A Graphical Interpretation of SISO $\mathcal{H}_\infty$ Controller Approximation*

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Abstract

Given a stabilising controller which satisfies a $\mathcal{H}_\infty$ performance bound on the closed loop transfer fuction, sufficient conditions for any other controller to be stabilising and satisfy the same $\mathcal{H}_\infty$ performance bound are presented in [4, 5, 6]. Such controllers are said to be (P,γ)-admissible, where P is the model of the plant under consideration and γ is the required level of prespecified $\mathcal{H}_\infty$ performance. The conditions are expressed as norm bounds on particular frequency weighted errors where the weights are selected to make a specific transfer function a contraction. Subject to this constraint, approximation is made easier if the weights are as large as possible.

Here we show that for scalar controllers the sufficient conditions of [4, 5, 6] have a natural frequency by frequency interpretation as either the inside or outside of a circle in the complex plane.

We show that given a multivariable (P,γ)-admissible controller maximising the product of the determinants of the weights always leads to a *direction* where the weights are the best possible. Further, we show that for scalar weights maximising the product of the traces of the weights always leads to a direction where the weights are the best possible.

Keywords : Controller Approximation, Frequency weighted model reduction, Graphical Methods, H-infinity.

1 Introduction

Consider $\mathcal{F}_l(\tilde{R},\Delta)$ as given in Figure 1.

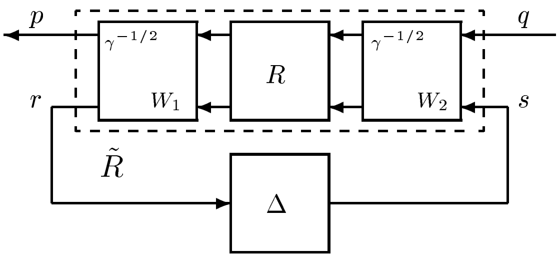

Figure 1: $\mathcal{F}_l(\tilde{R},\Delta)$

Given a nominal (P,γ)-admissible controller $K_0 = \tilde{V}_0^{-1}\tilde{U}_0$, an appropriately defined R, and stable, minimum phase and invertible weights which make $\tilde{R}$ contractive, it is shown in [5] that $K = K^0 + W_2\Delta W_1$ is (P,γ)-admissible if

$$\|W_2^{-1}(K-K^0)W_1^{-1}\|_\infty < 1. \qquad (1)$$

Similarly, in [6] it is shown that (for a different R and different weights)

$$\|W_2^{-1}(\left[\begin{array}{cc}\tilde{U} & \tilde{V}\end{array}\right] - \left[\begin{array}{cc}\tilde{U}_0 & \tilde{V}_0\end{array}\right])W_1^{-1}\|_\infty < 1 \qquad (2)$$

implies that $K = \tilde{V}^{-1}\tilde{U}$ is (P,γ)-admissible.

Selecting weights according to the following Lemma ensures that $\tilde{R}$ is contractive.

Lemma 1.1 ([4, 5, 6]) *Assume $\|R_{11}\|_\infty < \gamma$ and define*

$$\begin{aligned} L &= \left[\begin{array}{cc} L_1 & L_2 \\ L_2^\sim & L_3 \end{array}\right] \\ &\triangleq \mathcal{F}_l(\left[\begin{array}{cc:cc} 0 & -R_{22} & 0 & R_{21} \\ -R_{22}^\sim & 0 & R_{12}^\sim & 0 \\ \hdashline 0 & R_{12} & 0 & -R_{11} \\ R_{21}^\sim & 0 & -R_{11}^\sim & 0 \end{array}\right], \gamma^{-1}I). \end{aligned}$$

Define $X \triangleq (W_1^\sim W_1)^{-1}$ and $Y \triangleq (W_2 W_2^\sim)^{-1}$. Then $\tilde{R}$ is a contraction if and only if

$$\left[\begin{array}{cc} X & 0 \\ 0 & Y \end{array}\right] \geq \left[\begin{array}{cc} L_1 & L_2 \\ L_2^\sim & L_3 \end{array}\right]. \qquad (3)$$

Equations (1) and (2) have a straight forward interpretation as frequency weighted model reduction problems. The design of reduced order (P,γ)-admissible controllers by these techniques is therefore equivalent to finding the set of reduced order K or $\left[\begin{array}{cc}\tilde{U} & \tilde{V}\end{array}\right]$ that lie within a weighted $\mathcal{L}_\infty$ ball of left radius W_2 and right radius W_1 centred on K^0 or $\left[\begin{array}{cc}\tilde{U}_0 & \tilde{V}_0\end{array}\right]$.

The approximation is made easier if the weights are maximised in some sense. A procedure to maximise the product $\det(W_1^\sim W_1)\det(W_2 W_2^\sim)$ is given in [4, 5] and a procedure to maximise the product $\mathrm{trace}(W_1^\sim W_1)\mathrm{trace}(W_2 W_2^\sim)$ is given in [4, 6].

Lemma 1.2 ([4, 5]) *Define L as in proposition 1.1 and define $SS^\sim \triangleq L_1 > 0$, $T^\sim T \triangleq L_3 > 0$,*

$$Z_1^2 \triangleq S^{-1}L_2L_3^{-1}L_2^\sim S^{\sim-1} \qquad (4)$$

$$Z_2^2 \triangleq T^{\sim-1}L_2^\sim L_1^{-1}L_2T^{-1}, \qquad (5)$$

where $Z_1 = Z_1^\sim \geq 0$ and $Z_2 = Z_2^\sim \geq 0$. Then the weights W_1 and W_2 which maximise the product $\det(W_1^\sim W_1)\det(W_2W_2^\sim)$ subject to equation (3) satisfy

$$(W_1^\sim W_1)^{-1} = S(I+Z_1)S^\sim = L_1 + SZ_1S^\sim \qquad (6)$$

$$(W_2W_2^\sim)^{-1} = T^\sim(I+Z_2)T = L_3 + T^\sim Z_2 T \qquad (7)$$

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Lemma 1.3 ([4, 6]) *Define L as in Proposition 1.1 and define*

$$\alpha_{opt} \triangleq \sqrt{(trace(L_3)/trace(L_1))} \quad (8)$$

$$Z_1^2 \triangleq L_2^{\sim} L_2 \quad (9)$$

$$Z_2 \triangleq L_2 Z_1^{-1} L_2^{\sim} \quad (10)$$

where $Z_1 = Z_1^{\sim} > 0$ and $Z_2 = Z_2^{\sim} \geq 0$. Then the weights W_1 and W_2 which maximise the product $trace(W_1^{\sim} W_1) trace(W_2 W_2^{\sim})$ subject to equation (3) satisfy

$$(W_1^{\sim} W_1)^{-1} = L_1 + \alpha_{opt}^{-1} Z_2 \quad (11)$$

$$(W_2 W_2^{\sim})^{-1} = L_3 + \alpha_{opt} Z_1 \quad (12)$$

In Section 2 we show that taken frequency by frequency, for SISO controllers, equations (1) and (2) have a natural frequency by frequency interpretation as disks in the complex plane. Hence, if at every frequency a controller's Nyquist plot lies inside, or where appropriate, outside the circle which corresponds to the boundary of the disk at that frequency then the controller is guaranteed to be (P, γ)-admissible.

The discussion which follows is valid on a frequency by frequency basis. Each frequency corresponds to a particular point on a controller's Nyquist plot and to a particular circle. When we say controllers lying inside/outside of a circle are (P, γ)-admissible, we mean that at every frequency the point on the controller's Nyquist plot corresponding to that frequency lies inside/outside of the circle which corresponds to that frequency.

For brevity all proofs have been omitted. They may be found in [4].

2 A Graphical Approach

To show, for scalar controllers, the equivalence between the sufficient conditions of [5, 6] and circles in the complex plane, we must firstly show the equivalence between equations (1) and (2) respectively and the equation,

$$K = U + V \Upsilon W. \quad (13)$$

Notice that for constant U, V, $W \in \mathbb{C}^{1\times1}$, and $\Upsilon \in \mathbb{C}^{1\times1}$ varying such that $|\Upsilon| = 1$, equation (13) represents a circle in the complex plane with centre U and radius $|VW|$. Further, $|\Upsilon| < 1$ represents the inside of the circle and $|\Upsilon| > 1$ represents the outside of the circle. This allows us to use a graphical procedure to design (P, γ)-admissible controllers.

In the matrix case, to see that $\Delta = W_2^{-1}(K - K^0)W_1^{-1}$ may be written in the form of equation (13), simply define $U \triangleq K^0$, $V \triangleq W_2$, $W \triangleq W_1$ and $\Upsilon \triangleq \Delta$. $\|\Delta\|_\infty < 1$ implies K is a (P, γ)-admissible controller. Therefore, in the scalar case, (P, γ)-admissible controllers which satisfy the sufficient condition of equation (1) may be interpreted graphically as those controllers whose Nyquist plot lies within a circle of radius $|W_2 W_1|$ centred on K^0.

To show that equation (2) may be written in the form of equation (13) requires the following two Lemma's. Firstly notice that, in the matrix case, equation (2) is equivalent to,

$$K = \mathcal{F}_l\left(\begin{bmatrix} I & 0 \\ 0 & W_1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} K^0 \\ I \\ -K^0 \end{bmatrix} & \begin{bmatrix} \tilde{V}_0^{-1} \\ 0 \\ -\tilde{V}_0^{-1} \end{bmatrix} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W_2 \end{bmatrix}, \Delta\right) \quad (14)$$

$$\triangleq \mathcal{F}_l\left(\begin{bmatrix} E & F \\ S & T \end{bmatrix}, \Delta\right)$$

where $\Delta = \begin{bmatrix} \Delta_{\tilde{U}} & \Delta_{\tilde{V}} \end{bmatrix} = W_2^{-1}\left\{\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix} - \begin{bmatrix} \tilde{U}_0 & \tilde{V}_0 \end{bmatrix}\right\} W_1^{-1}$ and $\|\Delta\|_\infty < 1$.

Lemma 2.1 is applicable when $\bar{\sigma}(T) < 1$, while Lemma 2.2 is applicable when $\underline{\sigma}(T) > 1$.

Lemma 2.1 *Define*

$$K \triangleq \mathcal{F}_l\left(\begin{bmatrix} E & F \\ S & T \end{bmatrix}, \Delta\right) \in \mathbb{C}^{m\times p}$$

where E, F, S, T and Δ are appropriately dimensioned complex valued matrices, $\bar{\sigma}(\Delta) < 1$ and $\bar{\sigma}(T) < 1$. Then K may be written

$$K = U + V\Upsilon W$$

where $\bar{\sigma}(\Upsilon) < 1$ and

$$U = E + F\left\{(I - T'T)^{-1/2} T' (I - TT')^{-1/2}\right\} S \quad (15)$$

$$V = F(I - T'T)^{-1/2} \quad (16)$$

$$W = (I - TT')^{-1/2} S. \quad (17)$$

Lemma 2.2 *Define*

$$K \triangleq \mathcal{F}_l\left(\begin{bmatrix} E & F \\ S & T \end{bmatrix}, \Delta\right) \in \mathbb{C}^{p\times p}$$

where E, $F \in \mathbb{C}^{p\times p}$ with F invertible, S, T and Δ are appropriately dimensioned complex valued matrices, $\bar{\sigma}(\Delta) < 1$ and $\underline{\sigma}(T) > 1$. Assume $F(T'T - I)^{-1}F' > 0$ and $S'(TT' - I)^{-1}S > 0$. Then K may be written

$$K = U + V\Upsilon W$$

where $\underline{\sigma}(\Upsilon) > 1$ and

$$U = E - F(T'T - I)^{-1}T'S$$

$$VV' = F(T'T - I)^{-1}F'$$

$$W'W = S'(TT' - I)^{-1}S$$

We are now in a position to specialise the general matrix valued results of Lemmas 2.1 and 2.2 to the case of scalar controllers.

Proposition 2.3 *Consider the scalar valued LFT*

$$K \triangleq \mathcal{F}_l\left(\begin{bmatrix} E & F \\ S & T \end{bmatrix}, \Delta\right) \in \mathbb{C}^{1\times1} \quad (18)$$

where E, $F \in \mathbb{C}^{1\times1}$, S, $T \in \mathbb{C}^{2\times1}$, $\Delta \in \mathbb{C}^{1\times2}$ and Δ varies such that $\sigma(\Delta) < 1$[1].

[1] Note that in the scalar case Δ and T each have only one singular value.

i) If $\sigma(T) < 1$ then K describes the inside of a circle in the complex plane.

ii) If $\sigma(T) > 1$, F is invertible and $S'(TT'-I)^{-1}S > 0$ then K describes the outside of a circle in the complex plane.

iii) If $\sigma(T) = 1$ and $S'(TT'-I)^{\dagger}S > 0$ then K describes a half plane.

Proposition 2.3 shows that in the scalar case, equation (14), and hence equation (2), represents either the inside or outside of a circle in the complex plane. Experience indicates that $\sigma(T) > 1$ for low frequencies and $\sigma(T) < 1$ for medium and high frequencies. Hence in the coprime factor framework of [6] (P,γ)-admissible controllers may have to lie outside circles in one frequency range and inside circles in another. These ideas will become more concrete when we apply them to our example in Section 4.

In the example we will observe that at every frequency the circle which results from the coprime factor approache corresponds exactly with the boundary of the region of the complex plane where it is necessary for $K(j\omega)$ to lie for $\bar{\sigma}(\mathcal{F}_l(P(j\omega),K(j\omega))) < \gamma$. Note that there is no guarantee that a controller whose Nyquist plot satisfies $\bar{\sigma}(\mathcal{F}_l(P(j\omega),K(j\omega))) < \gamma\ \forall\omega$ will stabilise the closed loop.

3 Pareto Optimal Weights

In this section we show that for the additive uncertainty approach and a given K^0, (in the matrix case) maximising the determinant of the weights always leads to a *direction* where the weights are the best possible, and hence cannot be uniformly improved upon in the class considered (termed Pareto optimal). Further, in the scalar case, we show that maximising the trace of the weights always leads to a *direction* where the weights are the best possible.

We now define what we mean by the term *Pareto optimal* weights.

Definition 3.1 *W_1 and W_2 are Pareto optimal weights if $\exists\Delta \in \mathcal{RH}_\infty$ such that $\|\Delta\|_\infty = 1$ and $K = K^0 + W_2\Delta W_1$ such that $\|\mathcal{F}_l(P,K)\|_\infty = \gamma$.*

Definition 3.1 implies there exists a direction where our $\mathcal{L}_\infty$ ball cannot be made any larger without K losing (P,γ)-admissibility.

Proposition 3.2 *Consider $\mathcal{F}_l(\tilde{R},\Delta)$ as given in Figure 1. Define X, Y and L as in Proposition 1.1. If there exists a vector $v \neq 0$ such that*

$$v'\left\{L_1 - L_2(Y-L_3)^{-1}L_3(Y-L_3)^{-1}L_2^{\sim}\right\}v \quad = \quad 0,$$

then W_1 and W_2 are the best possible weights.

Lemma 3.3 *Assume weights for the additive perturbation approach of equation (1) are calculated according to Lemma 1.2, and let $v \triangleq L_1^{-1}L_2u$. Then $\forall u$,*

$$v'\left\{L_1 - L_2(Y-L_3)^{-1}L_3(Y-L_3)^{-1}L_2^{\sim}\right\}v \quad = \quad 0.$$

Hence W_1 and W_2 are Pareto optimal weights.

Remark 3.4 *Note that Lemma 3.3 holds for general matrix valued weights. Hence there always exists a* direction *where the $\mathcal{L}_\infty$ ball centred on K^0 with left radius W_2 and right radius W_1 is as large as possible subject to the closed loop remaining less than γ.*

Lemma 3.5 *Assume K^0 is a scalar controller and weights for the additive perturbation approach of equation (1) are calculated according to Lemma 1.3. Then $\forall v$,*

$$v'\left\{L_1 - L_2(Y-L_3)^{-1}L_3(Y-L_3)^{-1}L_2^{\sim}\right\}v \quad = \quad 0.$$

Hence W_1 and W_2 are Pareto optimal weights.

Lemma 3.3 and Lemma 3.5 imply that for scalar weights, the procedure to maximise the product of the determinants of the weights and the procedure to maximise the product of the traces of the weights are equivalent, although this is also obvious since in the scalar case the criteria are the same.

In the scalar case the Pareto optimal weights will in fact be best possible in the sense that the disk corresponding to using the additive uncertainty approach just touches the boundary of the region of the complex plane where it is necessary for $K(j\omega)$ to lie for $\bar{\sigma}(\mathcal{F}_l(P(j\omega),K(j\omega))) < \gamma$. Note again that there is no guarantee that a controller whose Nyquist plot satisfies $\bar{\sigma}(\mathcal{F}_l(P(j\omega),K(j\omega))) < \gamma\ \forall\omega$ will stabilise the closed loop.

4 Example

To demonstrate the ideas presented in the previous sections, and to compare the performance of standard frequency weighted model reduction to the proposed graphical approach, we consider controller synthesis for the SISO pitch loop (including performance weights) of the Generic VSTOL Aircraft Model (GVAM) as studied in [7]. We wish to design controllers such that $\|\mathcal{F}_l(P,K)\|_\infty < \gamma$ where

$$\mathcal{F}_l(P,K) \quad = \quad \left[\begin{array}{c} I \\ K \end{array}\right](I-GK)^{-1}\left[\begin{array}{cc} I & G \end{array}\right]$$

and G is a 17 state model which describes the GVAM in hover mode with reaction jets controlling the pitch attitude. The optimal $\gamma_{opt} = 2.12$. Here we use the suboptimal value $\gamma = 4$.

The results of standard frequency weighted model reduction (without any subsequent parameter optimisation) are presented in Table 1. The controller of least degree which satisfies the appropriate sufficient condition for the frequency weighted methods, or $\|\mathcal{F}_l(P,K)\|_\infty < 4$, for the unweighted methods, is presented.

The following abbreviations are used:

FWAdd	–	Additive uncertainty approach of [5].
FWLCFR	–	Coprime factor uncertainty approach of [6].
UOHNA	–	Unweighted optimal Hankel norm approx.
UBT	–	Unweighted balanced truncation.
LCFR	–	Reduction of controller left coprime factors.

K^0 is chosen to be a near optimal 17 state controller. The $\|(\text{weighted error})\|_\infty$ corresponds to the norm of equation (1) or (2) with the appropriate weights.

Reduction Technique	Controller Order	$\|(\text{weighted error})\|_\infty$	$\|\mathcal{F}_l(P,K)\|_\infty$
	17 (full)	–	2.12
FWAdd	2	0.84	3.44
FWLCFR	4	0.67	2.36
UOHNA	3	–	2.9
UBT	3	–	2.6
LCFR	3	–	2.7

Table 1: Frequency Weighted Model Reduction

Reduction Technique	Lower bound for $\|(\text{weighted error})\|_\infty$	Actual $\|(\text{weighted error})\|_\infty$
FWAdd	0.68	11.67
FWLCFR1	0.82	8.84

Table 2: Lower Bound for Zeroth Order Approximation

The results of Table 1 show that for the example considered and using frequency weighted model reduction without parameter optimisation, the smallest order (P,γ)-admissible controller calculated is second order.

However, Table 2 shows that the lower bound on the approximation error for a zeroth order controller (i.e. constant feedback) is less than unity. Although the lower bound may not be attainable, these results suggest the possible existence of a zeroth order (P,γ)-admissible controller. Unfortunately the zeroth order controllers obtained by frequency weighted model reduction without optimisation don't even stabilise the closed loop. Employing parameter optimisation – using the controller obtained from frequency weighted optimal Hankel norm model reduction as a starting point [8] – a zeroth order controller satisfying equation (1) for FWAdd was found. Unfortunately, for the FWLCFR approach no constant $\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix}$ satisfying equation (2) could be found.

Reduction Technique	Controller Order	$\|(\text{weighted error})\|_\infty$	$\|\mathcal{F}_l(P,K)\|_\infty$
FWAdd	0	0.956	2.89

Table 3: Zeroth Order (P,γ)-admissible Controllers

The results thus far have served to reinforce the belief that frequency weighted model reduction without optimisation may perform poorly and can be improved upon by including a parameter optimisation step. However, even with optimisation, no coprime factorisation of a zeroth order (P,γ)-admissible controller could be found which satisfied equation (2). We will now show that by using the graphical approach of Section 2 a zeroth order controller may be calculated using either the FWAdd or FWLCFR approach.

Using an appropriate frequency list, calculate U, V, W and Υ as required to write equations (1) and (2) in the form of equation (13). Plots of the calculated circles for the frequencies $\omega = 0.1, 1, 2, 4, 10, 13, 20, 50, \text{and} 1000\text{rad/s}$ are shown in Figure 2. (Note that in practice many more frequency points than this must be examined). Each plot contains two circles and the Nyquist plot for the nominal, near optimal, controller K^0 which has a '+' corresponding to the relevant frequency for that plot.

The dashed circles are for the FWAdd approach. Note that they are centered on the '+'. Since the '+' is inside the circle for each of these plots K^0 is (P,γ)-admissible – as it was designed to be. Observe that the region of the real axis between 0.6 and 0.78 (approximately) lies within the circles for every frequency. Constant feedback controllers with a value in this region therefore satisfy equation (1) and are guaranteed to be (P,γ)-admissible. This is easily confirmed by calculating the weighted error for such controllers and seeing it is less than unity.

The dotted circles are for the FWLCFR approach. For many of the plots a scale has been chosen such that only a (small) portion of the circle can be seen. However from careful examination of the plots it is a simple task to determine which part of the plane is enclosed. $\sigma(T) > 1$ for the plots up to and including $\omega = 13$rad/s, hence the Nyquist plots at these frequencies must lie outside the dotted circle. It is easy to verify from these plots that K^0 is (P,γ)-admissible.

For FWLCFR the region of the real axis between 0.62 and 1.1 (approximately) always lies in the appropriate region of the plane. Hence there exists a coprime factorisation $\begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix}$, such that $\tilde{V}^{-1}\tilde{U}$ takes a constant value between 0.62 and 1.1, which satisfies equation (2). However, we may not be able to explicitly calculate such a factorisation. Importantly, $K = \tilde{V}^{-1}\tilde{U}$ is known, and is guaranteed to be (P,γ)-admissible.

To obtain an indication of the possible conservatism in our weights we then calculated the region of the plane where it is necessary for $K(j\omega)$ to lie for $\bar{\sigma}(\mathcal{F}_l(P(j\omega), K(j\omega))) < 4$. We stress again that there is no guarantee that a controller whose Nyquist plot satisfies $K(j\omega)$ such that $\bar{\sigma}(\mathcal{F}_l(P(j\omega), K(j\omega))) < 4 \ \forall\omega$ will stabilise the closed loop.

We have a priori information about this necessary region. The region must always contain the sufficient regions which result from the FWAdd and FWLCFR approaches. Hence, when $\sigma(T) > 1$ the boundary of the necessary region must be contained within the dotted circles, and when $\sigma(T) < 1$ the necessary region must contain the dotted circles (and the dashed circles). Further, since we know that the dashed circles are as large as possible, they must always touch the necessary region at at least one point.

Surprisingly, the boundary of this region corresponds exactly with the circle derived from the FWLCFR approach. Further investigations showed that for randomly generated generalised plants, using the FWLCFR approach and maximising the product of the traces of the weights always led to an $\tilde{R}$ such that $\tilde{R}'\tilde{R} = I$. It would be tempting to conclude from this that the conditions of the FWLCFR approach are necessary in addition to being sufficient.

5 Conclusion

We have shown that for scalar controllers, equations (1) and (2) may be interpreted frequency by frequency as disks in the complex plane. Hence the sufficient conditions for the design of reduced order (P,γ)-admissible controllers by the techniques of [4, 5, 6] have a graph-

Figure 2: Plots for Graphical Design Procedure

ical interpretation. If at every frequency the appropriate point on a controller's Nyquist plot lies inside/outside of the circle which corresponds to the boundary of the disk at that frequency then the controller is (P, γ)-admissible.

We have proved that for the additive uncertainty approach and a given K^0, the procedure presented in [5] to maximise the product of the determinants of the weights results (in the general matrix case,) in a *direction* where the weights are the best possible. No other maximising criteria can do better in this direction. In the scalar case this implies that the best possible weights have been selected. We also proved that in the scalar case, maximising the product of the traces of the weights results in the best possible weights (when used with the additive uncertainty approach).

In the design example we saw that the proposed graphical approach was able to give a constant controller when this was not forthcoming from the frequency weighted model reduction.

That the circles for the FWLCFR approach are larger than those for the FWAdd approach over the entire frequency range is consistent with the extra degree of freedom afforded to the fractional representations by allowing unstable poles to move across the $j\omega$ axis. However, we have been yet to characterize when these circles coincide with the circles which correspond to the boundary of the region of necessity.

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