Controller Reduction: Weights for Stability and Performance Preservation.

P.J. Goddard*and K. Glover†
Department of Engineering
Trumpington Street
Cambridge CB2 1PZ
U.K.

Email: pjg@uk.ac.cam.eng, kg@uk.ac.cam.eng

Abstract

In this paper we investigate the design of reduced order controllers using an $\mathcal{H}_\infty$ framework. Provided there exists a stabilising controller, K^0, giving satisfactory closed loop performance we derive frequency weights such that if $\|W_2^{-1}(K-K^0)W_1^{-1}\|_\infty < 1$ then K is also a stabilising controller satisfying the same level of closed loop performance.

Keywords : Controller order reduction, Frequency weighted model reduction, H-infinity.

1 Introduction

The standard suboptimal $\mathcal{H}_\infty$ control problem is to find all stabilising controllers for a linear system such that the $\mathcal{H}_\infty$ norm of the closed loop transfer function is less than some prespecified performance level $\gamma > 0$. The state-space characterisation of all controllers solving the suboptimal problem is presented in [7]. However, to provide guaranteed bounds on closed loop performance via the techniques of [7] leads to controllers of high McMillan degree, typically equal to the sum of the plant and weighting function McMillan degrees. What is not generally known is whether there exist controllers with lower McMillan degree which are also stabilising and provide the same level of closed loop performance. Nor is it known how to characterise all such reduced order controllers if any exist.

The order of controllers may be reduced using one of many model reduction techniques (see references in [1]). Balance and truncate [4] and optimal Hankel norm approximation [5] methods have the advantage that $\mathcal{L}_\infty$ bounds on the approximation error may be calculated a priori as a function of the full order controller's Hankel singular values. They also allow for the incorporation of frequency weighting functions [4, 9, 14], which may be used to emphasise the properties of the high order controller which it is desirable for the reduced order controller to possess. However it should be emphasised that good performance for frequency weighted model reduction schemes is not guaranteed, which is in contrast to the unweighted case.

There are several approaches to setting up controller reduction problems as frequency weighted model reduction problems. In [1] weights are in terms of a given stabilising controller, that is, given a stabilising controller K^0 find a reduced order controller $\hat{K}$ such that $\|W_1(K^0-\hat{K})W_2\|_\infty$ is less than some specified quantity. In [1] weights are derived to give a sufficient condition for the reduced order controller to be stabilising. In [10] conditions are derived for the $\mathcal{H}_\infty$ norm of the closed loop with reduced order controller, to be less than γ, although the results are possibly conservative by a factor $\sqrt{2}$. The results of this paper are strongly influenced by the approach in [10], and again lead to sufficient conditions for closed loop performance but without the above $\sqrt{2}$ factor.

What follows relies heavily on manipulation of linear fractional transformations (LFTs) and the Redheffer star product. Readers unfamiliar with such manipulations are referred to [3].

2 Preliminaries

The notation used throughout this paper is standard and follows that of [7]. The following theorem, proved by Redheffer in [13] and presented by Glover in [6] for a state-space/LFT framework, is central to our results.

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Theorem 1 ([6], Theorem 2.3 (f) and (g)): Consider the feedback system of Fig. 1.

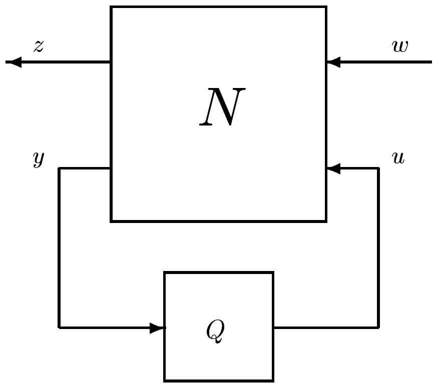

Figure 1: Lower Linear Fractional Transformation

The closed loop transfer function from w to z is referred to as the linear fractional transformation of Q with coefficient matrix N and is denoted,

$$T_{zw} = \mathcal{F}_l(N, Q) = N_{11} + N_{12}Q(I - N_{22}Q)^{-1}N_{21}$$

where

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}, \; N_{11} : p_1 \times m_1, \; N_{22} : p_2 \times m_2$$

and dimensions are compatible. Assume the feedback loop is well posed, therefore $\det(I - N_{22}(\infty)Q(\infty)) \neq 0$. Then

(i) If $\text{rank}(N_{21}(j\omega)) = p_2 \;\; \forall \omega \in \mathrm{R} \cup \infty$ with $N^{\sim}N = I$ then $\|\mathcal{F}_l(N, Q)\|_\infty < 1$ if and only if $\|Q\|_\infty < 1$.

(ii) If $\text{rank}(N_{12}(j\omega)) = m_2 \;\; \forall \omega \in \mathrm{R} \cup \infty$ with $NN^{\sim} = I$ then $\|\mathcal{F}_l(N, Q)\|_\infty < 1$ if and only if $\|Q\|_\infty < 1$. □

Theorem 1 says that assuming the well-posedness condition is satisfied, N_{21} has full row rank (N_{12} has full column rank) at all frequencies and N is inner (co-inner) then

$$\{\|\mathcal{F}_l(N, Q)\|_\infty < 1 \iff \|Q\|_\infty < 1\}.$$

2.1 Problem Setup

Consider the linear time invariant (LTI), multiple input multiple output (MIMO), LFT interconnection shown in Fig. 2, where P is a generalised plant incorporating the nominal plant plus any associated performance weights. Inputs w and u represent exogenous and control signals respectively. Outputs z and y represent error and measurement signals respectively. Assume that K is a stabilising controller such that $\|T_{zw}\|_\infty < \gamma$.

The parameterisation (provided certain assumptions are satisfied) of all such controllers is given in [7] as

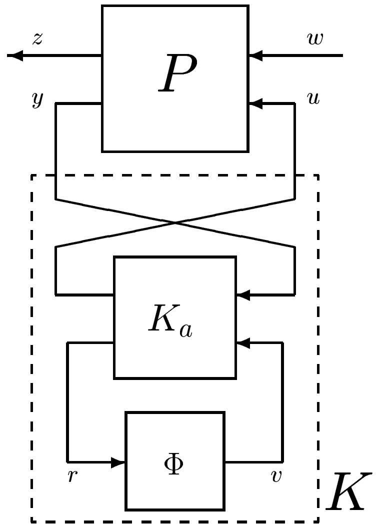

Figure 2: Plant and Controller LFT Connection

$K = \mathcal{F}_l(K_a, \Phi)$, where $\Phi \in \mathcal{RH}_\infty$, such that $\|\Phi\|_\infty < \gamma$ and K_a is of the form

$$K_a = \left[\begin{array}{c|cc} A_a & B_{1a} & B_{2a} \\ \hline C_{1a} & D_{11a} & D_{12a} \\ C_{2a} & D_{21a} & 0 \end{array}\right].$$

D_{12a}, D_{21a}, and hence K_a, are invertible. $A_a - B_{2a}D_{12a}^{-1}C_{1a}$ and $A_a - B_{1a}D_{21a}^{-1}C_{2a}$ are both stable. The controller K_a is derived via the solution of two Riccati equations.

Manipulation of $K = \mathcal{F}_l(K_a, \Phi)$ leads to $\Phi = \mathcal{F}_l(\bar{K}_a^{-1}, K)$ where

$$\bar{K}_a^{-1} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} K_a^{-1} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Now assume that controller K can also be represented by a nominal controller, K^0, subject to a weighted perturbation, $K = K^0 + W_2 \Delta W_1$, where W_1 and W_2 are stable, minimum phase and invertible and Δ is stable. This assumption imposes the restriction that K and K^0 possess the same right half plane poles, thus to some degree limiting the set of reduced order controllers attainable by our method. However, since we rely on balance and truncate or optimal Hankel norm approximation to calculate K as a weighted approximation to K^0 - where the unstable part of K^0 is copied directly into K - the assumption doesn't impose an unacceptable constraint.

Now notice that

$$\begin{aligned} \|\Phi\|_\infty < \gamma \quad &\iff \quad \|\mathcal{F}_l(\bar{K}_a^{-1}, K)\|_\infty < \gamma \\ &\iff \quad \|\mathcal{F}_l(\bar{K}_a^{-1}, K^0 + W_2 \Delta W_1)\|_\infty < \gamma \\ &\iff \quad \|\mathcal{F}_l(\tilde{R}, \Delta)\|_\infty < 1 \qquad (1) \end{aligned}$$

where

$$\tilde{R}=\begin{bmatrix} \gamma^{-1/2}I & 0 \\ 0 & W_1 \end{bmatrix}\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}\begin{bmatrix} \gamma^{-1/2}I & 0 \\ 0 & W_2 \end{bmatrix} \tag{2}$$

and R is given by the Redheffer star product [3],

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \mathcal{S}(\bar{K}_a^{-1}, \begin{bmatrix} K^0 & I \\ I & 0 \end{bmatrix}) \tag{3}$$

Remark 2: The system R is stable – hence $\tilde{R}$ is stable. This is shown by firstly observing that K^0 may be written $K^0 = \mathcal{F}_l(K_a, \Phi^0)$ for some $\Phi^0 \in \mathcal{RH}_\infty$, $\|\Phi^0\|_\infty < \gamma$ and substituting into equation (3) then using the properties of K_a.

Remark 3: The nominal controller K^0 may be designed using any methodology provided K^0 is stabilising and satisfies the condition $\|\mathcal{F}_l(P, K^0)\|_\infty < \gamma$.

The results of Theorem 1 can now be applied to equation (1). Suppose K^0 is some controller such that the closed loop $\mathcal{F}_l(P, K^0)$ is stable and $\|\mathcal{F}_l(P, K^0)\|_\infty < \gamma$. If stable, minimum phase and invertible weights W_1 and W_2 can be calculated such that $\tilde{R}$ is an isometry (note that this is not generally possible) then $\{\|\mathcal{F}_l(\tilde{R}, \Delta)\|_\infty < 1 \iff \|\Delta\|_\infty < 1\}$. Hence we obtain the condition that any controller, $K = K^0 + W_2\Delta W_1$ for some $\Delta \in \mathcal{RH}_\infty$, will also stabilise the closed loop and satisfy the same closed loop error bound $\|\mathcal{F}_l(P, K)\|_\infty < \gamma$ if and only if $\|W_2^{-1}(K - K^0)W_1^{-1}\|_\infty < 1$. This may be considered as a frequency weighted model reduction problem.

3 Weights for Sufficiency

Weights do not generally exist which satisfy the isometry condition on $\tilde{R}$. However it is always possible to find weights W_1 and W_2 which make $\tilde{R}$ a contraction and thus satisfy the sufficiency condition of the above theorem.

That is, suppose W_1 and W_2 make $\tilde{R}$ a contraction and that K^0 is some stabilising controller such that $\|\mathcal{F}_l(P, K^0)\|_\infty < \gamma$. Then K is a stabilising controller and $\|\mathcal{F}_l(P, K)\|_\infty < \gamma$ if $\|W_2^{-1}(K - K^0)W_1^{-1}\|_\infty < 1$.

Proposition 4: Assume $\|R_{11}\|_\infty < \gamma$ and define

$$\begin{aligned} L &= \begin{bmatrix} L_1 & L_2 \\ L_2^{\sim} & L_3 \end{bmatrix} \\ &\triangleq \mathcal{F}_l(\left[\begin{array}{cc|cc} 0 & -R_{22} & 0 & R_{21} \\ -R_{22}^{\sim} & 0 & R_{12}^{\sim} & 0 \\ \hline 0 & R_{12} & 0 & -R_{11} \\ R_{21}^{\sim} & 0 & -R_{11}^{\sim} & 0 \end{array}\right], \gamma^{-1}I). \end{aligned}$$

Then $\tilde{R}$ is a contraction iff weights W_1 and W_2 satisfy

$$\begin{bmatrix} (W_1^{\sim}W_1)^{-1} & 0 \\ 0 & (W_2W_2^{\sim})^{-1} \end{bmatrix} \geq \begin{bmatrix} L_1 & L_2 \\ L_2^{\sim} & L_3 \end{bmatrix} \tag{4}$$

□

Proof of proposition 4 requires the following theorem on positive semi-definite matrices.

Lemma 5 ([8], Theorem 7.7.6): Suppose that an Hermitian matrix is partitioned as $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ where $A > 0$ and $C = C^*$. This matrix is positive semi-definite if and only if $C - B^*A^{-1}B \geq 0$.

□

Proof (of proposition 4)**:** We require $\|\tilde{R}\|_\infty \leq 1$. Now

$$\begin{aligned} &\|\tilde{R}\|_\infty \leq 1 \\ &\iff \begin{bmatrix} I & \tilde{R} \\ \tilde{R}^{\sim} & I \end{bmatrix} \geq 0 \\ &\iff \left[\begin{array}{cc|cc} \gamma^{1/2}I & 0 & 0 & 0 \\ 0 & W_1^{-1} & 0 & 0 \\ \hline 0 & 0 & \gamma^{1/2}I & 0 \\ 0 & 0 & 0 & W_2^{\sim -1} \end{array}\right] \begin{bmatrix} I & \tilde{R} \\ \tilde{R}^{\sim} & I \end{bmatrix} \\ &\qquad \times \left[\begin{array}{cc|cc} \gamma^{1/2}I & 0 & 0 & 0 \\ 0 & W_1^{\sim -1} & 0 & 0 \\ \hline 0 & 0 & \gamma^{1/2}I & 0 \\ 0 & 0 & 0 & W_2^{-1} \end{array}\right] \geq 0 \\ &\iff \left[\begin{array}{cc|cc} \gamma I & 0 & R_{11} & R_{12} \\ 0 & (W_1^{\sim}W_1)^{-1} & R_{21} & R_{22} \\ \hline R_{11}^{\sim} & R_{21}^{\sim} & \gamma I & 0 \\ R_{12}^{\sim} & R_{22}^{\sim} & 0 & (W_2W_2^{\sim})^{-1} \end{array}\right] \geq 0 \\ &\iff \left[\begin{array}{cc|cc} \gamma I & R_{11} & 0 & R_{12} \\ R_{11}^{\sim} & \gamma I & R_{21}^{\sim} & 0 \\ \hline 0 & R_{21} & X & R_{22} \\ R_{12}^{\sim} & 0 & R_{22}^{\sim} & Y \end{array}\right] \geq 0 \end{aligned}$$

where $X \triangleq (W_1^{\sim}W_1)^{-1}$ and $Y \triangleq (W_2W_2^{\sim})^{-1}$. Note that

$$\|R_{11}\|_\infty < \gamma \;\Rightarrow\; \begin{bmatrix} \gamma I & R_{11} \\ R_{11}^{\sim} & \gamma I \end{bmatrix} > 0.$$

Therefore lemma 5 implies

$$\begin{aligned} &\|\tilde{R}\|_\infty \leq 1 \\ &\iff \begin{bmatrix} X & R_{22} \\ R_{22}^{\sim} & Y \end{bmatrix} \geq \begin{bmatrix} 0 & R_{21} \\ R_{12}^{\sim} & 0 \end{bmatrix} \begin{bmatrix} \gamma I & R_{11} \\ R_{11}^{\sim} & \gamma I \end{bmatrix}^{-1} \\ &\qquad\qquad \times \begin{bmatrix} 0 & R_{12} \\ R_{21}^{\sim} & 0 \end{bmatrix} \\ &\iff \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \geq \begin{bmatrix} L_1 & L_2 \\ L_2^{\sim} & L_3 \end{bmatrix}. \end{aligned}$$

□

To obtain the set of controllers K such that $\|W_2^{-1}(K - K^0)W_1^{-1}\|_\infty < 1$ is equivalent to finding the set of controllers K that lie within a weighted $\mathcal{L}_\infty$ ball of left radius W_2 and right radius W_1 centred on K^0. Hence we require both W_1 and W_2 to be large in some sense to make the resulting approximation as easy as possible. In the scalar case maximising $|W_1W_2|$ is appropriate, however in the matrix case the optimal choice is not clear. A satisfactory approach is to maximise $\det(W_1^\sim W_1)\det(W_2W_2^\sim)$.

Lemma 6: Define L as in proposition 4 and define $SS^\sim \triangleq L_1 > 0$, $T^\sim T \triangleq L_3 > 0$,

$$Z_1^2 \triangleq S^{-1}L_2L_3^{-1}L_2^\sim S^{\sim -1} \quad (5)$$

and

$$Z_2^2 \triangleq T^{\sim -1}L_2^\sim L_1^{-1}L_2T^{-1}, \quad (6)$$

where $Z_1 = Z_1^\sim \geq 0$ and $Z_2 = Z_2^\sim \geq 0$. Then the weights W_1 and W_2 which maximise $\det(W_1^\sim W_1)\det(W_2W_2^\sim)$ satisfy

$$(W_1^\sim W_1)^{-1} = S(I+Z_1)S^\sim = L_1 + SZ_1S^\sim \quad (7)$$

$$(W_2W_2^\sim)^{-1} = T^\sim(I+Z_2)T = L_3 + T^\sim Z_2T \quad (8)$$

□

Proof: Define $X = (W_1^\sim W_1)^{-1}$ and $Y = (W_2W_2^\sim)^{-1}$. We wish to minimise $\det(X)\det(Y)$ such that

$$\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \geq \begin{bmatrix} L_1 & L_2 \\ L_2^\sim & L_3 \end{bmatrix}$$

where the structure of K_a implies $L_1 = L_1^\sim > 0$ and $L_3 = L_3^\sim > 0$.

(a) Firstly assume L_2 has full row rank. Therefore $X > L_1$ and given an X the unique minimising Y is given by

$$Y = L_3 + L_2^\sim(X-L_1)^{-1}L_2.$$

Now define

$$f(X) \triangleq \frac{\det(X)\det(Y)}{\det(L_3)} = \frac{\det(X)\det(X-L_1+L_2L_3^{-1}L_2^\sim)}{\det(X-L_1)}.$$

Therefore, for $\Delta = \Delta^\sim$

$$f(X+\Delta) = \det(X)\det(I+X^{-1}\Delta) \times \frac{\det(X-L_1+L_2L_3^{-1}L_2^\sim)}{\det(X-L_1)} \times \frac{\det(I+(X-L_1+L_2L_3^{-1}L_2^\sim)^{-1}\Delta)}{\det(I+(X-L_1)^{-1}\Delta)}.$$

In order to find a stationary point observe that

$$\det(I+M\Delta) = 1+\mathrm{trace}(M\Delta)+o(\Delta)$$

and hence

$$f(X+\Delta) = f(X)\{\, 1+\mathrm{trace}([\, X^{-1}+(X-L_1+L_2L_3^{-1}L_2^\sim)^{-1} - (X-L_1)^{-1}\,]\Delta)+o(\Delta)\,\},$$

giving a stationary point if

$$\mathrm{trace}([\, X^{-1}+(X-L_1+L_2L_3^{-1}L_2^\sim)^{-1} - (X-L_1)^{-1}\,]\Delta) = 0 \qquad \forall \Delta = \Delta^\sim.$$

However for $M = M^\sim$,

$$\mathrm{trace}(M\Delta) = 0 \quad \forall \Delta = \Delta^\sim \quad \Rightarrow \quad M = M^\sim = 0.$$

Hence $f(X)$ is at a stationary point if

$$\begin{aligned} & (X-L_1+L_2L_3^{-1}L_2^\sim)^{-1} = (X-L_1)^{-1}L_1X^{-1} \\ \Rightarrow\ & X-L_1+L_2L_3^{-1}L_2^\sim = XL_1^{-1}(X-L_1) \\ \Rightarrow\ & XL_1^{-1}X-2X+L_1-L_2L_3^{-1}L_2^\sim = 0 \\ \Rightarrow\ & (S^{-1}XS^{\sim -1}-I)^2 = Z_1^2 \end{aligned}$$

and $\det(X)\det(Y)$ is minimised for

$$X = S(I+Z_1)S^\sim = L_1+SZ_1S^\sim.$$

Now note that $Y - L_3 = Y - T^\sim T$ and therefore for the above X

$$\begin{aligned} (T^{\sim -1}YT^{-1}-I)^2 &= (T^{\sim -1}(Y-T^\sim T)T^{-1})^2 \\ &= (T^{\sim -1}(Y-L_3)T^{-1})^2 \\ &= T^{\sim -1}L_2^\sim S^{\sim -1}Z_1^{-1}Z_1^2Z_1^{-1}S^{-1}L_2T^{-1} \\ &= T^{\sim -1}L_2^\sim L_1^{-1}L_2T^{-1} \\ &= Z_2^2 \end{aligned}$$

or equivalently

$$Y = T^\sim(I+Z_2)T = L_3+T^\sim Z_2T.$$

(b) If L_2 doesn't have full row rank then consider the associated augmented problem of minimising $\det(X)\det(Y_a)$ such that

$$\begin{bmatrix} X & 0 & 0 \\ 0 & Y_{a11} & Y_{a12} \\ 0 & Y_{a21}^\sim & Y_{a22} \end{bmatrix} \geq \begin{bmatrix} L_1 & L_2 & -\varepsilon I \\ L_2^\sim & L_3 & 0 \\ -\varepsilon I & 0 & I \end{bmatrix}$$

where $Y_a = Y_a^\sim > 0$, $L_1 = L_1^\sim > 0$ and $L_3 = L_3^\sim > 0$. This problem satisfies the full row rank assumption of (a) and hence the previous result applies. Then via a limiting argument it is straightforward to show that $\det(X)\det(Y)$ is minimised by X and Y_{a11} as $\varepsilon \to 0$ and that the limits are as in equations (5-8).

□

4 Computational Issues

In general it is not possible to exactly solve factorisations of the form of equations (5) and (6) parametrically. However, the following lemma shows that we may get arbitrarily close to an exact solution by using a rational approximation. More precisely, we calculate a suitable rational approximation $r(\Psi)$ to $g(\Psi) = (I+\Psi^{1/2})$ for $\Psi = \Psi^\sim \geq 0$ which can then be substituted directly into equations (7) and (8). Care must be taken when calculating this approximation to ensure that $r(\Psi) \geq g(\Psi)$. Therefore the resulting weights $\hat{W}_1$ and $\hat{W}_2$ satisfy $\hat{W}_1 \leq W_1$ and $\hat{W}_2 \leq W_2$ and our sufficient condition remains valid when weights $\hat{W}_1$ and $\hat{W}_2$ are substituted for W_1 and W_2.

Lemma 7: Given two scalar valued functions $f(\psi) = \sum_{i=0}^{\infty} \alpha_i \psi^i$ and $g(\psi) = \sum_{i=0}^{\infty} \beta_i \psi^i$ such that $\|f(\psi) - g(\psi)\|_\infty < \varepsilon$ for $\psi \in [a, b]$ then $\|f(\Psi) - g(\Psi)\|_\infty < \varepsilon$ for any matrix Ψ such that $a \leq \underline{\sigma}(\Psi) \leq x \leq \overline{\sigma}(\Psi) \leq b$. □

Proof: The result follows from considering a singular value decomposition of $h(\Psi) = f(\Psi) - g(\Psi)$. □

Hence by calculating a rational approximation, $r(\psi)$, to the function $g(\psi) = (1 + \sqrt{\psi})$ such that $r(\psi) \geq g(\psi)$ and $\|r(\psi) - g(\psi)\|_\infty$ is small (see [12] and the references therein for suitable approximation algorithms), then substituting Z_1^2 and Z_2^2 for ψ, the approximations $r(Z_1) = (I + \hat{Z}_1) \geq (I + Z_1)$ and $r(Z_2) = (I + \hat{Z}_2) \geq (I + Z_2)$ are obtained. These approximations may then be directly substituted into equation (7) before solving for $\hat{W}_1$ and into equation (8) before solving for $\hat{W}_2$.

A drawback of the proposed procedure is that the state dimension of Z_1^2 and Z_2^2 and hence the state dimension of $r(Z_1)$ and $r(Z_2)$ can be large even for an R of only moderate order. The subsequent spectral factorisations required to calculate $\hat{W}_1$ and $\hat{W}_2$ therefore become demanding. Hence a trade off exists between the accuracy with which the weights can be calculated and their McMillan degree. However, particularly when the Hankel singular values of R tail off quickly, appropriate model reduction may be performed at intermediate stages without introducing unacceptable errors.

5 Example

We now give a numerical example. Consider the interconnection shown in Fig. 3 where G is the 8 state, 6 input, 6 output, uncertainty model (including performance weights) for the longitudinal dynamics of the HIMAT advanced fighter aircraft as studied in [2].

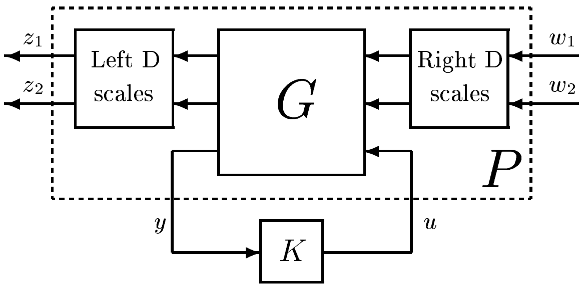

Figure 3: HIMAT LFT Interconnection

The D-scales are a product of three so called 'D-K' iterations resulting in a 20 state generalised plant. K^0 is chosen to be a $\mathcal{H}_\infty$ designed controller such that $\|\mathcal{F}_l(P, K)\|_\infty = 0.97$ and hence also has 20 states. Our aim is to maintain robust performance of the unscaled plant, i.e. $\|\mathcal{F}_l(P, K)\|_\infty < 1$. Here we look at two approaches to achieving this goal.

The following abbreviations are used throughout:

FWOHNA – Freq. weighted optimal Hankel norm approx.
FWBT – Freq. weighted balance and truncate. [1]
UOHNA – Unweighted optimal Hankel norm approx.
UBT – Unweighted balance and truncate.
LCFR – Reduction of controller left coprime factors.

For each approach an exchange algorithm [12] was employed to calculate a second order rational approximation $r(\psi)$ to $g(\psi) = (1 + \sqrt{\psi})$ such that $r(\psi) \geq g(\psi)$ and $\|r(\psi) - g(\psi)\|_\infty \approx 0.04$.

Approach I

Although acceptable controllers are those giving $\|\mathcal{F}_l(P, K)\|_\infty < 1$ experience has shown that weights calculated using $\gamma = 1$ do not necessarily lead to acceptable controllers of low McMillan degree and that better results can be obtained by relaxing γ slightly. Here we relax gamma to 1.5. In this way we expand the weighted ball around K^0 within which the search for reduced order controllers is conducted. Care must be taken in evaluating these results as for this approach $\|\hat{W}_2^{-1}(K - K^0)\hat{W}_1^{-1}\|_\infty < 1$ implies $\|\mathcal{F}_l(P, K)\|_\infty < 1.5$.

The results of model reduction are presented in Table 1. The controller order given is the least controller McMillan degree for which $\|\mathcal{F}_l(P, K)\|_\infty < 1$ is satisfied.

Reduction Technique	Controller Order	$\|\hat{W}_2^{-1}(K - K^0)\hat{W}_1^{-1}\|_\infty$	$\|\mathcal{F}_l(P, K)\|_\infty$
FWOHNA	11	0.09	0.98
FWBT	12	0.04	0.98
UOHNA	13	–	0.98
UBT	13	–	0.98
LCFR	13	–	0.98

Table 1: Results of model reduction

Table 1 shows that for the example considered, use of the proposed frequency weighted technique leads to controllers with lower McMillan degree than the unweighted techniques, while still satisfying our initial aim of robust performance.

Approach II

For the second approach we take $\gamma = 1$. Using FWOHNA or FWBT the controller of least McMillan degree giving $\|\hat{W}_2^{-1}(K - K^0)\hat{W}_1^{-1}\|_\infty < 1$ and hence $\|\mathcal{F}_l(P, K)\|_\infty < 1$ is 14. However the lower bound on the norm of the frequency weighted error for a seventh order controller was found to be 0.45. This suggests the possible existence of an acceptable seventh order controller.

[1] Here we truncate a balanced realisation of the stable part of $\hat{W}_2^{-1} K^0 \hat{W}_1^{-1}$ to obtain say $\overline{K}$ then construct K as the stable part of $\hat{W}_2 \overline{K} \hat{W}_1$.

Employing the convex optimisation procedure of [14] (which uses the FWOHNA parameterisation as a starting point) a seventh order controller satisfying $\|\hat{W}_2^{-1}(K - K^0)\hat{W}_1^{-1}\|_\infty < 1$ was found. This controller gave $\|\mathcal{F}_l(P, K)\|_\infty = 0.98$. Hence via the given procedure the nominal 20 state $\mathcal{H}_\infty$ designed controller has been reduced down to 7 states without loss of robust performance.

6 Conclusion

Given a stabilising controller K^0 such that $\|\mathcal{F}_l(P, K^0)\|_\infty < \gamma$ we have derived a sufficient condition for another controller, K, to also be stabilising and satisfy $\|\mathcal{F}_l(P, K)\|_\infty < \gamma$. The condition is formulated using an $\mathcal{H}_\infty$ framework and presented in terms of a norm bound on the frequency weighted error between K and K^0. This may be considered as a frequency weighted model reduction problem with K^0 being a high order controller and K a reduced order controller.

Although the method presented has worked well, in practise it is conservative. The approach taken will only find controllers K which lie within a weighted $\mathcal{L}_\infty$ ball of K^0 and which possess the same unstable poles as K^0. Hence different K^0 will lead to different K. There may also be stabilising controllers which lie arbitrarily far from K^0 yet still satisfy $\|\mathcal{F}_l(P, K)\|_\infty < \gamma$.

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