisation may be employed (see for instance [27]) which minimises γ subject to the constraint of equation (6.6) with $X > 0$.

Remark 6.2.9 *Using Schur complements [7, 10, 43], equation (6.6) may be written*

$$\begin{aligned} & {A_\rho}'X + XA_\rho + \gamma^{-2}{C_\rho}'C_\rho \\ & \quad + \quad (XB_\rho + \gamma^{-2}{C_\rho}'D_\rho)(I - \gamma^{-2}{D_\rho}'D_\rho)^{-1}({B_\rho}'X + \gamma^{-2}{D_\rho}'C_\rho) < 0. \end{aligned} \tag{6.7}$$

A difficulty with determining quadratic performance using Lemma 6.2.8 is the infinite number of constraints which must be satisfied. However, by making the restrictions that

(a) the state space matrices A_ρ, B_ρ, C_ρ and D_ρ depend affinely on $\rho(t)$, and

(b) that $\rho(t)$ varies in a convex polytope with a finite number of vertices $\{\rho_1, \rho_2, \ldots, \rho_N\}$

then there exists an X satisfying equation (6.6) for all $\rho(t)$ if and only if there exists an X satisfying equation (6.6) at the vertices of the polytope $\{\rho_1, \rho_2, \ldots, \rho_N\}$ [4, 7]. This results from the fact that if a solution is found which holds for a finite number of vertices, then the solution will hold for the set of points contained in the convex hull of the vertices. Hence to determine performance for a system satisfying (a) and (b) it is sufficient to satisfy a finite number of constraints. Given a convex decomposition of the current parameter value

$$\rho(t) = \alpha_1\rho_1 + \ldots + \alpha_N\rho_N \quad \alpha_i \geq 0 \quad \textstyle\sum_{i=1}^N \alpha_i = 1$$

then the state-space matrices of an affine LPV model are determined by

$$\begin{bmatrix} A(\rho(t)) & B(\rho(t)) \\ C(\rho(t)) & D(\rho(t)) \end{bmatrix} = \sum_{i=1}^{N} \alpha_i \begin{bmatrix} A(\rho_i) & B(\rho_i) \\ C(\rho_i) & D(\rho_i) \end{bmatrix}.$$

Hence the state-space matrices of the model are given by a convex interpolation of their value at the vertices.

6.3 Open Loop Approximation

In this section we shall extend the balanced truncation and optimal Hankel norm approximation techniques to LPV systems. For LTI systems, the balanced truncation (Section 3.2.2) and optimal Hankel norm (Section 3.2.3) approximation techniques are known to satisfy both upper and lower bounds on the approximation error. Further, they are easy to implement and simple to use. We will see that except for the lower bound, each of these desirable characteristics carries over to the LPV framework.

Firstly we define what is meant by a $\mathcal{Q}$-balanced realisation.

Definition 6.3.1 ($\mathcal{Q}$-balanced Realisations) *Assume $\mathcal{G}_\rho$ is $\mathcal{Q}$-stable. Then there exists a realisation for $\mathcal{G}_\rho$ where P and Q, the solutions to equations (6.4) and (6.5) respectively, are equal and diagonal, i.e. $P = Q = \Sigma = diag(\sigma_1 I, \ldots, \sigma_n I)$. Such a realisation is called $\mathcal{Q}$-balanced. Further, we call σ_i $\mathcal{Q}$-singular values. Analogous to the LTI convention, we write $\mathcal{Q}$-singular values in descending order of magnitude.*

Remark 6.3.2 *Unlike for LTI systems where P and Q are unique solutions to Lyapunov equalities, here P and Q are only required to satisfy an inequality. Hence there may be infinitely many $\mathcal{Q}$-balanced realisation for a given system and $\mathcal{Q}$-singular values are not unique.*

6.3.1 LPV Balanced Truncation

For LTI systems, balanced truncation corresponds to eliminating those states which are least controllable and observable. The procedure for balanced truncation of LPV systems is similar to that for LTI systems, however, due to the non-uniqueness of LPV $\mathcal{Q}$-balanced realisations caution must be shown when selecting a suitable $\mathcal{Q}$-balanced realisation to ensure that the *least important* states are those which are actually being eliminated.

Lemma 6.3.3 *Assume $\mathcal{G}_\rho = \left[\begin{array}{cc|c} A_{\rho 11} & A_{\rho 12} & B_{\rho 1} \\ A_{\rho 21} & A_{\rho 22} & B_{\rho 2} \\ \hline C_{\rho 1} & C_{\rho 2} & 0 \end{array}\right]$ of order n is $\mathcal{Q}$-balanced, and conformably partitioned with $P = Q = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$ where $\Sigma_1 = diag(\sigma_1, \ldots, \sigma_k)$, $\Sigma_2 = diag(\sigma_{k+1}, \ldots, \sigma_n)$ and $\sigma_k > \sigma_{k+1}$. Then $\hat{\mathcal{G}}_\rho \triangleq \left[\begin{array}{c|c} A_{\rho 11} & B_{\rho 1} \\ \hline C_{\rho 1} & 0 \end{array}\right]$ is $\mathcal{Q}$-stable and*

$$\|\mathcal{G}_\rho - \hat{\mathcal{G}}_\rho\|_{i,2} < 2 \sum_{j=k+1}^{n} \sigma_j.$$

The proof is a modification to a previously unpublished proof of Glover [33] for the standard LTI error bound. An equivalent result for balanced truncation of discrete $\mathcal{Q}$-balanced systems is presented in [91, 6].

Proof: We will initially consider the case where only those states corresponding to the smallest $\mathcal{Q}$-singular value are truncated. Since $\mathcal{G}_\rho$ is $\mathcal{Q}$-balanced we have

$$\begin{aligned} A_\rho \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \sigma I \end{bmatrix} + \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \sigma I \end{bmatrix} A_\rho' + B_\rho B_\rho' &< 0 \\ A_\rho' \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \sigma I \end{bmatrix} + \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \sigma I \end{bmatrix} A_\rho' + C_\rho' C_\rho &< 0. \end{aligned}$$

Conformably partition $\mathcal{G}_\rho$ with $\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \sigma I \end{bmatrix}$ such that

$$\mathcal{G}_\rho = \left[\begin{array}{cc|c} A_{\rho 11} & A_{\rho 12} & B_{\rho 1} \\ A_{\rho 21} & A_{\rho 22} & B_{\rho 2} \\ \hline C_{\rho 1} & C_{\rho 2} & 0 \end{array}\right]$$

and define

$$\hat{\mathcal{G}}_\rho \triangleq \left[\begin{array}{c|c} A_{\rho 11} & B_{\rho 1} \\ \hline C_{\rho 1} & 0 \end{array}\right].$$

Then it is trivial to show that Σ_1 satisfies both the controllability and observability Lyapunov inequalities for $\hat{\mathcal{G}}_\rho$ and hence $\hat{\mathcal{G}}_\rho$ is $\mathcal{Q}$-stable.

To prove the error bound, define

$$\begin{aligned} E_\rho &\triangleq \mathcal{G}_\rho - \hat{\mathcal{G}}_\rho \\ &= \left[\begin{array}{ccc|c} A_{\rho 11} & 0 & 0 & B_{\rho 1} \\ 0 & A_{\rho 11} & A_{\rho 12} & B_{\rho 1} \\ 0 & A_{\rho 21} & A_{\rho 22} & B_{\rho 2} \\ \hline -C_{\rho 1} & C_{\rho 1} & C_{\rho 2} & 0 \end{array}\right]. \end{aligned}$$

Then use the matrices

$$T^{-1} = \begin{bmatrix} \frac{1}{2}I & \frac{1}{2}I & 0 \\ \frac{1}{2}I & -\frac{1}{2}I & 0 \\ 0 & 0 & I \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} I & I & 0 \\ I & -I & 0 \\ 0 & 0 & I \end{bmatrix}$$

to perform a state transformation on E_ρ to obtain

$$\begin{aligned} E_\rho &= \left[\begin{array}{ccc|c} A_{\rho 11} & 0 & \frac{1}{2}A_{\rho 12} & B_{\rho 1} \\ 0 & A_{\rho 11} & -\frac{1}{2}A_{\rho 12} & 0 \\ A_{\rho 21} & -A_{\rho 21} & A_{\rho 22} & B_{\rho 2} \\ \hline 0 & -2C_{\rho 1} & C_{\rho 1} & 0 \end{array}\right] \\ &\triangleq \left[\begin{array}{c|c} A_{\rho e} & B_{\rho e} \\ \hline C_{\rho e} & D_{\rho e} \end{array}\right]. \end{aligned}$$

We need to show that $||E_\rho||_{i,2} < 2\sigma$, or equivalently, by Lemma 6.6, that there exists a matrix $P_e = P_e' > 0$ such that

$$A_{\rho e}P_e + P_e A'_{\rho e} + B_{\rho e}B'_{\rho e} + \frac{1}{4\sigma^2}P_e C'_{\rho e}C_{\rho e}P_e \;<\; 0. \tag{6.8}$$

We'll now show that a suitable choice is

$$P_e = \begin{bmatrix} \Sigma & 0 & 0 \\ 0 & \sigma^2\Sigma^{-1} & 0 \\ 0 & 0 & 2\sigma I \end{bmatrix}.$$

Firstly, note that with this choice of P_e

$$\frac{1}{2\sigma}P_e C'_{\rho e} \;=\; \begin{bmatrix} 0 \\ -\sigma\Sigma^{-1}C'_{\rho 1} \\ \sigma C'_{\rho 2} \end{bmatrix}.$$

Now, using matrix manipulation, it is simple to show that

$$\begin{aligned} A_{\rho e}P_e + P_e A_{\rho e} + B_{\rho e}B'_{\rho e} + \frac{1}{4\sigma^2}P_e C'_{\rho e}C_{\rho e}P_e \;&=\; \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}\left\{A_\rho P + PA'_\rho + B_\rho B'_\rho\right\}\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \\ &+\; \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}\left\{A'_\rho Q + QA_\rho + C_\rho C'_\rho\right\}\begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \end{aligned}$$

which by assumption is negative definite. Therefore the chosen P_e satisfies equation (6.8) and we have that

$$||\mathcal{G}_\rho - \hat{\mathcal{G}}_\rho||_{i,2} < 2\sigma.$$

The general error bound for truncation of more than just the states corresponding to the smallest $\mathcal{Q}$-singular value results from recursive use of the above procedure. ∎

Corollary 6.3.4 *$\hat{\mathcal{G}}_\rho$ obtained as in Lemma 6.3.3 is $\mathcal{Q}$-balanced.*

To ensure that the states of least importance are the ones actually being eliminated, extreme care must be taken when selecting the initial $\mathcal{Q}$-balanced realisation for $\mathcal{G}_\rho$. In [93], Wood et al. propose the following iterative scheme to reduce conservatism in the error bound.

Procedure 6.3.5

1. *Pre- and post-multiply equation (6.4) by P^{-1} and equation (6.5) by Q^{-1} to derive the equations*

$$P^{-1}A_\rho + A_\rho{}'P^{-1} + P^{-1}B_\rho B_\rho{}'P^{-1} \;<\; 0 \tag{6.9}$$
$$Q^{-1}A_\rho{}' + A_\rho Q^{-1} + Q^{-1}C_\rho{}'C_\rho Q^{-1} \;<\; 0 \tag{6.10}$$

2. *Solve equation (6.9) for P^{-1} which minimises the trace$(-P^{-1})$.*
3. *Solve equation (6.10) for Q^{-1} which minimises the trace$(-(P^{-1}Q^{-1}))$ where P^{-1} comes from step two or step four.*
4. *Solve equation (6.9) for P^{-1} which minimises the trace$(-(P^{-1}Q^{-1}))$ where Q^{-1} comes from step three.*
5. *Repeat steps three and four until the trace$(-(P^{-1}Q^{-1}))$ can be decreased no further.*

Once suitable P^{-1} and Q^{-1} have been selected they may be inverted to obtain P and Q which are then used to calculate the $\mathcal{Q}$-balancing transformation.

By solving equations (6.9) and (6.10) for P^{-1} and Q^{-1} rather than equations (6.4) and (6.5) for P and Q the need for any matrix inversion is deferred until the iterations are complete. This gives a numerically more efficient solution technique. Procedure 6.3.5 emphasises the smallest $\mathcal{Q}$-singular values, i.e. the ones which correspond to the subsystem to be truncated. By forcing σ_n to be small, a small approximation error is guaranteed.

6.3.2 Optimal Hankel Norm Approach

Motivated by the standard LTI optimal Hankel norm approximation procedure [34], here we propose a similar reduction scheme for LPV systems. However, we stress that unlike its LTI counterpart, the interpretation of this scheme as being optimal in any sense isn't clear. We retain this name because we use similar manipulations to the LTI counterpart and the result reduces to the LTI result when the plant under consideration is purely LTI.

Lemma 6.3.6 *Assume $\mathcal{G}_\rho$ with realisation $(\mathcal{P}, A_\rho, B_\rho, C_\rho, D_\rho)$ has state-dimension n, is $\mathcal{Q}$-stable and $\mathcal{Q}$-balanced such that*

$$\begin{aligned} A_\rho P + P{A_\rho}' + B_\rho {B_\rho}' &< 0 \\ {A_\rho}'Q + Q{A_\rho}' + {C_\rho}'C_\rho &< 0 \end{aligned}$$

where

$$P = Q = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \sigma I \end{bmatrix} > 0$$

with $\Sigma_1 \in \mathbb{R}^{k\times k}$ diagonal $(k < n)$ and $\sigma \neq 0$. Now augment B_ρ, C_ρ and D_ρ such that

$$\begin{aligned} B_{\rho a} &= \begin{bmatrix} B_\rho & \tilde{B}_{\rho a} \end{bmatrix} \\ C_{\rho a} &= \begin{bmatrix} C_\rho \\ \tilde{C}_{\rho a} \end{bmatrix} \end{aligned}$$

satisfy

$$\begin{aligned} A_\rho P + P{A_\rho}' + B_{\rho a}B'_{\rho a} &= 0 \\ {A_\rho}'Q + Q{A_\rho}' + C'_{\rho a}C_{\rho a} &= 0 \end{aligned}$$

and

$$D_{\rho a} \;=\; \begin{bmatrix} D_\rho & 0 \\ 0 & 0 \end{bmatrix}.$$

Conformably partition A_ρ*,* $B_{\rho a}$*,* $C_{\rho a}$ *with* P*, as*

$$A_\rho \;=\; \begin{bmatrix} A_{\rho 11} & A_{\rho 12} \\ A_{\rho 21} & A_{\rho 22} \end{bmatrix} \quad B_{\rho a} = \begin{bmatrix} B_{\rho a1} \\ B_{\rho a2} \end{bmatrix} \quad C_{\rho a} = \begin{bmatrix} C_{\rho a1} & C_{\rho a2} \end{bmatrix}$$

and define

$$\begin{aligned} \hat{A}_\rho &\triangleq \Gamma^{-1}(\sigma^2 A'_{\rho 11} + \Sigma_1 A_{\rho 11}\Sigma_1 - \sigma C'_{\rho a1} U_\rho B'_{\rho a1}) \\ \hat{B}_{\rho a} &\triangleq \Gamma^{-1}(\Sigma_1 B_{\rho a1} + \sigma C'_{\rho a1} U_\rho) \\ \hat{C}_{\rho a} &\triangleq C_{\rho a1}\Sigma_1 + \sigma U_\rho B'_{\rho a1} \\ \hat{D}_{\rho a} &\triangleq D_{\rho a} - \sigma U_\rho \end{aligned}$$

where U_ρ *is a unitary matrix satisfying*

$$B_{\rho a2} \;=\; -C'_{\rho a2} U_\rho$$

and Γ *is defined as*

$$\Gamma \;\triangleq\; \Sigma_1^2 - \sigma^2 I.$$

Then $\hat{\mathcal{G}}_\rho$ *with realisation* $(\mathcal{P}, \hat{A}_\rho, \hat{B}_\rho, \hat{C}_\rho, \hat{D}_\rho)$ *has state-dimension* k*, is* $\mathcal{Q}$*-stable and satisfies*

$$\|\mathcal{G}_\rho - \hat{\mathcal{G}}_\rho\|_{i,2} \;\le\; \sigma$$

where $\hat{B}_\rho$*,* $\hat{C}_\rho$ *and* $\hat{D}_\rho$ *are obtained from conformably partitioning,* $\hat{B}_{\rho a}$*,* $\hat{C}_{\rho a}$ *and* $\hat{D}_{\rho a}$ *with* $B_{\rho a}$ *and* $C_{\rho a}$ *as*

$$\hat{B}_{\rho a} = \begin{bmatrix} \hat{B}_\rho & \hat{B}_{\rho a2} \end{bmatrix} \quad \hat{C}_{\rho a} = \begin{bmatrix} \hat{C}_\rho \\ \hat{C}_{\rho a2} \end{bmatrix} \quad \hat{D}_{\rho a} = \begin{bmatrix} \hat{D}_\rho & \hat{D}_{\rho 12} \\ \hat{D}_{\rho 21} & \hat{D}_{\rho 22} \end{bmatrix}.$$

Proof: Theorem 6.3 of [34] gives that the error between the two systems $(\mathcal{P}, A_\rho, B_{\rho a}, C_{\rho a}, D_{\rho a})$ and $(\mathcal{P}, \hat{A}_\rho, \hat{B}_{\rho a}, \hat{C}_{\rho a}, \hat{D}_{\rho a})$ is equal to σ. The result follows by noticing that $\mathcal{G}_\rho - \hat{\mathcal{G}}_\rho$ is a submatrix of the above error. ∎

Remark 6.3.7 *In general, the matrix* U_ρ *will be a complex function of* ρ*. Hence the approximation will potentially be a complex function of* ρ *and difficult to write explicitly. Indeed, the complexity introduced by this formulation may exceed any gains obtained by reducing the order. However, if the matrices are defined on a convex polytope then approximation may be performed at each of the vertices (using the same* P *and* Q *at each vertex) and the general solution obtained using a convex interpolation of the solutions at each vertex. Note that the* $\hat{\mathcal{G}}_\rho$ *obtained in this way will be different from the* $\hat{\mathcal{G}}_\rho$ *resulting from Lemma 6.3.6.*

6.4 Closed Loop Approximation

Previous chapters have shown that techniques for approximating LTI systems are well developed, and that many techniques for LTI controller approximation give an a priori bound on degradation of the closed loop performance when a low order controller is substituted for a high order controller. Indeed, the bulk of the work in this dissertation has shown that it is possible to design LTI controllers of low order for LTI plants without any degradation of closed loop $\mathcal{H}_\infty$ performance.

Approximation techniques for LPV systems are nowhere near as mature. The natural question to ask is: *Given a high order $\mathcal{Q}$-stabilising LPV controller which satisfies a certain level of induced $\mathcal{L}_2$ gain, is it possible to reduce the state-dimension of the controller such that it remains $\mathcal{Q}$-stabilising and satisfies an a priori bound on induced $\mathcal{L}_2$ gain performance degradation*? Here we take one tentative step towards answering this question. Many of the techniques we employ have direct LTI analogies. However, the state-space now evolves with parameter $\rho(t)$.

To determine how performance is effected by substituting a reduced order controller into the closed loop, consider the interconnection of Figure 6.1. $\mathcal{H}(\mathcal{G}_\rho, \mathcal{K}_\rho)$ represents the closed loop operator from disturbances $w_i \in \mathcal{L}_{2e}$ to plant and controller inputs $e_i \in \mathcal{L}_{2e}$. It may be written

$$\mathcal{H}(\mathcal{G}_\rho, \mathcal{K}_\rho) = \begin{bmatrix} \mathcal{K}_\rho \\ I \end{bmatrix} (I - \mathcal{G}_\rho \mathcal{K}_\rho)^{-1} \begin{bmatrix} \mathcal{G}_\rho & I \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

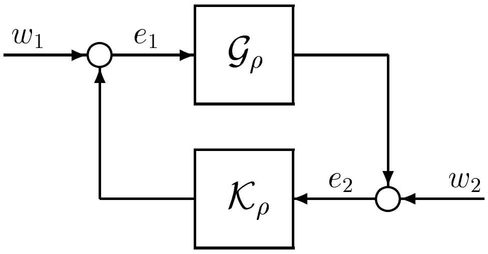


Figure 6.1: $\mathcal{H}(\mathcal{G}_\rho, \mathcal{K}_\rho)$

To analyse this system we require the notion of coprime factorisations for LPV models. The following definition is a generalisation of Definition 2.5.1 (coprime factorisations for LTI models).

Definition 6.4.1 *The pair* $\begin{bmatrix} \tilde{\mathcal{N}}_\rho & \tilde{\mathcal{M}}_\rho \end{bmatrix}$, *where* $\tilde{\mathcal{N}}_\rho$ *and* $\tilde{\mathcal{M}}_\rho$ *are both* $\mathcal{Q}$*-stable, constitute a left coprime factorisation of* $\mathcal{G}_\rho$ *if and only if for all* $\rho \in \mathcal{P}$

(a) $\tilde{\mathcal{M}}_\rho$ *is square and invertible*

(b) $\mathcal{G}_\rho = \tilde{\mathcal{M}}_\rho^{-1} \tilde{\mathcal{N}}_\rho$

(c) $\tilde{\mathcal{N}}_\rho$ *and* $\tilde{\mathcal{M}}_\rho$ *are left coprime, i.e., there exists* $\mathcal{V}_\rho$ *and* $\mathcal{U}_\rho$ *both* $\mathcal{Q}$*-stable such that* $\tilde{\mathcal{M}}_\rho \mathcal{V}_\rho + \tilde{\mathcal{N}}_\rho \mathcal{U}_\rho = I$.

Right coprime factorisations may be defined analogously.

Note that coprime factorisations exist if $\mathcal{G}_\rho$ is $\mathcal{Q}$-stabilisable and $\mathcal{Q}$-detectable and may be constructed analogously to their LTI formulations (see for instance [62, 101]).

Lemma 6.4.2 *Assume $\mathcal{K}_\rho$ is an LPV controller which $\mathcal{Q}$-stabilises the LPV plant $\mathcal{G}_\rho$ such that $\|\mathcal{H}(\mathcal{G}_\rho,\mathcal{K}_\rho)\|_{i,2} < \gamma$. Assume $\begin{bmatrix} \tilde{\mathcal{N}}_\rho & \tilde{\mathcal{M}}_\rho \end{bmatrix}$ is a left coprime factorisation for $\mathcal{G}_\rho = \tilde{\mathcal{M}}_\rho^{-1}\tilde{\mathcal{N}}_\rho$ and $\begin{bmatrix} \mathcal{U}_\rho \\ \mathcal{V}_\rho \end{bmatrix}$ is a right coprime factorisation for $\mathcal{K}_\rho = \mathcal{U}_\rho\mathcal{V}_\rho^{-1}$. Define $\delta \triangleq \|\Delta\|_{i,2}$ and $\hat{\mathcal{K}}_\rho \triangleq \hat{\mathcal{U}}_\rho\hat{\mathcal{V}}_\rho^{-1}$ where*

$$\Delta \triangleq \begin{bmatrix} \mathcal{U}_\rho \\ \mathcal{V}_\rho \end{bmatrix} - \begin{bmatrix} \hat{\mathcal{U}}_\rho \\ \hat{\mathcal{V}}_\rho \end{bmatrix}.$$

Also define

$$\epsilon \triangleq \|(\tilde{\mathcal{M}}_\rho\mathcal{V}_\rho - \tilde{\mathcal{N}}_\rho\mathcal{U}_\rho)^{-1}\begin{bmatrix} -\tilde{\mathcal{N}}_\rho & \tilde{\mathcal{M}}_\rho \end{bmatrix}\|_{i,2}.$$

Then if $\hat{\mathcal{K}}_\rho$ is $\mathcal{Q}$-stabilising and $\epsilon\delta < 1$ we have

$$\|\mathcal{H}(\mathcal{G}_\rho,\hat{\mathcal{K}}_\rho)\|_{i,2} \leq \gamma + (1+\gamma)\epsilon\delta + O(\delta^2)\epsilon. \tag{6.11}$$

Proof: We have

$$\begin{aligned} \mathcal{H}(\mathcal{G}_\rho,\hat{\mathcal{K}}_\rho) &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \hat{\mathcal{K}}_\rho \\ I \end{bmatrix}(I - \mathcal{G}_\rho\mathcal{K}_\rho)^{-1}\begin{bmatrix} \hat{\mathcal{G}}_\rho & I \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \hat{\mathcal{U}}_\rho \\ \hat{\mathcal{V}}_\rho \end{bmatrix}(\tilde{\mathcal{M}}_\rho\hat{\mathcal{V}}_\rho - \tilde{\mathcal{N}}_\rho\hat{\mathcal{U}}_\rho)^{-1}\begin{bmatrix} \tilde{\mathcal{N}}_\rho & \tilde{\mathcal{M}}_\rho \end{bmatrix}. \end{aligned} \tag{6.12}$$

Now note that

$$\begin{aligned} (\tilde{\mathcal{M}}_\rho\hat{\mathcal{V}}_\rho - \tilde{\mathcal{N}}_\rho\hat{\mathcal{U}}_\rho)^{-1} &= (\tilde{\mathcal{M}}_\rho\hat{\mathcal{V}}_\rho - \tilde{\mathcal{N}}_\rho\hat{\mathcal{U}}_\rho + \tilde{\mathcal{M}}_\rho\mathcal{V}_\rho - \tilde{\mathcal{M}}_\rho\mathcal{V}_\rho - \tilde{\mathcal{N}}_\rho\mathcal{U}_\rho + \tilde{\mathcal{N}}_\rho\mathcal{U}_\rho)^{-1} \\ &= (\tilde{\mathcal{M}}_\rho(\hat{\mathcal{V}}_\rho - \mathcal{V}_\rho) - \tilde{\mathcal{N}}_\rho(\hat{\mathcal{U}}_\rho - \mathcal{U}_\rho) + \tilde{\mathcal{M}}_\rho\mathcal{V}_\rho - \tilde{\mathcal{N}}_\rho\mathcal{U}_\rho)^{-1} \\ &= ((\tilde{\mathcal{M}}_\rho\mathcal{V}_\rho - \tilde{\mathcal{N}}_\rho\mathcal{U}_\rho) - \begin{bmatrix} -\tilde{\mathcal{N}}_\rho & \tilde{\mathcal{M}}_\rho \end{bmatrix}\Delta)^{-1} \\ &= (I - (\tilde{\mathcal{M}}_\rho\mathcal{V}_\rho - \tilde{\mathcal{N}}_\rho\mathcal{U}_\rho)^{-1}\begin{bmatrix} -\tilde{\mathcal{N}}_\rho & \tilde{\mathcal{M}}_\rho \end{bmatrix}\Delta)^{-1} \\ &\qquad \times(\tilde{\mathcal{M}}_\rho\mathcal{V}_\rho - \tilde{\mathcal{N}}_\rho\mathcal{U}_\rho)^{-1}. \end{aligned} \tag{6.13}$$

Observe that

$$\begin{aligned} \epsilon\delta < 1 &\iff \|(\tilde{\mathcal{M}}_\rho\mathcal{V}_\rho - \tilde{\mathcal{N}}_\rho\mathcal{U}_\rho)^{-1}\begin{bmatrix} -\tilde{\mathcal{N}}_\rho & \tilde{\mathcal{M}}_\rho \end{bmatrix}\|_{i,2}\|\Delta\|_{i,2} < 1 \\ &\Rightarrow \|(\tilde{\mathcal{M}}_\rho\mathcal{V}_\rho - \tilde{\mathcal{N}}_\rho\mathcal{U}_\rho)^{-1}\begin{bmatrix} -\tilde{\mathcal{N}}_\rho & \tilde{\mathcal{M}}_\rho \end{bmatrix}\Delta\|_{i,2} < 1 \end{aligned}$$

and for $\|\mathcal{X}\|_{i,2} < 1$

$$(I - \mathcal{X})^{-1} = I + \mathcal{X} + O(\mathcal{X}^2).$$

Hence if $\delta\epsilon < 1$ we have

$$(\tilde{\mathcal{M}}_\rho \hat{\mathcal{V}}_\rho - \tilde{\mathcal{N}}_\rho \hat{\mathcal{U}}_\rho)^{-1} = \{\, I + (\tilde{\mathcal{M}}_\rho \mathcal{V}_\rho - \tilde{\mathcal{N}}_\rho \mathcal{U}_\rho)^{-1} \begin{bmatrix} -\tilde{\mathcal{N}}_\rho & \tilde{\mathcal{M}}_\rho \end{bmatrix} \Delta \\ + \mathcal{O}(\delta^2) \,\} (\tilde{\mathcal{M}}_\rho \mathcal{V}_\rho - \tilde{\mathcal{N}}_\rho \mathcal{U}_\rho)^{-1} \tag{6.14}$$

where $\mathcal{O}(\delta^2)$ is a matrix with $\|\mathcal{O}(\delta^2)\|_{i,2}/\delta^2$ bounded as $\delta \to 0$. Substituting equation (6.14) into equation (6.12) then gives

$$\begin{aligned}
\mathcal{H}(\mathcal{G}_\rho, \hat{\mathcal{K}}_\rho) &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \left\{ \begin{bmatrix} \mathcal{U}_\rho \\ \mathcal{V}_\rho \end{bmatrix} - \Delta \right\} \Big\{ I + (\tilde{\mathcal{M}}_\rho \mathcal{V}_\rho - \tilde{\mathcal{N}}_\rho \mathcal{U}_\rho)^{-1} \\
&\quad \times \begin{bmatrix} -\tilde{\mathcal{N}}_\rho & \tilde{\mathcal{M}}_\rho \end{bmatrix} \Delta + \mathcal{O}(\delta^2) \Big\} (\tilde{\mathcal{M}}_\rho \mathcal{V}_\rho - \tilde{\mathcal{N}}_\rho \mathcal{U}_\rho)^{-1} \begin{bmatrix} \tilde{\mathcal{N}}_\rho & \tilde{\mathcal{M}}_\rho \end{bmatrix} \\
&= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathcal{U}_\rho \\ \mathcal{V}_\rho \end{bmatrix} (\tilde{\mathcal{M}}_\rho \mathcal{V}_\rho - \tilde{\mathcal{N}}_\rho \mathcal{U}_\rho)^{-1} \begin{bmatrix} \tilde{\mathcal{N}}_\rho & \tilde{\mathcal{M}}_\rho \end{bmatrix} \\
&\quad + \left\{ \begin{bmatrix} \mathcal{U}_\rho \\ \mathcal{V}_\rho \end{bmatrix} (\tilde{\mathcal{M}}_\rho \mathcal{V}_\rho - \tilde{\mathcal{N}}_\rho \mathcal{U}_\rho)^{-1} \begin{bmatrix} -\tilde{\mathcal{N}}_\rho & \tilde{\mathcal{M}}_\rho \end{bmatrix} - I \right\} \\
&\quad \times \Delta (\tilde{\mathcal{M}}_\rho \mathcal{V}_\rho - \tilde{\mathcal{N}}_\rho \mathcal{U}_\rho)^{-1} \begin{bmatrix} \tilde{\mathcal{N}}_\rho & \tilde{\mathcal{M}}_\rho \end{bmatrix} \\
&\quad + \{ \begin{bmatrix} \mathcal{U}_\rho \\ \mathcal{V}_\rho \end{bmatrix} \mathcal{O}(\delta^2) - \Delta (\tilde{\mathcal{M}}_\rho \mathcal{V}_\rho - \tilde{\mathcal{N}}_\rho \mathcal{U}_\rho)^{-1} \begin{bmatrix} -\tilde{\mathcal{N}}_\rho & \tilde{\mathcal{M}}_\rho \end{bmatrix} \Delta \\
&\quad - \Delta \mathcal{O}(\delta^2) \,\} (\tilde{\mathcal{M}}_\rho \mathcal{V}_\rho - \tilde{\mathcal{N}}_\rho \mathcal{U}_\rho)^{-1} \begin{bmatrix} \tilde{\mathcal{N}}_\rho & \tilde{\mathcal{M}}_\rho \end{bmatrix}. \\
&= \mathcal{H}(\mathcal{G}_\rho, \mathcal{K}_\rho) + \left\{ \mathcal{H}(\mathcal{G}_\rho, \mathcal{K}_\rho) \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} \right\} \\
&\quad \times \Delta (\tilde{\mathcal{M}}_\rho \mathcal{V}_\rho - \tilde{\mathcal{N}}_\rho \mathcal{U}_\rho)^{-1} \begin{bmatrix} \tilde{\mathcal{N}}_\rho & \tilde{\mathcal{M}}_\rho \end{bmatrix} \\
&\quad + \{ \begin{bmatrix} \mathcal{U}_\rho \\ \mathcal{V}_\rho \end{bmatrix} \mathcal{O}(\delta^2) - \Delta (\tilde{\mathcal{M}}_\rho \mathcal{V}_\rho - \tilde{\mathcal{N}}_\rho \mathcal{U}_\rho)^{-1} \begin{bmatrix} -\tilde{\mathcal{N}}_\rho & \tilde{\mathcal{M}}_\rho \end{bmatrix} \Delta \\
&\quad - \Delta \mathcal{O}(\delta^2) \,\} (\tilde{\mathcal{M}}_\rho \mathcal{V}_\rho - \tilde{\mathcal{N}}_\rho \mathcal{U}_\rho)^{-1} \begin{bmatrix} \tilde{\mathcal{N}}_\rho & \tilde{\mathcal{M}}_\rho \end{bmatrix}.
\end{aligned}$$

And since the triangular inequality holds

$$\|\mathcal{H}(\mathcal{G}_\rho, \hat{\mathcal{K}}_\rho)\|_{i,2} \leq \gamma + (1+\gamma)\delta\epsilon + O(\delta^2)\epsilon \tag{6.15}$$

where

$$O(\delta^2) = \left\| \left\{ \begin{bmatrix} \mathcal{U}_\rho \\ \mathcal{V}_\rho \end{bmatrix} \mathcal{O}(\delta^2) - \Delta (\tilde{\mathcal{M}}_\rho \mathcal{V}_\rho - \tilde{\mathcal{N}}_\rho \mathcal{U}_\rho)^{-1} \begin{bmatrix} -\tilde{\mathcal{N}}_\rho & \tilde{\mathcal{M}}_\rho \end{bmatrix} \Delta - \Delta \mathcal{O}(\delta^2) \right\} \right\|_{i,2}.$$

■

Assuming the error is small and second order terms can be neglected, Lemma 6.4.2 allows us to bound the degradation in closed loop performance when $\hat{\mathcal{K}}_\rho$ is substituted for $\mathcal{K}_\rho$. Unfortunately the results are conservative, which when combined with the inherent conservatism involved in bounding the quadratic performance severely limit their application.

6.5 Summary

In this chapter we have shown that the LTI techniques of balanced truncation and optimal Hankel norm approximation may be extended to an LPV framework. (Although the Hankel norm approach is no longer optimal in any sense.) These results complement many recent results on the synthesis of controllers for LPV plants. However, since $\mathcal{Q}$-balanced realisations are far from unique, caution must be shown when applying the techniques to ensure that the states being neglected correspond to small $\mathcal{Q}$-singular values and hence contribute little to the input-output characteristics.

Chapter 7

Design Examples

7.1 Introduction

In this chapter we discuss some of the practical aspects of the controller reduction techniques proposed in previous chapters by presenting three numerical examples.

The first two examples consider synthesis of reduced order $\mathcal{H}_\infty$ controllers for LTI systems. Given a generalised plant P, the objective in suboptimal $\mathcal{H}_\infty$ control is to calculate a controller K which stabilises the closed loop and satisfies the norm bound $\|\mathcal{F}_l(P,K)\|_\infty < \gamma$. The generalised plant incorporates both the actual plant which is to be controlled and weighting functions which specify performance objectives. The performance weights are selected to emphasise objectives such as disturbance attenuation at low frequency and robustness to modeling uncertainty. Selection of appropriate weights is non-trivial and often involves a certain amount of iteration. For a more detailed discussion on weight selection see [45, 63]. Here we assume that design objectives have been prespecified and that all (P,γ)-admissible controllers satisfy the required objectives. Hence we will concentrate purely on the problem of reducing controller complexity by designing low-order (P,γ)-admissible controllers.

The third example considers reducing the order of LPV controllers. Although we cannot even guarantee these techniques will lead to a reduced order stabilising controller we demonstrate that they can work well in practice.

The chapter is arranged as follows. In Section 7.2 we discuss some of the computational difficulties connected with calculating the frequency weights which make the $\tilde{R}$ of reduction approaches $\mathcal{A}(P/\Phi)$ and $\mathcal{CF}(P/\Phi)$ a contraction. In Section 7.3 we examine robust performance for the pitch loop of a highly manoeuverable experimental aircraft model. In Section 7.4 we examine tracking control in a Compact Disk mechanism. This has recently been studied by Wortelboer [94], and we compare his results to those obtained from our design techniques. Except when specifically stated otherwise, the results shown come from employing the one block frequency weighted optimal Hankel norm reduction procedure of Section 3.3.2. The results obtained using frequency weighted balanced truncation were similar and hence have been omitted. The final design example considers pitch control of a missile. This is presented in Section 7.5.

7.2 Computational Issues

There are three principle computational difficulties encountered in implementing the proposed LTI performance-preserving controller approximation schemes.

Firstly, it is not generally possible to exactly solve factorisations of the form required to calculate Z_1 and Z_2 (equations (4.30) and (4.31), or (4.34) and (4.35)) parametrically. However, we may get arbitrarily close to an exact solution by employing an appropriate rational approximation. More precisely, assume we wish to approximate $\Psi^{1/2}$ where $\Psi = \Psi^{\sim} \geq 0$[1] by the rational function $r(\Psi) = r^{\sim}(\Psi) \geq 0$ such that $\|r(\Psi) - \Psi^{1/2}\|_\infty < \epsilon$ for some $\epsilon > 0$. To accomplish this we may calculate a scalar rational approximation $r(\psi)$ to $\sqrt{\psi}$ where $\psi \geq 0$ such that $|r(\psi) - \sqrt{\psi}| < \epsilon$ then substitute Ψ for ψ.

Care must be taken when calculating this rational approximation to ensure that the resulting weights still make $\tilde{R}$ a contraction. This requires $r(\Psi) \geq \Psi^{1/2}$ which implies weights $\hat{W}_1$ and $\hat{W}_2$ satisfy $\hat{W}_1^{\sim}\hat{W}_1 \leq W_1^{\sim}W_1$ and $\hat{W}_2\hat{W}_2^{\sim} \leq W_2W_2^{\sim}$ and our sufficient condition remains valid when weights $\hat{W}_1$ and $\hat{W}_2$ are used in place of W_1 and W_2.

The following lemma justifies this approach.

Lemma 7.2.1 *Given two scalar valued functions* $f(\psi) = \sum_{i=0}^{\infty} \alpha_i \psi^i$ *and* $g(\psi) = \sum_{i=0}^{\infty} \beta_i \psi^i$ *such that* $|f(\psi) - g(\psi)| < \varepsilon$ *for* $0 \leq a \leq \psi \leq b$ *then* $\|f(\Psi) - g(\Psi)\|_\infty < \varepsilon$ *for any matrix* $\Psi = \Psi^{\sim} \geq 0$ *such that* $a \leq \underline{\sigma}(\Psi) \leq \psi \leq \overline{\sigma}(\Psi) \leq b$.

Proof: The result follows by substituting a singular value decomposition of Ψ into $h(\Psi) = f(\Psi) - g(\Psi)$. ■

Hence by calculating a rational approximation, $r(\psi)$ to $\sqrt{\psi}$ such that $r(\psi) \geq \sqrt{\psi}$ and $|r(\psi) - \sqrt{\psi}|$ is small , then substituting Z_1^2 and Z_2^2 for ψ, the approximations $r(Z_1) = \hat{Z}_1 \geq Z_1$ and $r(Z_2) = \hat{Z}_2 \geq Z_2$ are obtained. These approximations may then be directly substituted into the equations for $(W_1^{\sim}W_1)^{-1}$ and $(W_2W_2^{\sim})^{-1}$ when solving for $\hat{W}_1$ and $\hat{W}_2$. See [76] and the references therein for suitable rational approximation algorithms.

The second computational difficulty which arises is that the order of the weights may become large. Hence the weights become time consuming to calculate and demanding to manipulate. The growth in state dimension is as follows. Given a generalised plant of order n the order of K_a will be n, the order of K^0 will typically be n, and hence the order of R will be $2n$ and L will be of order $4n$. Therefore, the order of Z_1^2 and Z_2^2 may be up to $12n$ and hence the state dimension of $\hat{Z}_1$ and $\hat{Z}_2$ will be $12n$ times the degree of the scalar approximation considered appropriate to make $|r(\psi) - \sqrt{\psi}|$ small. In the examples to follow we have typically chosen a second order approximation. Consequently, even for a plant of moderate order the subsequent spectral factorisations required to calculate $\hat{W}_1$ and $\hat{W}_2$ become demanding. Hence a trade off exists between the order of the weights and the accuracy with which they can be calculated. However, particularly when the Hankel singular values of R tail off quickly, appropriate model reduction may be performed at intermediate stages without introducing unacceptable errors.

The third of the principle computational issues relates to the effectiveness of frequency weighted model reduction techniques. Experience has shown that they tend not to give

[1]Note that $(\bullet)^{1/2}$ represents a square root, not a spectral factorisation.

approximants which even come close to attaining the lower bound without employing some form of parameter optimisation. Experience indicates that although the lower bound is not necessarily attainable a combined model reduction/parameter optimisation approach using the results of frequency weighted model reduction as a starting point for optimisation works well. This will be borne out by our examples where we employ the combined Hankel norm/convex optimisation approach presented in [100] and outlined at the end of Section 3.3.2. All our optimisation was undertaken using the *Consol-Optcad*[2] package [19].

7.3 Pitch Control for the HIMAT Model

In this example we consider the design of reduced order robust controllers for the longitudinal dynamics of an experimental highly manoeuverable (HIMAT) aeroplane. Initially studied in [79] using LQG synthesis, a linearised model and detailed specifications for $\mathcal{H}_\infty$ design can be found in [5].

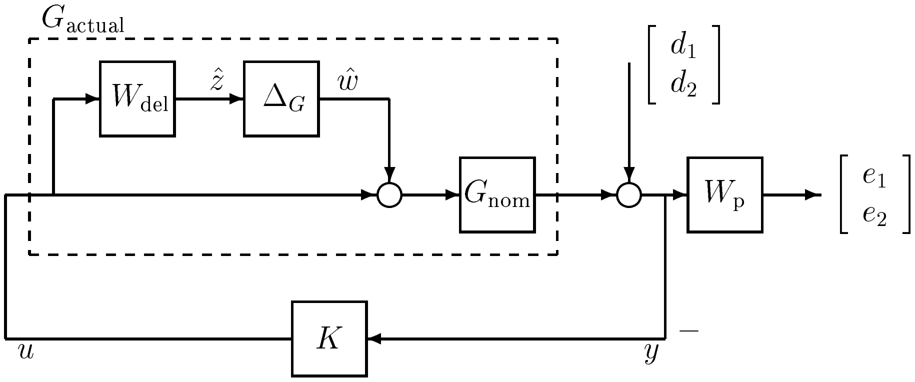


Figure 7.1: HIMAT Closed Loop Interconnection

The closed loop interconnection is shown in Figure 7.1. The actual aeroplane is represented by G_{actual}, with the combination of W_{del} and Δ_G reflecting the amount and type of uncertainty in the nominal model G_{nom}. G_{nom} has four states,

Name		Symbol
forward speed	–	v
angle-of-attack	–	α
pitch rate	–	q
pitch angle	–	θ

two control inputs ,

Name		Symbol
elevon command	–	δ_e
canard command	–	δ_c

and two measured outputs,

[2]Consol-Optcad is a trademark of the University of Maryland at College Park

Name		Symbol
angle-of-attack	–	α
pitch angle	–	θ.

The flight path angle is given by $\zeta = \theta - \alpha$.

In [5] a 20 state $\mathcal{H}_\infty$ controller is designed using μ-synthesis to give robust performance for the following three maneuvers:

Vertical Translation: Control the vertical velocity at constant θ (α varies). This implies that the attitude remains constant as the velocity vector rotates.

Pitch Pointing: Control the attitude at a constant flight path angle (i.e. $\theta = \alpha$). For this maneuver the velocity vector does not rotate.

Direct Lift: Control the flight path angle while holding the angle-of-attack constant (i.e. $\zeta = \theta$). This maneuver produces a normal acceleration without changing the angle-of-attack.

These performance objectives are captured in the weight W_{p}. They correspond to requiring the transfer function from disturbances $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ to errors $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ to be less than unity for all $\|\Delta_G\|_\infty < 1$.

The objective of this section is to design a reduced order controller which preserves robust performance. The first step is to convert the HIMAT interconnection of Figure 7.1 to the generalised plant and controller LFT interconnection required in our setup. This is shown in Figure 7.2. The *D scales* result from three so called *D-K iterations* used in the μ-synthesis controller design procedure.

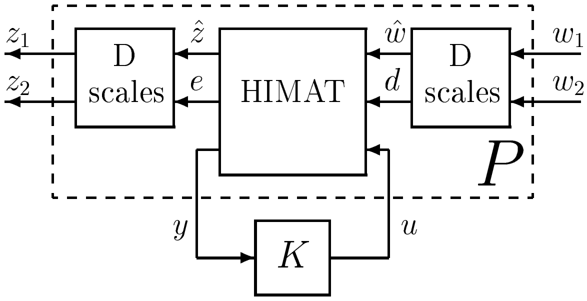


Figure 7.2: HIMAT Controller Synthesis LFT Interconnection

The generalised plant P has 20 states. K^0 is a 20 state (near optimal) controller such that $\|\mathcal{F}_l(P,K)\|_\infty = 0.97$. Our aim is to maintain robust performance of the unscaled plant, i.e $\|\mathcal{F}_l(P,K)\|_\infty < 1$.

The results of model reduction are presented in Table 7.1. The following abbreviations are used:

$\mathcal{A}(\Phi)$	–	Freq. weighted reduction using the $\mathcal{A}(\Phi)$ procedure and maximising the determinant of the weights.
$\mathcal{CF}(\Phi)$	–	Freq. weighted reduction using the $\mathcal{CF}(\Phi)$ procedure and maximising the trace of the weights.
$\mathcal{CF}(\Theta)$	–	Freq. weighted reduction using the $\mathcal{CF}(\Theta)$ procedure.
UOHNA	–	Unweighted optimal Hankel norm approximation.
UBT	–	Unweighted balance and truncate.
LCFR	–	Unweighted optimal Hankel norm reduction of controller left coprime factors.

For the $\mathcal{CF}(\Phi)$ approach, a normalised left coprime factorisation of K^0 was selected as the starting point. For the $\mathcal{CF}(\Theta)$ approach, the specific left coprime factorisation required by Proposition 4.4.3 was used. For the unweighted methods, the controller of least degree satisfying $\|\mathcal{F}_l(P,K)\|_\infty < 1$ is presented.

Reduction Approach	Controller Order	$\|(\text{weighted error})\|_\infty$	$\|\mathcal{F}_l(P,K)\|_\infty$
	20 (full)	–	0.99
$\mathcal{A}(\Phi)$	14	0.67	0.99
$\mathcal{CF}(\Phi)$	16	0.94	0.98
$\mathcal{CF}(\Theta)$	16	0.03	0.99
UOHNA	13	–	0.98
UBT	13	–	0.98
LCFR	13	–	0.98

Table 7.1: Results of model reduction (without optimisation)

The results of model reduction without optimisation are disappointing, with the unweighted approaches producing (P,γ)-admissible controllers of lower order than the frequency weighted approaches. However, this is in a large part due to the poor performance of the actual frequency weighted reduction algorithm rather than a deficiency of the proposed approaches. By studying the lower bound on the approximation error (Table 7.2) the advantage of the approaches we've presented are self-evident. They indicate the possible existence of a seventh order (P,γ)-admissible controller and give an initial starting point for parameter optimisation to find such a controller. However, we stress that since our conditions are only sufficient this by no means guarantees the existence of a seventh order (P,γ)-admissible controller. In fact, the singular values for the $\mathcal{CF}(\Theta)$ approach are so close together and near $1/\sqrt{2}$ that even a ninth order controller is unlikely to be calculated. Table 7.3 presents the achieved closed loop performance for the seventh order controllers resulting from the $\mathcal{A}(\Phi)$ and $\mathcal{CF}(\Phi)$ approaches, and the tenth order controller resulting from the $\mathcal{CF}(\Theta)$ approach, before undertaking parameter optimisation.

i	$\mathcal{A}(P/\Phi)$	$\mathcal{CF}(P/\Phi)$	$\mathcal{CF}(\Theta)$
1	22.01	10.05	0.87
2	10.01	8.41	0.73
3	7.58	6.94	0.71
4	3.65	3.76	0.71
5	2.50	2.02	0.71
6	1.44	1.81	0.71
7	1.08	1.05	0.71
8	0.44	0.64	0.70
9	0.30	0.50	0.70
10	0.26	0.41	0.69
11	0.19	0.26	0.59
12	0.15	0.13	0.56
13	0.06	0.19	0.48
14	0.06	0.06	0.41
15	0.04	0.01	0.27
16	.0	.0	0.03
17	.0	.0	0.02
18	.0	.0	.0

Table 7.2: $\sigma_i\left(\left[M_1^{\sim-1}K^0M_2^{\sim-1}\right]_+\right)$

Reduction Approach	Controller Order	Lower bound for $\\|(\text{weighted error})\\|_\infty$	Actual $\\|(\text{weighted error})\\|_\infty$	$\\|\mathcal{F}_l(P,K)\\|_\infty$
	20 (full)	–	–	0.98
$\mathcal{A}(\Phi)$	7	0.44	11.27	1.27
$\mathcal{CF}(\Phi)$	7	0.64	956	7.07
$\mathcal{CF}(\Theta)$	10	0.59	12.44	2.02

Table 7.3: Lower Bound for k^{th} Order Approximation

Optimisation[3] worked better for the $\mathcal{A}(\Phi)$ approach than for the $\mathcal{CF}(\Phi)$ or $\mathcal{CF}(\Theta)$ approaches. A seventh order controller satisfying the sufficient condition of approach $\mathcal{A}(\Phi)$ was found (see Table 7.4), but could not be calculated for the $\mathcal{CF}(\Phi)$ approach. Further, for the $\mathcal{CF}(\Theta)$ approach, a 10^{th} order controller couldn't be found which satisfied the relevant sufficient condition.

[3]Here we were required to switch between optimising over the B and the D matrices and the C and the D matrices of K. Both of these optimisations are convex.

Reduction Approach	Controller Order	$\|(\text{weighted error})\|_\infty$	$\|\mathcal{F}_l(P,K)\|_\infty$
$\mathcal{A}(\Phi)$	7	0.79	0.98

Table 7.4: Seventh Order (P,γ)-admissible Controllers

Hence the nominal 20 state (P,γ)-admissible controller has successfully been reduced down to a 7 state (P,γ)-admissible controller. Controller complexity has therefore been (dramatically) reduced without an associated loss of performance.

Figure 7.3 gives a plot of μ for the nominal controller K^0 and for the seventh order controller K. As designed, both controllers give $\mu < 1$ and hence perform robustly (for all $\Delta_G \in \mathcal{RH}_\infty$, $\|\Delta_G\|_\infty < 1$) with the unscaled plant. A plot of the closed loop maximum singular value for the each of the controllers is given in Figure 7.4 and of the controller condition numbers in Figure 7.5. No significant degradation has been introduced to any of these measures of performance.

Figures 7.6 and 7.7 show the output sensitivity functions for the nominal plant and a plant perturbed by $\bar{\Delta}_G = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.1 \end{bmatrix}$ respectively. Similarly, Figures 7.8 and 7.9 show the time responses (for the first two seconds) for a step input into the first channel, for the nominal plant and a plant perturbed by $\bar{\Delta}_G$ (above) respectively. These four figures show that the reduced order controller performs as well as the full order controller with nearly one third the number of states.

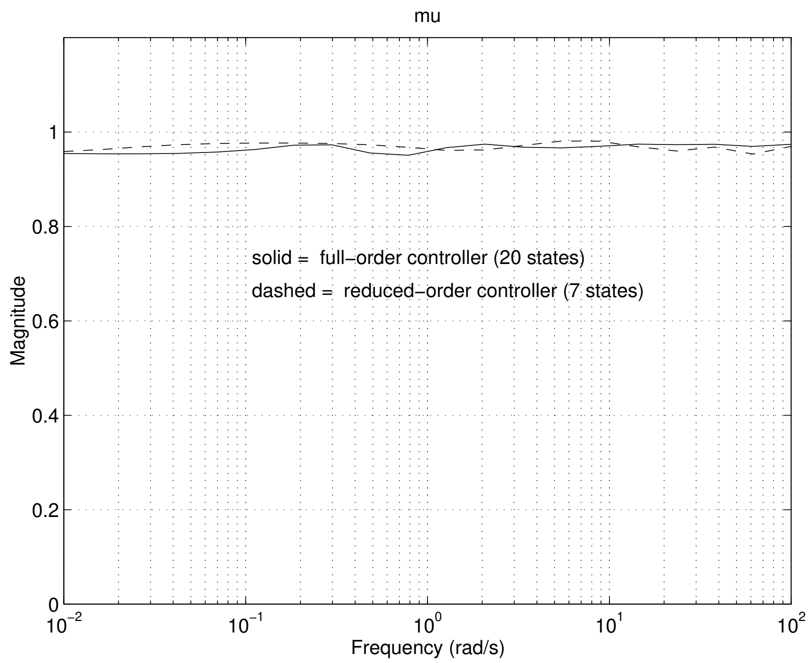


Figure 7.3: Mu

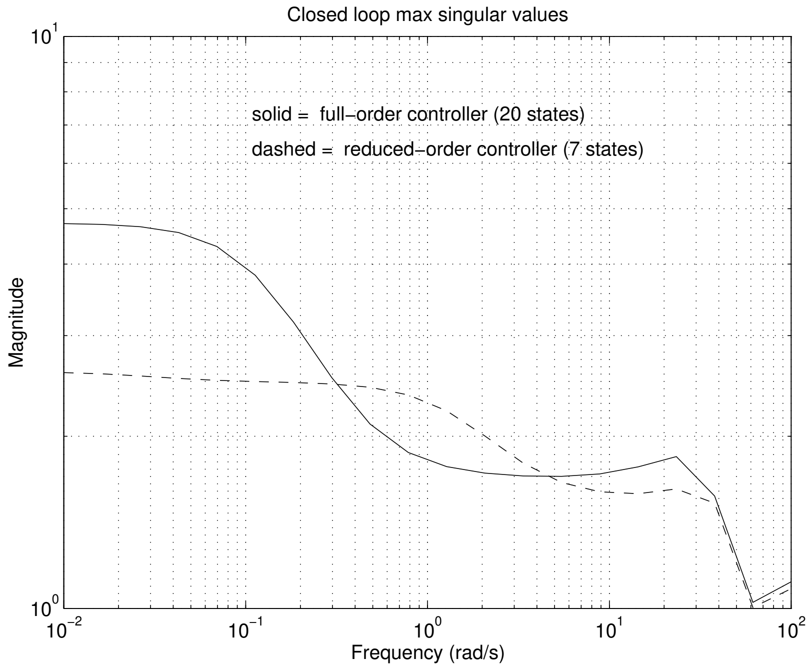


Figure 7.4: Closed Loop Singular Values

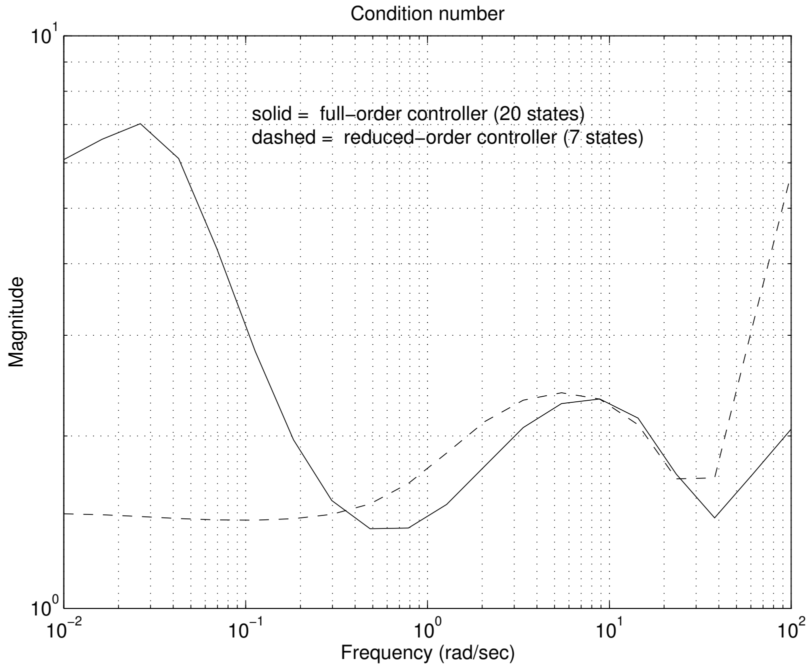


Figure 7.5: Controller Condition Number

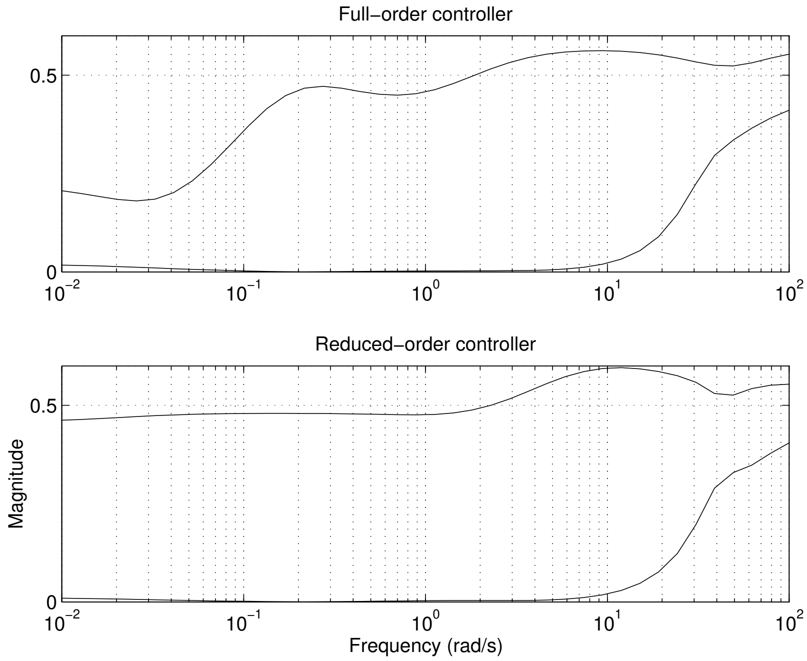


Figure 7.6: Output Sensitivity (Nominal Plant)

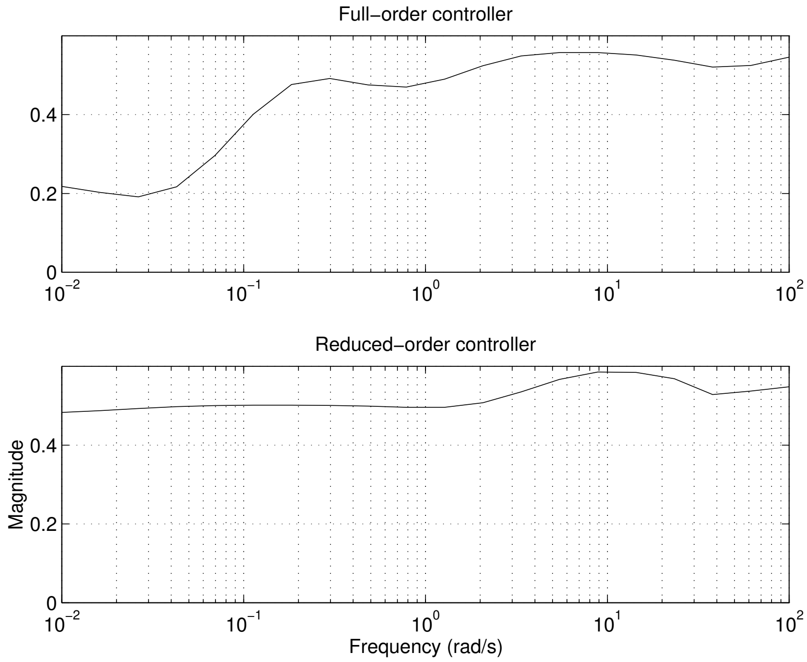


Figure 7.7: Output Sensitivity (Perturbed Plant)

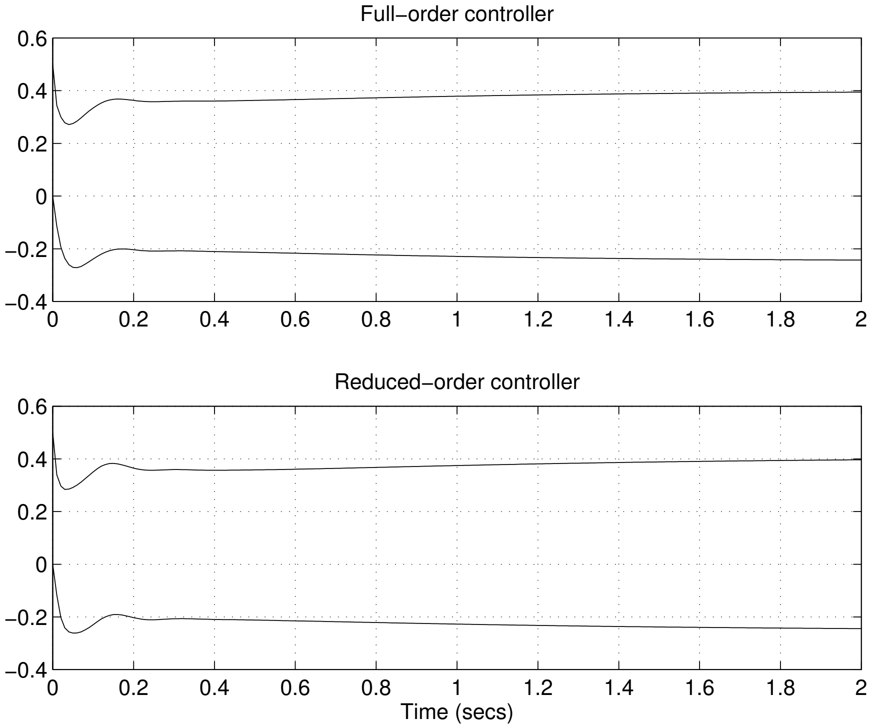


Figure 7.8: Step Input to Nominal Closed Loops

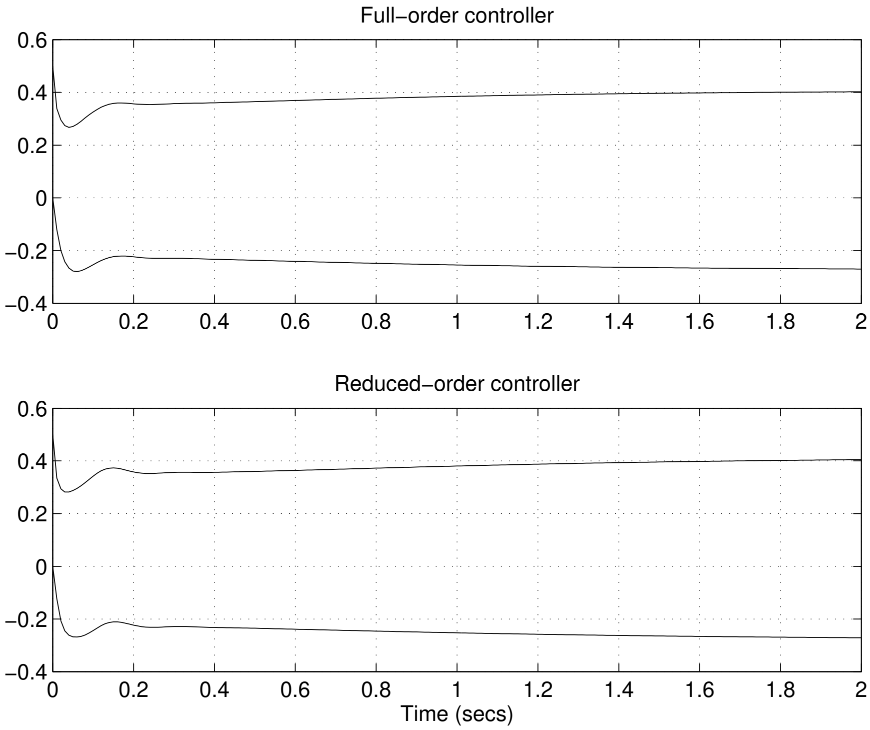


Figure 7.9: Step Input to Perturbed Closed Loops

7.4 Tracking Control for a Compact Disk Mechanism

In this example we consider the synthesis of $\mathcal{H}_\infty$ controllers for the track following mechanism of a compact disk player. This problem has been studied extensively by Wortelboer and his co-workers at Philips Research in Eindhoven (see [94, 87] and the references therein.) Here we compare controllers derived using the $\mathcal{A}(\Phi)$ approach with one derived by Wortelboer [94].

The principle control problems of a compact disk mechanism are those of tracking and focusing. The tracking function is performed by swinging a radial arm in a horizontal plane below the disk to keep the laser spot pointing at the track as the CD is rotating. Focusing is performed via a lens which is suspended from the radial arm and focuses the laser spot on the track. The goal is to produce good tracking and focusing in the presence of disturbances (particularly for CD players designed for outdoor applications) and model uncertainty. For mass production, it is particularly important for a controller to be robust to component tolerances.

Here we concentrate on the tracking problem and assume that focusing is guaranteed. This enables us to compare the controllers derived from the approaches proposed in this dissertation with controllers derived by Wortelboer [94] using his so called *frequency interval balanced reduction*. The underlying time domain specifications for the problem to be studied may be found in [87]. In the frequency domain these effectively translate to finding a controller to minimise

$$\left\| \begin{bmatrix} W_3KS & W_3KSG \\ W_4S & W_4SG \end{bmatrix} \right\|_\infty$$

where G is a model of the CD mechanism, W_3 and W_4 are appropriately chosen performance weights, and $S = (I - GK)^{-1}$ is the sensitivity function. We thank Pepijn Wortelboer [95] for supplying the data (and many of his MATLAB[4] scriptfiles), which may also be found in [94].

Initially a 120 state model G_{120}, which captures the vibration modes with natural frequencies less than 10kHz is available. Before the controller synthesis step, this is reduced to a 32 state model G_{32}. The weighting functions have a combined state-order of 5. Hence the generalised plant, P, required for $\mathcal{H}_\infty$ synthesis has 37 states.

We choose K^0 to be a 37 state near optimal controller giving $\|\mathcal{F}_l(P, K^0)\|_\infty \leq 5 = \gamma$. In [94], Wortelboer uses *frequency interval balanced reduction* to perform a closed loop approximation of K^0. This involves an iteration phase to obtain model reduction weights which emphasise the frequency regions of greatest interest, followed by a closed loop balanced truncation scheme. Closed loop balanced truncation of a controller involves writing the states of the closed loop as $x_{cl} = \begin{bmatrix} x_P' & x_K' \end{bmatrix}'$ where x_P are the states of the generalised plant and x_K are the states of the controller, then calculating a transformation which balances the lower right hand corner of the closed loop controllability Gramian (say $\mathcal{P}_{22}$) with the lower right hand corner of the closed loop observability Gramian (say $\mathcal{Q}_{22}$). Performing this transformation on K, then truncating the states which correspond to small eigenvalues of $\mathcal{P}_{22}\mathcal{Q}_{22}$, results in a reduced order controller. After a number of iterations to obtain

[4]MATLAB is a trademark of MATHWORKS Inc.

appropriate model reduction weights, this approach resulted is a four state controller K_{w4}. A posteriori it is checked that $\|\mathcal{F}_l(P, K_{w4})\|_\infty \leq 5$. Hence K_{w4} is (P,γ)-admissible. We will now design a (P,γ)-admissible controller using the $\mathcal{A}(\Phi)$ approach and compare its performance to K_{w4}.

The singular values of $\sigma_i\left(\left[M_1^{\sim -1} K^0 M_2^{\sim -1}\right]_+\right)$ where $M_1^\sim M_1 = \hat{W}_2 \hat{W}_2^\sim$ and $M_2 M_2^\sim = \hat{W}_1^\sim \hat{W}_1$ are presented in Table 7.5. Using frequency weighted optimal Hankel norm model reduction the lowest order controller which satisfied $\|\hat{W}_2^{-1}(K - K^0)\hat{W}_1^{-1}\|_\infty < 1$ was of fourth order. Call this controller K_{pp4}. However, Table 7.5 suggests the possible existence of a second or third order (P,γ)-admissible controller. Unfortunately, even using parameter optimisation no second or third order (P,γ)-admissible controllers could be calculated. Nevertheless, a fourth order controller has been found without the need to iteratively calculate appropriate model reduction weights. Further, since $\|\hat{W}_2^{-1}(K_{pp4} - K^0)\hat{W}_1^{-1}\|_\infty < 1$, we know a priori that K_{pp4} is (P,γ)-admissible. It is interesting to note that $\|\hat{W}_2^{-1}(K_{w4} - K^0)\hat{W}_1^{-1}\|_\infty = 101.8$. Hence, although K_{w4} is (P,γ)-admissible it would not have been found by our proposed approach.

i	$\mathcal{A}(\Phi)$
1	2189.86
2	1.45
3	0.96
4	0.66
5	0.16

Table 7.5: $\sigma_i\left(\left[M_1^{\sim -1} K^0 M_2^{\sim -1}\right]_+\right)$

Figures 7.10 to 7.13 compare various other performance criteria for the controllers K_{pp4} and K_{w4}. The maximum singular value of each controller is plotted in Figure 7.10, and the maximum singular value of each closed loop is plotted in Figure 7.11. The *sensitivity* and *complementary sensitivity* functions with K_{pp4} and K_{w4} are plotted in Figures 7.12 and 7.13 respectively. All four plots show that there is little to choose between K_{pp4} and K_{w4}. K_{pp4} makes the closed loop maximum singular value plot (marginally) *flatter* than K_{w4}, while K_{w4} gives a (marginally) closer match of the complementary sensitivity around the frequency $\omega \approx 2800$rad/s. For the particular problem discussed here, the main advantage of K_{pp4} over K_{w4} is the elimination of the iteration phase when calculating appropriate model reduction weights.

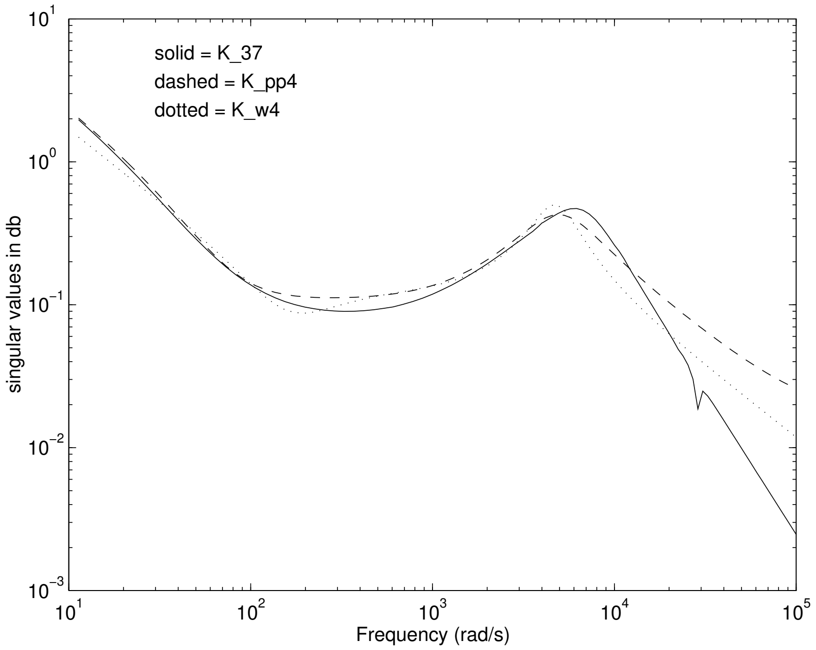


Figure 7.10: Maximum Singular Value of the Controllers

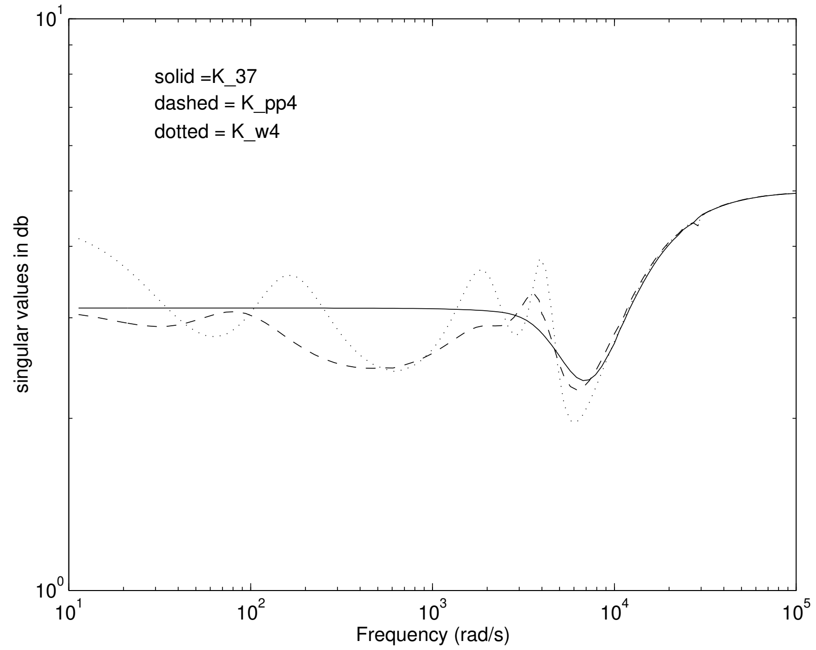


Figure 7.11: Maximum Singular Value of the Closed Loops

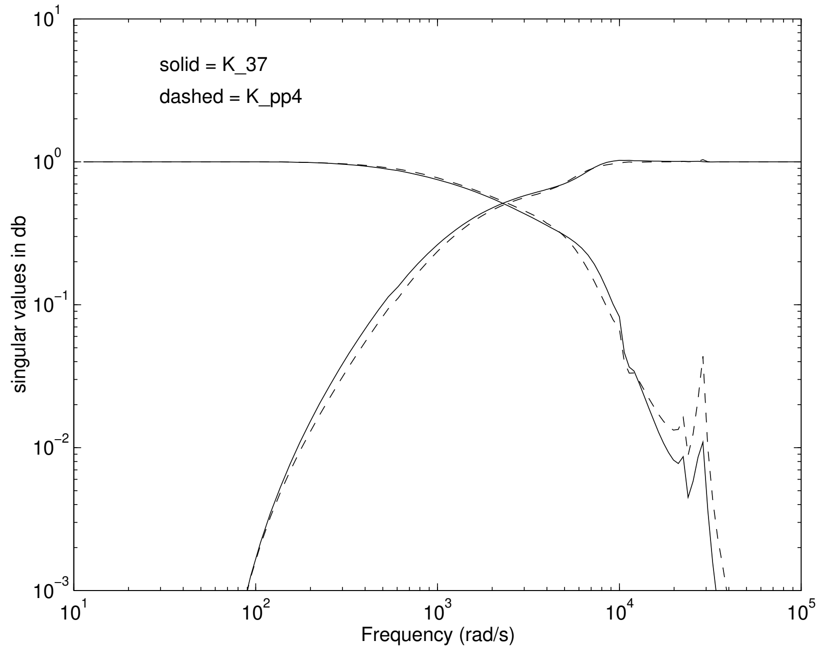


Figure 7.12: Sensitivity and Complementary Sensitivity

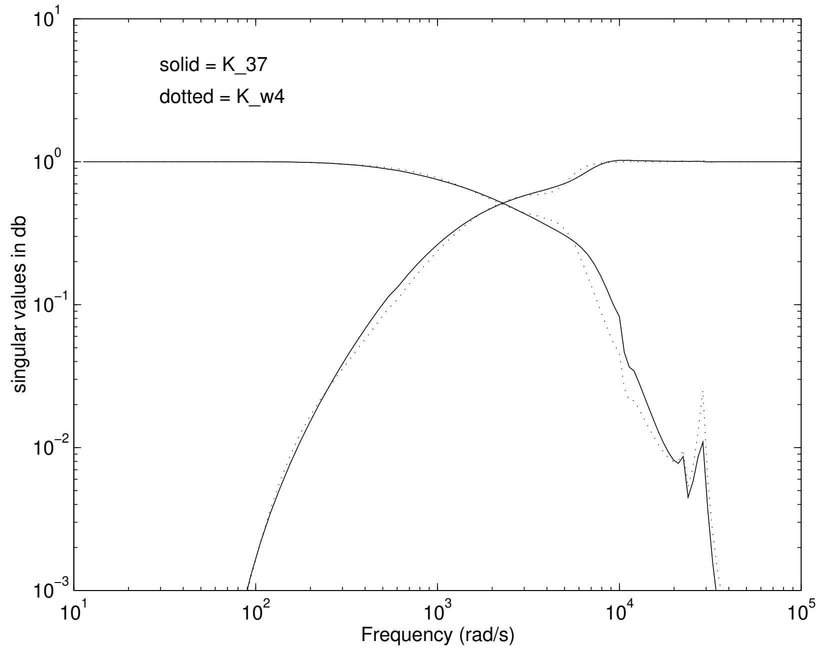


Figure 7.13: Sensitivity and Complementary Sensitivity

7.5 Pitch Axis Control of a Missile

In this example we examine reduced order controller synthesis for pitch axis control of a missile. The dynamics of the missile under consideration vary greatly as a function of speed (v), altitude (H) and angle-of-attack (α), hence a single LTI model and controller is insufficient for effective control. Here we'll consider an LPV model and controller. The model we will use has previously been studied by Gahinet et al. [27] for the synthesis of LPV controllers of state-dimension equal to the weighted model. The parameter dependent model is given by,

$$\mathcal{G}_\rho = \left[\begin{array}{cc|c} -Z_\alpha & 1 & 0 \\ -M_\alpha & 0 & 1 \\ \hline -1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

where Z_α and M_α are aerodynamic co-efficients which depend on v, H and α. The input to $\mathcal{G}_\rho$ is the fin deflection (δ_f). The two states are angle-of-attack (α) and pitch rate (q) respectively. The two outputs are the (normalised) vertical acceleration (a_{zv}) and pitch rate respectively.

We assume that v, H and α may all be measured in real time and vary over the following ranges.

		Lower Limit	Upper Limit
v (Mach#)	–	0.5	4
H (km)	–	0	18
α (degrees)	–	0	40

The combination of v, H and α imply that $Z_\alpha \in [0.5, 4]$ and $M_\alpha \in [0, 106]$. Note that Z_α and M_α enter the state-space matrices in an affine way, and may be considered as representing a convex polytope with four vertices.

The objective we wish to achieve is for the step response of a_{zv} to have a settling time of less than half a second for all variations of Z_α and M_α in the above range. Given the performance weights,

$$W_3 = \frac{2.01}{s + 0.201}$$
$$W_4 = \frac{96.78s^3 + 0.29s^2 + 0.0003s - 0.0039}{s^3 + 1.12 \times 10^5 s^2 + 1.05 \times 10^8 s + 1.07 \times 10^{11}}$$

the performance specification corresponds to designing a controller which achieves,

$$\left\| \begin{array}{c} W_3(I + \mathcal{G}_\rho \mathcal{K}_\rho)^{-1} \\ W_4 \mathcal{K}_\rho (I + \mathcal{G}_\rho \mathcal{K}_\rho)^{-1} \end{array} \right\|_\infty < 1. \tag{7.1}$$

A methodology for synthesis of LPV controllers satisfying equation (7.1) may be found in [7, 10, 27]. The resulting controllers have a state-dimension equal to that of the weighted model. Starting with the controller designed by Gahinet et al [27] (which we will call $\mathcal{K}_\rho^0$)

we now examine the effectiveness of the model approximation techniques of Chapter 6 for synthesis of satisfactory reduced order LPV controllers.

The approach we take is to reduce $\mathcal{K}_\rho^0$ using the open loop approximation techniques of Section 6.3, then a posteriori check the performance of the reduced order controller. Although there are no guarantees that reduced-order controllers obtained in this manner will even $\mathcal{Q}$-stabilise the closed loop, currently this appears to be the most tractable approach. Further, we will see that the approach can work well.

The polytope we are considering has four vertices. Hence, a single P and a single Q which satisfy the relevant Lyapunov inequalities (equations (6.4) and (6.5)) for $\mathcal{K}_\rho^0$ evaluated at each of the four vertices is required. Using Procedure 6.3.5 to iteratively calculate appropriate P and Q led to a $\mathcal{Q}$-balanced realisation with the $\mathcal{Q}$-singular values shown in Table 7.6. Ten iterations were required.

i	σ_i
1	7110.91
2	2703.76
3	2071.81
4	550.00
5	87.06
6	0.04

Table 7.6: $\mathcal{Q}$-singular values for $\mathcal{K}_\rho^0$

The $\mathcal{Q}$-singular values of $\mathcal{K}_\rho^0$ indicate that at least one state may be removed. Using the balanced truncation procedure of Section 6.3.1 we calculated a fifth order controller $\mathcal{K}_\rho^b$, and using the Hankel norm type procedure of Section 6.3.2 we calculated a fifth order controller $\mathcal{K}_\rho^h$. In Figures 7.14 to 7.16 we compare the performance of $\mathcal{K}_\rho^b$ and $\mathcal{K}_\rho^h$ to that of $\mathcal{K}_\rho^0$. (Eliminating more than one state resulted in reduced order controllers which gave unacceptably poor closed loop behaviour.) Note that due to the conservative nature in which the $\mathcal{L}_2$ gain is bounded we were not able to confirm that the error satisfied the known upper bounds ($2\sigma_6$ and σ_6 respectively).

For Figure 7.14 we generated fifteen random LTI plants within the polytope described by Z_α and M_α. The step response for each of these LTI plants was then calculated and plotted. There is no appreciable difference between the responses obtained using any of the three controllers and the settling time is well below the target figure of half a second.

As in [27] we then chose the particular spiral parameter trajectory,

$$\begin{aligned} Z_\alpha &= 2.25 + 1.70e^{-4t}\cos(100t) \\ M_\alpha &= 50 + 49e^{-4t}\sin(100t) \end{aligned}$$

to test the closed loop performance. This trajectory is shown in Figure 7.15. The controller's state-space data were constructed using a convex interpolation of their values at the four vertices. The step response to the trajectory is shown in Figure 7.16. The solid line is the response for both the full order controller and for $\mathcal{K}_\rho^h$. The dashed response is for the controller $\mathcal{K}_\rho^b$. They show that satisfactory reduced order controllers have been obtained from both approaches.

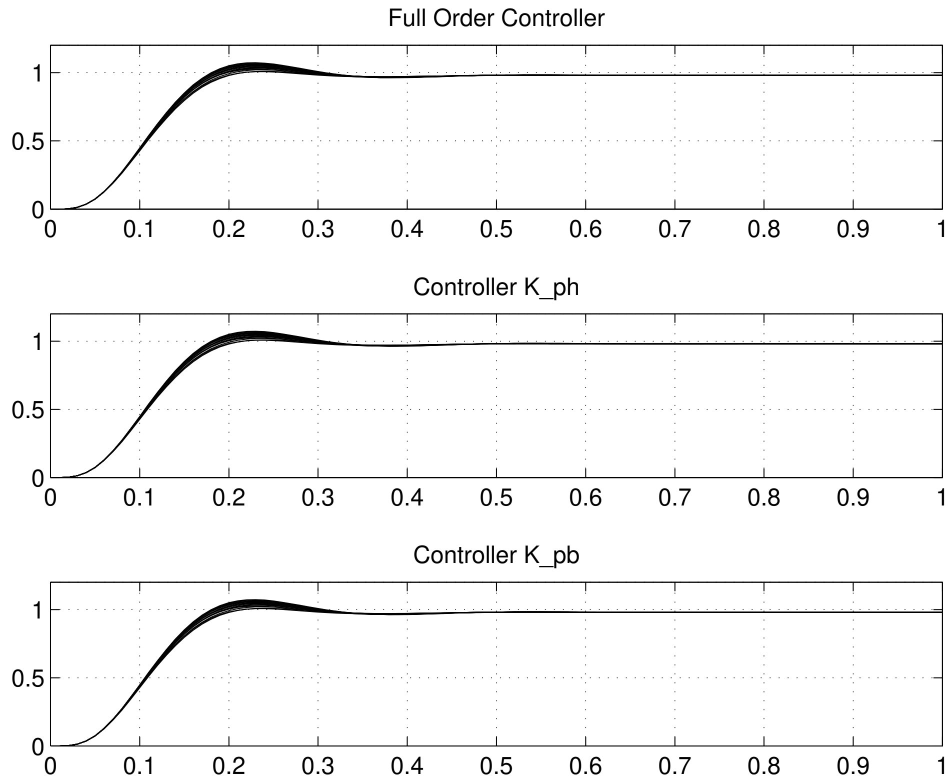


Figure 7.14: Closed Loop Step Responses

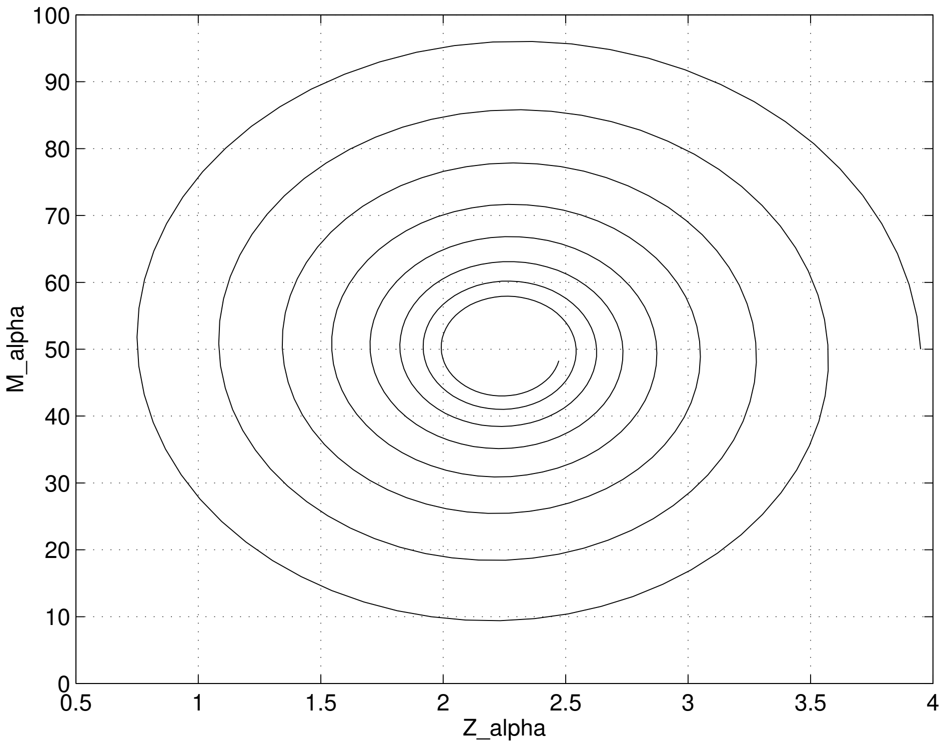


Figure 7.15: Parameter Trajectory: (0-1 second)

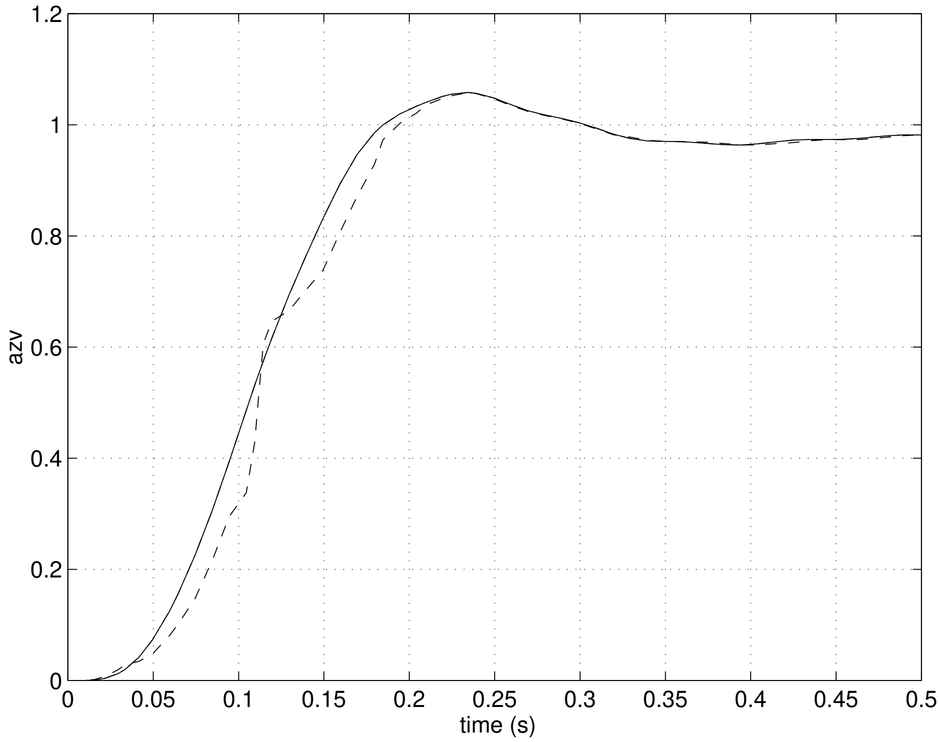


Figure 7.16: Step Response to Spiral Trajectory

7.6 Summary

The results presented in this chapter show the effectiveness of the proposed approaches for the design of reduced order controllers.

In each of the LTI examples the controller complexity has been considerably reduced without incurring any degradation in closed loop $\mathcal{H}_\infty$ performance. Given a high order (P, γ)-admissible controller, low order controllers which are guaranteed to be (P, γ)-admissible have been calculated. Because standard frequency weighted model reduction procedures have generally resulted in approximants which do not come close to achieving the lower approximation error bound we have employed a combined model reduction/parameter optimisation method. An advantage of this combined approach is that an order constraint may be imposed on the controller with some confidence that a controller of that order exists. However, we must stress again that there is no guarantee that this is always the case.

For the LPV example the controller order has been successfully reduced without incurring any unacceptable degradation in closed loop performance. However, for LPV controller reduction the closed loop performance must be checked a posteriori. For a general framework there are no a priori conditions which allow us to bound performance degradation.

Further investigations need to be undertaken to fully evaluate the proposed approaches. Results to date are encouraging, however the approaches are still in their infancy and require evaluation on many more problems to determine their full potential.

Chapter 8

Conclusions

By way of conclusion we now summarise the main contributions of this dissertation and outline future research in the area of controller approximation.

8.1 Contributions of this Dissertation

- Given a generalised plant and a nominal (P, γ)-admissible controller, we have derived sufficient conditions for any other controller to be (P, γ)-admissible. These conditions are conveniently expressed in terms of a norm bound on a frequency weighted error where the frequency weights must be chosen to make a specific transfer function a contraction.

- The derived sufficient conditions have a natural interpretation as frequency weighted model reduction problems. Hence, given a nominal (P, γ)-admissible controller and appropriate weights we have shown how standard frequency weighted model reduction procedures may be employed to synthesise reduced order (P, γ)-admissible controllers.

- Two systematic procedures for calculating appropriate frequency weights have been presented. Since the size of the weights correspond to the size of an $\mathcal{L}_\infty$ ball around the nominal (P, γ)-admissible controller (or a coprime factor representation of the nominal (P, γ)-admissible controller) in which other (P, γ)-admissible controllers (or a coprime factor representation of other (P, γ)-admissible controllers) may lie, it is beneficial for the weights to be as large as possible in some sense. We have characterised solutions which maximise either the product of the traces or the product of the determinants of the weights. Further, we have shown that for the coprime factor framework, maximising the determinant of the weights is not a sensible choice.

- We have shown that given an Hermitian function $Z^2 = (Z^2)^\sim \geq 0$, factorisations of the form $Z = Z^\sim \geq 0$ may be calculated to an arbitrary accuracy using a rational approximation technique.

- We have shown that the procedure to maximise the determinant of the weights always leads to a *direction* where the weights are as large as possible. For a given nominal controller no weights can be selected which give a bigger $\mathcal{L}_\infty$ ball in this direction. When the weights are scalar, we have shown that the procedure to maximise the trace of the weights always leads to a *direction* where the weights are as large as possible. Hence, for scalar weights, the procedure to maximise the trace of the weights and the procedure to maximise the determinant of the weights are equivalent. Further, the calculated weights are the best possible.

- We have shown that for scalar controllers the derived sufficient conditions may be interpreted graphically. This led to a graphical procedure for the design of scalar (P, γ)-admissible controllers.

- We have extended the LTI approximation techniques of balanced truncation and optimal Hankel norm approximation to an LPV framework. However, an interpretation of the Hankel norm approach remaining optimal in any sense is not clear. These results complement many recent results on the synthesis of controllers for linear, parameter dependent plants. A difficulty with approximation of LPV systems is that $\mathcal{Q}$-balanced realisations are not unique. Extreme care must be taken when selecting an initial $\mathcal{Q}$-balanced realisation to guarantee a small approximation error.

- Two design examples which illustrate the effectiveness of the proposed LTI performance-preserving controller approximation techniques, have been presented. The examples reinforce the belief that standard frequency weighted model reduction algorithms generally perform poorly – giving approximants well away from their (possible unachievable) lower bound. However it was demonstrated that good results may be obtained using a combined model reduction/convex optimisation approach.

- A design example which illustrates the proposed LPV approximation techniques has been presented. Although there are no guarantees that these approaches will result in reduced order controllers which give good performance it is seen that they can work well.

8.2 Recommendations for Future Work

- K^0 in the $\mathcal{A}(P/\Phi)$ and $\mathcal{CF}(P/\Phi)$ approaches may be any (P, γ)-admissible controller. Starting with different K^0 leads to different weights and hence different $\mathcal{L}_\infty$ balls in which to search for reduced order (P, γ)-admissible controllers. Presently the $\mathcal{CF}(\Theta)$ approach requires a particular parameterisation of the central controller as its starting point. A generalisation of this may be possible.

- In Chapter 5 we conjectured that for scalar controllers, the conditions of the $\mathcal{CF}(P/\Phi)$ approaches are necessary in addition to being sufficient. This requires to be proved or disproved.

- Although the rational approximation used to calculate Z_1 and Z_2 (equations (4.30) and (4.31) or (4.34) and (4.35)) has been seen to work well, the procedure used may lead to Z_i of high order. This makes the resulting spectral factorisations difficult and lengthy to perform. The derived model approximation approaches would be even more useful if an exact parametric solution could be calculated.

- We have seen that standard frequency weighted model reduction algorithms generally perform poorly. Hence we have often relied on parameter optimisation to enhance our results. Is it possible to improve current frequency weighted approximation techniques, or derive new approximation methods, to eliminate this reliance? The removal of this layer of complexity from the proposed design procedures would greatly streamline the derivation of reduced order (P, γ)-admissible controllers.

- The area of approximation for LPV systems is still very much in its infancy. Consequently there are many open problems. The inclusion of a rate limit on the parameter vector is presently an active research topic. An extension to include parameter dependent frequency weights would give the techniques wider application to controller approximation. However, there are many conceptual and computational difficulties to be overcome before LPV approaches receive widespread attention.

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