Approximation of Parameter Dependent Systems

P.J. Goddard[1]*, G.D. Wood[2]† and K. Glover[3]

[1,3](Corresponding Author: K. Glover)
Department of Engineering
University of Cambridge
Trumpington Street
Cambridge, CB2 1PZ
U.K.
Telephone: +44 1223 332752
Fax: +44 1223 332662
Email: pjg@eng.cam.ac.uk, kg@eng.cam.ac.uk

[2]Cambridge Control
Cambridge Business Park
Cowley Road
Cambridge, U.K.
Email: giles@camcontrol.co.uk

Abstract

Well known concepts for the approximation of LTI systems are extended to systems which exhibit a parameter dependency. For quadratically stable systems, generalisations of the LTI balanced truncation and optimal Hankel norm approximation methods are discussed. The methods are shown to preserve quadratic stability, and satisfy a (non-tight) upper error bound. The concept of the graph operator of a parameter dependent system is discussed. We show how, by approximating the graph operator, the state-dimension of models which are not quadratically stable may be reduced. Techniques for solving the necessary linear matrix inequalities are considered, and an example presented.

Keywords: Parameter Dependent Systems, Model Approximation

1 Introduction

The trade-off between model accuracy and model simplicity is a central theme of model-based control theory. For finite-dimensional, linear time invariant (LTI) systems a natural measure of complexity is the state-dimension of a minimal realisation representing the given system. Since optimal control techniques, such as LQG and H_∞ synthesis, generally produce controllers with the same state-dimension as the models from which they are designed it is evident that the use of

*The first author gratefully acknowledges financial support from The University Of Western Australia, The Gledden Foundation and Trinity College Cambridge.

†The second author gratefully acknowledges financial support from the Emanuel Bradlow Foundation and St. John's College Cambridge.

these design techniques on models having a high state-dimension will produce controllers having a high state-dimension.

From an implementation perspective this is far from ideal. *Simple* controllers are easier to implement, require less processing power and have higher integrity. For these reasons the problem of reducing the state-dimension of a mathematical model in such a way as to ensure that its input-output behaviour is not unduly compromised has received much attention in the control literature. See [1, 24] for a comprehensive list of references.

In this paper a number of approximation techniques are examined in the light of recent advances in the control of linear parameter-varying (LPV) systems [3, 4, 6, 9, 10]. A drawback of LPV controller synthesis is that the computational requirements increase rapidly as a function of the state-dimension [3], even more so than for LTI controller synthesis. Hence it is important that the state-dimension be kept as low as possible. This motivates the development of approximation techniques for LPV models.

In the sequel the concept of a balanced LPV model is introduced, and those approximation techniques which rely on system balancing – namely balanced truncation and optimal Hankel norm approximation – are studied. We demonstrate that many, although by no means all, of the concepts from approximation of LTI systems carry over into the LPV framework. For LPV model approximation we see that it is possible to bound the approximation error from above. However, unlike LTI approximation, the existence of a (non-zero) lower bound has yet to be proven. And although the Hankel norm approach considered here for LPV approximation has its foundations in LTI optimal Hankel norm approximation, an interpretation of it being optimal in any sense is no longer clear.

This paper is structured as follows: In Section 2 LPV systems and some of their important properties are defined. Then balanced realisations for LPV systems are introduced in Section 3. In Section 4 we extend the standard LTI technique of balanced trunctation and provide an error bound for the truncation procedure. In Section 5 a Hankel approach is motivated on the basis that it gives the same error bound. Frequency weighted approximation is studied in Section 6. The graph operator of an LPV system is considered in Section 7 where we show how it may be used for approximating LPV systems which are not quadratically stable. Important computational issues are discussed in Section 8 and a numerical example presented in Section 9.

With the exception of some proofs and Sections 6 and 9, this paper is substantially the same as [23] and is based on the results of [14] and [22].

2 Preliminaries

Linear parameter-varying systems are a special class of time-varying systems where the time dependence enters the state equations through at least one exogenous parameter. (For a comprehensive discussion of LPV systems and their properties see [3, 6, 9] and the references therein.) Consider a system which has a state-space realisation given by

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) + B(\rho(t))u(t) \\ y(t) &= C(\rho(t))x(t) + D(\rho(t))u(t) \end{aligned} \tag{1}$$

where $A : \mathbb{R}^s \mapsto \mathbb{R}^{n\times n}, B : \mathbb{R}^s \mapsto \mathbb{R}^{n\times m}, C : \mathbb{R}^s \mapsto \mathbb{R}^{p\times n}$ and $D : \mathbb{R}^s \mapsto \mathbb{R}^{p\times m}$ are continuous functions of the parameter vector $\rho \in \mathbb{R}^s$. Note that there is no assumption that the parameter dependence exhibited by the state-space matrices is linear.

Definition 2.1 *Define the set of feasible parameter trajectories F_ρ to be a subset of all piecewise continuous functions $C^0 : \mathbb{R} \mapsto \mathbb{R}^s$ according to*

$$F_\rho \triangleq \{\rho(t) : \mathbb{R} \mapsto \mathbb{R}^s, \rho_{i_{\min}} \le \rho_i \le \rho_{i_{\max}}, i = 1, 2 \ldots s\}.$$

Continuity of the state-space matrices implies boundedness on compact subsets of $\mathbb{R}^s$ thus ensuring that for each $\rho(t) \in F_\rho$ the state transition matrix, denoted $\Phi_\rho(t,\tau)$, is unique and continuous.

Definition 2.2 *Given a state-space realisation of an LPV system, a continuous matrix $T(\rho)$ such that $T^{-1}(\rho)$ exists for all $\rho \in F_\rho$, and let $T(\rho)$ have continuous partial derivatives with respect to the parameters in the parameter vector. When $T(\rho)$ satisfies these conditions define the parameter-varying state transformation $x(t) = T(\rho(t))\tilde{x}(t)$. This gives*

$$\begin{aligned}
\dot{\tilde{x}}(t) &= (T^{-1}(\rho(t))A(\rho(t))T(\rho(t)) - T^{-1}(\rho(t))\dot{T}(\rho(t)))\tilde{x}(t) + T^{-1}(\rho(t))B(\rho(t))u(t) \\
y(t) &= C(\rho(t))T(\rho(t))\tilde{x}(t) + D(\rho(t))u(t).
\end{aligned}$$

Definition 2.3 *Given a state-space representation of an LPV system the linear operator P_ρ : $L^+_{2,e} \mapsto L^+_{2,e}$, $u(t) \mapsto y(t)$ is defined as*

$$y(t) = \int_0^t C(\rho(t))\Phi_\rho(t,\tau)B(\rho(\tau))u(\tau)d\tau + D(\rho(t))u(t).$$

For this class of system the notions of quadratic stability, stabilisability and detectability may be defined as follows [3, 5, 6].

Definition 2.4 (Quadratic Stability) *The LPV system P_ρ with state-space matrices given by equation (1) is quadratically stable if there exists a real positive-definite matrix $P = P^T > 0$ such that*

$$A^T(\rho)P + PA(\rho) < 0 \qquad \forall\ \rho(t) \in F_\rho.$$

Definition 2.5 (Quadratically Stabilisable) *The LPV system P_ρ with state-space matrices given by equation (1) is quadratically stabilisable if there exists a continuous matrix function $F(\rho) : \mathbb{R}^s \mapsto \mathbb{R}^{m\times n}$, and a constant positive-definite matrix P, such that*

$$(A(\rho) + B(\rho)F(\rho))^T P + P(A(\rho) + B(\rho)F(\rho))) < 0\, \forall \rho(t) \in F_\rho.$$

Definition 2.6 (Quadratically Detectable) *The LPV system P_ρ with state-space matrices given by equation (1) is quadratically detectable if there exists an continuous matrix function $L(\rho) : \mathbb{R}^s \mapsto \mathbb{R}^{n\times p}$, and a constant positive-definite matrix P, such that*

$$(A(\rho) + L(\rho)C(\rho))P + P(A(\rho) + L(\rho)C(\rho))^T < 0\ \ \forall \rho(t) \in F_\rho.$$

The notion of performance for an LPV system is similarly defined.

Proposition 2.7 *The LPV system P_ρ with state-space matrices in equation (1) is quadratically stable and satisfies,*

$$\|P_\rho\|_{i,2} = \sup_{\rho(t)\in F_\rho} \sup_{u(t)\in L_2^+} \frac{\|P_\rho u\|_2}{\|u\|_2} \le \gamma,$$

if there exists a constant positive-definite matrix X such that

$$\begin{aligned}
&XA(\rho) + A^T(\rho)X + C^T(\rho)C(\rho) + (XB(\rho) + C^T(\rho)D(\rho)) \times \\
&\qquad (\gamma^2 I - D^T(\rho)D(\rho))^{-1}(B^T(\rho)X + D^T(\rho)C(\rho)) < 0, \quad \rho(t) \in F_\rho.
\end{aligned}$$

3 Balancing of LPV Systems

In [17], Moore introduced the concept of a balanced LTI system. Such representations are central to a number of popular LTI approximation methodologies as they give an indication of which subsystems contribute little to the input-output response and hence may be eliminated without significant model degradation.

Similarly, the concept of a balanced LPV system is central to the approximation results discussed in this paper. Two results are now presented which provide a geometric interpretation of LPV systems and an intuitive basis for their balancing. The results are a generalisation of those for LTI systems.

Theorem 3.1 ([22]) *Given a continuous state-space realisation of an LPV system P_ρ, an initial state $x(t_0) = x_0$ and a constant matrix $Q = Q^T > 0$ satisfying*

$$A^T(\rho)Q + QA(\rho) + C^T(\rho)C(\rho) < 0 \quad \forall \quad \rho(t) \in F_\rho, \tag{2}$$

then

i. the LPV system P_ρ is quadratically stable;

ii. when $u(t) = 0$ for all $t \geq 0$ the energy in the output signal is bounded, from above, according to

$$\|y\|_2^2 < x_0^T Q x_0.$$

Proof:

i. This follows from the Lyapunov function $V = x^T Q x$ which is positive definite, radially unbounded and decrescent. Since the derivative of this Lyapunov function is uniformly negative definite [1]

$$\frac{dV}{dt} \leq -\epsilon I,$$

the result is immediate.

ii. Given the initial state $x_0 = x(t_0)$, the output is given by

$$y \quad = \quad C(\rho)\Phi_\rho(t, t_0)x_0,$$

so the energy in the output signal is given by

$$\|y\|_2^2 \quad = \quad \int_{t_0}^{\infty} x_0^T \Phi_\rho^T(t, t_0) C^T(\rho) C(\rho) \Phi_\rho(t, t_0) x_0 dt,$$

where $\Phi_\rho(t, t_0)$ is the unique state transition matrix for the specified path $\rho(t) \in F_\rho$. Now consider equation (2): multiplying on the left by $\Phi_\rho^T(t, t_0)$ and on the right by $\Phi_\rho(t, t_0)$ gives

$$\Phi_\rho^T(t, t_0)A^T(\rho)Q\Phi_\rho(t, t_0) + \Phi_\rho^T(t, t_0)QA(\rho)\Phi_\rho(t, t_0) + \Phi_\rho^T(t, t_0)C^T(\rho)C(\rho)\Phi_\rho(t, t_0) \quad < \quad 0.$$

Using general properties of the state transition matrix then gives

$$\frac{d}{dt}\left(\Phi_\rho^T(t, t_0)Q\Phi_\rho(t, t_0)\right) + \Phi_\rho^T(t, t_0)C^T(\rho)C(\rho)\Phi_\rho(t, t_0) < 0.$$

[1] This follows from compactness of the feasible set of parameter trajectories.

The integrating the last expression from t_0 to ∞ gives

$$Q > \int_{t_0}^{\infty} \Phi_\rho^T(t,t_0) C^T(\rho) C(\rho) \Phi_\rho(t,t_0) dt,$$

from which the result follows readily. ■

Thus Q can be used to define a set of observability ellipsoids, the length of the principal axes of which provide an upper bound on the energy in the output signal given the initial state $x(t_0) = x_0$ and the initial parameter $\rho(t_0) = \rho_0$.

Next we consider the largest set of state vectors $x(0) = x_0$ reachable with a unit energy input $u(t) \in L_2^-$.

Theorem 3.2 ([22]) *Given a continuous state-space realisation of an LPV system P_ρ, an initial state $x(t_0) = x_0$ and a real constant matrix $P = P^T > 0$ satisfying*

$$A(\rho)P + PA^T(\rho) + B(\rho)B^T(\rho) < 0 \;\; \forall \;\; \rho(t) \in F_\rho, \tag{3}$$

then

i. the LPV system P_ρ is quadratically stable;

ii. the energy required to drive the system from $x(-\infty) = 0$ to $x(0) = x_0$ with the input $u(t) \in L_2^-$ is bounded, from below, according to

$$\|u\|_2^2 > x_0^T P^{-1} x_0.$$

Proof:

i. This follows by left and right multiplying equation (3) by P^{-1} and using the Lyapunov function $V = x^T P^{-1} x$. The time derivative of this Lyapunov function is uniformly negative definite and this implies exponential stability.

ii. Using the fact that

$$x_0 = \int_{-\infty}^{0} \Phi_\rho(0,\tau) B(\rho(\tau)) u(\tau) d\tau,$$

the energy in the input signal can be computed from

$$\|u\|_2^2 = \int_{-\infty}^{0} u^T(\tau) u(\tau) d\tau.$$

The minimum energy input $u(t)$ that achieves the desired state $x(0) = x_0$ is given by

$$\begin{aligned} u(t) &= B^T(\rho)\Phi_\rho^T(0,t) W_c^{-1} x_0 \quad \forall\, t \le 0, \\ \text{where} \qquad W_c &= \int_{-\infty}^{0} \Phi_\rho(0,\tau) B(\rho) B^T(\rho) \Phi_\rho^T(0,\tau) d\tau, \end{aligned}$$

from which it follows that

$$\|u\|_2^2 = x_0^T W_c^{-1} x_0.$$

To see that this is the minimum energy input, consider perturbing the input such that

$$\tilde{u}(t) = B^T(\rho)\Phi_\rho^T(0,t)W_c^{-1}x_0 + u_1(t) \qquad \text{and} \qquad \int_{-\infty}^{0} \Phi_\rho(0,t)B(\rho(t))u_1(t)dt = 0,$$

where the integral constraint is required to ensure that the condition $x(0) = x_0$ is met. It is easily shown that the energy in the perturbed input is given by

$$\|\tilde{u}\|_2^2 = \|u\|_2^2 + \|u_1\|_2^2.$$

Multiplying equation (3) on the left by $\Phi_\rho(0,t)$ and on the right by $\Phi_\rho^T(0,t)$ gives

$$\Phi_\rho(0,t)A(\rho)P\Phi_\rho^T(0,t) + \Phi_\rho(0,t)PA^T(\rho)\Phi_\rho^T(0,t) + \Phi_\rho(0,t)B(\rho)B^T(\rho)\Phi_\rho^T(0,t) \quad < \quad 0.$$

Using general properties of the state transition matrix, equation (4) may be written as

$$-\frac{d}{dt}(\Phi_\rho(0,t)P\Phi_\rho^T(0,t)) + \Phi_\rho(0,t)B(\rho)B^T(\rho)\Phi_\rho^T(0,t) < 0,$$

which on integrating over the semi-infinite time axis $(-\infty, 0]$, and noting that $\lim_{t\to-\infty}\Phi_\rho(0,t) = 0$ gives

$$P > W_c. \tag{4}$$

The result follows directly from equation (4). ■

Thus P^{-1} can be used to define a family of parameter-varying controllability ellipsoids, the length of the principal axes of which bound, from below, the energy required to reach an initial state $x(0) = x_0$.

Remark 3.3 *Non-strict inequalities could be used in equations (2) and (3) with $P \geq 0$ and $Q \geq 0$. However, as shown by Beck et al. [2], such a system may be reduced (by truncation of states) to one satisfying equations (2) and (3) without any reduction error. This may be thought of along similar lines to the LTI situation of obtaining a minimal realisation from one which is non-minimal by eliminating any uncontrollable or unobservable states.*

Henceforth we refer to a Q satisfying equation (2) as a parameter-varying observability Gramian and a P satisfying equation (3) as a parameter-varying controllability Gramian[2].

Lemma 3.4 *Given a continuous state-space realisation of an LPV system P_ρ, an observability Gramian Q, a controllability Gramian P and a constant state transformation matrix T then*

$$\begin{aligned} \tilde{Q} &= T^TQT \\ \tilde{P} &= T^{-1}PT^{-T} \end{aligned}$$

are observability and controllability Gramians for the transformed system respectively.

Definition 3.5 *Given an n-state quadratically stable LPV system P_ρ, with a parameter varying observability Gramian Q satisfying equation (2) and a parameter varying controllability Gramian P satisfying equation (3), then*

$$\sigma_i = \sqrt{\lambda_i(QP)} \quad \sigma_1 \geq \sigma_2 \geq \ldots \sigma_n > 0.$$

are defined to be $\mathcal{Q}_e$ singular values of P_ρ.

[2]Observe that we have taken care to emphasise the inherent lack of uniqueness in the solutions of equations (2) and (3). If Q satisfies equation (2) then so does αQ for any constant $\alpha \geq 1$. A similar conclusion holds for P.

For LTI systems, if the inequalities of equations (2) and (3) are replaced by equalities then the $\mathcal{Q}_e$ singular values are the unique input-output invariants referred to as Hankel singular values. Although not unique, the $\mathcal{Q}_e$ singular values do provide useful insight into the input-output mapping of a given LPV system; in particular, as shown by Theorems 3.1 and 3.2, they bound the mapping from past inputs to future outputs.

Proposition 3.6 *$\mathcal{Q}_e$ singular values bound, from above, the following norm of the operator P_ρ*

$$\|P_\rho\|_H = \sup_{\rho \in F_\rho} \sup_{u \in L_2^-} \frac{\|\Pi^+ P_\rho u\|_2}{\|u\|_2} \leq \sigma_1.$$

We can now define precisely what we mean by a balanced parameter-varying realisation.

Proposition 3.7 *Given a continuous state-space realisation of an LPV system P_ρ, an observability Gramian Q and a controllability Gramian P, then it is possible to find a constant state transformation matrix T such that the transformed Gramians $\tilde{P} = \tilde{Q} = \Sigma$. Σ is a diagonal matrix which has the $\mathcal{Q}_e$ singular values arranged along its diagonal in descending order $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \ldots \sigma_n > 0$.*

Remark 3.8 *Explicit rate bounds can be included in the model reduction formulation by making both P and Q parameter-dependent. In this case the existence of an analytic balancing transformation matrix[3] is less obvious. Details can be found in [22].*

Definition 3.9 *Given a continuous realisation of an LPV system and a balancing state transformation matrix T such that $\tilde{P} = \tilde{Q} = \Sigma$, define the balanced parameter-varying realisation as follows*

$$P_\rho \stackrel{s}{=} \left[\begin{array}{c|c} T^{-1}A(\rho)T & T^{-1}B(\rho) \\ \hline C(\rho)T & D(\rho) \end{array}\right].$$

4 LPV Balanced Truncation

The balanced realisation introduced in the preceding section leads immediately to a balanced truncation procedure for LPV systems.

Lemma 4.1 ([14]) *Assume P_ρ is an n-state, quadratically stable, balanced LPV system partitioned as follows*

$$P_\rho \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11}(\rho) & A_{12}(\rho) & B_1(\rho) \\ A_{21}(\rho) & A_{22}(\rho) & B_2(\rho) \\ \hline C_1(\rho) & C_2(\rho) & D(\rho) \end{array}\right],$$

where $A_{11} \in \mathbb{R}^{r\times r}, A_{12} \in \mathbb{R}^{r\times(n-r)}, A_{21} \in \mathbb{R}^{(n-r)\times r}, A_{22} \in \mathbb{R}^{(n-r)\times(n-r)}, B_1 \in \mathbb{R}^{r\times m}, B_2 \in \mathbb{R}^{(n-r)\times m}, C_1 \in \mathbb{R}^{p\times r}$ and $C_2 \in \mathbb{R}^{p\times(n-r)}$. Then

$$\hat{P}_\rho \stackrel{s}{=} \left[\begin{array}{c|c} A_{11}(\rho) & B_1(\rho) \\ \hline C_1(\rho) & D(\rho) \end{array}\right]$$

is an r-state, quadratically stable, balanced approximation to P_ρ and

$$\|P_\rho - \hat{P}_\rho\|_{i,2} \quad \leq \quad 2\sum_{j=r+1}^{n} \sigma_j.$$

[3] When rate bounds are included the transformation matrix must be differentiable.

An equivalent result for balanced truncation of discrete $\mathcal{Q}$-balanced systems is presented in [21, 2]. The proof follows similar lines to that presented in [25] for the LTI case.

Proof: Consider the case where only those states corresponding to the smallest $\mathcal{Q}_e$-singular value are truncated. Since P_ρ is balanced we have

$$\begin{aligned} A(\rho)\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \sigma I \end{bmatrix} + \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \sigma I \end{bmatrix} A^T(\rho) + B(\rho)B^T(\rho) &< 0 \\ A(\rho)^T\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \sigma I \end{bmatrix} + \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \sigma I \end{bmatrix} A(\rho) + C^T(\rho)C(\rho) &< 0. \end{aligned}$$

Conformably partition P_ρ with $\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \sigma I \end{bmatrix}$ such that

$$P_\rho \overset{s}{=} \left[\begin{array}{cc|c} A_{11}(\rho) & A_{12}(\rho) & B_1(\rho) \\ A_{21}(\rho) & A_{22}(\rho) & B_2(\rho) \\ \hline C_1(\rho) & C_2(\rho) & D(\rho) \end{array}\right]$$

and define

$$\hat{P}_\rho \triangleq \left[\begin{array}{c|c} A_{11}(\rho) & B_1(\rho) \\ \hline C_1(\rho) & D(\rho) \end{array}\right].$$

Then it is trivial to show that Σ_1 satisfies both the controllability and observability Lyapunov inequalities for $\hat{P}_\rho$ and hence $\hat{P}_\rho$ is quadratically stable.
To prove the error bound, define

$$\begin{aligned} E_\rho &\triangleq P_\rho - \hat{P}_\rho \\ &\overset{s}{=} \left[\begin{array}{ccc|c} A_{11}(\rho) & 0 & 0 & B_1(\rho) \\ 0 & A_{11}(\rho) & A_{12}(\rho) & B_1(\rho) \\ 0 & A_{21}(\rho) & A_{22}(\rho) & B_2(\rho) \\ \hline -C_1(\rho) & C_1(\rho) & C_2(\rho) & 0 \end{array}\right]. \end{aligned}$$

Then use the matrices

$$T^{-1} = \begin{bmatrix} \frac{1}{2}I & \frac{1}{2}I & 0 \\ \frac{1}{2}I & -\frac{1}{2}I & 0 \\ 0 & 0 & I \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} I & I & 0 \\ I & -I & 0 \\ 0 & 0 & I \end{bmatrix}$$

to perform a state transformation on E_ρ to obtain

$$\begin{aligned} E_\rho &\overset{s}{=} \left[\begin{array}{ccc|c} A_{11}(\rho) & 0 & \frac{1}{2}A_{12}(\rho) & B_1(\rho) \\ 0 & A_{11}(\rho) & -\frac{1}{2}A_{12}(\rho) & 0 \\ A_{21}(\rho) & -A_{21}(\rho) & A_{22}(\rho) & B_2(\rho) \\ \hline 0 & -2C_1(\rho) & C_1(\rho) & 0 \end{array}\right] \\ &\triangleq \left[\begin{array}{c|c} A_e(\rho) & B_e(\rho) \\ \hline C_e(\rho) & D_e(\rho) \end{array}\right]. \end{aligned}$$

We need to show that $\|E_\rho\|_{i,2} \leq 2\sigma$, or equivalently, by Lemma 2.7, that there exists a matrix $P_e = P_e^T > 0$ such that

$$A_e(\rho)P_e + P_e A_e^T(\rho) + B_e(\rho)B_e^T(\rho) + \frac{1}{4\sigma^2}P_e C_e^T(\rho)C_e(\rho)P_e < 0.$$

We'll now show that a suitable choice is

$$P_e = \begin{bmatrix} \Sigma & 0 & 0 \\ 0 & \sigma^2\Sigma^{-1} & 0 \\ 0 & 0 & 2\sigma I \end{bmatrix}.$$

Firstly, note that with this choice of P_e

$$\frac{1}{2\sigma}P_e C_e^T(\rho) \quad = \quad \begin{bmatrix} 0 \\ -\sigma\Sigma^{-1}C_1^T(\rho) \\ \sigma C_2^T(\rho) \end{bmatrix}.$$

Now, using matrix manipulation, it is simple to show that

$$\begin{aligned} & A_e(\rho)P_e + P_e A_e(\rho) + B_e(\rho)B_e^T(\rho) + \frac{1}{4\sigma^2}P_e C_e^T(\rho)C_e(\rho)P_e = \\ & \qquad \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \{A(\rho)P + PA^T(\rho) + B(\rho)B^T(\rho)\} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \\ & \qquad\qquad + \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix} \{A^T(\rho)Q + QA(\rho) + C^T(\rho)C(\rho)\} \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \end{aligned}$$

which by assumption is negative definite. Therefore the chosen P_e satisfies the requirements and we have

$$\|P_\rho - \hat{P}_\rho\|_{i,2} \leq 2\sigma.$$

The general error bound for truncation of more than just the states corresponding to the smallest $\mathcal{Q}_e$-singular value results from recursive use of the above procedure. ■

Corollary 4.2 *The balanced model approximant $\hat{P}_\rho$ of a quadratically stable LPV system P_ρ is itself balanced.*

The above procedure is amenable to the frequency weighting procedure proposed by Enns [7] for LTI systems. The weights can be used to filter the input and output spaces to emphasis specific frequency ranges of interest, see [22]. This is further discussed in Section 6.

5 Hankel Norm Approach

Motivated by the standard LTI optimal Hankel norm approximation procedure [12], we propose a similar reduction scheme for LPV systems. However, we stress that unlike its LTI counterpart, the interpretation of this scheme as being optimal in any sense isn't clear. We retain this name because we use similar manipulations to the LTI counterpart and the result reduces to the LTI result when the plant under consideration is purely LTI.

Lemma 5.1 ([14]) *Assume P_ρ, of state-dimension n, is quadratically stable and balanced such that*

$$\begin{aligned} A(\rho)P + PA^T(\rho) + B(\rho)B^T(\rho) &< 0 \\ A^T(\rho)Q + QA(\rho) + C^T(\rho)C(\rho) &< 0 \end{aligned}$$

where

$$P = Q = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \sigma I \end{bmatrix} \quad > \quad 0$$

with $\Sigma_1 \in \mathbb{R}^{k\times k}$ *diagonal* ($k < n$) *and* $\sigma \neq 0$. *Now augment* $B(\rho)$, $C(\rho)$ *and* $D(\rho)$ *such that*

$$\begin{aligned} B_a(\rho) &= \begin{bmatrix} B(\rho) & \tilde{B}_a(\rho) \end{bmatrix} \\ C_a(\rho) &= \begin{bmatrix} C(\rho) \\ \tilde{C}_a(\rho) \end{bmatrix} \end{aligned}$$

satisfy

$$\begin{aligned} A(\rho)P + PA^T(\rho) + B_a(\rho)B_a^T(\rho) &= 0 \\ A^T(\rho)Q + QA(\rho) + C_a^T(\rho)C_a(\rho) &= 0 \end{aligned}$$

and

$$D_a(\rho) = \begin{bmatrix} D(\rho) & 0 \\ 0 & 0 \end{bmatrix}.$$

Conformably partition $A(\rho)$, $B_a(\rho)$, $C_a(\rho)$ *with* P, *as*

$$\begin{aligned} A_\rho &= \begin{bmatrix} A_{11}(\rho) & A_{12}(\rho) \\ A_{21}(\rho) & A_{22}(\rho) \end{bmatrix} \quad B_a(\rho) = \begin{bmatrix} B_{a1}(\rho) \\ B_{a2}(\rho) \end{bmatrix} \\ C_a(\rho) &= \begin{bmatrix} C_{a1}(\rho) & C_{a2}(\rho) \end{bmatrix} \end{aligned}$$

and define

$$\begin{aligned} \hat{A}(\rho) &\triangleq \Gamma^{-1}(\sigma^2 A_{11}^T(\rho) + \Sigma_1 A_{11}(\rho)\Sigma_1 - \sigma C_{a1}^T(\rho)U(\rho)B_{a1}^T(\rho)) \\ \hat{B}_a(\rho) &\triangleq \Gamma^{-1}(\Sigma_1 B_{a1}(\rho) + \sigma C_{a1}^T(\rho)U(\rho)) \\ \hat{C}_a(\rho) &\triangleq C_{a1}(\rho)\Sigma_1 + \sigma U(\rho)B_{a1}^T(\rho) \\ \hat{D}_a(\rho) &\triangleq D_a(\rho) - \sigma U(\rho) \end{aligned}$$

where $U(\rho)$ *is a unitary matrix satisfying*

$$B_{a2}(\rho) = -C_{a2}^T(\rho)U(\rho)$$

and Γ *is defined as*

$$\Gamma \triangleq \Sigma_1^2 - \sigma^2 I.$$

Then $\hat{P}_\rho$ *with realisation* $(\hat{A}(\rho), \hat{B}(\rho), \hat{C}(\rho), \hat{D}(\rho))$ *has state-dimension* k, *is quadratically stable and satisfies*

$$\|P_\rho - \hat{P}_\rho\|_{i,2} \leq \sigma$$

where $\hat{B}(\rho)$, $\hat{C}(\rho)$ *and* $\hat{D}(\rho)$ *are obtained from conformably partitioning,* $\hat{B}_a(\rho)$, $\hat{C}_a(\rho)$ *and* $\hat{D}_a(\rho)$ *with* $B_a(\rho)$ *and* $C_a(\rho)$ *as*

$$\hat{B}_a(\rho) = \begin{bmatrix} \hat{B}(\rho) & \hat{B}_{a2}(\rho) \end{bmatrix} \quad \hat{C}_a(\rho) = \begin{bmatrix} \hat{C}(\rho) \\ \hat{C}_{a2}(\rho) \end{bmatrix}$$

$$\hat{D}_a(\rho) = \begin{bmatrix} \hat{D}(\rho) & \hat{D}_{12}(\rho) \\ \hat{D}_{21}(\rho) & \hat{D}_{22}(\rho) \end{bmatrix}.$$

Proof: Theorem 6.3 of [12] gives that the error between the two systems with state matrices $(A(\rho), B(\rho), C(\rho), D_a(\rho))$ and $(\hat{A}(\rho), \hat{B}_a(\rho), \hat{C}_a(\rho), \hat{D}_a(\rho))$ as σ. The result follows by noticing that $P_\rho - \hat{P}_\rho$ is a submatrix of the above error. ■

Remark 5.2 *In general, the matrix $U(\rho)$ will be a complex function of ρ. Hence the approximation will potentially be a complex function of ρ and difficult to write explicitly. Indeed, the complexity introduced by this formulation may exceed any gains obtained by reducing the order. However, if the matrices are defined on a convex polytope then approximation may be performed at each of the vertices (using the same P and Q at each vertex) and the general solution obtained using a convex interpolation of the solutions at each vertex. Note that the $\hat{P}_\rho$ obtained in this way will be different from the $\hat{P}_\rho$ resulting from Lemma 5.1.*

6 Frequency-Weighted Balanced Truncation

Balanced truncation of LTI systems produces zero error at infinite frequency because it does not influence the system's D matrix. However, from a control perspective it is generally better to obtain small approximation errors at low and intermediate frequencies.

In the case of LTI systems it is well known that there is an upper bound on the truncation error which is tight and actually achieved at DC when only one state is truncated [7]. In the general case, where more than one state is truncated, it has been observed that the maximum approximation error resulting from balanced truncation occurs at low frequency. This has led to the development of several frequency weighted approximation schemes which can be used to improve the appoximant for control purposes.

Of course parameter-varying systems do not have a frequency response, however we may draw on intuition gained from LTI systems in order to improve LPV approximation for the purpose of control design. In this regard the frequency-weighted technique introduced by Enns [7] proves particularly useful since it may be regarded as weighting (or filtering) the signal space to emphasise particular frequency ranges. In this section we examine Enns' frequency-weighted approximation technique, showing how it extends very naturally to the LPV case.

Consider introducing the weights W_i and W_o at the plant input and output respectively. Assume the following state-space realisations are given (there is no loss of generality by assuming P_ρ to be strictly proper)

$$P_\rho \stackrel{s}{=} \left[\begin{array}{c|c} A(\rho) & B(\rho) \\ \hline C(\rho) & 0 \end{array}\right] \qquad W_i \stackrel{s}{=} \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right] \qquad \text{and} \qquad W_o \stackrel{s}{=} \left[\begin{array}{c|c} A_o & B_o \\ \hline C_o & D_o \end{array}\right],$$

where P_ρ is assumed to have n states, then a state-space realisation of the interconnection is given by

$$P_w \stackrel{s}{=} \left[\begin{array}{c|c} \bar{A}(\rho) & \bar{B}(\rho) \\ \hline \bar{C}(\rho) & 0 \end{array}\right] \tag{5}$$

$$P_w \stackrel{s}{=} \left[\begin{array}{ccc|c} A(\rho) & 0 & B(\rho)C_i & B(\rho)D_i \\ B_oC(\rho) & A_o & 0 & 0 \\ 0 & 0 & A_i & B_i \\ \hline D_oC(\rho) & C_o & 0 & 0 \end{array}\right]. \tag{6}$$

Let $\bar{P}$ and $\bar{Q}$ be controllability and observability Gramians satisfying

$$\begin{aligned} \bar{A}(\rho)\bar{P} + \bar{P}\bar{A}^T(\rho) + \bar{B}(\rho)\bar{B}^T(\rho) &< 0 \quad \forall\ \rho(t) \in F_\rho \\ \bar{A}^T(\rho)\bar{Q} + \bar{Q}\bar{A}(\rho) + \bar{C}^T(\rho)\bar{C}(\rho) &< 0 \quad \forall\ \rho(t) \in F_\rho. \end{aligned}$$

The input-weighted controllability Gramian and the output-weighted observability Gramian are then defined as follows

$$P \triangleq \begin{bmatrix} I_n & 0 & 0 \end{bmatrix} \bar{P} \begin{bmatrix} I_n \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad Q \triangleq \begin{bmatrix} I_n & 0 & 0 \end{bmatrix} \bar{Q} \begin{bmatrix} I_n \\ 0 \\ 0 \end{bmatrix}.$$

Let $x(0) = [x_0 \;\; 0 \;\; 0]^T$ be partitioned in accordance with equation (6) with x_0 the initial state of P_ρ. Then the energy in the filtered output signal with $u(t) = 0$ for all $t \geq 0$ is bounded by

$$\|\tilde{y}\|_2^2 < x_0^T Q x_0.$$

Similarly with initial states given by $x(0) = [x_0 \;\; y_0 \;\; z_0]^T$ the energy required of a filtered input to reach this state is bounded by

$$\|u\|_2^2 > x_0^T P^{-1} X_0$$

for some y_0 and z_0. The intention is therefore to emphasis the frequency range of interest and effective results can be obtained. However since the initial states are not matched in the above (i.e. in the inequality bounding the input energy the states of the weights are not constrained whereas in inequality bounding the output energy the initial states of the weights are required to be zero) these singular values can be unreliable indicators of the frequency weighted error. Applying the balanced truncation technique using the weighted Gramians effectively removes any states that either do not contribute strongly to the input-output map or which only contribute strongly in a frequency range in which we have little interest.

The weighted approximation scheme is implemented by balancing the weighted controllability and weighted observability Gramians P and Q and then performing a standard balanced truncation. Simple algebraic manipulations show that the controllability Gramian P satisfies

$$\begin{bmatrix} A(\rho) & B(\rho)C_i \\ 0 & A_i \end{bmatrix} \begin{bmatrix} P & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} + \begin{bmatrix} P & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \begin{bmatrix} A(\rho) & B_iC_i \\ 0 & A_i \end{bmatrix}^T + \begin{bmatrix} B(\rho)D_i \\ B_i \end{bmatrix} \begin{bmatrix} D_i^T B^T(\rho) & B_i^T \end{bmatrix} < 0,$$

while the observability Gramian Q satisfies

$$\begin{bmatrix} A(\rho) & 0 \\ B_oC(\rho) & A_o \end{bmatrix}^T \begin{bmatrix} Q & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} + \begin{bmatrix} Q & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} A(\rho) & 0 \\ B_oC(\rho) & A_o \end{bmatrix}^T + \begin{bmatrix} C^T(\rho)D_o^T \\ C_o^T \end{bmatrix} \begin{bmatrix} D_oC(\rho) & C_o \end{bmatrix} < 0.$$

In general neither $V = x^T P^{-1} x$ nor $V = x^T Q x$ is a Lyapunov function for the unweighted system P_ρ, and hence cannot be used to guarantee stability of the full-order model, let alone a reduced-order approximant. In the special case where unity input weighting ($W_i = I$) is used the controllability Gramian satisfies

$$A(\rho)P + PA^T(\rho) + B(\rho)B^T(\rho) < 0 \;\; \forall\, \rho(t) \in F_\rho.$$

Thus $V = x^T P^{-1} x$ is a Lyapunov function for the original parameter-varying system and on performing a balancing transformation it is easily shown that the $(1,1)$ block of the balanced controllability Gramian satisfies

$$A_{11}(\rho)\Sigma + \Sigma A_{11}^T(\rho) + B_1(\rho)B_1^T(\rho) < 0 \;\; \forall\, \rho(t) \in F_\rho,$$

where $A_{11}(\rho)$, $B_1(\rho)$ and $C_1(\rho)$ are the state-space matrices of the output frequency-weighted model approximant. This shows that the weighted truncation scheme guarantees a quadratically stable model approximant with unity input weighting. A similar argument shows that the same stability result holds if unity output weighting ($W_o = I$) is used. For these two special cases we obtain quadratically stable approximants from quadratically stable systems when using weighted balanced truncation.

7 Approximation in the Graph Topology

A limitation with the techniques developed in previous sections is that, P_ρ, the model to be approximated, must be quadratically stable.

The extension of balanced truncation to approximation of unstable LTI systems by approximating the symbol of the system's graph has been studied by a number o f authors. In this section we extend some of the standard results on coprime factorisation of LTI systems to parameter-varying systems. A symbol for the system's graph is constructed which may be approximated using the machinery already developed. Although any coprime factorisation would suffice, we concentrate on what we call a contractive coprime factorisation (CCF). This is analogous to the normalised coprime factorisation of LTI systems and its extension to LTV systems in [20]. For LPV systems a normalisation condition for all feasible parameter trajectories can not be satisfied, hence the notion of a contractive coprime factorisation.

There is an attractive duality between the Gramians of a CCF and the solutions of certain Riccati inequalities. In [22] this duality is exploited to develop an efficient technique for reduced order controller synthesis.

Definition 7.1 *Let $\mathcal{S}_F$ denote the ring of all causal, quadratically stable, finite-dimensional LPV systems defined on the underlying feasible parameter set F_ρ. Denote by $\mathcal{S}_F^-$ the elements in $\mathcal{S}_F$ that have causal inverses.*

The objective is to obtain a coprime factorisation of the LPV system P_ρ over the ring $\mathcal{S}_F$.

Lemma 7.2 ([22]) *Let P_ρ have a continuous, quadratically stabilisable and quadratically detectable state-space realisation and let $F(\rho)$ and $L(\rho)$ be stabilising parameter-dependent state feedback and output injection matrices. Ignoring ρ dependence, define*

$$\begin{bmatrix} N_\rho & \tilde{Y}_\rho \\ M_\rho & \tilde{X}_\rho \end{bmatrix} \triangleq \left[\begin{array}{c|cc} A+BF & B & -L \\ \hline C+DF & D & I \\ F & I & 0 \end{array}\right]$$

$$\begin{bmatrix} X_\rho & Y_\rho \\ \tilde{M}_\rho & -\tilde{N}_\rho \end{bmatrix} \triangleq \left[\begin{array}{c|cc} A+LC & L & -(B+LD) \\ \hline F & 0 & I \\ C & I & -D \end{array}\right].$$

Then

$$\begin{bmatrix} X_\rho & Y_\rho \\ \tilde{M}_\rho & -\tilde{N}_\rho \end{bmatrix}\begin{bmatrix} N_\rho & \tilde{Y}_\rho \\ M_\rho & \tilde{X}_\rho \end{bmatrix} = I.$$

Proof: The product has a state-space realisation given by

$$\begin{bmatrix} X_\rho & Y_\rho \\ \tilde{M}_\rho & -\tilde{N}_\rho \end{bmatrix}\begin{bmatrix} N_\rho & \tilde{Y}_\rho \\ M_\rho & \tilde{X}_\rho \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{cc|cc} A+LC & LC-BF & -B & L \\ 0 & A+BF & B & -L \\ \hline F & F & I & 0 \\ C & C & 0 & I \end{array}\right].$$

Making use of the parameter-independent state transformation

$$T^{-1} = \begin{bmatrix} I & I \\ I & -I \end{bmatrix}, \qquad T = \begin{bmatrix} \frac{I}{2} & \frac{I}{2} \\ \frac{I}{2} & -\frac{I}{2} \end{bmatrix}, \qquad \text{gives}$$

$$\begin{bmatrix} X_\rho & Y_\rho \\ \tilde{M}_\rho & -\tilde{N}_\rho \end{bmatrix} \begin{bmatrix} N_\rho & \tilde{Y}_\rho \\ M_\rho & -\tilde{X}_\rho \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{cc|cc} A+LC & 0 & 0 & 0 \\ LC-BF & A+BF & -2B & 2L \\ \hline F & 0 & I & 0 \\ C & 0 & 0 & I \end{array}\right]$$

which is unobservable and uncontrollable. ■

Definition 7.3 *Let $N_\rho \in \mathcal{S}_F$, $M_\rho \in S_F^-$ have the same number of columns. The ordered pair $[N_\rho, M_\rho]$ represents a contractive right coprime factorisation (CRCF) of P_ρ over S_F, if*

i. $P_\rho = N_\rho M_\rho^{-1}$;

ii. there exist $X_\rho, Y_\rho \in \mathcal{S}_F$ such that $X_\rho N_\rho + Y_\rho M_\rho = I$;

iii. $[N_\rho^T \ \ M_\rho^T]^T$ *is a contraction in the following sense*

$$\sup_{\rho \in F_\rho} \sup_{\{u \in L_2^+ : \|u\|_2 \le 1\}} \left\| \begin{bmatrix} N_\rho \\ M_\rho \end{bmatrix} u \right\|_{i,2} \le 1.$$

Definition 7.4 *Define the contractive right graph symbol $\mathcal{G}_\rho : L_2^+ \mapsto L_2^+ \otimes L_2^+$ of the LPV system P_ρ as follows*

$$\mathcal{G}_\rho \triangleq \begin{bmatrix} N_\rho \\ M_\rho \end{bmatrix},$$

where $[N_\rho, M_\rho]$ is a CRCF of P_ρ.

Observe that $\mathcal{G}_\rho$ generates the set of all stable input-output pairs of the given LPV system by allowing $\mathcal{G}_\rho$ to act on the whole of L_2^+.

Theorem 7.5 *Let P_ρ have a continuous, quadratically stabilisable quadratically detectable realisation, then a contractive right graph symbol of P_ρ is given by*

$$\mathcal{G}_\rho \triangleq \left[\begin{array}{c|c} A(\rho)+B(\rho)F(\rho) & B(\rho)S^{-\frac{1}{2}}(\rho) \\ \hline C(\rho)+D(\rho)F(\rho) & D(\rho)S^{-\frac{1}{2}}(\rho) \\ F(\rho) & S^{-\frac{1}{2}}(\rho) \end{array}\right], \tag{7}$$

where $F(\rho) = -S^{-1}(\rho)(B^T(\rho)Z_1 + D^T(\rho)C(\rho))$, $S(\rho) = I + D^T(\rho)D(\rho)$, $R(\rho) = I + D(\rho)D^T(\rho)$ and Z_1 is a constant solution of the generalised control Riccati inequality (GCRI)

$$\begin{aligned}(A(\rho) - B(\rho)S^{-1}(\rho)D^T(\rho)C(\rho))^T Z_1 + Z_1(A(\rho) - B(\rho)S^{-1}(\rho) \times \\ D^T(\rho)C(\rho)) - Z_1 B(\rho)S^{-1}(\rho)B^T(\rho)Z_1(\rho) + \\ C^T(\rho)R^{-1}(\rho)C(\rho) < 0 \ \ \forall \rho(t) \in F_\rho.\end{aligned} \tag{8}$$

Proof: The quadratic dissipation function in Definition 2.7 is used to show that the system is $\mathcal{Q}_e$ stable and a contraction. Firstly reformulate the problem as follows

$$\begin{aligned}\inf\{\gamma : X(A+BF) + (A+BF)^T X + (C+DF)^T(C+DF) + F^T F + \\ (XBS^{-\frac{1}{2}} + (C+DF)^T DS^{-\frac{1}{2}} + F^T S^{-\frac{1}{2}})(\gamma^2 I - I)^{-1} \\ (XBS^{-\frac{1}{2}} + (C+DF)^T DS^{-\frac{1}{2}} + F^T S^{-\frac{1}{2}})^T < 0 \ \forall \rho(t) \in F_\rho.\}\end{aligned}$$

Then observe that by taking $X = Z_1$, we have

$$XBS^{-\frac{1}{2}} + (C+DF)^T DS^{-\frac{1}{2}} + F^T S^{-\frac{1}{2}} = (Z_1 B + C^T D + F^T(I + D^T D))S^{-\frac{1}{2}} = 0.$$

Hence the following inequality holds

$$Z_1(A+BF) + (A+BF)^T Z_1 + (C+DF)^T(C+DF) + F^T F < 0 \quad \forall\, \rho(t) \in F_\rho, \tag{9}$$

and by Proposition 2.7 the system is exponentially stable and has an induced norm which is bounded, from above, by $\gamma = 1 + \epsilon$ for arbitrary $\epsilon > 0$. Clearly $N_\rho \in \mathcal{S}_F$ and $M_\rho \in \mathcal{S}_F^-$. Finally, using the state-space formulas in Lemma 7.2, and the detectability of P_ρ, we can construct a left inverse for $\mathcal{G}_\rho$. ∎

Remark 7.6 *The results as stated here are for the right graph; dual results are obtainable for the left graph.*

Remark 7.7 *The Hermitian square root of $S(\rho)$ in equation (7) is usually obtained through singular value decomposition. If $D(\rho) \neq 0$ then it may be necessary to use the methods in [15] to ensure that $S^{\frac{1}{2}}(\rho)$ is continuous. That said, if the coprime factors are only being generated for the purpose of model reduction, careful examination of subsequent formulae shows that it is not necessary to explicitly compute $S^{\frac{1}{2}}(\rho)$.*

Lemma 7.8 *Let P_ρ have a continuous, quadratically stabilisable and a quadratically detectable realisation, and let the ordered pair $[N_\rho, M_\rho]$ represent the CRCF of P_ρ given in equation (7). Then*

$$Q = Z_1 \quad \text{and} \quad P = (I + Z_2 Z_1)^{-1} Z_2$$

are observability and controllability Gramians for the given realisation. Here Z_1 solves GCRI and Z_2 solves the generalised filtering Riccati inequality (GFRI) given by

$$\begin{aligned}(A(\rho) - B(\rho)D^T(\rho)R^{-1}(\rho)C(\rho))Z_2 + Z_2(A(\rho) - B(\rho)D^T(\rho) \times \\ R^{-1}(\rho)C(\rho))^T - Z_2 C^T(\rho)R^{-1}(\rho)C(\rho)Z_2 + \\ B(\rho)S^{-1}(\rho)B^T(\rho) < 0 \ \forall \rho(t) \in F_\rho,\end{aligned}$$

where $R(\rho) = I + D(\rho)D^T(\rho)$, $S(\rho) = I + D^T(\rho)D(\rho)$.

Proof: Starting with GCRI and setting $Z_1 = Q$, some algebraic manipulations allow GCRI to be written as

$$\begin{aligned}(A(\rho) + B(\rho)F(\rho))^T Q + Q(A(\rho) + B(\rho)F(\rho)) + (C(\rho) + D(\rho)F(\rho))^T(C(\rho) + D(\rho)F(\rho)) + \\ F^T(\rho)F(\rho) < 0 \ \forall \rho(t) \in F_\rho,\end{aligned}$$

which is recognised as the observability inequality for the realisation in equation (7). To show that $P = (I + Z_2 Z_1)^{-1} Z_2$ is a controllability Gramian we need to show that

$$(A(\rho) + B(\rho)F(\rho))P + P(A(\rho) + B(\rho)F(\rho))^T + B(\rho)S^{-1}(\rho)B^T(\rho) < 0 \;\; \forall \rho(t) \in F_\rho. \qquad (10)$$

Substituting $P = (I + Z_2 Z_1)^{-1} Z_2$ into equation (10) and performing some lengthy manipulations gives

$$\begin{aligned}(A(\rho) + B(\rho)F(\rho))P + P(A(\rho) + B(\rho)F(\rho))^T + B(\rho)S^{-1}(\rho)B^T(\rho) \\ = \mathrm{GFRI} + Z_2 \mathrm{GCRI} Z_2 < 0 \;\; \forall \;\; \rho(t) \in F_\rho.\end{aligned}$$

■

Remark 7.9 *This result provides a direct link between the solutions of GCRI and GFRI and balanced truncation. It has been used in [22] to develop an efficient technique for synthesising reduced order controllers for LPV systems.*

Definition 7.10 *Given the CRCF of the n-state system P_ρ in equation (7), a controllability Gramian P, an observability Gramian Q, together with a balancing state transformation matrix T such that the transformed Gramians $\tilde{P} = \tilde{Q} = \Sigma$ are diagonal, define a balanced parameter-varying CRCF as*

$$\begin{aligned}
\mathcal{G}_\rho &\stackrel{s}{=} \left[\begin{array}{c|c} \bar{A}(\rho) + \bar{B}(\rho)\bar{F}(\rho) & \bar{B}(\rho)S^{-\frac{1}{2}}(\rho) \\ \hline \bar{C}(\rho) + D(\rho)\bar{F}(\rho) & D(\rho)S^{-\frac{1}{2}}(\rho) \\ \bar{F}(\rho) & S^{-\frac{1}{2}}(\rho) \end{array}\right], \\
\bar{A}(\rho) &= T^{-1}A(\rho)T, \quad \bar{B}(\rho) = T^{-1}B(\rho) \\
\bar{C}(\rho) &= C(\rho)T, \quad \bar{F}(\rho) = -S^{-1}(\rho)(\bar{B}^T\Sigma + D^T(\rho)\bar{C}(\rho)).
\end{aligned}$$

Definition 7.11 *Ignoring ρ dependence, partition the balanced CRCF of the n-state graph symbol in definition 7.10 according to*

$$\mathcal{G}_\rho \stackrel{s}{=} \left[\begin{array}{cc|c} \bar{A}_{11} + \bar{B}_1\bar{F}_1 & \bar{A}_{12} + \bar{B}_1\bar{F}_2 & \bar{B}_1 S^{-\frac{1}{2}} \\ \bar{A}_{21} + \bar{B}_2\bar{F}_1 & \bar{A}_{22} + \bar{B}_2\bar{F}_2 & \bar{B}_2 S^{-\frac{1}{2}} \\ \hline \bar{C}_1 + D\bar{F}_1 & \bar{C}_2 + D\bar{F}_2 & DS^{-\frac{1}{2}} \\ \bar{F}_1 & \bar{F}_2 & S^{-\frac{1}{2}} \end{array}\right], \qquad (11)$$

where $\bar{A}_{11} \in \mathbb{R}^{r\times r}, \bar{A}_{12} \in \mathbb{R}^{r\times(n-r)}, \bar{A}_{21} \in \mathbb{R}^{(n-r)\times r}, \bar{A}_{22} \in \mathbb{R}^{(n-r)\times(n-r)}, \bar{B}_1 \in \mathbb{R}^{r\times m}, \bar{B}_2 \in \mathbb{R}^{(n-r)\times m}, \bar{C}_1 \in \mathbb{R}^{p\times r}, \bar{C}_2 \in \mathbb{R}^{p\times(n-r)}, \bar{F}_1 \in \mathbb{R}^{m\times r}$ and $\bar{F}_2 \in \mathbb{R}^{m\times(n-r)}$. Let $\hat{\mathcal{G}}_\rho$, of state-dimension r, be obtained by truncating $\mathcal{G}_\rho$ as follows

$$\hat{\mathcal{G}}_\rho \stackrel{s}{=} \left[\begin{array}{c|c} \bar{A}_{11} + \bar{B}_1\bar{F}_1 & \bar{B}_1 S^{-\frac{1}{2}} \\ \hline \bar{C}_1 + D\bar{F}_1 & DS^{-\frac{1}{2}} \\ \bar{F}_1 & S^{-\frac{1}{2}} \end{array}\right]. \qquad (12)$$

We define $\hat{\mathcal{G}}_\rho$ to be a balanced fractional approximant of the system's graph symbol $\mathcal{G}_\rho$, and we associate with $\hat{\mathcal{G}}_\rho$ the following r-state model

$$\hat{P}_\rho = \left[\begin{array}{c|c} \bar{A}_{11} & \bar{B}_1 \\ \hline \bar{C}_1 & D \end{array}\right].$$

Lemma 7.12 *Given the LPV system P_ρ, Z_1 satisfying GCRI and Z_2 satisfying GFRI, then we have already shown (Lemma 7.8) that $Q = Z_1$ and $P = (I + Z_2 Z_1)^{-1} Z_2$ are observability and controllability Gramians for the realisation of the CRCF given in equation (7). If the realisation in equation (7) is subsequently balanced and truncated to give the balanced fractional approximant in equation (12), then*

$$\hat{Z}_1 = \Sigma_1 \qquad \textit{and} \qquad \hat{Z}_2 = (I - \Sigma_1^2)^{-1}\Sigma_1$$

satisfy GCRI and GFRI for the reduced-order model given by

$$\hat{P}_\rho = \left[\begin{array}{c|c} \bar{A}_{11}(\rho) & \bar{B}_1(\rho) \\ \hline \bar{C}_1(\rho) & D(\rho) \end{array}\right]. \tag{13}$$

Proof: We need to show that the mappings $Q = Z_1$ and $P = (I + Z_2 Z_1)^{-1} Z_2$ are invertible. The result is trivial in the LTI case where the inequalities can be replaced with equalities: we now show that the result is still true in the LPV case. First replace GCRI and GFRI with equalities as follows

$$\begin{aligned} (A - BS^{-1}D^T C)^T Z_1 + Z_1(A - BS^{-1}D^T C) - Z_1 BS^{-1}B^T Z_1 + C^T R^{-1} C + E_1(\rho) &= 0 \quad (14) \\ (A - BD^T R^{-1} C) Z_2 + Z_2(A - BD^T R^{-1}C)^T - Z_2 C^T R^{-1} C Z_2 + BS^{-1}B^T + E_2(\rho) &= 0, (15) \end{aligned}$$

where $E_1(\rho), E_2(\rho) > 0 \ \forall \rho(t) \in F_\rho$. Putting $Z_1 = Q$ equation (14) is easily rearranged to give

$$(A + BF)^T Q + Q(A + BF) + (C + DF)^T (C + DF) + F^T F + E_1 = 0, \tag{16}$$

while left and right multiplying equation (14) by Z_2 and adding the result to equation (15) gives

$$(A + BF)P + P(A + BF)^T + BS^{-1}B^T + PE_1 P + (I + Z_2 Z_1)^{-1} E_2 (I + Z_1 Z_2)^{-1} = 0. \tag{17}$$

Let $T(\rho)$ be the balancing (naturally-balancing) transformation matrix such that $T^{-1}(\rho)P(\rho)T^{-T}(\rho) = \Sigma(\rho)$ and $T^T(\rho)Q(\rho)T(\rho) = \Sigma(\rho)$. Left multiplying equation (17) by T^{-1} and right multiplying by T^{-T} gives

$$\begin{aligned} &(\bar{A} + \bar{B}\bar{F})\Sigma + \Sigma(\bar{A} + \bar{B}\bar{F})^T + \\ &\qquad \bar{B}S^{-1}\bar{B}^T + \Sigma T^T E_1 T \Sigma + T^{-1}(I + Z_2 Z_1)^{-1} E_2 (I + Z_1 Z_2)^{-1} T^{-T} = 0, \end{aligned} \tag{18}$$

where $\bar{A}, \bar{B}, \bar{C}$ and $\bar{F}$ were defined in Definition 7.10. Left multiplying equation (16) by T^T and right multiplying by T gives

$$(\bar{A} + \bar{B}\bar{F})^T \Sigma + \Sigma(\bar{A} + \bar{B}\bar{F}) + (\bar{C} + D\bar{F})^T (\bar{C} + D\bar{F}) + \bar{F}^T \bar{F} + T^T E_1 T = 0. \tag{19}$$

Putting $\Sigma = \bar{Z}_1$ allows equation (19) to be written as

$$(\bar{A} - \bar{B}S^{-1}D^T\bar{C})^T \bar{Z}_1 + \bar{Z}_1(\bar{A} - \bar{B}S^{-1}D^T\bar{C}) - \bar{Z}_1\bar{B}S^{-1}\bar{B}^T\bar{Z}_1 + \bar{C}^T R^{-1}\bar{C} + T^T E_1 T = 0,$$

as desired. Finally, left and right multiplying equation (19) by Σ, subtracting the result from equation (18) and putting $\bar{Z}_2 = (I - \Sigma^2)^{-1}\Sigma$ gives

$$\begin{aligned} &-\dot{\bar{Z}}_2 + (\bar{A} - \bar{B}D^T R^{-1}\bar{C})\bar{Z}_2 + \bar{Z}_2(\bar{A} - \bar{B}D^T R^{-1}\bar{C})^T - \bar{Z}_2\bar{C}^T R^{-1}\bar{C}\bar{Z}_2 + \\ &\qquad \bar{B}S^{-1}\bar{B}^T + T^{-1}(I + Z_2 Z_1)^{-1} E_2 (I + Z_1 Z_2)^{-1} T^{-T} = 0. \end{aligned} \tag{20}$$

Extracting the $(1,1)$ blocks of equation (20) and (20) shows that $\hat{Z}_1$ and $\hat{Z}_2$ have the stated properties. ■

Corollary 7.13 *The r-state model approximant*

$$\hat{P}_\rho = \left[\begin{array}{c|c} \bar{A}_{11}(\rho) & \bar{B}_1(\rho) \\ \hline \bar{C}_1(\rho) & D(\rho) \end{array}\right]$$

is detectable and a stabilising output injection is given by $L = -(\bar{B}_1 D^T + \hat{Z}_2 \bar{C}_1^T)R^{-1}$.

Theorem 7.14 *Let* $\mathcal{G}_\rho$*, of state-dimension* n*, be balanced and let* $\hat{\mathcal{G}}_\rho$*, of state-dimension* r*, be obtained by truncating* $\mathcal{G}_\rho$*. Then* $\hat{\mathcal{G}}_\rho$ *is itself balanced and represents a CRCF over* $\mathcal{S}_F$ *of the r-state model*

$$\hat{P}_\rho = \left[\begin{array}{c|c} \bar{A}_{11}(\rho) & \bar{B}_1(\rho) \\ \hline \bar{C}_1(\rho) & D(\rho) \end{array}\right]. \tag{21}$$

Proof: The proof amounts to showing that $\hat{\mathcal{G}}_\rho$ is quadratically stable and a contraction. Showing that the factorisation is coprime requires a little more work and the results of Lemma 7.12. Since we begin with a balanced CRCF (equation (11)), the observability inequality is given by

$$(\bar{A}(\rho) + \bar{B}(\rho)\bar{F}(\rho))^T\Sigma + \Sigma(\bar{A}(\rho) + \bar{B}(\rho)\bar{F}) + (\bar{C}(\rho) + \bar{D}(\rho)\bar{F}(\rho))^T(\bar{C}(\rho) + \bar{D}(\rho)\bar{F}(\rho)) + \bar{F}^T(\rho)\bar{F}(\rho) < 0 \;\; \forall \rho(t) \in F_\rho,$$

and on extracting the $(1,1)$ block of this inequality we get

$$(\bar{A}_{11}(\rho) + \bar{B}_1(\rho)\bar{F}_1(\rho))^T\Sigma_1 + \Sigma_1(\bar{A}_{11}(\rho) + \bar{B}_1(\rho)\bar{F}_1(\rho)) + (\bar{C}_1(\rho) + \bar{D}(\rho)\bar{F}_1(\rho))^T \times (\bar{C}_1(\rho) + \bar{D}(\rho)\bar{F}_1(\rho)) + \bar{F}_1^T(\rho)\bar{F}_1(\rho) < 0 \, \forall \rho(t) \in F_\rho. \tag{22}$$

Equation (22) shows that the balanced fractional approximant is quadratically stable and together with the $(1,1)$ block of the controllability inequality, balanced. Using the fact that

$$\bar{F}_1(\rho) = -S^{-1}(\rho)(\bar{B}_1^T(\rho)\Sigma_1 + D^T(\rho)\bar{C}_1(\rho))$$

equation (22) can be written as

$$(A_{11}(\rho) - B_1(\rho)S^{-1}(\rho)D^T(\rho)C_1(\rho))^T\Sigma_1 + \Sigma_1(A_{11}(\rho) - B_1(\rho)S^{-1}(\rho)D^T(\rho)C_1(\rho)) - \Sigma_1 B_1(\rho)S^{-1}(\rho)B_1^T(\rho)\Sigma_1 + C_1^T(\rho)R^{-1}(\rho)C_1(\rho) < 0 \, \forall \rho(t) \in F_\rho.$$

Recalling Theorem 7.5, this shows that $\hat{\mathcal{G}}_\rho$ is a contractive factorisation of $\hat{P}_\rho$. ∎

Balanced truncation of the system's graph symbol has an immediate physical interpretation. Essentially, those states which do not contribute strongly to the input-output mapping of $\mathcal{G}_\rho$ are removed. This implies that the proposed approximation method is trying to maintain the graph of $\hat{P}_\rho$ close to the graph of P_ρ, which is arguably the correct setting for approximation since the concept of the graph topology and the gap between systems is not restricted to LTI systems ([8],[20]). In fact, for time-varying systems it is shown in [19] that the graph topology is still the weakest topology in which feedback is a robust property.

8 Computational Issues

In this section we outline a computational procedure for finding a controllability Gramian and an observability Gramian for a given parameter-dependent system. The objective of making the $\mathcal{Q}_e$ singular values small is shown to result in an optimisation problem that involves convex constraints and a non-convex cost function. In order to deal with this lack of convexity in the cost function we show how suitable LPV Gramians may be obtained by using LTI Gramians, as weights in weighted optimisation which is locally convex.

Unless the parameter dependence is polytopic then a solution to two linear matrix inequalities which satisfy an infinite number of constraints is required. The approach take taken is to satisfy the inequalities on a suitably dense grid covering the parameter space, then to exploit the continuity properties of the plant matrices to ensure that the inequalities are satisfied everywhere. Given bounds on the range of parameter variation the problem is to determine a constant matrix $P = P^T > 0$ and a constant matrix $Q = Q^T > 0$ satisfying

$$A^T(\rho)Q + QA(\rho) + C^T(\rho)C(\rho) < -\delta^2 I \ \ \forall\rho(t) \in F_\rho \tag{23}$$

$$A(\rho)P + PA^T(\rho) + B(\rho)B^T(\rho) < -\delta^2 I \ \ \forall\rho(t) \in F_\rho, \tag{24}$$

for some $\delta^2 > 0$. The small constant δ allows the feasibility of the solutions at the grid points to be extended to the entire parameter space. With P and Q constant, solution of equations (23) and (24) at a set of grid points becomes a convex feasibility problem for which there exist powerful numerical techniques ([6]) which are guaranteed to find a feasible solution.

However, from a model-reduction perspective a P and Q are required which do rather more than merely solve the Lyapunov inequalities. The solutions to equations (23) and (24) must simultaneously minimise the r smallest eigenvalues of the product PQ; or equivalently minimise the cost function $J = \mathrm{trace}(PQ)$. Minimising J subject to the constraints given by equations (23) and (24) constitutes a constrained optimisation with convex constraints but with a non-convex cost function.

The following iterative scheme is one possible approach to achieving this goal.

Procedure 8.1 .

i. Pre- and post multiply equation (23) by Q^{-1} and equation (24) by P^{-1} to get

$$\begin{bmatrix} Q^{-1}A^T(\rho) + A(\rho)Q^{-1} & Q^{-1}C(\rho)^T \\ C(\rho)Q^{-1} & -I \end{bmatrix} < 0 \tag{25}$$

$$\begin{bmatrix} P^{-1}A(\rho) + A^T(\rho)P^{-1} & P^{-1}B(\rho) \\ B^T(\rho)P^{-1} & -I \end{bmatrix} < 0 \tag{26}$$

i. Find Q_1 which satisfies equation (23) and minimises $J_1 = \sum_k \mathrm{trace}(Q_1 P_0(\rho_k))$, where $P_0(\rho_k)$ represents the LTI controllability Gramian at the grid point ρ_k.

ii. Solve for P_1 which satisfies equation (24) and minimises $J_2 = \mathrm{trace}(P_1 Q_1)$.

iii. Solve for Q_2 which satisfies equation (23) and minimises $J_3 = \mathrm{trace}(Q_2 P_1)$.

iv. Repeat steps ii. and iii. until the decrease in the cost function satisfies an appropriate convergence criterion.

For certain systems it may help to invert the LMIs so that instead of solving for P and Q we solve for their inverses. This is done to ensure that the smallest $\mathcal{Q}_e$ singular values contribute most to the cost function. For systems where the $\mathcal{Q}_e$ singular values vary over several orders of magnitude it is possible for the contribution made by the smallest singular values to be completely dominated by numerical errors. By inverting the Gramians this situation is avoided because the smallest singular values now make the largest contribution to the cost function. This ensures that the numerical procedure is consistent with the objective of producing the tightest upper bound on the truncation error.

Observe that in the first step of the iteration the cost function is weighted using the LTI controllability Gramian. In a sense this leads to solutions which satisfy the LPV Lyapunov inequalities and are close to the LTI Gramians. Computational experience has shown that this greatly increases the speed of convergence. The optimisation is locally convex (but not globally convex) so it is possible to find a global minimum at each step in the iteration before moving to the next step in the procedure. However, the procedure does not guarantee to find the global minimum of the non-convex cost function.

9 Pitch Axis Control of a Missile

In this example we examine reduced order controller synthesis for pitch axis control of a missile. The dynamics of the missile under consideration vary greatly as a function of speed (v), altitude (H) and angle-of-attack (α), hence a single LTI model and controller is insufficient for effective control. Here we'll consider an LPV model and controller. The model under consideration was originally studied by Gahinet et al. ([10]) for the synthesis of LPV controllers of state-dimension equal to the weighted model. The parameter dependent model is given by,

$$P_\rho = \left[\begin{array}{cc|c} -Z_\alpha & 1 & 0 \\ -M_\alpha & 0 & 1 \\ \hline -1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

where Z_α and M_α are aerodynamic co-efficients which depend on v, H and α. The input to P_ρ is the fin deflection (δ_f). The two states are angle-of-attack (α) and pitch rate (q) respectively. The two outputs are the (normalised) vertical acceleration (a_{zv}) and pitch rate respectively.

The parameters v, H and α are assumed to all be measurable in real time and vary over the following ranges:

		Lower Limit	Upper Limit
v (Mach#)	–	0.5	4
H (km)	–	0	18
α (degrees)	–	0	40

The combination of v, H and α imply that $Z_\alpha \in [0.5, 4]$ and $M_\alpha \in [0, 106]$. Note that Z_α and M_α enter the state-space matrices in an affine way, and may be considered as representing a convex polytope with four vertices.

The design objective is for the step response of a_{zv} to have a settling time of less than half a second for all variations of Z_α and M_α in the given range. Given the performance weights,

$$\begin{aligned} W_1 &= \frac{2.01}{s + 0.201} \\ W_2 &= \frac{96.78s^3 + 0.29s^2 + 0.0003s - 0.0039}{s^3 + 1.12 \times 10^5 s^2 + 1.05 \times 10^8 s + 1.07 \times 10^{11}} \end{aligned}$$

the performance specification corresponds to designing a controller which achieves,

$$\left\| \begin{array}{c} W_1(I + P_\rho K_\rho)^{-1} \\ W_2 K_\rho (I + P_\rho K_\rho)^{-1} \end{array} \right\|_\infty < 1. \tag{27}$$

A methodology for synthesis of LPV controllers satisfying equation (27) may be found in [3, 6, 10]. The resulting controllers have a state-dimension equal to that of the weighted model, i.e. 6 (4 from the weights and 2 from the plant). Starting with the controller designed by Gahinet et al [10] (which we will call $\mathcal{K}_\rho^0$) we now examine the effectiveness of the proposed LPV model approximation techniques for synthesis of satisfactory reduced order LPV controllers.

The approach taken is to reduce K_ρ^0 then a posteriori check the performance of the reduced order controller. Although there are no guarantees that reduced-order controllers obtained in this manner will even quadratically stabilise the closed loop, currently this appears to be the most tractable approach. Further, we will see that the approach can work well.

The polytope we are considering has four vertices. Hence, a constant P and a constant Q which satisfy the relevant Lyapunov inequalities for K_ρ^0 evaluated at each of the four vertices is required. Using Procedure 8.1 to iteratively calculate appropriate P and Q led to a balanced realisation with the $\mathcal{Q}_e$-singular values shown in Table 1.

i	σ_i
1	7110.91
2	2703.76
3	2071.81
4	550.00
5	87.06
6	0.04

Table 1: $\mathcal{Q}$-singular values for $\mathcal{K}_\rho^0$

The $\mathcal{Q}_e$-singular values of K_ρ^0 indicate that at least one state may be removed without seriously degrading the controller response. Using both the balanced truncation and the Hankel norm type procedure we calculated fifth order controllers, K_ρ^b and K_ρ^h respectively. A comparison of the performance of K_ρ^b and K_ρ^h to that of K_ρ^0 is given in Figure 1. (Eliminating more than one state resulted in reduced order controllers which gave unacceptably poor closed loop behaviour.)

For Figure 1(a) fifteen random LTI plants within the polytope described by Z_α and M_α were generated. The step response with each of these LTI plants was then calculated and plotted. There is no appreciable difference between the responses obtained using any of the three controllers and the settling time is well below the target figure of half a second.

As in [10] the particular spiral parameter trajectory,

$$\begin{aligned} Z_\alpha &= 2.25 + 1.70e^{-4t}\cos(100t) \\ M_\alpha &= 50 + 49e^{-4t}\sin(100t) \end{aligned}$$

was chosen to test the closed loop performance. This trajectory is shown in Figure 1(b). The controller's state-space data were constructed using a convex interpolation of their values at the four vertices. The step response to the trajectory is shown in Figure 1(c). The solid line is the response for both the full order controller and for K_ρ^h. The dashed response is for the controller K_ρ^b. They show that satisfactory reduced order controllers have been obtained from both approaches.

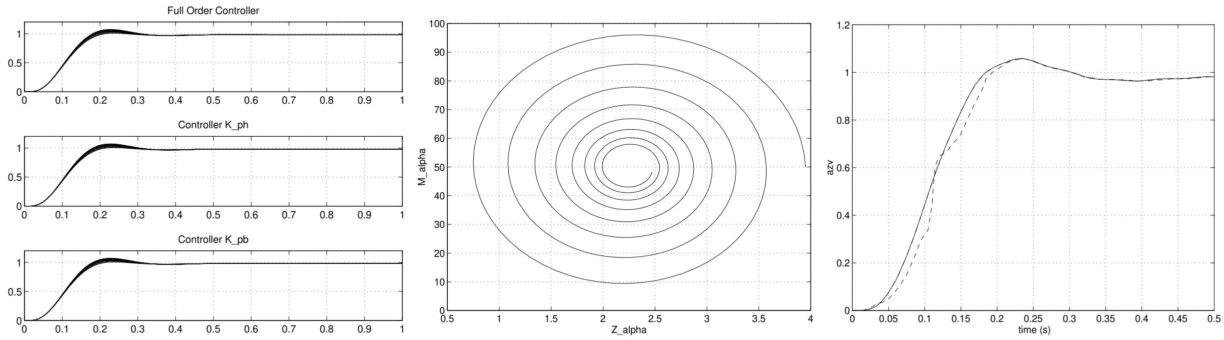

Figure 1:

(a) Closed Loop Step Response
(b) Spiral Parameter Trajectory: (0-1 sec)
(c) Step Response to Spiral Trajectory

10 Conclusions

In this paper we have shown that the LTI model approximation techniques of balanced truncation, optimal Hankel norm approximation and approximation of the graph operator may be extended to an LPV framework. (Although the Hankel approach is no longer optimal in any sense.) These results complement many recent results on the synthesis of controllers for LPV plants. However, since balanced parameter varying realisations are far from unique, caution must be shown when applying the techniques to ensure that the states being neglected do indeed contribute little to the input-output response.

References

[1] B.D.O. Anderson and Yi. Liu, *Controller Reduction: Concepts and Approaches*, IEEE. Trans. Automat. Control, Vol.34, No.8, pp 802-812, 1989.

[2] C. Beck, J.C. Doyle and K. Glover. *"Model Reduction of Multi-Dimensional and Uncertain Systems"*, Preprint, 1994.

[3] G.S. Becker, *Quadratic Stability and Performance of Linear Parameter Dependent Systems*, Phd dissertation, University of California at Berkeley, 1993.

[4] G. Becker and A. Packard, *Robust Performance of Linear Parametrically Varying Systems Using Parametrically-Dependent Linear Feedback* Systems and Control Letter, Vol. 23, 1994.

[5] S.Boyd and Q. Yang, *Structured and Simultaneous Lyapunov Functions for System Stability Problems*, Int. J. Control, Vol.49, No.6, pp 2215-2240, 1989.

[6] S. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM Studies in Applied Mathematics, 1994.

[7] D. Enns, *Model Reduction for Control Systems Design*, Phd dissertation, Stanford University, 1984.

[8] A. Feintuch, *The Gap Metric for Time-Varying Systems*, Systems and Control Letters, Vol. 16, pp 277-279, 1991.

[9] P. Gahinet and P. Apkarian, *A Linear Matrix Inequality Approach to H_∞ Control*, International Journal of Robust and Nonlinear Control, Vol.4, pp 421-448, 1994.

[10] P. Gahinet, A. Nemirovskii, A.J. Laub and M. Chilali, *The LMI Control Toolbox*, Beta-Release Version, The Mathworks Inc. Mass., November 1994.

[11] T.T. Georgiou and M.C. Smith, *Upper and Lower Bounds for Approximation in the Gap Metric*, IEEE. Trans. Automat. Control, Vol.38, pp 946-951, 1993.

[12] K. Glover, *All Optimal Hankel Norm Approximations of Linear Multivariable Systems and their L_∞ Error Bounds*, Int. J. Control, Vol.39, pp 1115-1193, 1984.

[13] K. Glover and D.C. MacFarlane, *Stabilisation of Normalised Coprime Factor Plant Descriptions with H_∞ Bounded Uncertainty*, IEEE. Trans. Automat. Control, Vol.34, No.8, pp 821-830, 1989.

[14] P.J. Goddard, *Performance-Preserving Controller Approximation*, PhD Dissertation, University of Cambridge, 1995.

[15] T. Kato, *Perturbation Theory for Linear Operators*, Springer Verlag, N.Y., 1966.

[16] D.G. Meyer, *Fractional Balanced Reduction: Model Reduction via Fractional Representations*, IEEE. Trans. Automat. Control, Vol.35, pp 1341-1345.

[17] B.C. Moore, *Principal Components Analysis in Linear Systems*, IEEE. Trans. Automat. Control, Vol.26, pp 17-32, 1981.

[18] R.J. Ober and D.C. MacFarlane, *Balanced Canonical Forms for Minimal Systems: a Normalised Coprime Factor Approach*, Linear Algebra and its Applications, 122-124: 23-64, 1989.

[19] A.M. Pascoal, R. Ravi and P.P. Khargonekar, *The Graph Topology for Linear Plants with Application to Nonlinear Robust Stabilisation*, IEEE. Trans. Automat. Control, Vol. 38, No. 2, 1993.

[20] R. Ravi, A.M. Pascoal and P.P. Khargonekar, *Normalised Coprime Factorisations for Linear Time Varying Systems*, Systems and Control Letters, Vol 18, pp 455-465, 1992.

[21] W. Wang, J.C. Doyle, C. Beck and K. Glover. *"Model Reduction of LFT Systems"*, Proceedings of the 30th IEEE CDC, Brighton, 1991.

[22] G.D. Wood, *Control of Parameter-Dependent Mechanical Systems*, PhD Dissertation, University of Cambridge, 1995.

[23] G.D. Wood, P.J. Goddard and K. Glover. *Approximation of Linear Parameter-Varying Systems*, Proceedings of the 35th IEEE CDC, Kobe, 1996.

[24] P.M.R. Wortleboer, *Frequency-Weighted Balanced Reduction of Closed-loop Mechanical Systems*, Phd dissertation, Delft University of Technology, 1994.

[25] K. Zhou, J.C. Doyle and K. Glover. Robust and Optimal Control, Prentice-Hall, 1996.