Performance-Preserving Frequency Weighted Controller Approximation: A Coprime Factorisation Approach*

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Abstract

Given a stabilising controller which satisfies a prespecified $\mathcal{H}_\infty$ performance bound on the closed loop we derive sufficient conditions for reduced order controllers to be stabilising and satisfy the same performance bound. The conditions are expressed as norm bounds on the frequency weighted error between coprime factor representations of the given controller and the reduced order controller. Reduced order controllers are then designed by interpreting the sufficient conditions as frequency weighted model reduction problems.

Keywords : Controller order reduction, Frequency weighted model reduction, Coprime factorisations, H-infinity.

1 Introduction

Consider the linear fractional transformation (LFT) interconnection of Figure 1 where P is the finite dimensional linear time invariant transfer function of a generalised plant and K is the transfer function of a controller to be designed.

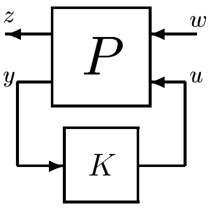

Figure 1: General Plant and Controller Interconnection

The $\mathcal{H}_\infty$ sub-optimal control problem is as follows: given $\gamma > 0$, find all controllers K such that the $\mathcal{H}_\infty$ norm of the closed loop transfer function, $\mathcal{F}_l(P,K)$, is less than γ, i.e. $\|\mathcal{F}_l(P,K)\|_\infty < \gamma$. Controllers satisfying the $\mathcal{H}_\infty$ sub-optimal control problem are said to be (P,γ)-admissible.

The solution to the $\mathcal{H}_\infty$ sub-optimal control problem is now well known [3, 8]. (P,γ)-admissible controllers exist, and may be represented as an LFT on a contractive stable free parameter, if and only if the unique stabilising solution to two Algebraic Riccati equations are positive definite and the spectral radius of their product is less than γ^2. The drawback of this approach is that the resulting controllers have a state dimension equal to the state dimension of the generalised plant.

Since models for physical systems are often of high order, controllers designed to satisfy the $\mathcal{H}_\infty$ sub-optimal control problem via these procedures will also be of high order. However in practice low order controllers are preferred to high order controllers. They are less complex and hence when implemented are less likely to run into either software or hardware problems. Hence some form of controller approximation is often employed to obtain *simpler* controllers.

It is accepted [1, 9] that controller reduction is best performed by incorporating frequency weights. This may be thought of in a frequency weighted model reduction framework and corresponds to requiring the modeling error to be smaller in selected frequency bands. Hence the salient features of the high order controller may be retained at the frequencies of most interest.

However it must be stressed that current techniques for performing frequency weighted model reduction [5, 12, 9, 19] do not have guaranteed performance (in the $\mathcal{L}_\infty$ norm sense), and upper and lower bounds are less tight than in the unweighted case.

Procedures for selecting weights depend on the properties which are to be retained [1, 10, 13, 14]. In [10] it was shown how to derive weights W_1 and W_2 such that if $\|W_2^{-1}(K - K^0)W_1^{-1}\|_\infty < 1$, with K^0 being a nominal (P,γ)-admissible controller, then K is also a (P,γ)-admissible controller satisfying the same level of closed loop performance. Weights W_1 and W_2 are chosen in such a way to make a certain transfer function a contraction.

The development here is similar to that of [10] and concentrates on preserving closed loop performance. The difference being, that whereas in [10] all controllers are represented as a stable weighted additive perturbation to a nominal controller, here we develop sufficient conditions in terms of stable weighted perturbations to controller coprime factors. Unlike the additive approach of [10], this approach gives the poles of K^0 freedom to move across the imaginary axis. This is particularly advantageous if K^0 has unstable poles.

In Section 2 some background results on inner transfer functions and parameterisations of all (P,γ)-admissible controllers are presented. Then Sections 3 and 4 consider deriving weights for frequency weighted model reduction. Sufficient conditions for reduced order controllers to be (P,γ)-admissible are then given in terms of these weights and a nominal controller. An example is presented in Section 5.

2 Preliminaries

The notation used throughout this paper is standard. Readers unfamiliar with LFT manipulation are referred to [4], while a more general discussion of fractional representations may be found in [15, 18]. All transfer function matrices are assumed to be real-rational and proper.

Firstly we define what is meant by a left coprime factorisation.

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Definition 2.1 ([15], definition 2.16)
The pair $\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix}$, *where* $\tilde{U}$, $\tilde{V} \in \mathcal{RH}_\infty$, *constitute a left coprime factorisation of* K *if and only if*

(a) $\tilde{V}$ *is square,* $det(\tilde{V}) \neq 0$

(b) $K = \tilde{V}^{-1}\tilde{U}$

(c) $\tilde{U}$ *and* $\tilde{V}$ *are left coprime. That is, there exists* M, $N \in \mathcal{RH}_\infty$ *such that* $\tilde{V}M - \tilde{U}N = I$.

The following theorem on inner matrices and LFTs is central to our development.

Theorem 2.2 ([7],Theorem 2.3 (f) and (g))
Consider the feedback system of Figure 2. The closed loop

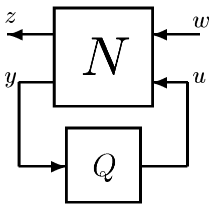

Figure 2: Lower Linear Fractional Transformation

transfer function from w *to* z *is referred to as the linear fractional transformation of* Q *with coefficient matrix* N *and is denoted,*

$$T_{zw} = \mathcal{F}_l(N, Q) = N_{11} + N_{12}Q(I - N_{22}Q)^{-1}N_{21} \qquad (1)$$

where

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix},\ N_{11} : p_1 \times m_1,\ N_{22} : p_2 \times m_2$$

and dimensions are compatible. Assume the feedback loop is well posed, therefore $\det(I - N_{22}(\infty)Q(\infty)) \neq 0$. *Then*

(i) If $rank(N_{21}(j\omega)) = p_2 \;\; \forall \omega \in \mathrm{R} \cup \infty$ *with* $N^{\sim}N = I$ *then* $\|\mathcal{F}_l(N, Q)\|_\infty < 1$ *if and only if* $\|Q\|_\infty < 1$.

(ii) If $rank(N_{12}(j\omega)) = m_2 \;\; \forall \omega \in \mathrm{R} \cup \infty$ *with* $NN^{\sim} = I$ *then* $\|\mathcal{F}_l(N, Q)\|_\infty < 1$ *if and only if* $\|Q\|_\infty < 1$.

Assuming the well-posedness condition is satisfied, N_{21} has full row rank (N_{12} has full column rank) at all frequencies, N is inner (co-inner) and $Q \in \mathcal{RH}_\infty$ then Theorem 2.2 implies that

$$\{\|\mathcal{F}_l(N, Q)\|_\infty < 1 \iff \|Q\|_\infty < 1\}.$$

We concluded this section with a brief discussion of two alternative parameterisations of all (P, γ)-admissible controllers.

Theorem 2.3 ([8]) *Provided certain technical assumptions are satisfied, all* (P, γ)*-admissible controllers may be represented by the lower LFT* $K = \mathcal{F}_l(K_a, \Phi)$, *where* $\Phi \in \mathcal{RH}_\infty$, *such that* $\|\Phi\|_\infty < \gamma$ *and* K_a *is of the form*

$$K_a \triangleq \begin{bmatrix} K_{a11} & K_{a12} \\ K_{a21} & K_{a22} \end{bmatrix}. \qquad (2)$$

K_a *is derived via the solution of two Algebraic Riccati equations and is invertible with* K_{a12}^{-1}, $K_{a21}^{-1} \in \mathcal{RH}_\infty$. K_{a11} *is called the central controller.*

The following lemma results from a trivial modification to Lemma 2.14 of [15]. Used in conjunction with equation (2), it gives an alternative parameterisation of all (P, γ)-admissible controllers.

Lemma 2.4 *Any LFT given by equation (1) such that* $\det(N_{12}) \neq 0$ *can also be written*

$$T_{zw} = (M_{22} + QM_{12})^{-1}(M_{21} + QM_{11})$$

where $\det(M_{22} + QM_{12}) \neq 0$ *and*

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \mathcal{F}_l\left(\left[\begin{array}{cc:c} N_{21} & 0 & -N_{22} \\ 0 & 0 & I \\ \hdashline N_{11} & I & 0 \end{array}\right], N_{12}^{-1}\right)$$

Remark 2.5 *The results presented here have been derived using left coprime factorisations. Analogous results are available by considering right coprime factorisations.*

3 Lower LFT Parameterisation

In this section we show how reduced order controller design may be considered as a frequency weighted model reduction problem. We show how to calculate weights which make a specific transfer function a contraction then use Theorem 2.2 to obtain a sufficient condition for reduced order controllers to be (P, γ)-admissible.

Consider the parameterisation of all (P, γ)-admissible controllers as given in Theorem 2.3 and represented in Figure 3.

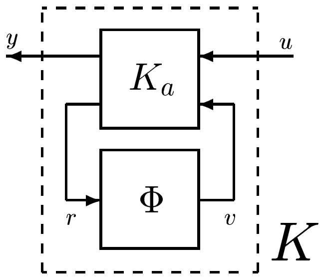

Figure 3: Controller LFT Configuration

Manipulation of $K = \mathcal{F}_l(K_a, \Phi)$ leads to the mapping from r to v as $\Phi = \mathcal{F}_u(K_a^{-1}, K)$. Therefore the norm bound condition on Φ in Theorem 2.3 is equivalent to a norm bound on $\mathcal{F}_u(K_a^{-1}, K)$, i.e.

$$\|\Phi\|_\infty < \gamma \iff \|\mathcal{F}_u(K_a^{-1}, K)\|_\infty < \gamma. \qquad (3)$$

Now assume that the left coprime factors of $K = \tilde{V}^{-1}\tilde{U}$ can be represented by a weighted perturbation to the left coprime factors of a nominal (P, γ)-admissible controller $K^0 = \tilde{V}_0^{-1}\tilde{U}_0$. That is,

$$\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix} = \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix} + W_2 \begin{bmatrix} \Delta_{\tilde{U}} & \Delta_{\tilde{V}} \end{bmatrix} W_1 \qquad (4)$$

where W_1 and W_2 are stable, minimum phase and invertible and $\Delta \triangleq \begin{bmatrix} \Delta_{\tilde{U}} & \Delta_{\tilde{V}} \end{bmatrix}$ is stable. Now substituting $\tilde{V}^{-1}\tilde{U}$ from equation (4) for K in equation (3) leads to the condition

$$\|\Phi\|_\infty < \gamma \iff \|\mathcal{F}_u(\tilde{R}, \Delta)\|_\infty < 1 \qquad (5)$$

where

$$\tilde{R}=\begin{bmatrix} W_1 & 0 \\ 0 & \gamma^{-1/2}I \end{bmatrix}\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}\begin{bmatrix} W_2 & 0 \\ 0 & \gamma^{-1/2}I \end{bmatrix} \quad (6)$$

and R is given by the Redheffer star product [4, 17],

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \mathcal{S}(\left[\begin{array}{c|c} 0 & I \\ -\tilde{V}_0^{-1} & -K^0 \\ \hline \tilde{V}_0^{-1} & K^0 \end{array}\right], K_a^{-1}). \quad (7)$$

Remark 3.1 *Both R and $\tilde{R}$ are stable. To show this, firstly substitute $K^0 = \mathcal{F}_l(K_a, \Phi^0)$ for some $\Phi^0 \in \mathcal{RH}_\infty$, $\|\Phi^0\|_\infty < \gamma$ into equation (7), then use the properties of K_a.*

Remark 3.2 *The nominal controller K^0 may be designed using any methodology provided it is (P,γ)-admissible.*

Comparing the right hand side of equation (5) with Theorem 2.2 tells us that if we could find weights W_1 and W_2 such that $\tilde{R}$ is inner then $\{\|\Delta\|_\infty < 1 \iff \|\Phi\|_\infty < \gamma\}$. Unfortunately such weights do not exist.

However it is possible to calculate weights such that $\tilde{R}$ is a contraction. This results in the sufficient condition $\{\|\Delta\|_\infty < 1 \Rightarrow \|\Phi\|_\infty < \gamma\}$.

Proposition 4 of [10], restated below in slightly modified form, gives weights such that the above $\tilde{R}$ is a contraction.

Proposition 3.3 *Assume $\|R_{22}\|_\infty < \gamma$ and define*

$$L = \begin{bmatrix} L_1 & L_2 \\ L_2^\sim & L_3 \end{bmatrix}$$
$$\triangleq \mathcal{F}_l(\left[\begin{array}{cc|cc} 0 & -R_{11} & 0 & R_{12} \\ -R_{11}^\sim & 0 & R_{21}^\sim & 0 \\ \hline 0 & R_{21} & 0 & -R_{22} \\ R_{12}^\sim & 0 & -R_{22}^\sim & 0 \end{array}\right], \gamma^{-1}I). \quad (8)$$

Then $\tilde{R}$ is a contraction if weights W_1 and W_2 satisfy

$$\begin{bmatrix} (W_1^\sim W_1)^{-1} & 0 \\ 0 & (W_2 W_2^\sim)^{-1} \end{bmatrix} \geq \begin{bmatrix} L_1 & L_2 \\ L_2^\sim & L_3 \end{bmatrix} \quad (9)$$

Using the weights of Proposition 3.3 we therefore obtain the sufficient condition,

$$\|W_2^{-1}\left\{\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix} - \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}\right\} W_1^{-1}\|_\infty < 1$$
$$\Rightarrow \quad K = \tilde{V}^{-1}\tilde{U} \quad \text{is } (P,\gamma)\text{-admissible.} \quad (10)$$

Equation (10) is in the form of a standard frequency weighted model reduction problem. We wish to find reduced order $\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix}$ which lie within an $\mathcal{L}_\infty$ ball (of right radius W_1 and left radius W_2,) of $\begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}$. $K = \tilde{V}^{-1}\tilde{U}$ is then a reduced order (P,γ)-admissible controller. The larger the $\mathcal{L}_\infty$ ball, the easier the approximation. Therefore, in addition to satisfying equation (9), we wish to make W_1 and W_2 large in some sense.

The results thus far have obvious similarities to those for the additive uncertainty case discussed in [10]. However there is one important difference. Equations (7) and (8) show that $L_1 \geq 0$ is rank deficient with kernel of dimension equal to the number of controller outputs. This is in contrast with the additive uncertainty case where $L_1 > 0$.

In [10] weights W_1 and W_2 were chosen to satisfy an equation similar to equation (9), only with $L_1 > 0$, and maximise the product $\det(W_1^\sim W_1)\det(W_2 W_2^\sim)$. However maximising $\det(W_1^\sim W_1)\det(W_2 W_2^\sim)$ is not a sensible choice in the present case as it can lead to *directions* where the modeling error must be zero.

To see this, define $X \triangleq (W_1^\sim W_1)^{-1}$, $Y \triangleq (W_2 W_2^\sim)^{-1}$ and notice that L can, after suitable congruence transformation, typically be rewritten as

$$L = \left[\begin{array}{cc|c} 0 & 0 & L_{21} \\ 0 & L_{13} & 0 \\ \hline L_{21}^\sim & 0 & L_3 \end{array}\right]$$

where $L_{13} = L_{13}^\sim > 0$, and $L_3 = L_3^\sim > 0$. We wish to maximise $\det(W_1^\sim W_1)\det(W_2 W_2^\sim)$, or equivalently to minimise $\det(X)\det(Y)$, subject to equation (9).

Given any $Y > L_3$ the unique minimising X is

$$\begin{aligned} X &= \begin{bmatrix} 0 & 0 \\ 0 & L_{13} \end{bmatrix} + \begin{bmatrix} L_{21} \\ 0 \end{bmatrix}(Y - L_3)^{-1}\begin{bmatrix} L_{21}^\sim & 0 \end{bmatrix} \\ &= \begin{bmatrix} L_{21}(Y-L_3)^{-1}L_{21}^\sim & 0 \\ 0 & L_{13} \end{bmatrix}. \end{aligned}$$

Therefore define

$$\begin{aligned} f(Y) &\triangleq \det(Y)\det(L_{21}(Y-L_3)^{-1}L_{21}^\sim) \\ &= \frac{\det(L_{21}L_{21}^\sim)}{\det(I - L_3 Y^{-1})} \end{aligned}$$

which reaches a minimum when $\det(X)\det(Y)$ is minimised and when $\det(I - L_3 Y^{-1})$ is maximised.

Now choose $Y = \mu L_3$ where $\mu \to \infty$. This choice maximises $\det(I - L_3 Y^{-1})$ and gives $X_{11} = L_{21}(\mu - 1)^{-1}L_3^{-1}L_{21}^\sim$. Therefore

$$\begin{aligned} \det(X)\det(Y) &= \det(L_{13})\det(L_{21}L_{21}^\sim)(\mu-1)^{-m_2}\mu^{m_2} \\ &\to \det(L_{13})\det(L_{21}L_{21}^\sim) \end{aligned}$$

where m_2 is the number of controller outputs.

Using the above weights, the required model reduction error is

$$\begin{aligned} &\|W_2^{-1}\left\{\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix} - \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}\right\} W_1^{-1}\|_\infty \\ &= \|\mu^{1/2}T\left\{\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix} - \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}\right\} \times \\ &\qquad \begin{bmatrix} (\mu-1)^{-1/2}S_1 & 0 \\ 0 & S_2 \end{bmatrix}\|_\infty < 1. \end{aligned} \quad (11)$$

where $T^\sim T = L_3$, $S_1 S_1^\sim = (L_{21}L_3^{-1}L_{21}^\sim)$ and $S_2 S_2^\sim = L_{13}$.

Equation (11) shows that as $\mu \to \infty$ the error $(\tilde{V} - \tilde{V}_0)$ must tend to zero for controller $K = \tilde{V}^{-1}\tilde{U}$ to satisfy the sufficient condition on the frequency weighted error.

We therefore require an alternative criterion for measuring the *size* of W_1 and W_2. A satisfactory choice is $\text{trace}((W_1^\sim W_1)^{-1})\text{trace}((W_2 W_2^\sim)^{-1})$.

Lemma 3.4 *Define L as in Proposition 3.3, and define*

$$\alpha_{opt} \triangleq \sqrt{(trace(L_3)/trace(L_1))} \quad (12)$$
$$Z_1^2 \triangleq L_2^\sim L_2 \quad (13)$$
$$Z_2 \triangleq L_2 Z_1^{-1} L_2^\sim \quad (14)$$

where $Z_1 = Z_1^\sim > 0$ and $Z_2 = Z_2^\sim \geq 0$. Then the weights W_1 and W_2 which minimise the product

$trace((W_1^\sim W_1)^{-1})trace((W_2 W_2^\sim)^{-1})$ subject to equation (9) satisfy

$$
\begin{aligned}
(W_1^\sim W_1)^{-1} &= L_1 + \alpha_{opt}^{-1} Z_2 \quad (15)\\
(W_2 W_2^\sim)^{-1} &= L_3 + \alpha_{opt} Z_1 \quad (16)
\end{aligned}
$$

Proof: Define $X \triangleq (W_1^\sim W_1)^{-1}$ and $Y \triangleq (W_2 W_2^\sim)^{-1}$. We wish to minimise $\mathrm{trace}(X)\mathrm{trace}(Y)$ subject to equation (9).

Given any $Y > L_3$ the unique minimising X is given by

$$X = L_1 + L_2(Y - L_3)^{-1} L_2^\sim .$$

Now define

$$f(Y) \triangleq \mathrm{trace}(X)\mathrm{trace}(Y)$$

and hence, for $\Delta = \Delta^\sim$

$$
\begin{aligned}
&f(Y+\Delta) = \mathrm{trace}(Y+\Delta)\\
&\qquad \times \mathrm{trace}(L_1 + L_2(Y - L_3 + \Delta)^{-1} L_2^\sim)\\
&= \{\mathrm{trace}(Y) + \mathrm{trace}(\Delta)\}\\
&\quad \times \big\{\mathrm{trace}(L_1 + L_2(Y - L_3 + \Delta)^{-1} L_2^\sim)\\
&\quad - \mathrm{trace}(L_2(Y-L_3)^{-1}\Delta(Y-L_3)^{-1}L_2^\sim) + o(\Delta)\big\}\\
&= f(Y) + \mathrm{trace}(\Delta)\mathrm{trace}(L_1 + L_2(Y-L_3)^{-1}L_2^\sim)\\
&\quad - \mathrm{trace}(Y)\mathrm{trace}((Y-L_3)^{-1}L_2^\sim L_2 (Y-L_3)^{-1}\Delta)\\
&\quad + o(\Delta).
\end{aligned}
$$

f(Y) is therefore at a stationary point if $\forall\Delta$

$$
\begin{aligned}
&\mathrm{trace}(\Delta)\mathrm{trace}(L_1 + L_2(Y-L_3)^{-1}L_2^\sim)\\
&\quad = \mathrm{trace}(Y)\mathrm{trace}((Y-L_3)^{-1}L_2^\sim L_2(Y-L_3)^{-1}\Delta)
\end{aligned}
$$

which implies that

$$(Y - L_3)^2 = \alpha^2 L_2^\sim L_2$$

for some scalar α.

Therefore

$$
\begin{aligned}
Y &= L_3 + \alpha Z_1\\
X &= L_1 + \alpha^{-1} Z_2
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{trace}(Y) &= \mathrm{trace}(L_3) + \alpha\mathrm{trace}(Z_1)\\
\mathrm{trace}(X) &= \mathrm{trace}(L_1) + \alpha^{-1}\mathrm{trace}(Z_1).
\end{aligned}
$$

Hence

$$
\begin{aligned}
f(Y) &= \mathrm{trace}(L_1)\mathrm{trace}(L_3) + (\mathrm{trace}(Z_1))^2\\
&\quad + \mathrm{trace}(Z_1)\left(\alpha\mathrm{trace}(L_1) + \alpha^{-1}\mathrm{trace}(L_3)\right)
\end{aligned}
$$

and is minimised when $\alpha = \alpha_{opt}$. Ξ

Remark 3.5 *It is not generally possible to calculate exact parametric solutions for Z_1 and Z_2. However, as in [10], an approximate solution may be calculated by substituting the right hand side of equations (13) and (14) into a scalar rational approximation to the square root. This results in weights $\hat{W}_1 \leq W_1$ and $\hat{W}_2 \leq W_2$ which still make $\tilde{R}$ a contraction and hence satisfy the sufficient condition.*

4 An Alternative Parameterisation

In the previous section we relied solely on the lower LFT parameterisation of all (P,γ)-admissible controllers as given by Theorem 2.3. Here we look at the parameterisation obtained by combining the results of Theorem 2.3 and Lemma 2.4. This leads to the alternative sufficient condition for reduced order controllers to be (P,γ)-admissible stated in Proposition 4.1.

Since K_{a12} is invertible, the following form of LFT can be used to parameterise all (P,γ)-admissible controllers:

$$K = (\theta_{22} + \Phi\theta_{12})^{-1}(\theta_{21} + \Phi\theta_{11}) = \tilde{V}^{-1}\tilde{U} \quad (17)$$

where $\|\Phi\|_\infty < \gamma$ and

$$
\begin{aligned}
\Theta &\triangleq \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix}\\
&= \mathcal{F}_l\left(\left[\begin{array}{cc:c} K_{a21} & 0 & -K_{a22} \\ 0 & 0 & I \\ \hdashline K_{a11} & I & 0 \end{array}\right], K_{a12}^{-1}\right). \quad (18)
\end{aligned}
$$

Observe that Θ is invertible and that both Θ and Θ^{-1} are stable.

Proposition 4.1 *Assume there exists a (P,γ)-admissible controller parameterised by equation (17), and define*

$$
\begin{aligned}
\Delta &= \begin{bmatrix} \Delta_{\hat{U}} & \Delta_{\hat{V}} \end{bmatrix}\\
&\triangleq \left\{\begin{bmatrix} \hat{U} & \hat{V} \end{bmatrix} - \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}\right\}\Theta^{-1}\begin{bmatrix} \gamma^{-1} I & 0 \\ 0 & I \end{bmatrix}
\end{aligned}
$$

where Θ is given by equation (18) and $\begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix} = \begin{bmatrix} \theta_{21} & \theta_{22} \end{bmatrix}$. Then the controller $\hat{K} = \hat{V}^{-1}\hat{U}$ is (P,γ)-admissible if $\|\Delta\|_\infty < 1/\sqrt{2}$.

Proof: Firstly observe that $\begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}$ is a left coprime factorisation of the central controller K_{a11}. Then represent the coprime factors of any other controller as the following weighted perturbation to these coprime factors,

$$
\begin{aligned}
\begin{bmatrix} \hat{U} & \hat{V} \end{bmatrix} &= \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix} + \begin{bmatrix} \gamma\Delta_{\hat{U}} & \Delta_{\hat{V}} \end{bmatrix}\Theta\\
&= \begin{bmatrix} \gamma\Delta_{\hat{U}} & I + \Delta_{\hat{V}} \end{bmatrix}\begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\begin{bmatrix} (I+\Delta_{\hat{V}})^{-1}\hat{U} & (I+\Delta_{\hat{V}})^{-1}\hat{V} \end{bmatrix} =\\
&\qquad \begin{bmatrix} \gamma(I+\Delta_{\hat{V}})^{-1}\Delta_{\hat{U}} & I \end{bmatrix}\begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix}. \quad (19)
\end{aligned}
$$

However from equation (17) we may write

$$\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix} = \begin{bmatrix} \Phi & I \end{bmatrix}\begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix}. \quad (20)$$

Therefore comparing equations (19) and (20) leads to,

$$
\begin{aligned}
\Phi &= \gamma(I + \Delta_{\hat{V}})^{-1}\Delta_{\hat{U}}\\
&= \gamma\left(I - \Delta\begin{bmatrix} 0 \\ -I \end{bmatrix}\right)^{-1}\Delta\begin{bmatrix} I \\ 0 \end{bmatrix}\\
&= \gamma\mathcal{F}_l\left(\left[\begin{array}{c:c} 0 & I/\sqrt{2} \\ \hdashline I & 0 \\ 0 & -I/\sqrt{2} \end{array}\right], \sqrt{2}\Delta\right). \quad (21)
\end{aligned}
$$

Hence, since the coefficient matrix in equation (21) is inner, Theorem 2.2 gives

$$\|\hat{\Delta}\|_\infty < 1/\sqrt{2} \Rightarrow \|\Phi\|_\infty < \gamma. \qquad \Xi$$

Remark 4.2 *Unlike the first approach, this approach requires* $\begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}$ *to be a specific coprime factorisation of the central controller.*

Next we shall establish a relationship between the weights of this approach and weights which make $\tilde{R}$ of the first approach (equation (6)) a contraction.

Lemma 4.3 *Define* $W_1^{-1} = \beta\Theta^{-1}\begin{bmatrix} \gamma^{-1}I & 0 \\ 0 & I \end{bmatrix}$ *and* $W_2^{-1} = \sqrt{2}/\beta$ *for* Θ *as given by equation (18) and some scalar* β. W_1 *and* W_2 *correspond to the weights required in Proposition 4.1. They make* $\tilde{R}$*, given by equation (6), a contraction if and only if* $\beta^2 = \gamma$.

Proof: Firstly observe from equation (18) that

$$\begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix} = \begin{bmatrix} K_{a12}^{-1}K_{a11} & K_{a12}^{-1} \end{bmatrix}.$$

Substitution into equation (7) then gives,

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} =$$

$$\begin{bmatrix} \begin{bmatrix} -K_{a21}^{-1}K_{a22} \\ K_{a11}K_{a21}^{-1}K_{a22} - K_{a12} \\ I \end{bmatrix} & \begin{bmatrix} K_{a21}^{-1} \\ -K_{a11}K_{a21}^{-1} \\ 0 \end{bmatrix} \end{bmatrix}.$$

Hence

$$\begin{bmatrix} L_1 & L_2 \\ L_2^{\sim} & L_3 \end{bmatrix} = \begin{bmatrix} R_{12}R_{12}^{\sim}/\gamma & -R_{11} \\ -R_{11}^{\sim} & I/\gamma \end{bmatrix}$$

and

$$\Theta^{-1} = \begin{bmatrix} R_{12} & -R_{11} \end{bmatrix}.$$

Also, since $W_2^{-1} = \sqrt{2}/\beta$, we have $Y \triangleq (W_2W_2^{\sim})^{-1} = 2/\beta^2$, and since

$$W_1^{-1} = \beta\Theta^{-1}\begin{bmatrix} \gamma^{-1}I & 0 \\ 0 & I \end{bmatrix}$$

we have

$$X \triangleq (W_1^{\sim}W_1)^{-1} = \beta^2\Theta^{-1}\begin{bmatrix} \gamma^{-2}I & 0 \\ 0 & I \end{bmatrix}\Theta^{\sim -1}.$$

We wish to calculate β such that

$$\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \geq \begin{bmatrix} L_1 & L_2 \\ L_2^{\sim} & L_3 \end{bmatrix}. \qquad (22)$$

Substituting for L, X and Y and using Theorem 2.2, equation (22) holds if and only if

$$\begin{bmatrix} R_{12} & -R_{11} \end{bmatrix}\left\{\begin{bmatrix} \frac{\beta^2}{\gamma^2}I & 0 \\ 0 & \beta^2 I \end{bmatrix} - \begin{bmatrix} \frac{1}{\beta}I & 0 \\ 0 & \frac{\beta^2\gamma}{2\gamma-\beta^2}I \end{bmatrix}\right\}\begin{bmatrix} R_{12}^{\sim} \\ -R_{11}^{\sim} \end{bmatrix} \geq 0$$

$$\iff \quad \begin{bmatrix} \frac{\beta^2-\gamma}{\beta^2}I & 0 \\ 0 & \frac{\beta^2(\gamma-\beta^2)}{2\gamma-\beta^2}I \end{bmatrix} \geq 0$$

$$\iff \quad \gamma \leq \beta^2 \ \text{ and } \ \gamma \geq \beta^2$$

$$\iff \quad \gamma = \beta^2. \qquad \Xi$$

5 Example

To illustrate the techniques of Sections 3 and 4 we will consider designing reduced order (P, γ)-admissible controllers for the longitudinal dynamics of the HIMAT advanced fighter aircraft as studied in [2]. Figure 4 shows the plant and controller interconnection where G is an 8 state, 6 input, 6 output, uncertainty model (including performance weights), and the D-scales result from μ-synthesis 'D-K' iterations. The generalised plant P has 20 states.

Robust performance of the unscaled plant is achieved if $\|\mathcal{F}_l(P,K)\|_\infty < 1$.

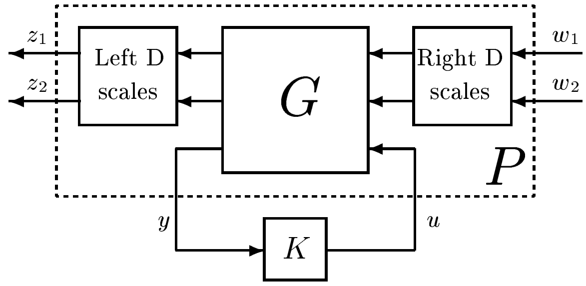

Figure 4: HIMAT LFT Interconnection

The following abbreviations are used throughout:

FWLCFR1	–	Freq. weighted approach of Section 3.
FWLCFR2	–	Freq. weighted approach of Section 4.
FWAdd	–	Freq. weighted approach of [10].
UOHNA	–	Unweighted optimal Hankel norm approx.
UBT	–	Unweighted balanced truncation.
LCFR	–	Reduction of controller left coprime factors.

A combined Hankel norm/optimisation procedure similar to that of [19][1], which uses a frequency weighted optimal Hankel norm approximation as a starting point for convex optimisation, was employed for FWLCFR1, FWLCFR2, and FWAdd.

For FWLCFR1, $\begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}$ was chosen to be a normalised left coprime factorisation of a near optimal 20 state $\mathcal{H}_\infty$ designed controller K^0 such that $\|\mathcal{F}_l(P,K^0)\|_\infty = 0.97$. The same K^0 was also used for FWAdd, UOHNA and UBT, with LCFR employing the same $\begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}$. For FWLCFR2, $\begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}$ must be the specific factorisation of the central controller as given in Proposition 4.1.

For FWLCFR1 and FWAdd a second order rational approximation was used to calculate $\hat{Z}_1$ and $\hat{Z}_2$.

The results of model reduction are presented in Table 1. For the frequency weighted methods, the calculated controller of least degree which satisfies equation (10) for FWLCFR1, Proposition 4.1 for FWLCFR2 and $\|\hat{W}_2^{-1}(K-K^0)\hat{W}_1^{-1}\|_\infty < 1$ for FWAdd is presented. For the unweighted methods, the controller of least degree satisfying $\|\mathcal{F}_l(P,K)\|_\infty < 1$ is presented.

Table 1 shows that the frequency weighted techniques presented here compare favorably with the additive uncertainty procedure of [10] and unweighted approximation. The procedure of FWLCFR2 is particularly attractive because of the ease with which the relevant weight is calculated.

[1] Here we were required to switch between optimising over the B and the D terms and the C and the D terms of $\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix}$ (for FWLCFR1 and FWLCFR2) and K (for FWAdd). Both of these optimisations are convex.

Reduction Technique	Controller Order	$\|(\text{weighted error})\|_\infty$	$\|\mathcal{F}_l(P,K)\|_\infty$
FWLCFR1	7	0.54	0.98
FWLCFR2	10	0.94	0.98
FWAdd	7	0.79	0.98
UOHNA	13	–	0.98
UBT	13	–	0.98
LCFR	13	–	0.98

Table 1: Results of model reduction

Remark 5.1 *It is possible to use a graphical technique to design reduced order SISO (P,γ)-admissible controllers which satisfy the sufficient conditions of FWLCFR1, FWLCFR2 and FWAdd. As noted in Section 3, finding the set of controllers K which satisfy the sufficient conditions for FWLCFR1 and FWLCFR2 is equivalent to finding the set of $\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix}$ that lie within a weighted $\mathcal{L}_\infty$ ball of left radius W_2 and right radius W_1 centred on $\begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}$. Similarly, finding the set of controllers which satisfy the sufficient condition for FWAdd is equivalent to finding the set of controllers K that lie within a weighted $\mathcal{L}_\infty$ ball of K^0. Taken frequency by frequency, for SISO controllers this weighted $\mathcal{L}_\infty$ ball represents a circle in the complex plane. Hence if at every frequency the appropriate point on a controller's Nyquist plot lies within the circle corresponding to that frequency then the controller is (P,γ)-admissible.*

6 Conclusions

Given that there exists a (P,γ)-admissible controller K^0, we have presented sufficient conditions for any other controller K to also be (P,γ)-admissible. These conditions were expressed as bounds on the norm of the frequency weighted error between left coprime representations of K^0 and K. Although not presented here, similar results are available for the right coprime factorisation case.

The above conditions are considered in a frequency weighted model reduction framework. The *larger* the weights the easier the approximation. Hence we have shown how to calculate weights which are large in some sense. This was achieved by minimising $\text{trace}((W_1^\sim W_1)^{-1})\text{trace}((W_2 W_2^\sim)^{-1})$. Further, it was shown that maximising $\det(W_1^\sim W_1)\det(W_2 W_2^\sim)$ is not a suitable criterion to consider since when the weights approach their optimal value they tend to give *directions* where the reduction error must be zero.

The main advantage of the present techniques is twofold. Firstly, the lower bound on the norm of the frequency weighted error for a controller of specified degree being less than one gives an indication that a reduced order controller of that degree may exist. Although again it must be stressed that frequency weighted model reduction procedures do not have guaranteed performance. Secondly, use of a coprime factor representation allows unstable poles in K_0 to be moved, thus giving additional freedom to the reduction procedure.

However the procedures presented here will only find reduced order controllers which have a coprime factorisation that lies within a weighted $\mathcal{L}_\infty$ ball centred on $\begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}$. There may be (P,γ)-admissible controllers which do not possess such a factorisation and hence will not be found.

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