Approximation of Linear Parameter-Varying Systems

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Abstract

Two well known LTI approximation methods – balanced truncation and optimal Hankel norm approximation – are extended to the LPV framework. Quadratic stability of the approximants and generalisations of known LTI error bounds are examined. The concept of the graph operator of an LPV system is discussed. We show how, by approximating the graph operator, the state-dimension of models that are not quadratically stable may be reduced.

1 Introduction

The trade-off between model accuracy and model simplicity is a central theme of model-based control theory. For linear finite-dimensional systems a natural measure of complexity is the state-dimension of a minimal realisation representing the given system. Since optimal control techniques, such as LQG and H_∞ synthesis, generally produce controllers with the same state-dimension as the models from which they are designed it is evident that the use of these design techniques on models having a high state-dimension will produce controllers having a high state-dimension.

In this paper we examine a number of approximation techniques in the light of recent advances in the control of linear parameter-varying (LPV) systems [2, 3, 4, 7, 8]. A drawback of LPV controller synthesis is that the computational requirements increase rapidly as a function of the state-dimension [2], even more so than for LTI controller synthesis. Hence it is important that the state-dimension be kept as low as possible. This motivates the development of approximation techniques for LPV models.

In the sequel we introduce the concept of a balanced LPV model, and study those approximation techniques which rely on system balancing – namely balanced truncation and optimal Hankel norm approximation. We demonstrate that many, although by no means all, of the concepts from approximation of LTI systems carry over into the LPV framework. For LPV model approximation we see that it is possible to bound the approximation error from above. However, unlike LTI approximation, the existence of a (non-zero) lower bound has yet to be proven. And although the Hankel norm approach considered here for LPV approximation has its foundations in LTI optimal Hankel norm approximation, an interpretation of it being optimal in any sense is no longer clear.

We begin by defining LPV systems and some of their important properties in Section 2. Balanced realisations for LPV systems are introduced in Section 3. In Sections 4 we extend the standard LTI technique of balanced trunctation and provide an error bound for the truncation procedure. In Section 5 a Hankel approach is motivated on the basis that it gives the same error bound. Then in Section 6 we study the graph operator of an LPV system, showing how it may be used for approximating LPV systems which are not quadratically stable. Important computational issues are discussed in Section 7. Where results are stated without proof, the proof may be found in [15].

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2 Preliminaries

Linear parameter-varying systems are a special class of time-varying systems where the time dependence enters the state equations through one, possibly more, exogenous parameters. (For a comprehensive discussion of LPV systems and their properties see [2, 4, 7] and the references therein.) Consider a system which has a state-space realisation given by

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) + B(\rho(t))u(t) \\ y(t) &= C(\rho(t))x(t) + D(\rho(t))u(t) \end{aligned} \tag{1}$$

where $A : \mathbb{R}^s \mapsto \mathbb{R}^{n\times n}, B : \mathbb{R}^s \mapsto \mathbb{R}^{n\times m}, C : \mathbb{R}^s \mapsto \mathbb{R}^{p\times n}$ and $D : \mathbb{R}^s \mapsto \mathbb{R}^{p\times m}$ are continuous functions of the parameter vector $\rho \in \mathbb{R}^s$. Note that there is no assumption that the parameter dependence exhibited by the state-space matrices is linear.

Definition 2.1 *Define the set of feasible parameter trajectories F_ρ to be a subset of all piecewise continuous functions $C^0 : \mathbb{R} \mapsto \mathbb{R}^s$ according to*

$$F_\rho \triangleq \{\rho(t) : \mathbb{R} \mapsto \mathbb{R}^s, \rho_{i_{\min}} \le \rho_i \le \rho_{i_{\max}}, i = 1, 2 \ldots s\}.$$

Continuity of the state-space matrices implies boundedness on compact subsets of $\mathbb{R}^s$ and this ensures that for each $\rho(t) \in F_\rho$ the state transition matrix, denoted $\Phi_\rho(t,\tau)$, is unique and continuous.

Definition 2.2 *Given a state-space representation of an LPV system the linear operator $P_\rho : L^+_{2,e} \mapsto L^+_{2,e},\ u(t) \mapsto y(t)$ is defined as*

$$y(t) = \int_0^t C(\rho(t))\Phi_\rho(t,\tau)B(\rho(\tau))u(\tau)d\tau + D(\rho(t))u(t).$$

Definition 2.3 *The LPV system P_ρ with state-space matrices given by equation (1) is quadratically stable if there exists a real positive-definite matrix $P = P^T > 0$ such that*

$$A^T(\rho)P + PA(\rho) < 0 \qquad \forall \ \rho(t) \in F_\rho.$$

Next we define the notions of quadratic stabilisability, detectability and performance for an LPV system.

Definition 2.4 *The LPV system P_ρ with state-space matrices given by equation (1) is quadratically stabilisable if there exists a continuous matrix function $F(\rho) : \mathbb{R}^s \mapsto \mathbb{R}^{m\times n}$, and a constant positive-definite matrix P, such that*

$$(A(\rho) + B(\rho)F(\rho))^T P + P(A(\rho) + B(\rho)F(\rho))) < 0 \ \forall\, \rho(t) \in F_\rho.$$

The dual definition of quadratic detectability is made in terms of a stabilising output injection.

Proposition 2.5 *The LPV system P_ρ with state-space matrices in equation (1) is quadratically stable and satisfies,*

$$\|P_\rho\|_{i,2} = \sup_{\rho(t)\in F_\rho} \sup_{u(t)\in L_2^+} \frac{\|P_\rho u\|_2}{\|u\|_2} \leq \gamma,$$

if there exists a constant positive-definite matrix X such that

$$XA(\rho)+A^T(\rho)X+C^T(\rho)C(\rho)+(XB(\rho)+C^T(\rho)D(\rho))\times$$
$$(\gamma^2 I-D^T(\rho)D(\rho))^{-1}(B^T(\rho)X+D^T(\rho)C(\rho))<0, \quad \rho(t)\in F_\rho.$$

3 Balancing of LPV systems

We now present two results which provide a geometric interpretation of LPV systems and an intuitive basis for their balancing.

Theorem 3.1 *Given a continuous state-space realisation of an LPV system P_ρ, an initial state $x(t_0)=x_0$ and a constant matrix $Q=Q^T>0$ satisfying*

$$A^T(\rho)Q+QA(\rho)+C^T(\rho)C(\rho)<0 \quad \forall \quad \rho(t)\in F_\rho, \qquad (2)$$

then

i. the LPV system P_ρ is quadratically stable;

ii. when $u(t)=0$ for all $t\geq 0$ the energy in the output signal is bounded, from above, according to

$$\|y\|_2^2 < x_0^T Q x_0.$$

Next we consider the largest set of state vectors $x(0)=x_0$ reachable with a unit energy input $u(t)\in L_2^-$.

Theorem 3.2 *Given a continuous state-space realisation of an LPV system P_ρ, an initial state $x(t_0)=x_0$ and a real constant matrix $P=P^T>0$ satisfying*

$$A(\rho)P+PA^T(\rho)+B(\rho)B^T(\rho)<0 \;\; \forall \;\; \rho(t)\in F_\rho, \qquad (3)$$

then

i. the LPV system P_ρ is quadratically stable;

ii. the energy required to drive the system from $x(-\infty)=0$ to $x(0)=x_0$ with the input $u(t)\in L_2^-$ is bounded, from below, according to

$$\|u\|_2^2 > x_0^T P^{-1} x_0.$$

Remark 3.3 *Non-strict inequalities could be used in equations (2) and (3) with $P\geq 0$ and $Q\geq 0$. However, as shown by Beck et al. [1], such a system may be reduced (by truncation of states) to one satisfying equations (2) and (3) without any reduction error.*

Henceforth we refer to a Q satisfying equation (2) as a parameter-varying observability Gramian and a P satisfying equation (3) as a parameter-varying controllability Gramian[1].

Lemma 3.4 *Given a continuous state-space realisation of an LPV system P_ρ, an observability Gramian Q, a controllability Gramian P and a constant state transformation matrix T then*

$$\begin{aligned}\tilde{Q} &= T^TQT\\ \tilde{P} &= T^{-1}PT^{-T}\end{aligned}$$

are observability and controllability Gramians for the transformed system respectively.

[1] Observe that we have taken care to emphasise the inherent lack of uniqueness in the solutions of equations (2) and (3). If Q satisfies equation (2) then so does αQ for any constant $\alpha\geq 1$. A similar conclusion holds for P.

Definition 3.5 *Given an n-state quadratically stable LPV system P_ρ, together with the Gramians P and Q, define*

$$\sigma_i = \sqrt{\lambda_i(QP)} \quad \sigma_1\geq\sigma_2\geq\ldots\sigma_n>0.$$

to be $\mathcal{Q}_e$ singular values.

For LTI systems if we replace the inequalities by equalities then the $\mathcal{Q}_e$ singular values are the unique input-output invariants referred to as Hankel singular values. Although not unique, the $\mathcal{Q}_e$ singular values do provide useful insight into the input-output mapping of a given LPV system; in particular, they bound the mapping from past inputs to future outputs.

Proposition 3.6 *The $\mathcal{Q}_e$ singular values bound, from above, the following norm of the operator P_ρ*

$$\|P_\rho\|_H = \sup_{\rho\in F_\rho}\sup_{u\in L_2^-}\frac{\|\Pi^+P_\rho u\|_2}{\|u\|_2}\leq\sigma_1.$$

We can now define precisely what we mean by a balanced parameter-varying realisation.

Proposition 3.7 *Given a continuous state-space realisation of an LPV system P_ρ, an observability Gramian Q and a controllability Gramian P, then it is possible to find a constant state transformation matrix T such that the transformed Gramians $\tilde{P}=\tilde{Q}=\Sigma$. Σ is a diagonal matrix which has the $\mathcal{Q}_e$ singular values arranged along its diagonal in descending order $\sigma_1\geq\sigma_2\geq\sigma_3\geq\ldots\sigma_n>0$.*

Remark 3.8 *Explicit rate bounds can be included in the model reduction formulation by making both P and Q parameter-dependent. In this case the existence of an analytic balancing transformation matrix[2] is less obvious. Details can be found in [15].*

Definition 3.9 *Given a continuous realisation of an LPV system and a balancing state transformation matrix T such that $\tilde{P}=\tilde{Q}=\Sigma$, define the balanced parameter-varying realisation as follows*

$$P_\rho \stackrel{s}{=} \left[\begin{array}{c|c} T^{-1}A(\rho)T & T^{-1}B(\rho)\\ \hline C(\rho)T & D(\rho)\end{array}\right].$$

4 LPV Balanced Truncation

The balanced realisation introduced in the preceding section leads immediately to a balanced truncation procedure for parameter-varying systems.

Lemma 4.1 *Assume P_ρ is an n-state, quadratically stable, balanced LPV system partitioned as follows*

$$P_\rho \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11}(\rho) & A_{12}(\rho) & B_1(\rho)\\ A_{21}(\rho) & A_{22}(\rho) & B_2(\rho)\\ \hline C_1(\rho) & C_2(\rho) & D(\rho)\end{array}\right],$$

where $A_{11}\in\mathbb{R}^{r\times r}, A_{12}\in\mathbb{R}^{r\times(n-r)}, A_{21}\in\mathbb{R}^{(n-r)\times r}, A_{22}\in\mathbb{R}^{(n-r)\times(n-r)}, B_1\in\mathbb{R}^{r\times m}, B_2\in\mathbb{R}^{(n-r)\times m}, C_1\in\mathbb{R}^{p\times r}$ and $C_2\in\mathbb{R}^{p\times(n-r)}$. Then

$$\hat{P}_\rho \stackrel{s}{=} \left[\begin{array}{c|c} A_{11}(\rho) & B_1(\rho)\\ \hline C_1(\rho) & D(\rho)\end{array}\right]$$

is an r-state, quadratically stable, balanced approximation to P_ρ and

$$\|P_\rho-\hat{P}_\rho\|_{i,2} \;\;\leq\;\; 2\sum_{j=r+1}^{n}\sigma_j.$$

[2] When rate bounds are included the transformation matrix must be differentiable.

An equivalent result for balanced truncation of discrete $\mathcal{Q}$-balanced systems is presented in [14, 1]. The proof follows similar lines to that presented in [16] for the LTI case.
Proof: Consider the case where only those states corresponding to the smallest $\mathcal{Q}_e$-singular value are truncated. Since P_ρ is balanced we have

$$\begin{aligned} A(\rho)\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \sigma I \end{bmatrix} + \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \sigma I \end{bmatrix} A^T(\rho) + B(\rho)B^T(\rho) &< 0 \\ A(\rho)^T\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \sigma I \end{bmatrix} + \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \sigma I \end{bmatrix} A(\rho) + C^T(\rho)C(\rho) &< 0. \end{aligned}$$

Conformably partition P_ρ with $\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \sigma I \end{bmatrix}$ such that

$$P_\rho = \left[\begin{array}{cc|c} A_{11}(\rho) & A_{12}(\rho) & B_1(\rho) \\ A_{21}(\rho) & A_{22}(\rho) & B_2(\rho) \\ \hline C_1(\rho) & C_2(\rho) & D(\rho) \end{array}\right]$$

and define

$$\hat{P}_\rho \triangleq \left[\begin{array}{c|c} A_{11}(\rho) & B_1(\rho) \\ \hline C_1(\rho) & D(\rho) \end{array}\right].$$

Then it is trivial to show that Σ_1 satisfies both the controllability and observability Lyapunov inequalities for $\hat{P}_\rho$ and hence $\hat{P}_\rho$ is quadratically stable.
To prove the error bound, define

$$\begin{aligned} E_\rho &\triangleq P_\rho - \hat{P}_\rho \\ &= \left[\begin{array}{ccc|c} A_{11}(\rho) & 0 & 0 & B_1(\rho) \\ 0 & A_{11}(\rho) & A_{12}(\rho) & B_1(\rho) \\ 0 & A_{21}(\rho) & A_{22}(\rho) & B_2(\rho) \\ \hline -C_1(\rho) & C_1(\rho) & C_2(\rho) & 0 \end{array}\right]. \end{aligned}$$

Then use the matrices

$$T^{-1} = \begin{bmatrix} \frac{1}{2}I & \frac{1}{2}I & 0 \\ \frac{1}{2}I & -\frac{1}{2}I & 0 \\ 0 & 0 & I \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} I & I & 0 \\ I & -I & 0 \\ 0 & 0 & I \end{bmatrix}$$

to perform a state transformation on E_ρ to obtain

$$\begin{aligned} E_\rho &= \left[\begin{array}{ccc|c} A_{11}(\rho) & 0 & \frac{1}{2}A_{12}(\rho) & B_1(\rho) \\ 0 & A_{11}(\rho) & -\frac{1}{2}A_{12}(\rho) & 0 \\ A_{21}(\rho) & -A_{21}(\rho) & A_{22}(\rho) & B_2(\rho) \\ \hline 0 & -2C_1(\rho) & C_1(\rho) & 0 \end{array}\right] \\ &\triangleq \left[\begin{array}{c|c} A_e(\rho) & B_e(\rho) \\ \hline C_e(\rho) & D_e(\rho) \end{array}\right]. \end{aligned}$$

We need to show that $\|E_\rho\|_{i,2} \le 2\sigma$, or equivalently, by Lemma 2.5, that there exists a matrix $P_e = P_e^T > 0$ such that

$$A_e(\rho)P_e + P_eA_e^T(\rho) + B_e(\rho)B_e^T(\rho) + \frac{1}{4\sigma^2}P_eC_e^T(\rho)C_e(\rho)P_e < 0.$$

We'll now show that a suitable choice is

$$P_e = \begin{bmatrix} \Sigma & 0 & 0 \\ 0 & \sigma^2\Sigma^{-1} & 0 \\ 0 & 0 & 2\sigma I \end{bmatrix}.$$

Firstly, note that with this choice of P_e

$$\frac{1}{2\sigma}P_eC_e^T(\rho) = \begin{bmatrix} 0 \\ -\sigma\Sigma^{-1}C_1^T(\rho) \\ \sigma C_2^T(\rho) \end{bmatrix}.$$

Now, using matrix manipulation, it is simple to show that

$$\begin{aligned} &A_e(\rho)P_e + P_eA_e(\rho) + B_e(\rho)B_e^T(\rho) + \frac{1}{4\sigma^2}P_eC_e^T(\rho)C_e(\rho)P_e = \\ &\begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}\left\{A(\rho)P + PA^T(\rho) + B(\rho)B^T(\rho)\right\}\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \\ &+\begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}\left\{A^T(\rho)Q + QA(\rho) + C^T(\rho)C(\rho)\right\}\begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \end{aligned}$$

which by assumption is negative definite. Therefore the chosen P_e satisfies the requirements and we have

$$\|P_\rho - \hat{P}_\rho\|_{i,2} \le 2\sigma.$$

The general error bound for truncation of more than just the states corresponding to the smallest $\mathcal{Q}_e$-singular value results from recursive use of the above procedure. ■

Corollary 4.2 *The balanced model approximant $\hat{P}_\rho$ of a quadratically stable LPV system P_ρ is itself balanced.*

Remark 4.3 *The above procedure is amenable to the frequency weighting procedure proposed by Enns [5] for LTI systems. The weights can be used to filter the input and output spaces to emphasis specific frequency ranges of interest, see [15].*

5 Hankel Norm Approach

Motivated by the standard LTI optimal Hankel norm approximation procedure [9], we propose a similar reduction scheme for LPV systems. However, we stress that unlike its LTI counterpart, the interpretation of this scheme as being optimal in any sense isn't clear. We retain this name because we use similar manipulations to the LTI counterpart and the result reduces to the LTI result when the plant under consideration is purely LTI.

Lemma 5.1 *Assume P_ρ, of state-dimension n, is quadratically stable and balanced such that*

$$\begin{aligned} A(\rho)P + PA^T(\rho) + B(\rho)B^T(\rho) &< 0 \\ A^T(\rho)Q + QA(\rho) + C^T(\rho)C(\rho) &< 0 \end{aligned}$$

where

$$P = Q = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \sigma I \end{bmatrix} > 0$$

with $\Sigma_1 \in \mathbb{R}^{k\times k}$ diagonal ($k < n$) and $\sigma \ne 0$. Now augment $B(\rho)$, $C(\rho)$ and $D(\rho)$ such that

$$\begin{aligned} B_a(\rho) &= \begin{bmatrix} B(\rho) & \tilde{B}_a(\rho) \end{bmatrix} \\ C_a(\rho) &= \begin{bmatrix} C(\rho) \\ \tilde{C}_a(\rho) \end{bmatrix} \end{aligned}$$

satisfy

$$\begin{aligned} A(\rho)P + PA^T(\rho) + B_a(\rho)B_a^T(\rho) &= 0 \\ A^T(\rho)Q + QA(\rho) + C_a^T(\rho)C_a(\rho) &= 0 \end{aligned}$$

and

$$D_a(\rho) = \begin{bmatrix} D(\rho) & 0 \\ 0 & 0 \end{bmatrix}.$$

Conformably partition $A(\rho)$, $B_a(\rho)$, $C_a(\rho)$ with P, as

$$\begin{aligned} A_\rho &= \begin{bmatrix} A_{11}(\rho) & A_{12}(\rho) \\ A_{21}(\rho) & A_{22}(\rho) \end{bmatrix} \quad B_a(\rho) = \begin{bmatrix} B_{a1}(\rho) \\ B_{a2}(\rho) \end{bmatrix} \\ C_a(\rho) &= \begin{bmatrix} C_{a1}(\rho) & C_{a2}(\rho) \end{bmatrix} \end{aligned}$$

and define

$$\hat{A}(\rho) \triangleq \Gamma^{-1}(\sigma^2 A_{11}^T(\rho) + \Sigma_1 A_{11}(\rho)\Sigma_1 - \sigma C_{a1}^T(\rho)U(\rho)B_{a1}^T(\rho))$$
$$\hat{B}_a(\rho) \triangleq \Gamma^{-1}(\Sigma_1 B_{a1}(\rho) + \sigma C_{a1}^T(\rho)U(\rho))$$
$$\hat{C}_a(\rho) \triangleq C_{a1}(\rho)\Sigma_1 + \sigma U(\rho)B_{a1}^T(\rho)$$
$$\hat{D}_a(\rho) \triangleq D_a(\rho) - \sigma U(\rho)$$

where $U(\rho)$ is a unitary matrix satisfying

$$B_{a2}(\rho) = -C_{a2}^T(\rho)U(\rho)$$

and Γ is defined as

$$\Gamma \triangleq \Sigma_1^2 - \sigma^2 I.$$

Then $\hat{P}_\rho$ with realisation $(\hat{A}(\rho), \hat{B}(\rho), \hat{C}(\rho), \hat{D}(\rho))$ has state-dimension k, is quadratically stable and satisfies

$$\|P_\rho - \hat{P}_\rho\|_{i,2} \le \sigma$$

where $\hat{B}(\rho)$, $\hat{C}(\rho)$ and $\hat{D}(\rho)$ are obtained from conformably partitioning, $\hat{B}_a(\rho)$, $\hat{C}_a(\rho)$ and $\hat{D}_a(\rho)$ with $B_a(\rho)$ and $C_a(\rho)$ as

$$\hat{B}_a(\rho) = \begin{bmatrix} \hat{B}(\rho) & \hat{B}_{a2}(\rho) \end{bmatrix} \qquad \hat{C}_a(\rho) = \begin{bmatrix} \hat{C}(\rho) \\ \hat{C}_{a2}(\rho) \end{bmatrix}$$
$$\hat{D}_a(\rho) = \begin{bmatrix} \hat{D}(\rho) & \hat{D}_{12}(\rho) \\ \hat{D}_{21}(\rho) & \hat{D}_{22}(\rho) \end{bmatrix}.$$

Proof: Theorem 6.3 of [9] gives that the error between the two systems with state matrices $(A(\rho), B(\rho), C(\rho), D_a(\rho))$ and $(\hat{A}(\rho), \hat{B}_a(\rho), \hat{C}_a(\rho), \hat{D}_a(\rho))$ as σ. The result follows by noticing that $P_\rho - \hat{P}_\rho$ is a submatrix of the above error. ■

Remark 5.2 *In general, the matrix $U(\rho)$ will be a complex function of ρ. Hence the approximation will potentially be a complex function of ρ and difficult to write explicitly. Indeed, the complexity introduced by this formulation may exceed any gains obtained by reducing the order.*

6 Approximation in the graph topology

A limitation with the techniques developed thus far is that the model to be approximated must be quadratically stable. If P_ρ is not quadratically stable then it cannot be approximated using the machinery of previous sections. The extension of balanced truncation to approximation of unstable LTI systems by approximating the symbol of the system's graph has been studied by a number of authors. In this section we extend some of the standard results on coprime factorisation of LTI systems to parameter-varying systems. In so doing we construct a symbol for the system's graph which can be approximated using the machinery already developed. For this purpose any coprime factorisation would suffice but we concentrate on what we call a contractive coprime factorisation (CCF). This is analogous to the normalised coprime factorisation of LTI systems and its extension to LTV systems in [13]. For LPV systems we cannot satisfy a normalisation condition for all feasible parameter trajectories, hence the notion of a contractive coprime factorisation. We mention that there is an attractive duality between the Gramians of a CCF and the solutions of certain Riccati inequalities. This duality has been exploited to develop an efficient technique for reduced order controller synthesis in [15].

Definition 6.1 *Let $\mathcal{S}_F$ denote the ring of all causal, quadratically stable, finite-dimensional LPV systems defined on the underlying feasible parameter set F_ρ. Denote by $\mathcal{S}_F^-$ the elements in $\mathcal{S}_F$ that have causal inverses.*

Our objective is to obtain a coprime factorisation of the LPV system P_ρ over the ring $\mathcal{S}_F$.

Lemma 6.2 *Let P_ρ have a continuous, quadratically stabilisable and quadratically detectable state-space realisation and let $F(\rho)$ and $L(\rho)$ be stabilising parameter-dependent state feedback and output injection matrices. Ignoring ρ dependence define*

$$\begin{bmatrix} N_\rho & \tilde{Y}_\rho \\ M_\rho & \tilde{X}_\rho \end{bmatrix} \triangleq \left[\begin{array}{c|cc} A+BF & B & -L \\ \hline C+DF & D & I \\ F & I & 0 \end{array}\right]$$
$$\begin{bmatrix} X_\rho & Y_\rho \\ \tilde{M}_\rho & -\tilde{N}_\rho \end{bmatrix} \triangleq \left[\begin{array}{c|cc} A+LC & L & -(B+LD) \\ \hline F & 0 & I \\ C & I & -D \end{array}\right],$$

then

$$\begin{bmatrix} X_\rho & Y_\rho \\ \tilde{M}_\rho & -\tilde{N}_\rho \end{bmatrix}\begin{bmatrix} N_\rho & \tilde{Y}_\rho \\ M_\rho & \tilde{X}_\rho \end{bmatrix} = I.$$

Definition 6.3 *Let $N_\rho \in \mathcal{S}_F$, $M_\rho \in S_F^-$ have the same number of columns. The ordered pair $[N_\rho, M_\rho]$ represents a contractive right coprime factorisation (CRCF) of P_ρ over S_F, if*

i. $P_\rho = N_\rho M_\rho^{-1}$;

ii. there exist $X_\rho, Y_\rho \in \mathcal{S}_F$ such that $X_\rho N_\rho + Y_\rho M_\rho = I$;

iii. $[N_\rho^T \ \ M_\rho^T]^T$ is a contraction in the following sense

$$\sup_{\rho \in F_\rho} \sup_{\{u \in L_2^+ : \|u\|_2 \le 1\}} \left\| \begin{bmatrix} N_\rho \\ M_\rho \end{bmatrix} u \right\|_{i,2} \le 1.$$

Definition 6.4 *Define the contractive right graph symbol $\mathcal{G}_\rho : L_2^+ \mapsto L_2^+ \otimes L_2^+$ of the LPV system P_ρ as follows*

$$\mathcal{G}_\rho \triangleq \begin{bmatrix} N_\rho \\ M_\rho \end{bmatrix},$$

where $[N_\rho, M_\rho]$ is a CRCF of P_ρ.

It should be immediately obvious that $\mathcal{G}_\rho$ generates the set of all stable input-output pairs of the given LPV system by allowing $\mathcal{G}_\rho$ to act on the whole of L_2^+.

Theorem 6.5 *Let P_ρ have a continuous, quadratically stabilisable quadratically detectable realisation, then a contractive right graph symbol of P_ρ is given by*

$$\mathcal{G}_\rho \triangleq \left[\begin{array}{c|c} A(\rho)+B(\rho)F(\rho) & B(\rho)S^{-\frac{1}{2}}(\rho) \\ \hline C(\rho)+D(\rho)F(\rho) & D(\rho)S^{-\frac{1}{2}}(\rho) \\ F(\rho) & S^{-\frac{1}{2}}(\rho) \end{array}\right], \tag{4}$$

where $F(\rho) = -S^{-1}(\rho)(B^T(\rho)Z_1 + D^T(\rho)C(\rho))$, $S(\rho) = I + D^T(\rho)D(\rho)$, $R(\rho) = I + D(\rho)D^T(\rho)$ *and* Z_1 *is a constant solution of the generalised control Riccati inequality (GCRI)*

$$(A(\rho)-B(\rho)S^{-1}(\rho)D^T(\rho)C(\rho))^T Z_1+Z_1(A(\rho)-B(\rho)S^{-1}(\rho)\times D^T(\rho)C(\rho)) - Z_1B(\rho)S^{-1}(\rho)B^T(\rho)Z_1(\rho) + C^T(\rho)R^{-1}(\rho)C(\rho) < 0 \quad \forall\rho(t) \in F_\rho. \quad (5)$$

Remark 6.6 *The Hermitian square root of* $S(\rho)$ *in equation (4) is usually obtained through singular value decomposition. If* $D(\rho) \neq 0$ *then it may be necessary to use the methods in [10] to ensure that* $S^{\frac{1}{2}}(\rho)$ *is continuous. That said, if the coprime factors are only being generated for the purpose of model reduction, careful examination of subsequent formulae shows that it is not necessary to explicitly compute* $S^{\frac{1}{2}}(\rho)$.

Lemma 6.7 *Let* P_ρ *have a continuous, quadratically stabilisable and a quadratically detectable realisation, and let the ordered pair* $[N_\rho, M_\rho]$ *represent the CRCF of* P_ρ *given in equation (4). Then*

$$Q = Z_1 \quad \text{and} \quad P = (I + Z_2Z_1)^{-1}Z_2$$

are observability and controllability Gramians for the given realisation. Here Z_1 *solves GCRI and* Z_2 *solves the generalised filtering Riccati inequality (GFRI) given by*

$$(A(\rho)-B(\rho)D^T(\rho)R^{-1}(\rho)C(\rho))Z_2+Z_2(A(\rho)-B(\rho)D^T(\rho)\times R^{-1}(\rho)C(\rho))^T - Z_2C^T(\rho)R^{-1}(\rho)C(\rho)Z_2 + B(\rho)S^{-1}(\rho)B^T(\rho) < 0 \quad \forall\rho(t) \in F_\rho,$$

where $R(\rho) = I + D(\rho)D^T(\rho)$, $S(\rho) = I + D^T(\rho)D(\rho)$.

Definition 6.8 *Given the CRCF of the n-state system* P_ρ *in equation (4), a controllability Gramian* P*, an observability Gramian* Q*, together with a balancing state transformation matrix* T *such that the transformed Gramians* $\tilde{P} = \tilde{Q} = \Sigma$ *are diagonal, define a balanced parameter-varying CRCF as*

$$\begin{aligned}
\mathcal{G}_\rho &\stackrel{s}{=} \left[\begin{array}{c|c} \bar{A}(\rho) + \bar{B}(\rho)\bar{F}(\rho) & \bar{B}(\rho)S^{-\frac{1}{2}}(\rho) \\ \hline \bar{C}(\rho) + D(\rho)\bar{F}(\rho) & D(\rho)S^{-\frac{1}{2}}(\rho) \\ \bar{F}(\rho) & S^{-\frac{1}{2}}(\rho) \end{array}\right], \\
\bar{A}(\rho) &= T^{-1}A(\rho)T, \qquad \bar{B}(\rho) = T^{-1}B(\rho) \\
\bar{C}(\rho) &= C(\rho)T, \qquad \bar{F}(\rho) = -S^{-1}(\rho)(\bar{B}^T\Sigma + D^T(\rho)\bar{C}(\rho)).
\end{aligned}$$

Definition 6.9 *Ignoring* ρ *dependence, partition the balanced CRCF of the n-state graph symbol in definition 6.8 according to*

$$\mathcal{G}_\rho \stackrel{s}{=} \left[\begin{array}{cc|c} \bar{A}_{11} + \bar{B}_1\bar{F}_1 & \bar{A}_{12} + \bar{B}_1\bar{F}_2 & \bar{B}_1S^{-\frac{1}{2}} \\ \bar{A}_{21} + \bar{B}_2\bar{F}_1 & \bar{A}_{22} + \bar{B}_2\bar{F}_2 & \bar{B}_2S^{-\frac{1}{2}} \\ \hline \bar{C}_1 + D\bar{F}_1 & \bar{C}_2 + D\bar{F}_2 & DS^{-\frac{1}{2}} \\ \bar{F}_1 & \bar{F}_2 & S^{-\frac{1}{2}} \end{array}\right], \quad (6)$$

where $\bar{A}_{11} \in \mathbb{R}^{r\times r}, \bar{A}_{12} \in \mathbb{R}^{r\times(n-r)}, \bar{A}_{21} \in \mathbb{R}^{(n-r)\times r}, \bar{A}_{22} \in \mathbb{R}^{(n-r)\times(n-r)}, \bar{B}_1 \in \mathbb{R}^{r\times m}, \bar{B}_2 \in \mathbb{R}^{(n-r)\times m}, \bar{C}_1 \in \mathbb{R}^{p\times r}, \bar{C}_2 \in \mathbb{R}^{p\times(n-r)}, \bar{F}_1 \in \mathbb{R}^{m\times r}$ *and* $\bar{F}_2 \in \mathbb{R}^{m\times(n-r)}$. *Let* $\hat{\mathcal{G}}_\rho$*, of state-dimension* r*, be obtained by truncating* $\mathcal{G}_\rho$ *as follows*

$$\hat{\mathcal{G}}_\rho \stackrel{s}{=} \left[\begin{array}{c|c} \bar{A}_{11} + \bar{B}_1\bar{F}_1 & \bar{B}_1S^{-\frac{1}{2}} \\ \hline \bar{C}_1 + D\bar{F}_1 & DS^{-\frac{1}{2}} \\ \bar{F}_1 & S^{-\frac{1}{2}} \end{array}\right]. \quad (7)$$

We define $\hat{\mathcal{G}}_\rho$ *to be a balanced fractional approximant of the system's graph symbol* $\mathcal{G}_\rho$*, and we associate with* $\hat{\mathcal{G}}_\rho$ *the following r-state model*

$$\hat{P}_\rho = \left[\begin{array}{c|c} \bar{A}_{11} & \bar{B}_1 \\ \hline \bar{C}_1 & D \end{array}\right].$$

Lemma 6.10 *Given the LPV system* P_ρ*,* Z_1 *satisfying GCRI and* Z_2 *satisfying GFRI, then we have already shown (Lemma 6.7) that* $Q = Z_1$ *and* $P = (I + Z_2Z_1)^{-1}Z_2$ *are observability and controllability Gramians for the realisation of the CRCF given in equation (4). If the realisation in equation (4) is subsequently balanced and truncated to give the balanced fractional approximant in equation (7), then*

$$\hat{Z}_1 = \Sigma_1 \qquad \text{and} \qquad \hat{Z}_2 = (I - \Sigma_1^2)^{-1}\Sigma_1$$

satisfy GCRI and GFRI for the reduced-order model given by

$$\hat{P}_\rho = \left[\begin{array}{c|c} \bar{A}_{11}(\rho) & \bar{B}_1(\rho) \\ \hline \bar{C}_1(\rho) & D(\rho) \end{array}\right]. \quad (8)$$

Corollary 6.11 *The r-state model approximant*

$$\hat{P}_\rho = \left[\begin{array}{c|c} \bar{A}_{11}(\rho) & \bar{B}_1(\rho) \\ \hline \bar{C}_1(\rho) & D(\rho) \end{array}\right]$$

is detectable and a stabilising output injection is given by $L = -(\bar{B}_1D^T + \hat{Z}_2\bar{C}_1^T)R^{-1}$.

Theorem 6.12 *Let* $\mathcal{G}_\rho$*, of state-dimension* n*, be balanced and let* $\hat{\mathcal{G}}_\rho$*, of state-dimension* r*, be obtained by truncating* $\mathcal{G}_\rho$*. Then* $\hat{\mathcal{G}}_\rho$ *is itself balanced and represents a CRCF over* $\mathcal{S}_F$ *of the r-state model*

$$\hat{P}_\rho = \left[\begin{array}{c|c} \bar{A}_{11}(\rho) & \bar{B}_1(\rho) \\ \hline \bar{C}_1(\rho) & D(\rho) \end{array}\right]. \quad (9)$$

Once again, balanced truncation of the system's graph symbol has an immediate physical interpretation. Essentially, we are removing those states which do not contribute strongly to the input-output mapping of $\mathcal{G}_\rho$. This implies that we are trying to keep the graph of $\hat{P}_\rho$ close to the graph of P_ρ, which is arguably the correct setting for approximation since the concept of the graph topology and the gap between systems is not restricted to LTI systems ([6],[13]). In fact, for time-varying systems it is shown in [12] that the graph topology is still the weakest topology in which feedback is a robust property.

7 Computational issues

In this section we outline a computational procedure for finding a controllability Gramian and an observability Gramian for a given parameter-dependent system. The objective of making the $\mathcal{Q}_e$ singular values small is shown to result in an optimisation problem that involves convex constraints and a non-convex cost function. In order to deal with the lack of convexity in the cost function we show how the LPV Gramians can be obtained by starting with the LTI Gramians, the latter being used as weights in weighted optimisation which is locally convex. Unless the parameter dependence is polytopic we are required to find a solution to two linear matrix inequalities while satisfying an infinite number of constraints. The approach we take is to satisfy the inequalities on a suitably dense grid covering the parameter space, which then allows us to exploit the continuity properties of the plant matrices to ensure that the inequalities are satisfied everywhere. Given bounds on the range of parameter variation we need to find a constant matrix $P = P^T > 0$ and a constant matrix $Q = Q^T > 0$ satisfying

$$A^T(\rho)Q + QA(\rho) + C^T(\rho)C(\rho) < -\delta^2 I \quad \forall \rho(t) \in F_\rho \qquad (10)$$
$$A(\rho)P + PA^T(\rho) + B(\rho)B^T(\rho) < -\delta^2 I \quad \forall \rho(t) \in F_\rho, \qquad (11)$$

for some $\delta^2 > 0$. The small constant δ allows the feasibility of the solutions at the grid points to be extended to the entire parameter space. With P and Q constant, solution of equation (10) and equation (11) at a set of grid points becomes a convex feasibility problem for which there exist powerful numerical techniques [4] that are guaranteed to find a feasible solution. From a model-reduction perspective we want to do rather more than just find solutions to the Lyapunov inequalities. What we really want is to find solutions to equation (10) and equation (11) and simultaneously minimise the r smallest eigenvalues of PQ. Equivalently we would like to minimise $J = \text{trace}(PQ)$. Minimising J subject to the constraints given by equation (10) and equation (11) constitutes a constrained optimisation with convex constraints but with a non-convex cost function. As an alternative we suggest the following iterative scheme.

Procedure 7.1 .

i. Find Q_1 which satisfies equation (10) and minimises $J_1 = \sum_k trace(Q_1 P_0(\rho_k))$, where $P_0(\rho_k)$ represents the LTI controllability Gramian at the grid point ρ_k.

ii. Solve for P_1 which satisfies equation (11) and minimises $J_2 = trace(P_1 Q_1)$.

iii. Solve for Q_2 which satisfies equation (10) and minimises $J_3 = trace(Q_2 P_1)$.

iv. Repeat steps ii. and iii. until the decrease in the cost function satisfies an appropriate convergence criterion.

For certain systems it may help to invert the LMIs so that instead of solving for P and Q we solve for their inverses. The reason for doing this is to ensure that the smallest $\mathcal{Q}_e$ singular values contribute most to the cost function. Observe that in the first step of the iteration the cost function is weighted using the LTI controllability Gramian. So in a sense we are looking for Gramians that satisfy the LPV Lyapunov inequalities and are close the the LTI Gramians. Computational experience has shown that this greatly increases the speed of convergence. The optimisation is locally convex (but not globally convex) so it is possible to find a global minimum at each step in the iteration before moving to the next step in the procedure. However, we cannot guarantee that the procedure will find the global minimum of the non-convex cost function.

8 Conclusions

In this paper we have shown that the LTI model approximation techniques of balanced truncation, optimal Hankel norm approximation and approximation of the graph operator may be extended to an LPV framework. (Although the Hankel approach is no longer optimal in any sense.) These results complement many recent results on the synthesis of controllers for LPV plants. However, since balanced parameter varying realisations are far from unique, caution must be shown when applying the techniques to ensure that the states being neglected do indeed contribute little to the input-output response.

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