Performance-Preserving Controller Approximation

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Abstract

This paper investigates the design of reduced order controllers using an $\mathcal{H}_\infty$ framework. Given a stabilising controller which satisfies a prespecified level of closed loop $\mathcal{H}_\infty$ performance sufficient conditions are derived for another controller to be stabilising and satisfy this level of $\mathcal{H}_\infty$ performance. Such controllers are said to be (P,γ)-admissible, where P is the model of the plant under consideration and γ is the required level of prespecified $\mathcal{H}_\infty$ performance. The conditions are expressed as norm bounds on particular frequency weighted errors where the weights are selected to make a specific transfer function a contraction. The design of reduced order (P,γ)-admissible controllers is then formulated as a frequency weighted model reduction problem.

It is advantageous for the required weights to be large in some sense. Solutions which maximise either the trace or the determinant of the weights are characterised. We show that the procedure for maximising the determinant of the weights always gives a direction where the weights are the best possible.

To conclude, we demonstrate by way of a numerical example, that when used in conjunction with a combined model reduction/convex optimisation scheme, the proposed design procedures are effective in substantially reducing controller complexity.

Keywords: Controller Approximation, Frequency Weighted Model Reduction, H-infinity Control.

1 Introduction

Controller complexity may be measured in terms of the number of calculations required per unit time to implement the controller, the number of parameters required to be stored and updated, and by the size of the controller state-dimension. For practical reasons it is desirable to design simple controllers. Not only are the dynamics of such controllers easy to understand and manipulate, they require less powerful hardware and simpler software to implement.

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The problem of reducing controller complexity via reducing the state-dimension has received much attention over the last 30 years (see [1, 38] and the references therein). Often techniques such as balanced truncation [6] and optimal Hankel norm approximation [11] can give effective reduction schemes when applied to the controller in isolation, as long as the error remains small. For reduction to lower dimensions the desired closed-loop performance needs to be incorporated. Approximation of coprime factors of the controller is one step in this direction [1]. In the case of robust stabilisation in the gap metric [10], or equivalently $\mathcal{H}_\infty$ loop shaping [28] controller reduction in the gap metric gives a very comparable set of results between the desired performance and controller reduction criterion [37, 29].

For more general problems several approaches have recently been proposed which incorporate closed loop considerations into the controller reduction problem. These generally involve posing the problem in terms of frequency weighted model reduction, with the weights chosen to emphasise the properties of the high order controller which it is desirable for the reduced order controller to possess. Standard frequency weighted model reduction tools ([6, 24, 15, 39]) may then be used to perform the reduction. However, it must be stressed that current techniques for performing frequency weighted model reduction do not have guaranteed performance (in the $\mathcal{L}_\infty$ sense), and upper and lower bounds on the approximation error are less tight than in the unweighted case.

In [1] weights are derived to give a sufficient condition for reduced order controllers to be stabilising. With these weights, and given a stabilising controller K^0, if $\|W_1(K^0 - K)W_2\|_\infty$ is less than some specified quantity then K is a reduced order stabilising controller. This approach is based on the (reasonable) premise that if a reduced order controller is close (in some sense) to a high order controller which is known to be stabilising then the reduced order controller will also be stabilising. An advantage of this approach is that a controller satisfying the required performance specification is known to exist, and hence it is simply a matter of calculating a low order controller which gives a similar response. A drawback is that there may be satisfactory controllers of the same or lower order which have a frequency response markedly different to the nominal high order controller.

In this paper we are concerned with the issue of reducing the state-dimension of controllers designed in an $\mathcal{H}_\infty$ framework such that there is no degradation in closed loop $\mathcal{H}_\infty$ performance. Consider the standard interconnection of Figure 1 where P is a finite dimensional, linear, time-invariant system with the following minimal state-space realisation:

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{1}$$

where $B_1 \in \mathbb{R}^{n\times m_1}$, $B_2 \in \mathbb{R}^{n\times m_2}$, $C_1 \in \mathbb{R}^{p_1\times n}$ and $C_2 \in \mathbb{R}^{p_2\times n}$.

$P(s)$ is called the *generalised plant.* It includes both the actual plant to be controlled and any frequency dependent weights used in the controller synthesis process. The standard suboptimal $\mathcal{H}_\infty$ control problem is to find all stabilising controllers for the generalised plant, P, such that the $\mathcal{H}_\infty$ norm of the closed loop transfer function, $T_{zw} = \mathcal{F}_l(P, K)$, is less than some prespecified performance level $\gamma > 0$. As in [32], we use the following definition for controllers which satisfy the $\mathcal{H}_\infty$ performance problem.

Definition 1.1 *A controller K is said to be (P, γ)-admissible if K stabilises P and* $\|T_{zw}\|_\infty = \|\mathcal{F}_l(P, K)\|_\infty < \gamma$.

We are concerned here with the problem of finding reduced order (P, γ)-admissible controllers. The starting point for the techniques to be developed in subsequent sections is the existence of a nominal (high order) (P, γ)-admissible controller. There are two commonly applied techniques

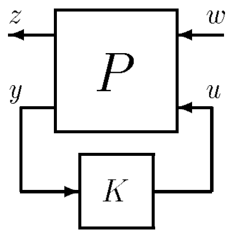

Figure 1: General Plant and Controller Interconnection

to determine if any (P, γ)-admissible controllers exist, and if they do, how to parameterise them. The first relies on the solution to two Riccati equations ([4, 14]), the second on the solution to two Linear Matrix Inequalities (LMIs) ([8]). In each case certain conditions (e.g. stabilisability and detectablity) must be satisfied, and they are assumed here.

The two Riccati equation approach is formulated by Doyle et al. in [4, 14]. In this approach all (P, γ)-admissible controllers are parameterised as an LFT on a stable, norm bounded, but otherwise free parameter, i.e. $K = \mathcal{F}_l(K_a, \Phi)$, where $\Phi \in \mathcal{RH}_\infty$, such that $\|\Phi\|_\infty < \gamma$ and K_a is an appropriately dimensioned n-state transfer function.

How best to exploit the freedom allowed in selecting the parameter Φ is an open question. Often Φ is chosen to be zero. This is equivalent to choosing K_{a11} which is called the central, or minimum entropy ([31]), controller. However, by judicious selection of Φ it is possible to derive reduced order (P, γ)-admissible controllers.

In [20], Gu et al. propose a procedure to select Φ so as to introduce p_2 unobservable modes into $K = \mathcal{F}_l(K_a, \Phi)$. These unobservable modes may then be truncated to obtain an $n - p_2$ state reduced order (P, γ)-admissible controller. Similar state cancellation approaches have been studied by Li and Chang [27] and Davidson [3]. Unfortunately, the selection of the state-space matrices of Φ to achieve the required state cancellations is not straightforward. Furthermore, since p_2 is generally small compared with n, the degree of reduction attainable with these methods is generally small.

Alternative to the Riccati equation based solution, in [8] necessary and sufficient conditions for the existence of (P, γ)-admissible controllers are given in terms of LMIs. Here the particular choice of (P, γ)-admissible controller depends on which solutions to three LMIs are selected. An advantage of this framework is that unlike the Riccati equation based solution, *singular* problems may be solved using the same machinery.

In the LMI formulation it is possible to include an additional rank constraint on the LMI solutions which enable (P, γ)-admissible controllers of order $k < n$ to be computed. However, due to the inclusion of the rank constraint, the design of reduced order (P, γ)-admissible controllers in this way is no longer convex. A method to approach this loss of convexity is proposed in [8].

This paper is organised as follows: In Section 2 background results on inner functions and the manipulation of linear fractional transformation (LFTs) and the Redheffer star product are presented. In Section 3 three new procedures for the design of reduced order controllers which are (P, γ)-admissible are proposed. Given a nominal (P, γ)-admissible controller K^0, we derive weights such that,

$$\|W_2^{-1}(K - K^0)W_1^{-1}\|_\infty < 1 \quad \Rightarrow \quad \|\mathcal{F}_l(P, K)\|_\infty < \gamma$$

or (for different weights),

$$\|W_2^{-1}(\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix} - \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix})W_1^{-1}\|_\infty < 1 \quad \Rightarrow \quad \|\mathcal{F}_l(P, K)\|_\infty < \gamma$$

where $K = \tilde{V}^{-1}\tilde{U}$ and $K^0 = \tilde{V}_0^{-1}\tilde{U}_0$ are coprime factorisations. This is equivalent to finding the set of K or $\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix}$ which lie within a weighted $\mathcal{L}_\infty$ ball centred on K^0 or $\begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}$ respectively.

For two of the proposed procedures the weights are required to be calculated such that a specific transfer function is a contraction. Appropriate weights to achieve this are derived in Section 4. Subject to this constraint, the approximation is made easier if the above weighted $\mathcal{L}_\infty$ balls are made as large as possible. Hence the weights should be made large in some sense. Procedures for maximising the trace or the determinant of the weights are presented in Section 5. In Section 6 conditions are given under which the weight maximising procedures of Section 5 are optimal. Computational issues are discussed in Section 7 and a design example, which highlights the effectiveness of the proposed procedures, is presented in Section 8.

The results of this paper appear in [16].

2 Preliminaries

The notation used throughout this paper is reasonably standard and follows that of [41]. Also it is assumed that the reader is familiar with basic results on $\mathcal{H}_\infty$ control, linear fractional transformation (LFT) manipulation, inner functions [5, 4, 34] and coprime factorisations [36].

LFTs provide a convenient framework in which to formulate most control problems. They are particularly useful in $\mathcal{H}_\infty$ control because of the ease with which controllers and/or uncertainty perturbations may be incorporated into the problem setup. Figures 2(a) and 2(b) show alternative, but equivalent, forms of LFTs. The closed loop transfer function from w to z is referred to as the LFT of Q with coefficient matrix N. For appropriately dimensioned N, and when the inverse exists,

$$\begin{aligned} T_{zw} &= \mathcal{F}_l(N,Q) \stackrel{\triangle}{=} N_{11} + N_{12}Q(I - N_{22}Q)^{-1}N_{21} \\ T_{zw} &= \mathcal{F}_u(N,Q) \stackrel{\triangle}{=} N_{22} + N_{21}Q(I - N_{11}Q)^{-1}N_{12}. \end{aligned}$$

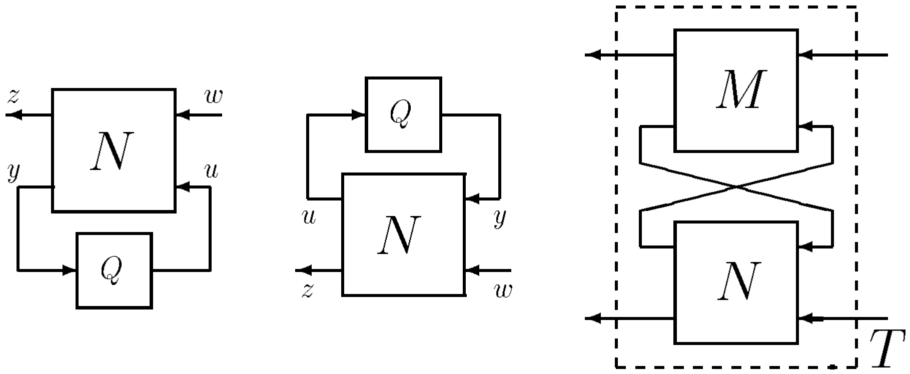

Figure 2: (a) $\mathcal{F}_l(N,Q)$, (b) $\mathcal{F}_u(N,Q)$ and (c) $\mathcal{S}(M,N)$

In the sequel we make extensive use of the following lemma on LFT inversion.

Lemma 2.1 ([41]) *Using the above notation, assume $T_{zw} = \mathcal{F}_l(N,Q)$ and N are proper,* $\det N(\infty) \neq 0$*,* $\det(N - \begin{bmatrix} T_{zw} & 0 \\ 0 & 0 \end{bmatrix})(\infty) \neq 0$*, and N_{12} and N_{21} are square and invertible. Then Q is proper and $Q = \mathcal{F}_u(N^{-1}, T_{zw})$.*

An important property of LFTs is that any interconnection of LFTs is also an LFT. A general form of interconnection is called the Redheffer star product [34].

Definition 2.2 *Consider the interconnection shown in Figure 2(c). Then the Redheffer star product of M with N is given by,*

$$
\begin{aligned}
T &\triangleq \mathcal{S}(M,N) = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \\
&= \begin{bmatrix} M_{11} + M_{12}N_{11}(I - M_{22}N_{11})^{-1}M_{21} & M_{12}(I - N_{11}M_{22})^{-1}N_{12} \\ N_{21}(I - M_{22}N_{11})^{-1}M_{21} & N_{22} + N_{21}M_{22}(I - N_{11}M_{22})^{-1}N_{12} \end{bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{F}_l(M, \mathcal{F}_l(N, \bullet)) &= \mathcal{F}_l(T, \bullet) \\
\mathcal{F}_u(N, \mathcal{F}_u(M, \bullet)) &= \mathcal{F}_u(T, \bullet).
\end{aligned}
$$

For more extensive results on the properties of LFTs and their manipulation see [5, 41].

Inner transfer functions are used extensively in both $\mathcal{H}_\infty$ synthesis ([4, 28]) and model approximation ([11, 12, 23]).

Definition 2.3 *A stable, $p \times m$ transfer function G, where $p \geq m$, is said to be inner if $G^\sim G = I$.*

Definition 2.4 *A stable, $p \times m$ transfer function G, where $p \leq m$, is said to be co-inner if $GG^\sim = I$.*

The importance of inner functions stems from their norm preserving properties. If the input to an inner function has bounded energy then the output will be bounded by the same energy level.

The following theorem on inner matrices and LFTs is central to our development.

Theorem 2.5 ([12],Theorem 2.3 (f) and (g)) *Consider the feedback system of Figure 2(a) where $N_{11} : p_1 \times m_1$, $N_{22} : p_2 \times m_2$ and all other dimensions are compatible. Assume the feedback loop is well posed, i.e.* $\det(I - N_{22}(\infty)Q(\infty)) \neq 0$. *Then*

(i) If rank$(N_{21}(j\omega)) = p_2$ $\forall\omega \in \mathbb{R} \cup \infty$ with $N^\sim N = I$ then $\|\mathcal{F}_l(N,Q)\|_\infty < 1$ if and only if $\|Q\|_\infty < 1$.

(ii) If rank$(N_{12}(j\omega)) = m_2$ $\forall\omega \in \mathbb{R} \cup \infty$ with $NN^\sim = I$ then $\|\mathcal{F}_l(N,Q)\|_\infty < 1$ if and only if $\|Q\|_\infty < 1$.

In subsequent sections the N of Theorem 2.5 will be stable. In this case, assuming the well-posedness condition is satisfied, N_{21} has full row rank (N_{12} has full column rank) at all frequencies and N is inner (co-inner) then Theorem 2.5 implies that

$$
\{\mathcal{F}_l(N,Q) \in \mathcal{RH}_\infty,\ \|\mathcal{F}_l(N,Q)\|_\infty < 1\} \iff \{Q \in \mathcal{RH}_\infty,\ \|Q\|_\infty < 1\}.
$$

2.1 Weights for Stability Preservation

To motivate the interpretation of controller approximation as a frequency weighted model reduction problem we now present a new derivation of a well known sufficient condition for controllers to be stabilising ([1, 6]). The condition is expressed as a norm bound on the weighted error between a nominal stabilising controller and another given controller, where the weight is chosen according to a specific scheme. The technique developed here will be extended in Section 3 to derive conditions for controllers to be (P,γ)-admissible. (In Section 3 the nominal controller must be (P,γ)-admissible.)

The following lemma is required.

Lemma 2.6 ([13]) *Assume* $R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \in \mathcal{RH}_\infty^{(p_1+p_2)\times(m_1+m_2)}$ *and* $\Delta \in \mathcal{RH}_\infty^{(m_1\times p_1)}$, *then*

$$\|R_{11}\Delta\|_\infty < 1 \quad \Rightarrow \quad \mathcal{F}_u(R,\Delta) \in \mathcal{RH}_\infty.$$

We begin by presenting a parameterisation of all stabilising controllers. Assume F and L are appropriately dimensioned matrices such that $\mathrm{Re}(\lambda_i(A + B_2F)) < 0$ and $\mathrm{Re}(\lambda_i(A + LC_2)) < 0$, then all controllers which stabilise $\mathcal{F}_l(P,K)$ are parameterised as $K = \mathcal{F}_l(J,Q)$ where

$$J = \left[\begin{array}{c|cc} A + B_2F + LC_2 & -L & B_2 \\ \hline F & 0 & I \\ -C_2 & I & 0 \end{array}\right]$$

and $Q \in \mathcal{RH}_\infty$ [41]. There are two important observations to be made about this parameterisation: firstly that $K = \mathcal{F}_l(J,Q)$ satisfies the assumptions of Lemma 2.1, which implies Q may be written as $Q = \mathcal{F}_u(J^{-1},K)$; and secondly that Q is any free parameter satisfying $Q \in \mathcal{RH}_\infty$. Combining these two observations implies that K stabilises P if and only if $\mathcal{F}_u(J^{-1},K) \in \mathcal{RH}_\infty$.

Now if we assume that K may be represented by a nominal stabilising controller, K^0, subject to a stable perturbation Δ, i.e. $K = K^0 + \Delta$, and define $R \triangleq \mathcal{S}(\begin{bmatrix} 0 & I \\ I & K^0 \end{bmatrix}, J^{-1})$ then we have that

$$Q \in \mathcal{RH}_\infty \quad \Longleftrightarrow \quad \mathcal{F}_u(R,\Delta) \in \mathcal{RH}_\infty.$$

This result is shown pictorially in Figure 3.

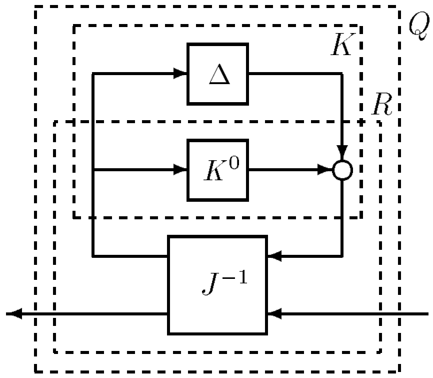

Figure 3: Calculation of Weights to Preserve Stability

Therefore, since R is stable for all stabilising K^0, and Δ is stable by design, from Lemma 2.6 we have that

$$\begin{aligned} \|R_{11}\Delta\|_\infty < 1 \quad &\Rightarrow \quad \mathcal{F}_u(R,\Delta) \in \mathcal{RH}_\infty \\ &\Longleftrightarrow \quad \mathcal{F}_l(P,K) \in \mathcal{RH}_\infty. \end{aligned}$$

This reduces to the well known condition ([1, 6]),

$$\|[I - P_{22}K^0]^{-1}P_{22}[K - K^0]\|_\infty < 1 \quad \Rightarrow \quad \mathcal{F}_l(P,K) \in \mathcal{RH}_\infty.$$

Therefore, letting $W_i = I$ and $W_o = [I - P_{22}K^0]^{-1}P_{22}$ in either the weighted balanced truncation [6], or weighted optimal Hankel norm approximation [24, 15] schemes, gives a tractable frequency weighted approximation problem for the design of reduced order stabilising controllers. Alternatively, this may be written with $W_i = P_{22}[I - K^0P_{22}]^{-1}$ and $W_o = I$ by using the condition $\|\Delta R_{11}\|_\infty < 1$.

3 Synthesis of Reduced Order (P, γ)-admissible Controllers

Given a (P,γ)-admissible controller K^0, in this section we present sufficient conditions for another controller K to be (P,γ)-admissible. The conditions are presented in terms of a norm bound on a particular frequency weighted error, which, for the purpose of reduced order controller synthesis, may be considered as a frequency weighted model reduction problem.

Three approaches, to be denoted $\mathcal{A}(P)$, $\mathcal{CF}(P)$ and $\mathcal{CF}(\Theta)$, are proposed. The reason for this choice of notation will become clear as we progress. A sufficient condition for reduced order controllers to be (P,γ)-admissible, expressed as a norm bound on a particular frequency weighted error, is derived for each approach.

Approaches $\mathcal{A}(P)$ and $\mathcal{CF}(P)$ are reminiscent of that taken by Lenz et al. in [25, 26]. However, whereas the Lenz approach led to a sufficient condition for reduced order controllers to be stabilising and guarantee performance degradation of at most a factor of $\sqrt{2}$, the sufficient conditions derived here eliminate the $\sqrt{2}$ factor, thus leading to reduced order (P,γ)-admissible controllers. In both cases we represent all controllers as a nominal (P,γ)-admissible controller subjected to a weighted perturbation. Although we study additive ($\mathcal{A}(P)$) and coprime factor ($\mathcal{CF}(P)$) perturbations, other perturbation classes could also be considered.

For the $\mathcal{A}(P)$ approach, the sufficient condition is expressed in terms of the weighted error between a (reduced order) controller K and a nominal (P,γ)-admissible controller K^0, e.g.

$$\|W_2^{-1}(K-K^0)W_1^{-1}\|_\infty < 1 \quad \Rightarrow \quad \|\mathcal{F}_l(P,K)\|_\infty < \gamma. \tag{2}$$

For the $\mathcal{CF}(P)$ and $\mathcal{CF}(\Theta)$ approaches, the sufficient conditions are expressed in terms of the weighted error between left coprime factors of a (reduced order) controller and a nominal (P,γ)-admissible controller, e.g.

$$\|W_2^{-1}(\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix} - \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix})W_1^{-1}\|_\infty < 1 \quad \Rightarrow \quad \|\mathcal{F}_l(P,K)\|_\infty < \gamma \tag{3}$$

where $K = \tilde{V}^{-1}\tilde{U}$ and $K^0 = \tilde{V}_0^{-1}\tilde{U}_0$. Note that dual results for right coprime factor perturbations are possible although not discussed here.

Equations (2) and (3) have a natural interpretation as frequency weighted model reduction problems. Given appropriate weights, the goal is to calculate reduced order K or $\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix}$ which lie within an $\mathcal{L}_\infty$ ball (of right radius W_1 and left radius W_2,) of K^0 or $\begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}$ respectively. $K = \tilde{V}^{-1}\tilde{U}$ is then a reduced order (P,γ)-admissible controller. The larger the $\mathcal{L}_\infty$ ball, the easier the approximation. We therefore wish to make W_1 and W_2 large in some sense, and procedures to achieve this are derived in Section 5.

The sufficient conditions for approaches $\mathcal{A}(P)$ and $\mathcal{CF}(P)$ are expressed in terms of either $\mathcal{F}_l(\tilde{R},\Delta)$ or $\mathcal{F}_u(\tilde{R},\Delta)$, where $\tilde{R}$ and Δ are different for each approach. The reason for using similar notation will become clear in Section 4 where we present a procedure for calculating weights such that $\tilde{R}$ is a contraction. This procedure is valid irrespective of which of the two approaches is being considered.

3.1 $\mathcal{A}(P)$ – The Additive Perturbation Case

The approach taken here is a variant of a method originally proposed by Lenz in [25, 26]. Section 3.1 considers the $\mathcal{A}(P)$ approach, while the $\mathcal{CF}(P)$ approach is considered in Section 3.2.

Assume that any controller K may be represented by a nominal (P,γ)-admissible controller K^0, subjected to a weighted additive perturbation, i.e. $K = K^0 + W_2\Delta W_1$, where W_1 and W_2 are stable, minimum phase and invertible and Δ is stable. This assumption imposes the restriction

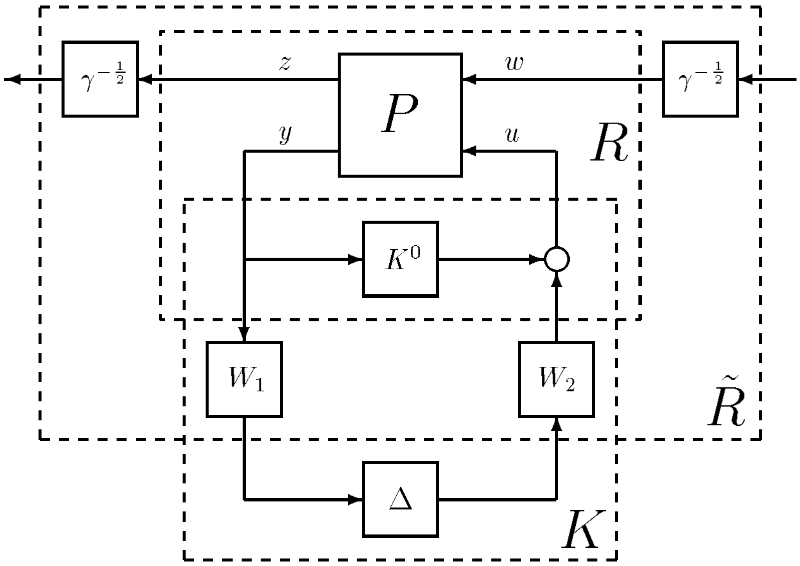

Figure 4: Plant and Controller Interconnection for $\mathcal{A}(P)$

that the unstable part of K^0 is copied directly into K thus to some degree limiting the set of reduced order controllers attainable by our method.
Consider Figure 4. Observe that $\mathcal{F}_l(P,K) = \gamma \mathcal{F}_l(\tilde{R},\Delta)$ and hence

$$\|\mathcal{F}_l(P,K)\|_\infty < \gamma \quad \Longleftrightarrow \quad \|\mathcal{F}_l(\tilde{R},\Delta)\|_\infty < 1 \tag{4}$$

where

$$\tilde{R} = \begin{bmatrix} \gamma^{-1/2}I & 0 \\ 0 & W_1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} \gamma^{-1/2}I & 0 \\ 0 & W_2 \end{bmatrix} \tag{5}$$

and R is given by the Redheffer star product [5, 34],

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \mathcal{S}(P, \begin{bmatrix} K^0 & I \\ I & 0 \end{bmatrix}). \tag{6}$$

The results of Theorem 2.5 can now be applied to equation (4). If stable, minimum phase and invertible weights W_1 and W_2 can be calculated such that $\tilde{R}$ is inner and $\tilde{R}_{21}$ has full row rank (note that this is not generally possible [25]) then

$$\begin{aligned} \|\Delta\|_\infty < 1 \quad & \Longleftrightarrow \quad \|W_2^{-1}(K-K^0)W_1^{-1}\|_\infty < 1 \\ & \Longleftrightarrow \quad \|\mathcal{F}_l(\tilde{R},\Delta)\|_\infty < 1 \\ & \Longleftrightarrow \quad \|\mathcal{F}_l(P,K)\|_\infty < \gamma. \end{aligned}$$

Hence (with weights such that $\tilde{R}$ is inner and $\tilde{R}_{21}$ has full row rank) we obtain the condition that any controller, $K = K^0 + W_2 \Delta W_1$ will also be (P,γ)-admissible if and only if $\|W_2^{-1}(K-K^0)W_1^{-1}\|_\infty < 1$.

Lenz shows that in general it is not possible to calculate weights such that $\tilde{R}$ is inner. However, weights satisfying

$$R_{21}^\sim W_1^\sim W_1 R_{21} = I - R_{11}^\sim R_{11} \tag{7}$$

and

$$W_2 W_2^\sim = (R_{12}^\sim R_{12} + R_{22}^\sim W_1^\sim W_1 R_{22})^{-1} \tag{8}$$

make the columns of $\tilde{R}$, ($\begin{bmatrix} \tilde{R}_{11} \\ \tilde{R}_{21} \end{bmatrix}$ and $\begin{bmatrix} \tilde{R}_{12} \\ \tilde{R}_{22} \end{bmatrix}$ respectively,) inner and lead to the result that

$$\|W_2^{-1}(K - K^0)W_1^{-1}\|_\infty < \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \|\mathcal{F}_l(P, K)\|_\infty < \sqrt{2}\gamma. \tag{9}$$

Equation (9) represents a sufficient condition for controller K to stabilise the closed loop and give a performance bound of $\|\mathcal{F}_l(P, K)\|_\infty < \sqrt{2}\gamma$. This represents a (possible) degradation in performance of $\sqrt{2}$.

The new method proposed in this paper relies on the observation that although it is not possible to calculate weights such that $\tilde{R}$ is inner, it is possible to calculate weights which make $\tilde{R}$ a contraction. Using such weights gives the sufficient condition,

$$\|W_2^{-1}(K - K^0)W_1^{-1}\|_\infty < 1 \quad \Rightarrow \quad \|\mathcal{F}_l(P, K)\|_\infty < \gamma \tag{10}$$

Calculation of appropriate weights is discussed in Section 4.

Remark 3.1 *Notice that if the weights were chosen so that $\tilde{R}$ is expansive (i.e. $\tilde{R}^\sim \tilde{R} \geq I$) with $\tilde{R}_{21}$ full column rank, then*

$$\|\mathcal{F}_l(\tilde{R}, \Delta)\|_\infty < 1 \quad \Rightarrow \quad \|\Delta\|_\infty < 1, \tag{11}$$

which is a necessary condition for K to be (P, γ)-admissible. However, this choice of weights will only be possible if the 1-1 block of $\tilde{R}^{-1}$ is contractive. Since this is not guaranteed, the above approach is not suitable for the derivation of necessary conditions.

Remark 3.2 *In equation (7) and Remark 3.1, in order for $\tilde{R}^\sim \tilde{R} \geq I$ with $\|\tilde{R}_{11}\|_\infty < 1$ requires certain rank, and hence dimension conditions to be satisfied. In particular, since*

$$R_{21}^\sim W_1^\sim W_1 R_{21} \geq I - R_{11}^\sim R_{11} > 0,$$

it is necessary that R_{21} has full row rank, which implies that $p_2 \geq m_1$. This is typically not the case.

An alternative approach which overcomes this rank problem is to consider the LFT parameterisation of all (P, γ)-admissible controllers, $K = \mathcal{F}_l(K_a, \Phi)$ where $\Phi \in \mathcal{H}_\infty$ and $\|\Phi\|_\infty < \gamma$, derived in [4, 14]. In [16] it is shown that

$$\begin{aligned} \Phi &= \mathcal{F}_u(K_a^{-1}, K) \\ &= \mathcal{F}_u(K_a^{-1}, K^0 + W_2 \Delta W_1) \\ &= \gamma \mathcal{F}_u(\tilde{S}, \Delta) \end{aligned} \tag{12}$$

where

$$\tilde{S} = \begin{bmatrix} W_1 & 0 \\ 0 & \gamma^{-1/2} I \end{bmatrix} \mathcal{S}\left(\begin{bmatrix} 0 & I \\ I & K^0 \end{bmatrix}, K_a^{-1} \right) \begin{bmatrix} W_2 & 0 \\ 0 & \gamma^{-1/2} I \end{bmatrix}. \tag{13}$$

Hence

$$\|\Phi\|_\infty < \gamma \quad \Longleftrightarrow \quad \|\mathcal{F}_u(\tilde{S}, \Delta)\|_\infty < 1. \tag{14}$$

From equation (13) both $\tilde{S}_{12}$ and $\tilde{S}_{21}$ will be square. Hence the rank constraint inherent in the original problem is not present in this formulation. The sufficient condition that any controller, $K = K^0 + W_2 \Delta W_1$ for some $\Delta \in \mathcal{RH}_\infty$ and weights which make $\tilde{S}$ a contraction with $\tilde{S}_{12}$ full row rank will also be (P,γ)-admissible if $\|W_2^{-1}(K - K^0)W_1^{-1}\|_\infty < 1$, may therefore be used as an alternative to that used in this paper. It is shown in [16], however, that the two conditions are in fact equivalent.

3.2 $\mathcal{CF}(P)$ – The Coprime Factor Perturbation Case

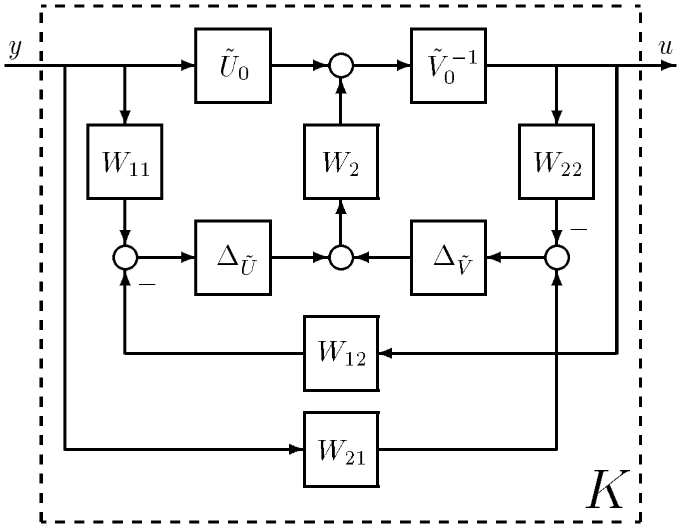

Figure 5: Weighted Controller for Approaches $\mathcal{CF}(P)$ and $\mathcal{CF}(\Phi)$

The development here is similar to that of the previous section, however, whereas for the $\mathcal{A}(P)$ approach all controllers K were represented by a stable weighted additive perturbation to a nominal (P,γ)-admissible controller, here we express all controllers K as a stable weighted perturbation to the coprime factors of a nominal (P,γ)-admissible controller.

The advantage of such coprime factor perturbations is that they do not restrict K and K^0 to possess the same number of unstable poles. This gives freedom to the chosen model reduction procedure to allow poles to move across the $j\omega$ axis. However there is a disadvantage. Since coprime factorisations are only unique to within multiplication by a *unit* in $\mathcal{RH}_\infty$ it is not clear if any particular factorisation will give better results than any other choice. This makes the selection of an initial factorisation of K^0 somewhat arbitrary. In fact, each choice would lead to a different set of possible (P,γ)-admissible controllers.

To begin, assume that the left coprime factors of $K = \tilde{V}^{-1}\tilde{U}$ may be represented by a weighted perturbation to the left coprime factors of a nominal controller $K^0 = \tilde{V}_0^{-1}\tilde{U}_0$, i.e.

$$\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix} = \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix} + W_2 \begin{bmatrix} \Delta_{\tilde{U}} & \Delta_{\tilde{V}} \end{bmatrix} W_1 \tag{15}$$

where W_1 and W_2 are stable, minimum phase and invertible and $\Delta \stackrel{\triangle}{=} \begin{bmatrix} \Delta_{\tilde{U}} & \Delta_{\tilde{V}} \end{bmatrix}$ is stable. This is shown in Figure 5 where

$$W_1 = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}.$$

Substituting the $\tilde{V}^{-1}\tilde{U}$ from Figure 5 (equation (15)) for K in Figure 4 leads to the condition

$$\|\mathcal{F}_l(P,K)\|_\infty < \gamma \quad \Longleftrightarrow \quad \|\mathcal{F}_l(\tilde{R},\Delta)\|_\infty < 1 \tag{16}$$

where

$$\tilde{R} = \begin{bmatrix} \gamma^{-1/2}I & 0 \\ 0 & W_1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} \gamma^{-1/2}I & 0 \\ 0 & W_2 \end{bmatrix} \tag{17}$$

and R is given by the Redheffer star product [5, 34],

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \mathcal{S}(P, \left[\begin{array}{c:c} K^0 & \tilde{V}_0^{-1} \\ \hdashline I & 0 \\ -K^0 & -\tilde{V}_0^{-1} \end{array}\right]). \tag{18}$$

Equation (17) shows that $\tilde{R}_{21}$ can never have full row rank. Hence the necessity part of Theorem 2.5 can never be satisfied. However, we can always calculate weights which make $\tilde{R}$ a contraction. The sufficiency part of Theorem 2.5 then leads to the condition,

$$\begin{aligned} \|\Delta\|_\infty < 1 \quad &\Longleftrightarrow \quad \|W_2^{-1}\{[\ \tilde{U} \quad \tilde{V}\ ] - [\ \tilde{U}_0 \quad \tilde{V}_0\ ]\}W_1^{-1}\|_\infty < 1 \\ &\Rightarrow \quad \|\mathcal{F}_l(\tilde{R}, \Delta)\|_\infty < 1 \\ &\Longleftrightarrow \quad \|\mathcal{F}_l(P, K)\|_\infty < \gamma. \end{aligned}$$

A procedure to calculate appropriate weights is given in Section 4.

Remark 3.3 *For both the $\mathcal{A}(P)$ and $\mathcal{CF}(P)$ approaches, R and $\tilde{R}$ are stable. This follows from R having the same A-matrix as $\mathcal{F}_l(P, K^0) \in \mathcal{RH}_\infty$.*

Remark 3.4 *The nominal controller K^0 may be designed using any methodology provided it is (P,γ)-admissible. Further, for the coprime factor case, any left coprime factorisation of K^0 may be chosen.*

3.3 $\mathcal{CF}(\Theta)$ – An Alternative LFT Approach

In the previous section standard LFT manipulations were employed to derive sufficient conditions for a given controller to be (P,γ)-admissible. These conditions were expressed in terms of a frequency weighted error where the weights were required to be selected to make a certain transfer function, $\tilde{R}$, a contraction.

Here we take an alternative approach which also leads to a sufficient condition for a given controller to be (P,γ)-admissible, expressed as a frequency weighted error. However, in this approach there is no freedom in the selection of weights. They are completely specified by the coefficient matrix K_a.

Firstly, we require the following alternative form of LFT. It results from a trivial modification to Lemma 2.14 of [28].

Lemma 3.5 *Any LFT $\mathcal{F}_l(N,Q)$ such that $\det(N_{12}) \neq 0$ can also be written*

$$T_{zw} = (M_{22} + QM_{12})^{-1}(M_{21} + QM_{11})$$

where $\det(M_{22} + QM_{12}) \neq 0$,

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \mathcal{F}_l(\left[\begin{array}{cc:c} N_{21} & 0 & N_{22} \\ 0 & 0 & -I \\ \hdashline N_{11} & I & N_{12} - D \end{array}\right], -D^{-1})$$

and D is any appropriately dimensioned invertible matrix.

Since K_{a12} is invertible [4, 14], the following form of LFT may be used to parameterise all (P,γ)-admissible controllers.

$$K = (\theta_{22} + \Phi\theta_{12})^{-1}(\theta_{21} + \Phi\theta_{11}) = \tilde{V}^{-1}\tilde{U} \tag{19}$$

where $\|\Phi\|_\infty < \gamma$ and

$$\begin{aligned} \Theta &\triangleq \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} \\ &= \mathcal{F}_l\left(\left[\begin{array}{cc:c} K_{a21} & 0 & K_{a22} \\ 0 & 0 & -I \\ \hdashline K_{a11} & I & K_{a12} - D_{a12} \end{array}\right], -D_{a12}^{-1}\right). \end{aligned} \tag{20}$$

Remark 3.6 *From equation (20) it is simple to show that the A-matrix of Θ is given by $A_a - B_{a2}D_{a12}^{-1}C_{a1}$ and the A-matrix of Θ^{-1} is given by $A_a - B_{a1}D_{a21}^{-1}C_{a2}$. These correspond to the A-matrices of K_{a12} and K_{a21} which are guaranteed to be stable [4, 14]. Hence Θ is stable and minimum phase.*

Proposition 3.7 *Assume there exists a (P,γ)-admissible controller parameterised by equation (19), and define*

$$\Delta = \{[\,\hat{U} \quad \hat{V}\,] - [\,\tilde{U}_0 \quad \tilde{V}_0\,]\}\,\Theta^{-1}\begin{bmatrix} \gamma^{-1}I & 0 \\ 0 & I \end{bmatrix} \in \mathcal{RH}_\infty$$

where $[\,\tilde{U}_0 \quad \tilde{V}_0\,] = [\,\theta_{21} \quad \theta_{22}\,]$. Then the controller $\hat{K} = \hat{V}^{-1}\hat{U}$ will be (P,γ)-admissible if $\|\Delta\|_\infty < 1/\sqrt{2}$. That is,

$$\begin{aligned} \|\{[\,\hat{U} \quad \hat{V}\,] - [\,\tilde{U}_0 \quad \tilde{V}_0\,]\}\,\Theta^{-1}\begin{bmatrix} \gamma^{-1}I & 0 \\ 0 & I \end{bmatrix}\|_\infty < 1/\sqrt{2} \quad &\Rightarrow \quad \|\Phi\|_\infty < \gamma \\ &\Longleftrightarrow \quad \mathcal{F}_l(P,\hat{K}) < \gamma. \end{aligned}$$

Proof: By assumption $\Delta \in \mathcal{RH}_\infty$ with $\|\Delta\|_\infty < 1/\sqrt{2}$. Now

$$[\,\hat{U} \quad \hat{V}\,] = [\,\tilde{U}_0 \quad \tilde{V}_0\,] + \Delta\begin{bmatrix} \gamma I & 0 \\ 0 & I \end{bmatrix}\Theta$$

which implies

$$[\,\hat{U} \quad \hat{V}\,] = [\,\gamma\Delta_1 \quad I + \Delta_2\,]\,\Theta$$

where $[\,\Delta_1 \quad \Delta_2\,] \triangleq \Delta$. Hence

$$\begin{aligned} [\,(I+\Delta_2)^{-1}\hat{U} \quad (I+\Delta_2)^{-1}\hat{V}\,] &= [\,\gamma(I+\Delta_2)^{-1}\Delta_1 \quad I\,]\,\Theta \\ &\triangleq [\,\tilde{U} \quad \tilde{V}\,] \end{aligned}$$

is a coprime factorisation of $\hat{K}$ such that

$$[\,\tilde{U} \quad \tilde{V}\,] = [\,\Phi \quad I\,]\,\Theta. \tag{21}$$

Now

$$
\begin{aligned}
\Phi &= \gamma(I+\Delta_2)^{-1}\Delta_1 \\
&= \gamma\mathcal{F}_l(\left[\begin{array}{c|c} 0 & I/\sqrt{2} \\ \hline I & 0 \\ 0 & -I/\sqrt{2} \end{array}\right], \sqrt{2}\Delta).
\end{aligned} \tag{22}
$$

and by Theorem 2.5,

$$
\|\Delta\|_\infty < 1/\sqrt{2} \quad \Rightarrow \quad \|\Phi\|_\infty < \gamma.
$$

■

Remark 3.8 *Unlike the previous approaches, $\mathcal{CF}(\Theta)$ requires $\left[\begin{array}{cc} \tilde{U}_0 & \tilde{V}_0 \end{array}\right]$ to be a specific coprime factorisation of the central controller.*

4 Weights for Sufficiency

For approaches $\mathcal{A}(P)$ and $\mathcal{CF}(P)$, weights do not generally exist which satisfy the inner requirement on $\tilde{R}$ (equation (5) or (17)). However it is always possible to find weights W_1 and W_2 which make $\tilde{R}$ a contraction and thus satisfy the sufficiency part of Theorem 2.5. (This gives the sufficient conditions derived in previous sections.)

Proposition 4.1 *Assume $\|R_{11}\|_\infty < \gamma$ and define*

$$
L = \left[\begin{array}{cc} L_1 & L_2 \\ L_2^\sim & L_3 \end{array}\right] \triangleq \mathcal{F}_l(\left[\begin{array}{cc|cc} 0 & -R_{22} & 0 & R_{21} \\ -R_{22}^\sim & 0 & R_{12}^\sim & 0 \\ \hline 0 & R_{12} & 0 & -R_{11} \\ R_{21}^\sim & 0 & -R_{11}^\sim & 0 \end{array}\right], \gamma^{-1}I). \tag{23}
$$

Define $X \triangleq (W_1^\sim W_1)^{-1}$ and $Y \triangleq (W_2 W_2^\sim)^{-1}$. Then $\tilde{R}$ (equation (5) or (17)) is a contraction if and only if

$$
\left[\begin{array}{cc} X & 0 \\ 0 & Y \end{array}\right] \geq \left[\begin{array}{cc} L_1 & L_2 \\ L_2^\sim & L_3 \end{array}\right]. \tag{24}
$$

Proof of Proposition 4.1 requires the following theorem on positive semi-definite matrices.

Lemma 4.2 ([21], Theorem 7.7.6) *Suppose that an Hermitian matrix is partitioned as $\left[\begin{array}{cc} A & B \\ B^* & C \end{array}\right]$ where $A = A^*$ and $C = C^* > 0$. This matrix is positive semi-definite if and only if $A - BC^{-1}B^* \geq 0$.*

Proof (of proposition 4.1): We require $\|\tilde{R}\|_\infty \leq 1$. Now

$$
\begin{aligned}
& \|\tilde{R}\|_\infty \leq 1 \\
\Longleftrightarrow \quad & \left[\begin{array}{cc} I & \tilde{R} \\ \tilde{R}^\sim & I \end{array}\right] \geq 0 \\
\Longleftrightarrow \quad & \left[\begin{array}{cc|cc} \gamma^{1/2}I & 0 & 0 & 0 \\ 0 & W_1^{-1} & 0 & 0 \\ \hline 0 & 0 & \gamma^{1/2}I & 0 \\ 0 & 0 & 0 & W_2^{\sim -1} \end{array}\right] \left[\begin{array}{cc} I & \tilde{R} \\ \tilde{R}^\sim & I \end{array}\right] \left[\begin{array}{cc|cc} \gamma^{1/2}I & 0 & 0 & 0 \\ 0 & W_1^{\sim -1} & 0 & 0 \\ \hline 0 & 0 & \gamma^{1/2}I & 0 \\ 0 & 0 & 0 & W_2^{-1} \end{array}\right] \geq 0
\end{aligned}
$$

$$\Longleftrightarrow \quad \left[\begin{array}{cc:cc} \gamma I & 0 & R_{11} & R_{12} \\ 0 & (W_1^\sim W_1)^{-1} & R_{21} & R_{22} \\ \hdashline R_{11}^\sim & R_{21}^\sim & \gamma I & 0 \\ R_{12}^\sim & R_{22}^\sim & 0 & (W_2 W_2^\sim)^{-1} \end{array}\right] \geq 0$$

and by interchanging the second and third rows and columns

$$\Longleftrightarrow \quad \left[\begin{array}{cc:cc} \gamma I & R_{11} & 0 & R_{12} \\ R_{11}^\sim & \gamma I & R_{21}^\sim & 0 \\ \hdashline 0 & R_{21} & X & R_{22} \\ R_{12}^\sim & 0 & R_{22}^\sim & Y \end{array}\right] \geq 0$$

where $X \stackrel{\triangle}{=} (W_1^\sim W_1)^{-1}$ and $Y \stackrel{\triangle}{=} (W_2 W_2^\sim)^{-1}$. Note that

$$\|R_{11}\|_\infty < \gamma \quad \Longleftrightarrow \quad \begin{bmatrix} \gamma I & R_{11} \\ R_{11}^\sim & \gamma I \end{bmatrix} > 0.$$

Therefore Lemma 4.2 implies

$$\begin{aligned} & \|\tilde{R}\|_\infty \leq 1 \\ \Longleftrightarrow \quad & \begin{bmatrix} X & R_{22} \\ R_{22}^\sim & Y \end{bmatrix} \geq \begin{bmatrix} 0 & R_{21} \\ R_{12}^\sim & 0 \end{bmatrix} \begin{bmatrix} \gamma I & R_{11} \\ R_{11}^\sim & \gamma I \end{bmatrix}^{-1} \begin{bmatrix} 0 & R_{12} \\ R_{21}^\sim & 0 \end{bmatrix} \\ \Longleftrightarrow \quad & \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \geq \begin{bmatrix} L_1 & L_2 \\ L_2^\sim & L_3 \end{bmatrix}. \end{aligned}$$

■

Remark 4.3 *Since $R_{11} = \mathcal{F}_l(P, K^0)$ and K^0 is a (P, γ)-admissible controller, it is guaranteed that $\|R_{11}\|_\infty < \gamma$.*

Remark 4.4 *Note from equation (23) that*

$$L = \begin{bmatrix} \gamma R_{21}(\gamma^2 I - R_{11}^\sim R_{11})^{-1} R_{21}^\sim & -(R_{22} + R_{21} R_{11}^\sim (\gamma^2 I - R_{11} R_{11}^\sim)^{-1} R_{12}) \\ -(R_{22}^\sim + R_{12}^\sim R_{11} (\gamma^2 I - R_{11}^\sim R_{11})^{-1} R_{21}^\sim) & \gamma R_{12}^\sim (\gamma^2 I - R_{11} R_{11}^\sim)^{-1} R_{12} \end{bmatrix}.$$

Hence L has the following dimensions and sign (semi-)definiteness.
Approach $\mathcal{A}(P)$: $L_1 \in \mathcal{RL}_\infty^{p_2 \times p_2} > 0$, $L_3 \in \mathcal{RL}_\infty^{m_2 \times m_2} > 0$. To see this notice that from equation (6),

$$R = \begin{bmatrix} P_{11} + P_{12} K^0 (I - P_{22} K^0)^{-1} P_{21} & P_{12}(I - K^0 P_{22})^{-1} \\ (I - P_{22} K^0)^{-1} P_{21} & P_{22}(I - K^0 P_{22})^{-1} \end{bmatrix}.$$

The dimensions of P and required assumptions on the rank of D_{12} and D_{21} ([4, 14]) then give the required result.
Approach $\mathcal{CF}(P)$: $L_1 \in \mathcal{RL}_\infty^{(p_2+m_2)\times(p_2+m_2)} \geq 0$, $L_3 \in \mathcal{RL}_\infty^{m_2 \times m_2} > 0$. To see this notice that from equation (18),

$$R = \begin{bmatrix} P_{11} + P_{12} K^0 (I - P_{22} K^0)^{-1} P_{21} & P_{12}(I - K^0 P_{22})^{-1} \tilde{V}_0^{-1} \\ \begin{bmatrix} I \\ -K^0 \end{bmatrix} (I - P_{22} K^0)^{-1} P_{21} & \begin{bmatrix} 0 \\ -\tilde{V}_0^{-1} \end{bmatrix} + \begin{bmatrix} I \\ -K^0 \end{bmatrix} P_{22}(I - K^0 P_{22})^{-1} \tilde{V}_0^{-1} \end{bmatrix}.$$

As above, the result follows from the dimensions of P and the rank assumptions of D_{12} and D_{21}. From this we can also determine that the rank of L_1 is p_2.

5 Weight Maximisation

In preceding sections, the proposed conditions for designing reduced order (P,γ)-admissible controllers, are expressed as a norm bound on a stable transfer function Δ, where, depending on the problem formulation

$$\Delta \;=\; \begin{cases} W_2^{-1}(K - K^0)W_1^{-1} & \mathcal{A}(P) \text{ approach} \\ W_2^{-1}\left(\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix} - \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}\right)W_1^{-1} & \mathcal{CF}(P) \text{ and } \mathcal{CF}(\Theta) \text{ approaches.} \end{cases}$$

To obtain the set of controllers K such that $\|\Delta\|_\infty < 1$ is equivalent to finding the set of controllers K that lie within a weighted $\mathcal{L}_\infty$ ball of left radius W_2 and right radius W_1 centred on K^0 or $\begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}$ respectively. Hence we require both W_1 and W_2 to be large in some sense to make any subsequent approximation as easy as possible. For the $\mathcal{CF}(\Theta)$ approach the weights are completely specified in the problem formulation. However, for the $\mathcal{A}(P)$ and $\mathcal{CF}(P)$ approaches there is freedom to select the weights subject to the transfer function $\tilde{R}$ being contractive.

In the scalar case maximising $|W_1W_2|$ is appropriate, however in the matrix case the optimal choice is not clear. Here we investigate two satisfactory approaches: firstly to maximise the product $\mathrm{trace}(W_1^\sim W_1)\mathrm{trace}(W_2W_2^\sim)$; and secondly to maximise the product $\det(W_1^\sim W_1)\det(W_2W_2^\sim)$.

Lemma 5.1 (Maximising trace$(W_1^\sim W_1)$trace$(W_2W_2^\sim)$) *Define L as in Proposition 4.1 and assume $L_1 \geq 0$, with trace$(L_1) \neq 0$, and $L_3 > 0$. Also define*

$$\alpha_{opt} \;\triangleq\; \sqrt{(trace(L_3)/trace(L_1))} \tag{25}$$

$$Z_1^2 \;\triangleq\; L_2^\sim L_2 \tag{26}$$

$$Z_2 \;\triangleq\; L_2 Z_1^{-1} L_2^\sim \tag{27}$$

where $Z_1 = Z_1^\sim > 0$ and $Z_2 = Z_2^\sim \geq 0$. Then weights W_1 and W_2 which maximise the product trace$(W_1^\sim W_1)$trace$(W_2W_2^\sim)$ subject to equation (24) satisfy

$$(W_1^\sim W_1)^{-1} \;=\; L_1 + \alpha_{opt}^{-1} Z_2 \tag{28}$$

$$(W_2W_2^\sim)^{-1} \;=\; L_3 + \alpha_{opt} Z_1 \tag{29}$$

Proof: Define $X \triangleq (W_1^\sim W_1)^{-1}$ and $Y \triangleq (W_2W_2^\sim)^{-1}$. We wish to minimise $\mathrm{trace}(X)\mathrm{trace}(Y)$ subject to equation (24).

Given any $Y > L_3$ the unique minimising X is given by

$$X \;=\; L_1 \;+\; L_2(Y - L_3)^{-1}L_2^\sim.$$

Now define

$$f(Y) \;\triangleq\; \mathrm{trace}(X)\mathrm{trace}(Y)$$

and hence, for $\Psi = \Psi^\sim$

$$f(Y+\Psi) \;=\; \mathrm{trace}(Y+\Psi)\mathrm{trace}(L_1 + L_2(Y - L_3 + \Psi)^{-1}L_2^\sim)$$

which, making use of the series expansion $(I + M\Psi)^{-1} = I - M\Psi + o(\Psi)$,

$$\begin{aligned} &= (\mathrm{trace}(Y) + \mathrm{trace}(\Psi))\big(\mathrm{trace}(L_1 + L_2(Y-L_3)^{-1}L_2^\sim) \\ &\qquad - \mathrm{trace}(L_2(Y-L_3)^{-1}\Psi(Y-L_3)^{-1}L_2^\sim)\big) + o(\Psi) \\ &= f(Y) \;+\; \mathrm{trace}(\Psi)\mathrm{trace}(L_1 + L_2(Y-L_3)^{-1}L_2^\sim) \\ &\qquad - \mathrm{trace}(Y)\mathrm{trace}((Y-L_3)^{-1}L_2^\sim L_2(Y-L_3)^{-1}\Psi) \;+\; o(\Psi). \end{aligned}$$

$f(Y)$ is therefore at a stationary point if $\forall \Psi = \Psi^{\sim}$,

$$\text{trace}(\Psi)\text{trace}(L_1 + L_2(Y - L_3)^{-1}L_2^{\sim}) \;=\; \text{trace}(Y)\text{trace}((Y - L_3)^{-1}L_2^{\sim}L_2(Y - L_3)^{-1}\Psi).$$

By successively considering $\Psi = e_i e_i'$, $\Psi = (e_i e_j' + e_j e_i')$ then $\Psi = \sqrt{-1}(e_i e_j' + e_j e_i')$ where e_i are the standard unit vectors with a 1 as the i^{th} element and zeros elsewhere, gives that

$$(Y - L_3)^{-1}L_2^{\sim}L_2(Y - L_3)^{-1} = \alpha^{-2}I$$

for some scalar $\alpha > 0$. Therefore noting that $L_2(L_2^{\sim}L_2)^{-1/2}L_2^{\sim} = (L_2L_2^{\sim})^{1/2}$ for L_2 with full column rank, we have

$$\begin{aligned} Y &= L_3 + \alpha Z_1 \\ X &= L_1 + \alpha^{-1}Z_2 \end{aligned}$$

and

$$\begin{aligned} \text{trace}(Y) &= \text{trace}(L_3) + \alpha\text{trace}(Z_1) \\ \text{trace}(X) &= \text{trace}(L_1) + \alpha^{-1}\text{trace}(Z_1). \end{aligned}$$

Hence

$$f(Y) \;=\; \text{trace}(L_1)\text{trace}(L_3) + (\text{trace}(Z_1))^2 + \text{trace}(Z_1)\left(\alpha\text{trace}(L_1) + \alpha^{-1}\text{trace}(L_3)\right)$$

and is minimised when $\alpha = \alpha_{opt}$. ■

Lemma 5.2 (Maximising $\det(W_1^{\sim}W_1)\det(W_2W_2^{\sim})$**)** *Define L as in Proposition 4.1 and assume $L_1 > 0$ and $L_3 > 0$. Also define $SS^{\sim} \triangleq L_1 > 0$, $T^{\sim}T \triangleq L_3 > 0$,*

$$Z_1^2 \;\triangleq\; S^{-1}L_2L_3^{-1}L_2^{\sim}S^{\sim -1} \tag{30}$$

and

$$Z_2^2 \;\triangleq\; T^{\sim -1}L_2^{\sim}L_1^{-1}L_2T^{-1}, \tag{31}$$

where $Z_1 = Z_1^{\sim} \geq 0$ and $Z_2 = Z_2^{\sim} \geq 0$. Then weights W_1 and W_2 which maximise $\det(W_1^{\sim}W_1)\det(W_2W_2^{\sim})$ *satisfy*

$$(W_1^{\sim}W_1)^{-1} = S(I + Z_1)S^{\sim} = L_1 + SZ_1S^{\sim} \tag{32}$$

$$(W_2W_2^{\sim})^{-1} = T^{\sim}(I + Z_2)T = L_3 + T^{\sim}Z_2T \tag{33}$$

Proof: Define $X = (W_1^{\sim}W_1)^{-1}$ and $Y = (W_2W_2^{\sim})^{-1}$. We wish to minimise $\det(X)\det(Y)$ subject to equation (24).

(a) Firstly assume L_2 has full column rank. Therefore $Y > L_3$ and given a Y the unique minimising X is given by

$$X = L_1 + L_2(Y - L_3)^{-1}L_2^{\sim}.$$

Now define

$$\begin{aligned} f(Y) &\triangleq \frac{\det(X)\det(Y)}{\det(L_1)} \\ &= \frac{\det(Y - L_3 + L_2^{\sim}L_1^{-1}L_2)\det(Y)}{\det(Y - L_3)}. \end{aligned}$$

Therefore, for $\Psi = \Psi^\sim$

$$\begin{aligned} f(Y+\Psi) &= \det(Y)\det(I+Y^{-1}\Psi) \\ &\quad \times \frac{\det(Y-L_3+L_2^\sim L_1^{-1}L_2)}{\det(Y-L_3)} \\ &\quad \times \frac{\det(I+(Y-L_3+L_2^\sim L_1^{-1}L_2)^{-1}\Psi)}{\det(I+(Y-L_3)^{-1}\Psi)}. \end{aligned}$$

In order to find a stationary point observe that

$$(I+M\Psi)^{-1} = I - M\Psi + o(\Psi)$$

and

$$\det(I+M\Psi) = 1 + \mathrm{trace}(M\Psi) + o(\Psi).$$

Hence

$$\begin{aligned} f(Y+\Psi) &= f(Y)\{\, 1 + \mathrm{trace}([\, Y^{-1} + (Y-L_3+L_2^\sim L_1^{-1}L_2)^{-1} \\ &\quad -(Y-L_3)^{-1}\,]\,\Psi) + o(\Psi)\,\}, \end{aligned}$$

giving a stationary point if

$$\mathrm{trace}([\, Y^{-1} + (Y-L_3+L_2^\sim L_1^{-1}L_2)^{-1} - (Y-L_3)^{-1}\,]\,\Psi) = 0 \quad \forall\, \Psi = \Psi^\sim.$$

However for $M = M^\sim$,

$$\mathrm{trace}(M\Psi) = 0 \ \ \forall \Psi = \Psi^\sim \Rightarrow \ M = M^\sim = 0.$$

Hence $f(Y)$ is at a stationary point if

$$\begin{aligned} & (Y-L_3+L_2^\sim L_1^{-1}L_2)^{-1} &= (Y-L_3)^{-1}L_3Y^{-1} \\ \Rightarrow\ & Y-L_3+L_2^\sim L_1^{-1}L_2 &= YL_3^{-1}(Y-L_3) \\ \Rightarrow\ & YL_3^{-1}Y - 2Y + L_3 - L_2^\sim L_1^{-1}L_2 &= 0 \\ \Rightarrow\ & (T^{\sim -1}YT^{-1}-I)^2 &= Z_2^2. \end{aligned}$$

Noting that we require $Y - L_3 > 0$, $\det(X)\det(Y)$ is therefore minimised for

$$Y = T^\sim(I+Z_2)T = L_3 + T^\sim Z_2 T.$$

Now note that $X - L_1 = X - SS^\sim$ and therefore for the above Y

$$\begin{aligned} (S^{-1}XS^{\sim -1}-I)^2 &= (S^{-1}(X-SS^\sim)S^{\sim -1})^2 \\ &= (S^{-1}(X-L_1)S^{\sim -1})^2 \\ &= S^{-1}L_2T^{-1}Z_2^{-1}Z_2^2Z_2^{-1}T^{\sim -1}L_2^\sim S^{\sim -1} \\ &= S^{-1}L_2L_3^{-1}L_2^\sim S^{\sim -1} \\ &= Z_1^2 \end{aligned}$$

or equivalently

$$X = S(I+Z_1)S^\sim = L_1 + SZ_1S^\sim.$$

(b) If L_2 doesn't have full column rank but does have full row rank then $X > L_1$ and appropriate weights can be calculated by working with the function

$$f(X) \triangleq \frac{\det(X)\det(Y)}{\det(L_3)}$$

instead of $f(Y)$.

(c) If L_2 has neither full column rank nor full row rank then consider the associated augmented problem of minimising $\det(X_a)\det(Y)$ such that

$$\begin{bmatrix} X_{a11} & X_{a12} & 0 \\ X_{a12}^{\sim} & X_{a22} & 0 \\ 0 & 0 & Y \end{bmatrix} \geq \begin{bmatrix} L_1 & 0 & L_2 \\ 0 & I & -\varepsilon I \\ L_2^{\sim} & -\varepsilon I & L_3 \end{bmatrix}$$

where $X_a = X_a^{\sim} > 0$, $L_1 = L_1^{\sim} > 0$ and $L_3 = L_3^{\sim} > 0$. This problem satisfies the full column rank assumption of (a) and hence the previous result applies. Then via a limiting argument it is straightforward to show that $\det(X)\det(Y)$ is minimised by X_{a11} and Y as $\varepsilon \to 0$ and that the limits are as in equations (30-33). ■

Remark 5.3 *In general it is not possible to exactly solve transfer function factorisations of the form required to calculate Z_1 and Z_2. However a satisfactory technique is presented in Section 8.*

Remark 5.4 *It is shown in [16, 18] that for approach $\mathcal{CF}(P)$, where coprime factor perturbations are employed, maximising $\det(W_1^{\sim}W_1)\det(W_2W_2^{\sim})$ is not a good choice. This follows from observing that since in this case L_1 is generically rank deficient,the optimal weights require the error $(\tilde{V} - \tilde{V}_0)$ to tend to zero for controller $K = \tilde{V}^{-1}\tilde{U}$ to satisfy the sufficient condition for (P, γ) admissibility.*

6 Pareto Optimal Weights

In this section we show that for the $\mathcal{A}(P)$ approach and a given K^0, maximising the determinant of the weights always leads to a *direction* where the weights are the best possible, and hence cannot be uniformly improved upon in the class considered. This is termed *Pareto* optimal. In [16, 19] it is shown, that in the scalar case, maximising the trace of the weights also leads to a *direction* where the weights are the best possible. However, this is to be expected since in the scalar case the trace and the determinant are equivalent.

Definition 6.1 *W_1 and W_2 are Pareto optimal weights if $\exists \Delta \in \mathcal{RH}_\infty$ such that $\|\Delta\|_\infty = 1$ and $K = K^0 + W_2\Delta W_1$ such that $\|\mathcal{F}_l(P, K)\|_\infty = \gamma$.*

Definition 6.1 implies there exists a direction where our $\mathcal{L}_\infty$ ball cannot be made any larger without K losing (P, γ)-admissibility.

Proposition 6.2 *Consider $\mathcal{F}_l(\tilde{R}, \Delta)$ as given in Figure 6. Define X, Y and L as in Proposition 4.1. If there exists a vector $v \neq 0$ such that*

$$v' \left\{ L_1 - L_2(Y - L_3)^{-1} L_3 (Y - L_3)^{-1} L_2^{\sim} \right\} v = 0,$$

then W_1 and W_2 are Pareto optimal weights.

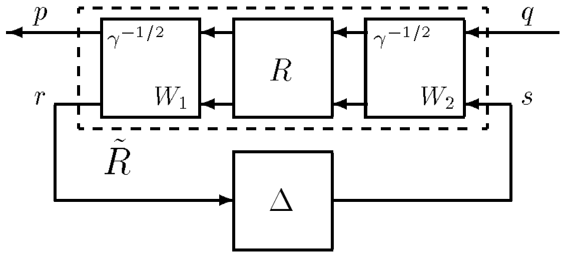

Figure 6: $\mathcal{F}_l(\tilde{R},\Delta)$

Proof: This proof is valid on a frequency by frequency basis. We need to show that $\forall\omega$, $\bar{\sigma}(\tilde{R}) = 1$ and $\exists\ \Delta$ and q such that $|r|^2 = |s|^2$ and $|p|^2 = |q|^2$.

That $\bar{\sigma}(\tilde{R}) = 1$ is guaranteed by selecting weights according to either Proposition 5.1 or Proposition 5.2. Hence we need only show that $\exists\ \Delta$ and q such that $|r|^2 = |s|^2$ (since $|p|^2 = |q|^2$ follows immediately).

Choose q such that $\begin{bmatrix} q \\ s \end{bmatrix}$ is an eigenvector of $\tilde{R}'\tilde{R}$ with unit eigenvalue. Then

$$\begin{bmatrix} q \\ s \end{bmatrix} = \tilde{R}' \begin{bmatrix} p \\ r \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} p \\ r \end{bmatrix} = \tilde{R} \begin{bmatrix} q \\ s \end{bmatrix},$$

or equivalently,

$$\begin{bmatrix} I & \tilde{R} \\ \tilde{R}' & I \end{bmatrix} \left[\begin{array}{c} p \\ r \\ \hdashline -q \\ -s \end{array}\right] = 0. \tag{34}$$

Now, using manipulations similar to the proof of Proposition 4.1, equation (34) implies,

$$\left[\begin{array}{cc:cc} \gamma I & R_{11} & 0 & R_{12} \\ R_{11}' & \gamma I & R_{21}' & 0 \\ \hdashline 0 & R_{21} & X & R_{22} \\ R_{12}' & 0 & R_{22}' & Y \end{array}\right] \left[\begin{array}{c} \bar{p} \\ \bar{r} \\ \hdashline -\bar{q} \\ -\bar{s} \end{array}\right] = 0 \tag{35}$$

where $\begin{bmatrix} \bar{p}' & \bar{r}' & -\bar{q}' & -\bar{s}' \end{bmatrix}' = \begin{bmatrix} \gamma^{1/2}p' & W_1'r' & -\gamma^{-1/2}q' & -W_2s' \end{bmatrix}'$. Therefore,

$$|r|^2 = |s|^2 \quad\iff\quad \bar{r}'X\bar{r} = \bar{s}'Y\bar{s}.$$

Now, premultiplying equation (35) by

$$\begin{bmatrix} I & 0 \\ \begin{bmatrix} 0 & R_{21} \\ R_{12}' & 0 \end{bmatrix} \begin{bmatrix} \gamma & R_{11} \\ R_{11}' & \gamma \end{bmatrix}^{-1} & I \end{bmatrix}$$

implies

$$\left[\begin{array}{cc:cc} \gamma I & R_{11} & 0 & R_{12} \\ R_{11}' & \gamma I & R_{21}' & 0 \\ \hdashline 0 & 0 & X - L_1 & -L_2 \\ 0 & 0 & -L_2' & Y - L_3 \end{array}\right] \left[\begin{array}{c} \bar{p} \\ -\bar{q} \\ \hdashline \bar{r} \\ -\bar{s} \end{array}\right] = 0.$$

Hence, for $v \neq 0$, choose $X = L_1 + L_2(Y - L_3)^{-1} L_2'$ and

$$\begin{bmatrix} \bar{r} \\ -\bar{s} \end{bmatrix} = \begin{bmatrix} I \\ (Y - L_3)^{-1} L_2' \end{bmatrix} v.$$

Now

$$\begin{aligned} \bar{r}' X \bar{r} - \bar{s}' X \bar{s} &= v' \left\{ X - L_2(Y - L_3)^{-1} Y (Y - L_3)^{-1} L_2' \right\} v \\ &= v' \left\{ L_1 - L_2(Y - L_3)^{-1} L_3 (Y - L_3)^{-1} L_2' \right\} v \end{aligned}$$

But by assumption $\forall \omega$ there exists a $v \neq 0$ such that

$$v' \left\{ L_1 - L_2(Y - L_3)^{-1} L_3 (Y - L_3)^{-1} L_2' \right\} v = 0.$$

Hence the result follows. ■

Lemma 6.3 *Assume weights for the $\mathcal{A}(P)$ approach are calculated according to Lemma 5.2 (maximising the product* $\det(W_1^\sim W_1)\det(W_2^\sim W_2)$*), and let $v \triangleq L_1^{-1} L_2 u$. Then $\forall u$,*

$$v' \left\{ L_1 - L_2(Y - L_3)^{-1} L_3 (Y - L_3)^{-1} L_2^\sim \right\} v = 0.$$

Hence, W_1 and W_2 are Pareto optimal weights.

Proof: Assume L_2 has full column rank. (If not, then perform an augmentation similar to that used in the proof of Lemma 5.2.) Using the notation of Lemma 5.2 we have,

$$\begin{aligned} L_1 - L_2(Y - L_3)^{-1} L_3 (Y - L_3)^{-1} L_2^\sim &= L_1 - L_2(T^\sim Z_2 T)^{-1} L_3 (T^\sim Z_2 T)^{-1} L_2^\sim \\ &= L_1 - L_2 T^{-1} Z_2^{-1} T^{\sim -1} L_3 T^{-1} Z_2^{-1} T^{\sim -1} L_2^\sim \\ &= L_1 - L_2 T^{-1} Z_2^{-2} T^{\sim -1} L_2^\sim \\ &= L_1 - L_2 (L_2^\sim L_1^{-1} L_2)^{-1} L_2^\sim \\ &= S \left\{ I - S^{-1} L_2 (L_2^\sim S^{\sim -1} S^{-1} L_2)^{-1} L_2^\sim S^{\sim -1} \right\} S^\sim \\ &= S \left\{ I - F(F^\sim F)^{-1} F^\sim \right\} S^\sim \end{aligned}$$

where $F = S^{-1} L_2$. Now, since $(I - F(F^\sim F)^{-1} F^\sim) F = 0$, letting $v = L_1^{-1} L_2 u$, implies

$$v' \left\{ L_1 - L_2(Y - L_3)^{-1} L_3 (Y - L_3)^{-1} L_2^\sim \right\} v = 0$$

for all vectors u. Hence, by Proposition 6.2, W_1 and W_2 are Pareto optimal weights. ■

Remark 6.4 *Note that Lemma 6.3 holds for general matrix valued weights. Hence there always exists a* direction *where the $\mathcal{L}_\infty$ ball centred on K^0 with left radius W_2 and right radius W_1 is as large as possible such that $\tilde{R}$ remains a contraction.*

7 Computational Issues

There are three principle computational difficulties encountered in implementing the proposed controller approximation schemes. These difficulties are connected with calculating the frequency weights which make the $\tilde{R}$ of reduction approaches $\mathcal{A}(P)$ and $\mathcal{CF}(P)$ a contraction.

Firstly, it is not generally possible to exactly solve factorisations of the form required to calculate Z_1 and Z_2 (equations (26) and (27), or (30) and (31)) parametrically. However, we may get arbitrarily close to an exact solution by employing an appropriate rational approximation. More precisely, assume we wish to approximate $\Psi^{1/2}$ where $\Psi = \Psi^{\sim} \geq 0$[1] by the rational function $r(\Psi) = r^{\sim}(\Psi) \geq 0$ such that $\|r(\Psi) - \Psi^{1/2}\|_{\infty} < \epsilon$ for some $\epsilon > 0$. To accomplish this we may calculate a scalar rational approximation $r(\psi)$ to $\sqrt{\psi}$ where $\psi \geq 0$ such that $|r(\psi) - \sqrt{\psi}| < \epsilon$ then substitute Ψ for ψ.

Care must be taken when calculating this rational approximation to ensure that the resulting weights still make $\tilde{R}$ a contraction. This requires $r(\Psi) \geq \Psi^{1/2}$ which implies weights $\hat{W}_1$ and $\hat{W}_2$ satisfy $\hat{W}_1^{\sim}\hat{W}_1 \leq W_1^{\sim}W_1$ and $\hat{W}_2\hat{W}_2^{\sim} \leq W_2W_2^{\sim}$ and our sufficient condition remains valid when weights $\hat{W}_1$ and $\hat{W}_2$ are used in place of W_1 and W_2.

The following lemma justifies this approach.

Lemma 7.1 *Given two scalar valued functions* $f(\psi) = \sum_{i=0}^{\infty} \alpha_i \psi^i$ *and* $g(\psi) = \sum_{i=0}^{\infty} \beta_i \psi^i$ *such that* $|f(\psi) - g(\psi)| < \varepsilon$ *for* $0 \leq a \leq \psi \leq b$ *then* $\|f(\Psi) - g(\Psi)\|_{\infty} < \varepsilon$ *for any matrix* $\Psi = \Psi^{\sim} \geq 0$ *such that* $a \leq \underline{\sigma}(\Psi) \leq \psi \leq \overline{\sigma}(\Psi) \leq b$.

Proof: The result follows by substituting a singular value decomposition of Ψ into $h(\Psi) = f(\Psi) - g(\Psi)$. ∎

Hence by calculating a rational approximation, $r(\psi)$ to $\sqrt{\psi}$ such that $r(\psi) \geq \sqrt{\psi}$ and $|r(\psi) - \sqrt{\psi}|$ is small , then substituting Z_1^2 and Z_2^2 for ψ, the approximations $r(Z_1) = \hat{Z}_1 \geq Z_1$ and $r(Z_2) = \hat{Z}_2 \geq Z_2$ are obtained. These approximations may then be directly substituted into the equations for $(W_1^{\sim}W_1)^{-1}$ and $(W_2W_2^{\sim})^{-1}$ when solving for $\hat{W}_1$ and $\hat{W}_2$. See [33] and the references therein for suitable rational approximation algorithms.

The second computational difficulty which arises is that the order of the weights may become large. Hence the weights become time consuming to calculate and demanding to manipulate. The growth in state dimension is as follows. Given a generalised plant of order n the order of K_a will be n, the order of K^0 will typically be n, and hence the order of R will be $2n$ and L will be of order $4n$. Therefore, the order of Z_1^2 and Z_2^2 may be up to $12n$ and hence the state dimension of $\hat{Z}_1$ and $\hat{Z}_2$ will be $12n$ times the degree of the scalar approximation considered appropriate to make $|r(\psi) - \sqrt{\psi}|$ small. In the examples to follow second order approximations have been selected. Consequently, even for a plant of moderate order the subsequent spectral factorisations required to calculate $\hat{W}_1$ and $\hat{W}_2$ become demanding. Hence a trade off exists between the order of the weights and the accuracy with which they can be calculated. However, particularly when the Hankel singular values of R tail off quickly, appropriate model reduction may be performed at intermediate stages without introducing unacceptable errors.

The third of the principle computational issues relates to the effectiveness of frequency weighted model reduction techniques. Experience has shown that they tend not to give approximants which even come close to attaining the lower bound without employing some form of parameter optimisation. Experience indicates that although the lower bound is not necessarily attainable a combined model reduction/parameter optimisation approach using the results of frequency weighted model reduction as a starting point for optimisation works well. This will be borne out by our example (Section 8) where the combined Hankel norm/convex optimisation reduction approach presented in [39] is employed. All our optimisation was undertaken using the *Consol-Optcad*[2] package [7].

[1]Note that $(\bullet)^{1/2}$ represents a square root, not a spectral factorisation.

[2]Consol-Optcad is a trademark of the University of Maryland at College Park

8 Design Example

In this section the effectiveness of the proposed approximation techniques are demonstrated by way of a numerical example. Except when specifically stated otherwise, the results shown come from employing a one block frequency weighted optimal Hankel norm reduction procedure.

Here we consider the design of reduced order robust controllers for the longitudinal dynamics of an experimental highly manoeuverable (HIMAT) aeroplane. Initially studied in [35] using LQG synthesis, a linearised model and detailed specifications for $\mathcal{H}_\infty$ design can be found in [2].

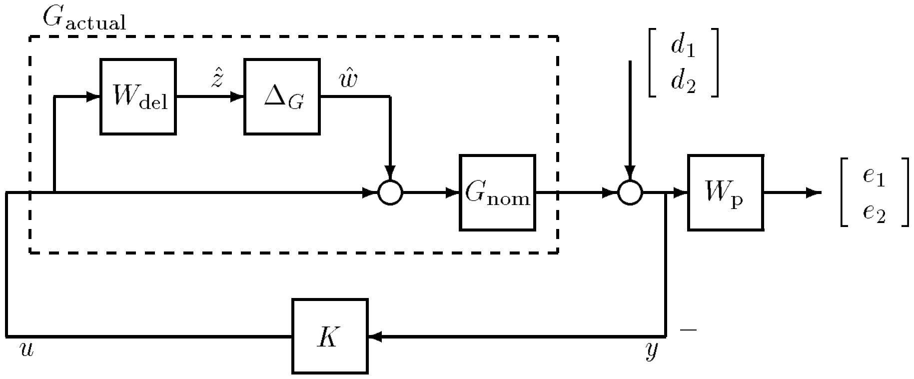

Figure 7: HIMAT Closed Loop Interconnection

The closed loop interconnection is shown in Figure 7. The actual aeroplane is represented by G_{actual}, with the combination of W_{del} and Δ_G reflecting the amount and type of uncertainty in the nominal model G_{nom}. G_{nom} has four states, two control inputs and two measured outputs (angle-of-attack (α) and pitch angle (θ)).

In [2] a 20 state $\mathcal{H}_\infty$ controller is designed using μ-synthesis to give robust performance for the following three maneuvers: Vertical Translation; Pitch Pointing; and Direct Lift. The performance objectives are captured in the weight W_{p}. They correspond to requiring the transfer function from disturbances $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ to errors $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ to be less than unity for all $\|\Delta_G\|_\infty < 1$.

The objective of this section is to design a reduced order controller which preserves robust performance. The first step is to convert the HIMAT interconnection of Figure 7 to the generalised plant and controller LFT interconnection required in our setup. This is shown in Figure 8. The *D scales* result from three so called *D-K iterations* used in the μ-synthesis controller design procedure.

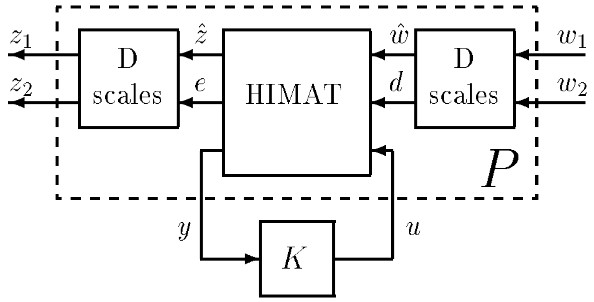

Figure 8: HIMAT Controller Synthesis LFT Interconnection

The generalised plant P has 20 states. K^0 is a 20 state (near optimal) controller such that $\|\mathcal{F}_l(P, K)\|_\infty = 0.97$. Our aim is to maintain robust performance of the unscaled plant, i.e

$\|\mathcal{F}_l(P,K)\|_\infty < 1$.

The results of model reduction are presented in Table 1. The following abbreviations are used:

$\mathcal{A}(P)$	–	Freq. weighted reduction using the $\mathcal{A}(P)$ procedure and maximising the determinant of the weights.
$\mathcal{CF}(P)$	–	Freq. weighted reduction using the $\mathcal{CF}(P)$ procedure and maximising the trace of the weights.
$\mathcal{CF}(\Theta)$	–	Freq. weighted reduction using the $\mathcal{CF}(\Theta)$ procedure.
UOHNA	–	Unweighted optimal Hankel norm approximation.
UBT	–	Unweighted balance and truncate.
LCFR	–	Unweighted optimal Hankel norm reduction of controller left coprime factors.

For the $\mathcal{CF}(\Phi)$ approach, a normalised left coprime factorisation of K^0 was selected as the starting point. For the $\mathcal{CF}(\Theta)$ approach, the specific left coprime factorisation required by Proposition 3.7 was used. For the unweighted methods, the controller of least degree satisfying $\|\mathcal{F}_l(P,K)\|_\infty < 1$ is presented.

Reduction Approach	Controller Order	$\|(\text{weighted error})\|_\infty$	$\|\mathcal{F}_l(P,K)\|_\infty$
	20 (full)	–	0.99
$\mathcal{A}(P)$	14	0.67	0.99
$\mathcal{CF}(P)$	16	0.94	0.98
$\mathcal{CF}(\Theta)$	16	0.03	0.99
UOHNA	13	–	0.98
UBT	13	–	0.98
LCFR	13	–	0.98

Table 1: Results of model reduction (without optimisation)

The results of model reduction without optimisation are disappointing, with the unweighted approaches producing (P,γ)-admissible controllers of lower order than the frequency weighted approaches. However, this is in a large part due to the poor performance of the actual frequency weighted reduction algorithm rather than a deficiency of the proposed approaches. By studying the lower bound on the approximation error (Table 2) the advantage of the approaches we've presented are self-evident. They indicate the possible existence of a seventh order (P,γ)-admissible controller and give an initial starting point for parameter optimisation to find such a controller. However, we stress that since our conditions are only sufficient this by no means guarantees the existence of a seventh order (P,γ)-admissible controller. In fact, the singular values for the $\mathcal{CF}(\Theta)$ approach are so close together and near $1/\sqrt{2}$ that even a ninth order controller is unlikely to be calculated. Table 3 presents the achieved closed loop performance for the seventh order controllers resulting from the $\mathcal{A}(P)$ and $\mathcal{CF}(P)$ approaches, and the tenth order controller resulting from the $\mathcal{CF}(\Theta)$ approach, before undertaking parameter optimisation.

Reduction Approach	Controller Order	Lower bound for $\|(\text{weighted error})\|_\infty$	Actual $\|(\text{weighted error})\|_\infty$	$\|\mathcal{F}_l(P,K)\|_\infty$
	20 (full)	–	–	0.98
$\mathcal{A}(P)$	7	0.44	11.27	1.27
$\mathcal{CF}(P)$	7	0.64	956	7.07
$\mathcal{CF}(\Theta)$	10	0.59	12.44	2.02

i	$\mathcal{A}(P)$	$\mathcal{CF}(P)$	$\mathcal{CF}(\Theta)$
1	22.01	10.05	0.87
2	10.01	8.41	0.73
3	7.58	6.94	0.71
4	3.65	3.76	0.71
5	2.50	2.02	0.71
6	1.44	1.81	0.71
7	1.08	1.05	0.71
8	0.44	0.64	0.70
9	0.30	0.50	0.70
10	0.26	0.41	0.69
11	0.19	0.26	0.59
12	0.15	0.13	0.56
13	0.06	0.19	0.48
14	0.06	0.06	0.41
15	0.04	0.01	0.27
16	.0	.0	0.03
17	.0	.0	0.02
18	.0	.0	.0

Table 2: Lower Bound for Approximation Error

Table 3: Lower Bound for k^{th} Order Approximation

Optimisation[3] worked better for the $\mathcal{A}(P)$ approach than for the $\mathcal{CF}(P)$ or $\mathcal{CF}(\Theta)$ approaches. A seventh order controller satisfying the sufficient condition of approach $\mathcal{A}(P)$ was found (see Table 4), but could not be calculated for the $\mathcal{CF}(P)$ approach. Further, for the $\mathcal{CF}(\Theta)$ approach, a 10^{th} order controller couldn't be found which satisfied the relevant sufficient condition.

Reduction Approach	Controller Order	$\|(\text{weighted error})\|_\infty$	$\|\mathcal{F}_l(P,K)\|_\infty$
$\mathcal{A}(P)$	7	0.79	0.98

Table 4: Seventh Order (P,γ)-admissible Controllers

Hence the nominal 20 state (P,γ)-admissible controller has successfully been reduced down to a 7 state (P,γ)-admissible controller. Controller complexity has therefore been (dramatically) reduced without an associated loss of performance.

Figure 9 gives a plot of μ for the nominal controller K^0 and for the seventh order controller K. As designed, both controllers give $\mu < 1$ and hence perform robustly (for all $\Delta_G \in \mathcal{RH}_\infty$, $\|\Delta_G\|_\infty < 1$) with the unscaled plant. A comparison of frequency domain characteristics and time domain responses for the full and reduced order control laws is given in [16]. They show that no significant degradation is introduced into any of the performance measures.

[3] Here we were required to switch between optimising over the B and the D matrices and the C and the D matrices of K. Both of these optimisations are convex.

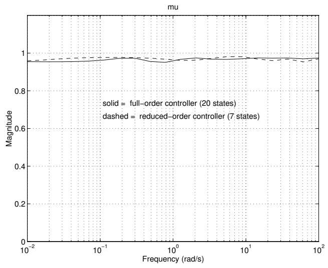

Figure 9: Mu

In [16] the reduction procedures proposed here are successfully applied to controller design for a compact disc mechanism.Also, a comparative study in [40] gave similar conclusions, with the $\mathcal{CF}(\Theta)$ method performing well.

9 Conclusions

Given a (P, γ)-admissible controller K^0, sufficient conditions for any other controller K, to be (P, γ)-admissible have been derived. The conditions are presented in terms of a norm bound on a particular frequency weighted error, which, for the purpose of reduced order controller synthesis, may be considered as a frequency weighted model reduction problem. If this norm bound is satisfied then it is possible to reduce the state-dimension of $\mathcal{H}_\infty$ controllers without degrading performance. However, the approaches are conservative; only searching for (P, γ)-admissible controllers which lie within a weighted $\mathcal{L}_\infty$ ball of the chosen K^0. Different K^0 lead to different K, and there may be (P, γ)-admissible controllers which lie arbitrarily far from K^0.

Two of the proposed approaches require weights to be calculated which make a specific transfer function a contraction. Any weights which meet this constraint may be selected. However, it is advantageous to select weights which are in some sense *large*. Hence procedures to maximise the product of the traces, or the product of the determinants of the weights have been presented. Further, we have shown that for certain cases these procedures lead to weights which are Pareto optimal.

A drawback of these procedures is that a non-standard transfer function factorization is required as an intermediate step. At present this factorisation can't be performed exactly, however a satisfactory solution, which can be made arbitrarily close to the exact solution, is presented.

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