Linear Periodic Systems: Robustness Analysis and Sampled-Data Control

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Abstract

The primary purpose of this dissertation is to present various results pertaining to the study of continuous-time systems that exhibit linear, periodically-time-varying (LPTV) behaviour. The original motivation for this work stems from the periodic nature of many sampled-data (SD) control-systems and a desire to characterise sensible frameworks for design and robustness analysis of such systems.

Towards the development of a sensible framework for SD, control-system design, a new notion of frequency response is established for LPTV systems. It is defined in terms of the average-power of the asymptotic response to sinusoidal signals of a single frequency and can be characterised in terms of the singular values of a frequency-dependent, finite-dimensional matrix. The performance indicating properties of this new notion of frequency response are intuitive and it is used to derive bounds that facilitate the design of parametric weights employed in a new H^∞-loopshaping based procedure proposed for SD, control-system development.

Quantitative and qualitative results that characterise the uncertainty types to which LPTV, closed-loop systems can be desensitised are established in terms of the gap metric, which is a measure of the distance between the graphs of two systems. To this end, a formula for the directed-gap between two LPTV systems is obtained in terms of an optimisation over H^∞. This plays a crucial role in the development of the main quantitative robustness result, which identifies the largest gap-ball of LPTV plants, centred at a nominal plant, that a nominal, stabilising controller is guaranteed to stabilise. A similar result that deals with simultaneous gap-perturbations to the plant and controller is also given. Qualitatively, it is established that the topology induced by the gap metric on quite a general class of LPTV systems, is the weakest with respect to which closed-loop performance varies continuously and closed-loop stability is a robust property. All of the results accommodate infinite-dimensional input/output spaces and apply to the special case of LPTV, SD control-systems.

The existence of particular representations of a class of LPTV systems is central to the framework used to obtain robustness results. Specifically, it is shown that the graph of a stabilisable, LPTV system can be expressed as the range and kernel of stable, LPTV systems that are respectively, left and right invertible by stable, LPTV systems. These representations of the graph resemble the coprime-factor representations known to exist for linear, time-invariant systems and lead to a useful characterisation of closed-loop stability. This in turn, yields a Youla-style parameterisation of stabilising controllers.

A numerical procedure is developed for computing the gap, to any desired accuracy, between LPTV systems that admit stabilisable and detectable, state-space realisations. This involves determining the existence of a solution to a related linear, shift-invariant, discrete-time, full-information, $\ell^2_{\mathbb{Z}+}(\cdot)$-synthesis problem, for which computationally-tractable necessary and sufficient conditions are obtained.

To complete this work, a new, compact derivation of state-space formula for H^∞, SD synthesis is presented. Related numerical issues are discussed and it is shown how to restructure the formulae derived for numerical robustness.

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As required by University Statute, I hereby declare that this dissertation is not substantially the same as any that I have submitted for a degree at any other University, is the result of my own work, and includes nothing which is the outcome of work done in collaboration.

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Contents

Notation

Symbols

$\mathbb{R}$ $(\mathbb{R}^+)$	set of real numbers (non-negative reals)
$\mathbb{Z}$ $(\mathbb{Z}^+)$	set of integers (non-negative integers)
$\mathbb{C}$	set of complex numbers
$\mathbb{D}$	open unit disc in the complex plane
$\mathbb{T}$	unit circle in the complex plane
$\mathbb{E}$	complement of the open unit disc; $\mathbb{E} := \mathbb{C} - \mathbb{D}$
$\mathbb{H}$	the interval $[0, h)$ of the real line
$]\cdot[$	the integer part of a real number
$\in$	is an element of
$\subset$	is a subset of
$\cup$	union of two sets
$\cap$	intersection of two sets
$\times$	Cartesian product of two sets
$:=$	left-hand side is defined by right-hand side
$=:$	right-hand side is defined by left-hand side
$\Leftrightarrow, \Updownarrow$	if and only if
$\mathscr{A}, \mathscr{F}$	this font is used to denote general sets
$\mathcal{B}, \mathcal{W}$	this font is used to denote general Banach Spaces
$\mathcal{H}, \mathcal{U}$	this font is used to denote general Hilbert spaces
$\Vert \cdot \Vert_{\mathscr{V}}$	norm on the vector space $\mathscr{V}$
$\langle \cdot, \cdot \rangle_{\mathcal{H}}$	inner product on the Hilbert space $\mathcal{H}$
$\mathcal{H} \ominus \mathcal{U}$ (or $\mathcal{U}^{\perp}$)	orthogonal complement of $\mathcal{U}$ in $\mathcal{H}$
$\mathcal{U} \oplus \mathcal{Y}$	orthogonal direct sum of $\mathcal{U}$ and $\mathcal{Y}$
$\mathbf{\Pi}_{\mathcal{U}}$	orthogonal projection onto $\mathcal{U}$
$\mathbf{\Pi}_{\mathcal{G} \Vert \mathcal{F}}$	parallel projection onto $\mathcal{G}$ along $\mathcal{F}$
$\mathcal{B}_{\mathcal{U} \to \mathcal{Y}}$	the set of bounded, linear operators mapping $\mathcal{U}$ to $\mathcal{Y}$
$\mathcal{D}.$	domain of a linear operator
$\mathcal{R}.$	range of a linear operator
$\mathcal{K}.$	kernel of a linear operator
$\mathcal{G}.$ $(\mathcal{G}'.)$	graph (inverse-graph) of a linear operator
$\cdot\vert_{\mathcal{H}}$	restriction of an operator to $\mathcal{H}$
$.^{-1}$	inverse of an operator
$.^{*}$	adjoint of a bounded, linear operator
$\mathrm{spec}(\cdot)$	spectrum of a bounded, linear operator
$\mathrm{rad}(\cdot)$	spectral radius of a bounded, linear operator
$\sigma(\cdot)$	singular values of a bounded, linear operator
$\overline{\sigma}\{\cdot\}$	maximum singular value of a compact operator
$\underline{\sigma}\{\cdot\}$	minimum singular value of a compact operator

Symbols ctd.

$\mathscr{Y}_{\mathbb{S}}(\mathcal{B})$	vector space of $\mathcal{B}$-valued signals on the time interval of definition $\mathbb{S}$
$\mathbf{P}_\tau$	the projection that truncates a signal to zero after $t = \tau$
$\mathbf{U}_\tau$	unilateral shift by τ on $\mathscr{Y}_{\mathbb{R}^+}(\cdot)$
$\mathbf{S}$	unilateral shift on $\mathscr{Y}_{\mathbb{Z}^+}(\cdot)$
$\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{H})$	the Hilbert space of $\mathcal{H}$-valued, (Lebesgue) square-integrable signals in $\mathscr{Y}_{\mathbb{R}^+}(\mathcal{H})$ (i.e. the finite-energy, continuous-time signals)
$\mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{H})$	the extended space corresponding to $\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{H})$
$\ell^2_{\mathbb{Z}^+}(\mathcal{H})$	the Hilbert space of $\mathcal{H}$-valued, square-summable functions in $\mathscr{Y}_{\mathbb{Z}^+}(\mathcal{H})$ (i.e. the finite-energy, discrete-time signals)
$\ell^{2,\mathrm{e}}_{\mathbb{Z}^+}(\mathcal{H})$	the extended space corresponding to $\ell^2_{\mathbb{Z}^+}(\mathcal{H})$
$\mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$	the Hardy space of $\mathcal{H}$-valued functions of a complex variable, that are analytic in $\mathbb{D}$ and (Lebesgue) square-integrable on $\mathbb{T}$
$\mathbf{Z}$	the $\mathbf{Z}$-Transform (isomorphism between $\ell^2_{\mathbb{Z}^+}(\mathcal{H})$ and $\mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$)
$\mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{U}\to\mathcal{Y}})$	the Hardy space of $\mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$-valued functions of a complex variable, that are analytic and bounded in $\mathbb{D}$
$\lVert\cdot\rVert_\infty$	the norm on $\mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{U}\to\mathcal{Y}})$
$\mathbf{M}_\cdot$	the multiplication operator with symbol $\cdot$
$\mathrm{L}^2_{\mathbb{H}}(\mathcal{H})$	the Hilbert space of (Lebesgue) square-integrable, $\mathcal{H}$-valued functions defined on the finite-horizon $\mathbb{H}$
$\mathbf{W}$	the time-lifting operator (isomorphism between $\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{H})$ and $\ell^2_{\mathbb{Z}^+}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{H}))$)
$\mathbf{A_D}$ and $\mathbf{D_A}$	ideal, sampling device and zero-order, hold device respectively
$\mathscr{P}_{\mathcal{U}\to\mathcal{Y}}$	the set of causal, linear, periodically-time-varying, continuous-time systems, which at each point in time map $\mathcal{U}$ to $\mathcal{Y}$
$\mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$	the subset of systems in $\mathscr{P}_{\mathcal{U}\to\mathcal{Y}}$ that are causally extendible to (time) locally, Lipschitz-continuous systems on extended space
$\mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U}\to\mathcal{Y}}$	the subset of systems in $\mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ that are strongly causal
$\mathscr{D}_{\mathcal{U}\to\mathcal{Y}}$	the set of causal, linear, shift-invariant, discrete-time systems, which at each point in time map $\mathcal{U}$ to $\mathcal{Y}$
P, A, u, y	this font is used to denote anything associated with discrete-time systems
$\vec{\cdot}$	is used to denote that $\cdot$ is in $\mathrm{L}^2_{\mathbb{H}}(\cdot)$
$\hat{\cdot}$	is used to denote frequency-domain transfer operator

Acronyms

LPTV	linear periodically-time-varying
LSI	linear shift-invariant (discrete-time)
LTI	linear time-invariant (continuous-time)
SD	sampled-data
TPBVP	two-point, boundary-value problem

1

Introduction

1.1 Motivation and Background

Control theory is the study of dynamical systems and techniques that allow their behaviour to be modified by manipulating accessible inputs. The auxiliary system used to determine the actions required to achieve a desired change in a system's behaviour is called a controller and as a whole, the original system and controller combined is called a control system. Given an *exact* representation of a system (including any disturbances acting upon it) and an *achievable* desired behaviour, it is possible to design a controller to achieve the desired change in behaviour without using measurable information.[1] This is referred to as *open-loop* control. The problem with this approach is that a system is most readily represented by a mathematical model, which can at best, only approximate the real behaviour of a physical system. The notion of uncertainty arising from the mismatch between physical processes (the system to be controlled, disturbances acting on it and the controller itself) and the mathematical models used to represent them was central to the development of *feedback* control theory, in which measurable information is exploited in the control strategy. An important property of closed-loop (or feedback) control-systems is that the effects of uncertainty arising in this manner can be significantly reduced. The study of uncertainty types to which closed-loop systems can be desensitised and the development of suitable mathematical frameworks to exploit this property, is called *robust control* theory.

Classifying systems according to way their behaviour varies with time (for example, invariantly or periodically) imparts additional structure to the mathematical frameworks used to study them. It is for this reason that linear, time-invariant (LTI) systems are well-understood and robust control

[1]Embedded in the notion of achievable behaviour is the assumption that such a controller corresponds exactly to a physically realisable system.

theory relatively mature for this class of systems [Zam81, Fra86, ZDG95]. The work presented in this dissertation arises from a desire to characterise sensible frameworks for design and robustness analysis of closed-loop systems that exhibit linear, periodically-time-varying (LPTV) behaviour. The primary motivation for this stems from the wide-spread use of digital hardware (computers) to implement control strategies for continuous-time systems, which invariably leads to time-varying, closed-loop behaviour, that is often periodic in nature. Since digital hardware can only process discrete-time information, any digital controller employed to control a continuous-time system in closed-loop incorporates a sampling device at the input, which is used to convert continuous-time signals into discrete-time signals, and a hold device at the output, which is used to convert discrete-time signals into continuous-time signals. Systems like this, in which there is a hybrid mixture of continuous-time and discrete-time signals, are called sampled-data (SD) systems. In practice, it is common for the sampling device to operate at a fixed rate (period) and for the hold device to be synchronised with the sampler. In this situation, even if the discrete-time control-strategy implemented in digital hardware does not exhibit time-varying behaviour, the controller as a whole does. This is because the sampling process is periodically time-varying in the following sense: given a continuous-time input u to a (fixed-rate) sampling device and corresponding discrete-time output y, the output of the sampler in response to a time-shifted version of u corresponds to a time-shifted version of y if and only if the time-shift is an integer multiple of the sampling period. As further motivation, it is interesting to note that SD systems are not the only periodically-time-varying systems encountered in engineering and science. For example, given a time-invariant, nonlinear model of a system and a desired time-periodic trajectory, a linearisation about this gives rise to an LPTV model. Such models arise naturally for systems that involve reciprocating (rotating) masses and in celestial mechanics, for example.

The origins of robust control theory lie with the Small Gain Theorem [Zam66] and the seminal work of Zames [Zam81]. Further advances, in terms of qualitative results characterising the uncertainty types to which closed-loop systems can be desensitised, were made with the definition of the graph topology, in which two systems are considered similar (or close) if any reasonable controller for the first achieves similar closed-loop behaviour with the second [ZES80, VSF83, Vid84]. Quantitative results followed for LTI systems with the development of several metrics that induce the graph topology for such systems: the gap metric [ZES80, Geo88]; the graph metric [Vid84]; and the ν-gap metric [Vin93a]. Of specific interest in this dissertation is the gap metric, which for LTI systems, as shown in [GS90], is intimately related to the notion of coprime-factor uncertainty [VK86, MG90]. This relationship is particularly important, because it is known that the mathematical framework used to study robustness to coprime-factor uncertainty has useful performance-indicating properties. These properties were exploited in [MG90] in the development of a robust, control-system, design procedure for LTI systems, using notions of classical, frequency-domain loopshaping. The main goal of the work presented in this dissertation is to establish results analogous to some of those mentioned above for LPTV systems.

The equivalence of LPTV, continuous-time systems to linear, shift-invariant (LSI), discrete-time systems (cf. [BPFT91, BP92]) is of fundamental importance in this dissertation. By virtue of this equivalence, although the analysis is somewhat more involved (with tractable technical difficulties that arise concerning causality and infinite-dimensionality), periodic time-variation can be exploited in a similar manner to time-invariance. This yields powerful results for LPTV systems that are analogous to many results known to hold for LTI systems. Specifically, using the gap metric, qualitative and quantitative results are obtained by which the robustness of LPTV, closed-loop systems can be characterised. To this end, the existence of particular range and kernel representations of the graph of a stabilisable, LPTV system is established. These representations resemble the coprime-factor representations known to exist for certain classes of LTI systems [Vid85]. Towards developing a framework for design, a new notion of frequency response is developed for LPTV systems. This new notion of frequency response can be used to derive frequency-by-frequency bounds on performance-indicating, closed-loop operators, which facilitate the design of parametric weights employed in a new H^∞-loopshaping based procedure proposed for SD, control-system development.

Mathematical analysis is of very little use from an engineering perspective unless computational procedures exist to exploit it for specific classes of systems. Hence the wide-spread use of state-space tools in robust control theory [DGKF89, ZDG95, GL95]. As such, a considerable component of this dissertation is devoted to developing numerical algorithms for automated analysis and synthesis. More specifically: a computational procedure is developed for calculating the new notion of frequency response introduced; it also is shown how to compute the gap between any two LPTV systems that admit stabilisable and detectable, state-space realisations; finally, new formulae are derived for H^∞, SD synthesis, with discussion of numerical issues that arise and how they should be handled.

1.2 Overview of Contents

Chapter 2: In this chapter, material is collected to facilitate understanding of the main component of this dissertation. In Section 2.2, the underlying mathematical structure of all signal and system spaces is defined. General signal and system theory is discussed in Section 2.3. In Section 2.4, state-space tools used in the development of numerical algorithms are presented. Finally, fundamental issues involved in stability and induced-norm analysis of SD control-systems are briefly discussed in Section 2.5.

Chapter 3: A new notion of frequency response for LPTV systems is developed in this chapter. It is defined in terms of the average-power of the asymptotic response of an LPTV system to sinusoids of a single frequency. The performance-indicating properties of this new, frequency-domain, analysis tool are intuitive and it is used to derive frequency-by-frequency bounds on particular performance-indicating, closed-loop operator-gains, which facilitate the design of parametric weights used in a new H^∞-loopshaping based design procedure proposed for SD, control-system development.

Chapter 4: The primary purpose of this chapter is to present qualitative and quantitative results that characterise the uncertainty types to which LPTV, closed-loop systems can be desensitised. These results are obtained in terms of the gap metric, which is a measure of the distance between the graphs of two systems. Towards establishing the desired robustness results, so-called strong-right and strong-left representations of the graphs of stabilisable, LPTV systems are shown to exist. It is established that these representations can be used to characterise closed-loop stability, which leads to a Youla-style parameterisation of stabilising controllers. A formula for the directed-gap between two LPTV systems is then derived. The formula obtained is essentially a generalisation of that in [Geo88] for LTI systems and it is crucial to the development of the robustness results obtained. Specifically, the largest gap-ball of LPTV plants, centred at a nominal plant, that a nominal, stabilising controller is guaranteed to stabilise, is identified. Further to this, the maximally-tolerable, simultaneous gap-perturbation to both the nominal plant and controller is characterised. Qualitatively, it is shown that the topology induced by the gap metric on a general class of LPTV systems, is the weakest with respect to which closed-loop stability is a robust property and closed-loop performance varies continuously. That is, the gap metric on LPTV systems induces the graph topology. All of the results derived accommodate infinite-dimensional, input/output spaces and importantly, apply to the special case of LPTV, SD control-systems.

Chapter 5: In this chapter a numerical procedure is developed for computing the gap (to any desired accuracy) between LPTV systems that admit stabilisable and detectable, state-space realisations. Much of the chapter is devoted to obtaining finite-dimensional and hence, computationally-tractable, necessary and sufficient conditions for the existence of a solution to a related LSI, full-information, $\ell^2_{\mathbb{Z}^+}(\cdot)$-synthesis problem, used to establish bounds on the directed gap. Two-point, boundary-value problems (TPBVPs) are used extensively in this chapter, to characterise particular combinations of finite-horizon integral-operators that arise in this approach.

Chapter 6: A new, compact derivation of state-space formulae is presented in this chapter for H^∞, SD synthesis. Numerical issues are discussed and it is shown how to restructure the formulae obtained to give a numerically robust procedure for computing the state-space matrices of the so-called SD-equivalent generalised-plant.

Chapter 7: In this chapter the main contributions of this work are summarised and potential directions for future research identified.

2

Preliminaries

2.1 Introduction

The purpose of this chapter is to collect preliminary results that will facilitate the understanding of material presented in the main body of this dissertation. The basic structure of the chapter is as follows: fundamental results from functional analysis, which are used throughout, are given in Section 2.2; in Section 2.3, the signal and system spaces considered in this dissertation are defined and fundamental results concerning the stability of closed-loop systems presented; state-space tools for LSI system analysis and synthesis are presented in Section 2.4; finally, fundamental results concerning stability and closed-loop, induced-norm analysis for SD control-systems expressed in linear-fractional form are presented in Section 2.5.

2.2 Functional Analysis

In this section the underlying mathematical structure of all signal and system spaces considered in this dissertation is defined. This includes notation to be used throughout. For further details on the material presented here, the reader is referred to [Kre89b, Bol90, KF75, Sut93, DS88, Rud91, Hal82].

2.2.1 Topological, Metric, Vector (Linear) and Hilbert Spaces

A set $\mathscr{A}$ is a collection of elements with a common property P. Given another property P_1, the set $\mathscr{A}_1 := \{x \in \mathscr{A} : x \text{ has property } P_1\}$ is said to be a subset of $\mathscr{A}$. It is standard to denote this by $\mathscr{A}_1 \subset \mathscr{A}$. Two sets $\mathscr{A}_1$ and $\mathscr{A}_2$ are said to be equal, denoted $\mathscr{A}_1 = \mathscr{A}_2$, if $\mathscr{A}_1 \subset \mathscr{A}_2$ and $\mathscr{A}_2 \subset \mathscr{A}_1$. The notation $x \in \mathscr{A}$ is used to mean that x is an element of $\mathscr{A}$ and $x \notin \mathscr{A}$ to mean that x is *not* an element of $\mathscr{A}$. Given two subsets $\mathscr{A}_1 := \{x \in \mathscr{A} : x \text{ has property } P_1\}$ and

$\mathscr{A}_2 := \{x \in \mathscr{A} : x$ has property $P_2\}$ of a set $\mathscr{A}$, the <u>union</u> of $\mathscr{A}_1$ and $\mathscr{A}_2$ is defined to be

$$\mathscr{A}_1 \cup \mathscr{A}_2 := \{x \in \mathscr{A} : x \text{ has property } P_1 \underline{\text{ or }} \text{ property } P_2\}$$

and the <u>intersection</u> to be

$$\mathscr{A}_1 \cap \mathscr{A}_2 := \{x \in \mathscr{A} : x \text{ has property } P_1 \underline{\text{ and }} \text{ property } P_2\}.$$

The <u>Cartesian product</u> of two sets $\mathscr{A}_1$ and $\mathscr{A}_2$ is defined by

$$\mathscr{A}_1 \times \mathscr{A}_2 := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \in \mathscr{A}_1,\ y \in \mathscr{A}_2 \right\}.$$

In the sequel, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ the set of real numbers and $\mathbb{C}$ the set of *ordered* pairs $\{(\sigma, \omega) : \sigma, \omega \in \mathbb{R}\}$. Elements of $\mathbb{C}$ are usually called complex numbers and are expressed as $z = \sigma + j\omega$, where $j^2 := -1$. The symbol $\mathbb{Z}^+$ denotes the subset of non-negative integers, $\mathbb{R}^+$ the subset of non-negative reals, $[\tau_1, \tau_2) := \{t \in \mathbb{R} : \tau_1 \leq t < \tau_2\}$ and $[\tau_1, \tau_2] := \{t \in \mathbb{R} : \tau_1 \leq t \leq \tau_2\}$. For notational convenience, given a real number $h > 0$, the symbol $\mathbb{H}$ denotes the interval $[0, h)$. The *open* unit disc in $\mathbb{C}$ is denoted by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ denotes the unit circle and $\mathbb{E} := \mathbb{C} - \mathbb{D}$ denotes the exterior of the *open* unit disc.

A <u>mapping</u> (or function) between two set $\mathscr{A}_1$ and $\mathscr{A}_2$, denoted by $f : \mathscr{A}_1 \to \mathscr{A}_2$, relates each element of $\mathscr{A}_1$ to a *single* element of $\mathscr{A}_2$. The mapping f is said to be <u>injective</u> (or one-to-one) if for every $y \in \mathscr{F} := \{x \in \mathscr{A}_2 : x = f(u),$ for some $u \in \mathscr{A}_1\}$ there is a unique $u \in \mathscr{A}_1$ that satisfies $y = f(u)$. The mapping is said to be <u>surjective</u> (or onto) if $\mathscr{A}_2 = \mathscr{F}$ and it is called <u>bijective</u>, if it is both injective and surjective. If f is injective then $f^{-1} : \mathscr{F} \to \mathscr{A}_1$, defined to relate each element $y \in \mathscr{F}$ to the elements $u \in \mathscr{A}_1$ which satisfy $y = f(u)$, is well defined as a mapping in that it relates each element in $\mathscr{F}$ to a single element in $\mathscr{A}_1$. f^{-1} is called the (set theoretic) <u>inverse</u> of f and if f is bijective, then f^{-1} is a well-defined mapping on the whole of $\mathscr{A}_2$.

For a given set $\mathscr{A}$ and mapping $f : \mathscr{A} \to \mathbb{R}$, a number $\alpha \in \mathbb{R}$ is said to be an <u>upper bound</u> of $\mathscr{F} := \{x \in \mathbb{R} : x = f(a),$ for some $a \in \mathscr{A}\}$ if $f(a) \leq \alpha$ for all $a \in \mathscr{A}$. An upper bound $\underline{\alpha} \in \mathbb{R}$ of $\mathscr{F}$ is the least upper bound or <u>supremum</u>, denoted $\sup_{a \in \mathscr{A}} f(a)$, in the sense that any number strictly less than $\underline{\alpha}$ is not an upper bound of $\mathscr{F}$, if and only if for all $\epsilon > 0$ there exists an $a \in \mathscr{A}$ such that $f(a) > \alpha - \epsilon$. Similarly, a number $\beta \in \mathbb{R}$ is said to be a <u>lower bound</u> of $\mathscr{F}$ if $\beta \leq f(a)$ for all $a \in \mathscr{A}$. A lower bound $\overline{\beta} \in \mathbb{R}$ of $\mathscr{F}$ is the greatest lower bound or <u>infimum</u>, denoted $\inf_{a \in \mathscr{A}} f(a)$, in the sense that any number strictly greater that $\overline{\beta}$ is not a lower bound of $\mathscr{F}$, if and only if for all $\epsilon > 0$ there exists an $a \in \mathscr{A}$ such that $f(a) < \overline{\beta} + \epsilon$.

A topology $\boldsymbol{\tau}$ on a set $\mathscr{A}$ is a collection of subsets of $\mathscr{A}$, called open sets, such that the empty set $\emptyset \in \boldsymbol{\tau}$, $\mathscr{A} \in \boldsymbol{\tau}$ and arbitrary unions and finite intersections of sets in $\boldsymbol{\tau}$ are in $\boldsymbol{\tau}$. A subset $\mathscr{A}_1 \subset \mathscr{A}$ is said to be a closed set (with respect to $\boldsymbol{\tau}$) if its complement

$$\mathscr{A}_1^{\mathrm{c}} := \{x \in \mathscr{A} \ : \ x \notin \mathscr{A}_1\},$$

is a set in $\boldsymbol{\tau}$. That is, if its complement is open. A topology $\boldsymbol{\tau}$ on $\mathscr{A}$ is said to be Hausdorff if for any two elements $x, y \in \mathscr{A}$, there are two open sets $\mathscr{U}_x$ and $\mathscr{U}_y$ with empty intersection, such that $x \in \mathscr{U}_x$ and $y \in \mathscr{U}_y$. That is, it is possible to distinguish between any two points in a set equipped with a Hausdorff topology. Given two topologies $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ on a set $\mathscr{A}$, $\boldsymbol{\tau}_1$ is said to be weaker than $\boldsymbol{\tau}_2$ if all sets in $\boldsymbol{\tau}_1$ are contained in $\boldsymbol{\tau}_2$.

It is often convenient to specify a topology by giving a basis for it. A basis for a topology $\boldsymbol{\tau}$, is a sub-collection $\boldsymbol{\sigma}$ of subsets of $\mathscr{A}$ such that $\boldsymbol{\sigma} \subset \boldsymbol{\tau}$ and every set in $\boldsymbol{\tau}$ is a union of sets in $\boldsymbol{\sigma}$. A given collection $\boldsymbol{\sigma}$ of subsets of $\mathscr{A}$ is a basis for a topology on $\mathscr{A}$ if and only if every element of $\mathscr{A}$ is in some set in $\boldsymbol{\sigma}$ and for any $\mathscr{A}_1, \mathscr{A}_2 \in \boldsymbol{\sigma}$, $\mathscr{A}_1 \cap \mathscr{A}_2$ is a union of sets in $\boldsymbol{\sigma}$.

A topological space is defined to be a set equipped with a topology. Given a set $\mathscr{A}$ and a topology $\boldsymbol{\tau}$ on $\mathscr{A}$, $\mathfrak{T}(\mathscr{A}, \boldsymbol{\tau})$ denotes the corresponding topological space. If the topology $\boldsymbol{\tau}$ is Hausdorff then $\mathfrak{T}(\mathscr{A}, \boldsymbol{\tau})$ is called a Hausdorff space. Given two topological spaces $\mathfrak{T}(\mathscr{A}_1, \boldsymbol{\tau}_1)$ and $\mathfrak{T}(\mathscr{A}_2, \tau_2)$, the product topology on the Cartesian product $\mathscr{A}_1 \times \mathscr{A}_2$ is the topology generated by the basis $\{\mathscr{U}_1 \times \mathscr{U}_2 \ : \ \mathscr{U}_1 \in \boldsymbol{\tau}_1, \ \mathscr{U}_2 \in \boldsymbol{\tau}_2\}$. Thus, a subset $\mathscr{A} \subset \mathscr{A}_1 \times \mathscr{A}_2$ is an open set in the product topology if and only if for every $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathscr{A}$ there exist open sets $\mathscr{U}_1 \in \boldsymbol{\tau}_1$ and $\mathscr{U}_2 \in \boldsymbol{\tau}_2$ such that $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathscr{U}_1 \times \mathscr{U}_2 \subset \mathscr{A}$.

A subset $\mathscr{N}(x) \subset \mathscr{A}$ is said to be a neighbourhood of a point x in a topological space $\mathfrak{T}(\mathscr{A}, \boldsymbol{\tau})$, if there exists an open set $\mathscr{U} \in \boldsymbol{\tau}$ such that $x \in \mathscr{U} \subset \mathscr{N}(x)$. A sequence $\{x_k\}_{k=0}^{\infty}$ in a topological space $\mathfrak{T}(\mathscr{A}, \boldsymbol{\tau})$ is said to converge to a point $x \in \mathfrak{T}(\mathscr{A}, \boldsymbol{\tau})$, denoted $x_k \to x$ (or $\lim_{k\to\infty} x_k = x$), if for every neighbourhood $\mathscr{N}(x)$ of x there exists a positive integer K such that $x_k \in \mathscr{N}(x)$ for all $k \geq K$. If $\mathfrak{T}(\mathscr{A}, \boldsymbol{\tau})$ is a Hausdorff space then any convergent sequence converges to a unique point. Given two topological spaces $\mathfrak{T}_1(\mathscr{A}_1, \boldsymbol{\tau}_1)$ and $\mathfrak{T}_2(\mathscr{A}_2, \boldsymbol{\tau}_2)$, a mapping $f : \mathscr{A}_1 \to \mathscr{A}_2$ is called continuous (with respect to $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$) if $\mathscr{Y} \in \boldsymbol{\tau}_2$ implies that $f^{-1}(\mathscr{Y}) \in \boldsymbol{\tau}_1$, where $f^{-1}(\mathscr{Y})$ denotes the inverse image of $\mathscr{Y}$ under f.

Given a set $\mathscr{A}$, a mapping $d : \mathscr{A} \times \mathscr{A} \to \mathbb{R}^+$ is called a metric on $\mathscr{A}$ if:

(M1) for $x, y \in \mathscr{A}$, $d(x, y) = 0 \ \Leftrightarrow \ x = y$;

(M2) $d(x,y) = d(y,x)$ for all $x, y \in \mathscr{A}$;

(M3) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in \mathscr{A}$.

A metric space is defined to be a set equipped with a metric. Given a set $\mathscr{A}$ and a metric d on $\mathscr{A}$, $\mathscr{X}(\mathscr{A}, d)$ denotes the corresponding metric space. For a metric space $\mathscr{X}(\mathscr{A}, d)$ the open balls

$$\mathscr{B}_d(x,r) := \{y \in \mathscr{X}(\mathscr{A}, d) \ : \ d(x,y) < r\} \quad (x \in \mathscr{X}(\mathscr{A}, d),\ r > 0),$$

form a basis for a topology on $\mathscr{X}(\mathscr{A}, d)$. This topology is said to be induced by the metric d and is called the metric topology. As such, every metric space can be considered to be a topological space. In fact, every metric space equipped with the metric topology is Hausdorff. In a metric space, convergence (with respect to the metric topology) of a sequence $\{x_k\}_{k=0}^{\infty} \subset \mathscr{X}(\mathscr{A}, d)$ can be expressed in the (probably) more familiar terms: $x_k \to x$ if and only if for all $\epsilon > 0$ there exists a positive integer K such that $d(x, x_k) < \epsilon$ for all $k \geq K$. Similarly, a function $f : \mathscr{A}_1 \to \mathscr{A}_2$, where $\mathscr{A}_1$ is equipped with the metric d_1 and $\mathscr{A}_2$ with d_2, is continuous if and only if for every $x \in \mathscr{A}_1$ and $\epsilon > 0$ there exists a $\delta > 0$ such that $d_2(f(x), f(u)) < \epsilon$ for all $u \in \mathscr{A}_1$ that satisfy $d_1(x,u) < \delta$. A sequence $\{x_k\}_{k=0}^{\infty} \subset \mathscr{X}(\mathscr{A}, d)$ is said to be Cauchy, if for for every $\epsilon > 0$ there exists a positive integer N such that $d(x_m, x_n) < \epsilon$ for all $m, n > N$. Note that every convergent sequence in a metric space is Cauchy. If every Cauchy sequence in a metric space $\mathscr{X}(\mathscr{A}, d)$ is convergent to a point in $\mathscr{X}(\mathscr{A}, d)$, then $\mathscr{X}(\mathscr{A}, d)$ is called a complete metric space.

A vector (or linear) space $\mathscr{V}$ over $\mathbb{F}$ (where $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$) is a set of elements called vectors, for which two operations called vector addition and scalar multiplication are defined with the following algebraic properties:

(V1) For all vectors x, y and z in $\mathscr{V}$ there corresponds a vector $x+y \in \mathscr{V}$, where the correspondence is such that: $x + y = y + x$ and $x + (y + z) = (x + y) + z$; $\mathscr{V}$ contains a unique zero vector 0 such that $x + 0 = x$ for all $x \in \mathscr{V}$; and for every $x \in \mathscr{V}$ there corresponds a unique vector $-x \in \mathscr{V}$ such that $x - x := x + (-x) = 0$.

(V2) For each $x \in \mathscr{V}$ and $\alpha \in \mathbb{F}$, there is a corresponding vector $\alpha x \in \mathscr{V}$, where the correspondence is such that: $1x = x$, where 1 denotes the unit in $\mathbb{F}$; $\alpha(\beta x) = (\alpha\beta)x$ (where $\beta \in \mathbb{F}$); and for all $x, y \in \mathscr{V}$, $\alpha(x + y) = \alpha x + \alpha y$ and $(\alpha + \beta)x = \alpha x + \beta x$.

A subset $\mathscr{V}_1 \subset \mathscr{V}$ is called a (linear) subspace of $\mathscr{V}$ if it is closed under the vector addition and scalar multiplication operations defined on $\mathscr{V}$. A norm on a vector space $\mathscr{V}$, denoted by $\|\cdot\|_{\mathscr{V}}$, is a mapping from $\mathscr{V}$ to $\mathbb{R}^+$, which for all $x, y \in \mathscr{V}$ satisfies:

(N1) $\|x\|_{\mathscr{V}} = 0 \ \Leftrightarrow \ x = 0$;

(N2) $\|\alpha x\|_{\mathscr{V}} = |\alpha| \cdot \|x\|_{\mathscr{V}}$ for all $\alpha \in \mathbb{F}$;

(N3) $\|x + y\|_{\mathscr{V}} \leq \|x\|_{\mathscr{V}} + \|y\|_{\mathscr{V}}$.

Property (N3) is often called the triangle inequality. A vector space $\mathscr{V}$ equipped with a norm $\|\cdot\|_{\mathscr{V}}$, is called a normed space. The norm on a normed space $\mathscr{V}$ induces the following metric:

$$d_{\mathscr{V}}(x, y) := \|x - y\|_{\mathscr{V}} \quad \text{for all } x, y \in \mathscr{V}.$$

If $\mathscr{V}$ is complete with respect to $d_{\mathscr{V}}$ then it is called a Banach space. In the sequel the Computer Modern Caligraphic font ($\mathcal{B}, \mathcal{W}$ for example) is used to identify a Banach space. A subspace $\mathscr{V}_1$ of a normed space $\mathscr{V}$ is closed (with respect to the topology induced by $d_{\mathscr{V}}$, also known as the norm topology) if every sequence in $\mathscr{V}_1$ that converges in $\mathscr{V}$, converges to a point in $\mathscr{V}_1$. A closed subspace of a Banach space is therefore complete.

An inner product on a vector space $\mathscr{V}$ over $\mathbb{F}$, denoted by $\langle \cdot, \cdot \rangle_{\mathscr{V}}$, is a mapping from the Cartesian product $\mathscr{V} \times \mathscr{V}$ to $\mathbb{F}$ such that for all $x, y, z \in \mathscr{V}$ and each $\mu \in \mathbb{F}$, the following hold:

(IP1) $\langle x, y \rangle_{\mathscr{V}} = \overline{\langle y, x \rangle}_{\mathscr{V}}$;

(IP2) $\langle \mu x, y \rangle_{\mathscr{V}} = \mu \langle x, y \rangle_{\mathscr{V}}$;

(IP3) $\langle x + y, z \rangle_{\mathscr{V}} = \langle x, z \rangle_{\mathscr{V}} + \langle y, z \rangle_{\mathscr{V}}$;

(IP4) $\langle x, x \rangle_{\mathscr{V}} \geq 0$ and $\langle x, x \rangle_{\mathscr{V}} = 0 \Leftrightarrow x = 0$,

where the over-bar in (IP1) denotes complex conjugation. An inner product induces the following norm on $\mathscr{V}$: $\|x\|_{\mathscr{V}} := \sqrt{\langle x, x \rangle_{\mathscr{V}}}$ for all $x \in \mathscr{V}$. A vector space equipped with an inner product is called an inner-product space and can be considered a normed space (and hence, a metric and topological space). If an inner-product space is complete with respect to the metric induced by the inner product (via the norm defined above), then it is called a Hilbert space. In the sequel, the Euler Script font ($\mathscr{H}, \mathscr{U}$ for example) is used to identify a Hilbert space, unless it is a special Hilbert space such as those define in Sub-Section 2.3.1.

Let $\mathscr{H}$ be a Hilbert space and $\mathscr{U}$ and $\mathscr{Y}$ be subspaces of $\mathscr{H}$. The orthogonal complement of $\mathscr{U}$ in $\mathscr{H}$ is defined to be the subspace

$$\mathscr{H} \ominus \mathscr{U} := \{x \in \mathscr{H} \ : \ \langle u, x \rangle_{\mathscr{H}} = 0 \text{ for all } u \in \mathscr{U}\}.$$

When it is clear from context what $\mathscr{H}$ is, the short hand $\mathscr{U}^{\perp}$ is used to denote $\mathscr{H} \ominus \mathscr{U}$. If $\mathscr{U}$ is a closed subspace then each $x \in \mathscr{H}$ can be uniquely decomposed as $x = u + x_1$ where $u \in \mathscr{U}$ and

$x_1 \in \mathcal{U}^{\perp}$. Bearing this in mind, the mapping of each $x \in \mathcal{H}$ to the corresponding $u \in \mathcal{U}$ is called an orthogonal projection and is denoted by the symbol $\mathbf{\Pi}_{\mathcal{U}}$. In the sequel, the symbol $\oplus$ denotes the orthogonal direct sum, by which $\mathcal{Y} \oplus \mathcal{U}$ is identified with the Cartesian product $\mathcal{Y} \times \mathcal{U}$ and has inner product defined for all $y_1, y_2 \in \mathcal{Y}$ and $u_1, u_2 \in \mathcal{U}$ by

$$\left\langle \begin{bmatrix} y_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} y_2 \\ u_2 \end{bmatrix} \right\rangle_{\mathcal{Y}\oplus\mathcal{U}} := \langle y_1, y_2 \rangle_{\mathcal{Y}} + \langle u_1, u_2 \rangle_{\mathcal{U}}.$$

The notation $\mathcal{H}^n$ is used to denote the Hilbert space $\mathcal{H}^n := \underbrace{\mathcal{H} \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n \text{ times}}$.

A set of vectors $\{e_k\}_{k\in\mathscr{K}} \subset \mathcal{H}$ (where $\mathscr{K}$ is a (possibly uncountable) index set) is said to be orthonormal if $\|e_k\|_{\mathcal{H}} = 1$ and $\langle e_k, e_l \rangle_{\mathcal{H}} = 0$ for $k \neq l$ in $\mathscr{K}$. If the only vector in $\mathcal{H}$ orthogonal to all e_k's is the zero vector, then such a set is called a complete, orthonormal basis for $\mathcal{H}$. Given an orthonormal basis $\{e_k\}_{k\in\mathscr{K}}$ for a Hilbert space $\mathcal{H}$, any vector $x \in \mathcal{H}$ can be expressed as

$$x = \sum_{k\in\mathscr{K}} \langle x, e_k \rangle_{\mathcal{H}} e_k.$$

The set $\{\langle x, e_k \rangle_{\mathcal{H}}\}_{k\in\mathscr{K}}$ uniquely (with respect to the basis $\{e_k\}_{k\in\mathscr{K}}$) represents x. Given an x and y in $\mathcal{H}$ with expansions $\sum_{k\in\mathscr{K}} \alpha_k e_k$ and $\sum_{k\in\mathscr{K}} \beta_k e_k$ respectively, then

$$\langle x, y \rangle_{\mathcal{H}} = \sum_{k\in\mathscr{K}} \alpha_k \overline{\beta}_k.$$

Furthermore, if $\{\alpha_k\}_{k\in\mathscr{K}}$ is a subset of $\mathbb{C}$ such that $\sum_{k\in\mathscr{K}} |\alpha_k|^2 < \infty$, then $\sum_{k\in\mathscr{K}} \alpha_k e_k$ is in $\mathcal{H}$. In the sequel, any non-specific, Hilbert space considered is assumed to have countable basis. Such Hilbert spaces are said to be separable.

2.2.2 Operator Theory

Given two vector spaces $\mathscr{V}$ and $\mathscr{Y}$ over $\mathbb{F}$ (where $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$), a mapping $P : \mathscr{V} \to \mathscr{Y}$ is called an operator and it is said to be linear if for every $v_1, v_2 \in \mathscr{V}$ and $\alpha, \beta \in \mathbb{F}$,

$$P(\alpha v_1 + \beta v_1) = \alpha P v_1 + \beta P v_2.$$

All operators considered in the sequel are linear and almost all are defined on Hilbert spaces. Let $\mathcal{U}$ and $\mathcal{Y}$ be two Hilbert spaces and $P : \mathcal{D}_P \subset \mathcal{U} \to \mathcal{Y}$ be a linear operator where

$$\mathcal{D}_P := \{u \in \mathcal{U} \; : \; Pu \in \mathcal{Y}\}$$

is called the domain of P, which is a subspace of $\mathcal{U}$. If $\mathcal{H}$ is a subspace of $\mathcal{D}_P$ then $P|_{\mathcal{H}} : \mathcal{H} \to \mathcal{Y}$ denotes the restriction of P to $\mathcal{H}$ and maps vectors from $\mathcal{H}$ to $\mathcal{Y}$ as P does. The range of P is defined to be

$$\mathcal{R}_P := \{Pu : u \in \mathcal{D}_P\}$$

and the kernel to be

$$\mathcal{K}_P := \{u \in \mathcal{D}_P : Pu = 0\}.$$

A linear operator $P : \mathcal{D}_P \subset \mathcal{U} \to \mathcal{Y}$ is said to be bounded if the induced norm

$$\|P\| := \sup_{\substack{u \in \mathcal{D}_P \\ u \neq 0}} \frac{\|Pu\|_{\mathcal{Y}}}{\|u\|_{\mathcal{U}}} < \infty.$$

$\mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$ is used to denote the set of all *bounded, linear* operators $P : \mathcal{U} \to \mathcal{Y}$ and it is a Banach space when equipped with the induced norm.

The graph of a linear operator $P : \mathcal{D}_P \subset \mathcal{U} \to \mathcal{Y}$ is defined to be the set of all bounded input-output pairs

$$\mathcal{G}_P := \begin{bmatrix} I \\ P \end{bmatrix} \mathcal{D}_P \subset \mathcal{U} \oplus \mathcal{Y},$$

which since P is linear, is a subspace of $\mathcal{U} \oplus \mathcal{Y}$. For notational convenience the inverse graph is defined to be $\mathcal{G}'_P := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \mathcal{G}_P$. Note that any linear subspace $\mathcal{G} \subset \mathcal{U} \oplus \mathcal{Y}$ corresponds to the graph of a linear operator if and only if $\begin{bmatrix} 0 \\ y \end{bmatrix} \in \mathcal{G} \Rightarrow y = 0$. The graph is a very useful characterisation of an operator and is used extensively in the sequel. An operator P is said to be closed if its graph is closed with respect to the product topology on $\mathcal{U} \oplus \mathcal{Y}$. A useful result is the Closed Graph Theorem, which states that a closed, linear operator $P : \mathcal{D}_P \subset \mathcal{U} \to \mathcal{Y}$ is bounded if $\mathcal{D}_P$ is closed with respect to the norm topology on $\mathcal{U}$. Furthermore, if it is known that a linear operator $P : \mathcal{D}_P \subset \mathcal{U} \to \mathcal{Y}$ is bounded and that $\mathcal{D}_P$ is closed, then $\mathcal{G}_P$ is closed with respect to the product topology on $\mathcal{U} \oplus \mathcal{Y}$.

Let $\mathcal{G}$ and $\mathcal{F}$ be two (linear) subspaces of a Hilbert space $\mathcal{H}$ that are closed with respect to the norm topology on $\mathcal{H}$. $\mathcal{G}$ and $\mathcal{F}$ are said to induce a co-ordinatisation of $\mathcal{H}$ if $\mathcal{G} \cap \mathcal{F} = \{0\}$ and $\mathcal{G} + \mathcal{F} = \mathcal{H}$. Given a co-ordinatisation $\{\mathcal{G}, \mathcal{F}\}$ of $\mathcal{H}$, any $x \in \mathcal{H}$ can be uniquely decomposed as the sum $x = g + f$ where $g \in \mathcal{G}$ and $f \in \mathcal{F}$ (cf. [DGS93]). With this decomposition in mind, the operator $\mathbf{\Pi}_{\mathcal{G}\|\mathcal{F}} : \mathcal{H} \to \mathcal{G}$ defined by

$$\mathbf{\Pi}_{\mathcal{G}\|\mathcal{F}} := x \mapsto g,$$

is called the parallel projection onto $\mathcal{G}$ along $\mathcal{F}$. Correspondingly, $\mathbf{\Pi}_{\mathcal{F}\|\mathcal{G}} := x \mapsto f$ is called the parallel projection onto $\mathcal{F}$ along $\mathcal{G}$. Since $\mathcal{G}$ and $\mathcal{F}$ are closed, linear subspaces, it follows that the graphs of the parallel projection operators are linear and closed. Hence, by the Closed Graph Theorem, the parallel projections are bounded. Note also, that $\|\mathbf{\Pi}_{\cdot\|\cdot}\| \geq 1$.

Given a bounded, linear operator $P \in \mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$, there exists a unique operator $P^* \in \mathcal{B}_{\mathcal{Y}\to\mathcal{U}}$ such that for all $u \in \mathcal{U}$ and $y \in \mathcal{Y}$

$$\langle Pu, y\rangle_{\mathcal{Y}} = \langle u, P^* y\rangle_{\mathcal{U}}.$$

The operator P^* is called the adjoint operator and $\|P\|^2 = \|P^*\|^2 = \|P^*P\|$. A useful result is that $\mathcal{K}_P = (\mathcal{R}_{P^*})^\perp$. A linear operator $P \in \mathcal{B}_{\mathcal{U}\to\mathcal{U}}$ is called self-adjoint if $P = P^*$. Such an operator is said to be positive definite (respectively positive semidefinite), denoted $P > 0$ (respectively $P \geq 0$), if there exists a real number $\alpha > 0$ such that for all $u \in \mathcal{U}$, $\langle u, Pu\rangle_{\mathcal{U}} \geq \alpha\langle u, u\rangle_{\mathcal{U}}$ (respectively $\langle u, Pu\rangle_{\mathcal{U}} \geq 0$). Similarly, a self-adjoint operator $P \in \mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$ is said to be negative definite (respectively negative semidefinite), denoted $P < 0$ (respectively $P \leq 0$), if $-P > 0$ (respectively $-P \geq 0$). A self-adjoint operator P is said to be idempotent if $P^2 = P$.

An operator $P \in \mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$, where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, is said to be an isometry if $\langle Pu, Pu\rangle_{\mathcal{Y}} = \langle u, u\rangle_{\mathcal{U}}$ for all $u \in \mathcal{U}$ and unitary if $\mathcal{R}_P = \mathcal{Y}$ and P is an isometry. $P \in \mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$ is called a co-isometry if $\mathcal{R}_P = \mathcal{Y}$ and P^* is an isometry. Two Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$ are said to be isomorphic if there exists a bijective operator $Y \in \mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$, called an isomorphism, such that

$$\langle u_1, u_2\rangle_{\mathcal{U}} = \langle Yu_1, Yu_2\rangle_{\mathcal{Y}}$$

for all $u_1, u_2 \in \mathcal{U}$. Two operators P_1 and P_2 defined on Hilbert spaces are said to be equivalent if there exists an isomorphism Y such that $P_1Y = YP_2$.

Given a Hilbert spaces $\mathcal{U}$ and an operator $P : \mathcal{D}_P \subset \mathcal{U} \to \mathcal{U}$ that is injective (or equivalently $\mathcal{K}_P = \{0\}$), then P has a well defined, set-theoretic inverse defined on $\mathcal{R}_P$. The inverse operator is denoted by P^{-1} and the following equalities hold: $P^{-1}P = I|_{\mathcal{D}_P}$ and $PP^{-1} = I|_{\mathcal{R}_P}$, where I denotes the identity map (that is, the operator which maps each element of a vector space to itself). It follows by the Open Mapping Theorem (cf. [Bol90, pg 80] for example), that if $P \in \mathcal{B}_{\mathcal{U}\to\mathcal{U}}$ is bijective then $P^{-1} \in \mathcal{B}_{\mathcal{U}\to\mathcal{U}}$. In the sequel, an operator $P \in \mathcal{B}_{\mathcal{U}\to\mathcal{U}}$ is said to be (boundedly) invertible if $P^{-1} \in \mathcal{B}_{\mathcal{U}\to\mathcal{U}}$. The spectrum of an operator $P \in \mathcal{B}_{\mathcal{U}\to\mathcal{U}}$ is defined to be the set

$$\operatorname{spec}(P) := \{\lambda \in \mathbb{C} \ : \ (\lambda I - P) \text{ is not invertible in } \mathcal{B}_{\mathcal{U}\to\mathcal{U}}\}$$

and the point spectrum to be the subset of $\operatorname{spec}(P)$ for which $(\lambda I - P)$ is not injective. The point spectrum is a set of discrete points in $\mathbb{C}$ and its elements are often called the eigenvalues of P. If $\mathcal{U}$

or $\mathcal{Y}$ is finite-dimensional (in the sense that there is a finitely-countable, complete basis for them), then all $\lambda \in \operatorname{spec}(P)$ are eigenvalues. The spectral radius of a linear operator $P \in \mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$ is defined by

$$\operatorname{rad}(P) := \sup_{\lambda \in \operatorname{spec}(P)} |\lambda| = \lim_{n\to\infty} \|P^n\|^{\frac{1}{n}} \leq \|P\|.$$

If P is self-adjoint, then its spectrum is real and $\operatorname{rad}(P) = \|P\|$.

A bounded, linear operator $P \in \mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$ is said to be compact if for every bounded sequence $\{u_k\}_{k=0}^{\infty}$ in $\mathcal{U}$ there is a subsequence $\{u_{k_n}\}_{n=0}^{\infty}$, such that Pu_{k_n} converges in $\mathcal{Y}$ when equipped with the norm topology. If $\mathcal{U}$ is finite-dimensional then P is compact. A compact operator $P \in \mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$ has the following important properties:

(C1) for any $R \in \mathcal{B}_{\mathcal{Y}\to\mathcal{H}}$, $RP \in \mathcal{B}_{\mathcal{U}\to\mathcal{H}}$ is compact and for any $R \in \mathcal{B}_{\mathcal{H}\to\mathcal{U}}$, $PR \in \mathcal{B}_{\mathcal{H}\to\mathcal{Y}}$ is compact;

(C2) the eigenvalues of P constitute a countable set;

(C3) the adjoint operator P^* is compact.

Another important property of compact operators is that every non-zero spectral value of a compact operator is an eigenvalue and the only possible accumulation point[1] of the spectrum is 0.

A number $\sigma \in \mathbb{R}^+$ is said to be a singular value of a bounded, linear operator $P \in \mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$ if σ^2 is an eigenvalue of $P^*P \in \mathcal{B}_{\mathcal{U}\to\mathcal{U}}$. The set of singular values of an operator $P \in \mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$ is denoted by

$$\sigma(P) := \{\sigma \in \mathbb{R}^+ \;:\; \sigma^2 \text{ is an eigenvalue of } P^*P\}.$$

If $P \in \mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$ is compact then so is P^*P and hence, the spectrum of P^*P is completely characterised by the countable set of singular values $\sigma(P)$. Then since P^*P is self-adjoint, it follows that given a compact operator $P \in \mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$

$$\|P\| = \sqrt{\|P^*P\|} = \sqrt{\operatorname{rad}(P^*P)} = \overline{\sigma}\{P\},$$

where

$$\overline{\sigma}\{P\} := \max_{\sigma\in\sigma(P)} \sigma$$

is called the maximum singular-value. The minimum singular-value of a compact, bounded, linear operator is defined similarly by

$$\underline{\sigma}\{P\} := \min_{\sigma\in\sigma(P)} \sigma.$$

[1] A point x in a metric space $\mathscr{X}(\mathscr{A}, d)$ is said to be an accumulation point of a subset $\mathscr{M} \subset \mathscr{X}(\mathscr{A}, d)$ if every neighbourhood of x in the metric topology contains at least one point in $\mathscr{M}$ that is distinct from x.

2.2.3 Analytic Functions

In this short sub-section the notion of an analytic function is defined and a few properties of analytic functions assembled. For further details on this material the reader is referred to [Ahl79, Rud86, Hof62].

A norm can be defined on all sets considered in this sub-section and as such convergence and continuity (although not explicitly stated) is with respect to the norm topology on the sets concerned. Let $\mathbb{U}$ be a connected,[2] open subset of the complex plane and consider a function $f : \mathbb{U} \to \mathcal{W}$, where $\mathcal{W}$ is a Banach space. If for a given point $z_0 \in \mathbb{U}$ there exists an $f_0 \in \mathcal{W}$ such that $f(z_i)$ converges to (the same) f_0 for every sequence $\{z_i\}_{i=0}^{\infty} \subset \mathbb{U}$ that converges to z_0, then the limit below is said to exist

$$\lim_{z \to z_0} f(z) := f_0.$$

If at a point $z_0 \in \mathbb{U}$,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, then $f(z)$ is said to be <u>differentiable</u> at z_0. A function $f : \mathbb{U} \to \mathcal{W}$ is said to be analytic in a connected, open subset $\mathbb{U}_1 \subset \mathbb{U}$ if f is differentiable at all points in $\mathbb{U}_1$. If f is analytic in $\mathbb{U}$, then it is also continuous on $\mathbb{U}$ and can be expressed in terms of a power series in the following sense: for every open ball $\mathscr{B}_{\mathbb{C}}(z_0, r) := \{z \in \mathbb{C} \ : \ |z - z_0| < r\} \subset \mathbb{U}$, there corresponds a power series

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n \qquad (\{c_n\}_{n=0}^{\infty} \subset \mathcal{W}),$$

which converges[3] to $f(z)$ for all $z \in \mathscr{B}_{\mathbb{C}}(z_0, r)$. This property is often used to define when a function is analytic.

Using Cauchy's Integral Formula, it can be shown that any function $f : \mathbb{U} \to \mathcal{W}$ that is analytic in $\mathbb{U}$ satisfies the mean value property

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{j\theta}) \, d\theta,$$

for all real $r > 0$ such that $\mathscr{B}_{\mathbb{C}}(z, r) \subset \mathbb{U}$. Using this and the triangle inequality, it follows that for any function $f : \mathbb{U} \to \mathcal{W}$ which is analytic in $\mathbb{U}$,

$$\|f(z)\|_{\mathcal{W}} \leq \frac{1}{2\pi} \int_0^{2\pi} \|f(z + re^{j\theta})\|_{\mathcal{W}} \, d\theta.$$

[2] A subset $\mathbb{U} \subset \mathbb{C}$ is said to be connected if $\mathbb{U}$ and $\emptyset$ are the only subsets of $\mathbb{U}$ that are both open and closed.

[3] Convergence of the series is taken to mean $\lim_{N \to \infty} \sum_{n=0}^{N} c_n (z - z_0)^n$ exists.

Furthermore, since for such a function the norm $\|f(z)\|_{\mathcal{W}}$ is continuous on any open subset of $\mathbb{U}$, it follows that $\|f(z)\|_{\mathcal{W}}$ satisfies a maximum principle on any connected, open subset of $\mathbb{U}_1 \subset \mathbb{U}$ in the following sense: $\|f(z)\|_{\mathcal{W}}$ has no maximum on $\mathbb{U}_1$ or it is constant on $\mathbb{U}_1$ (cf. [Ahl79, pg. 135] for example).

2.3 Signals and Systems Theory

In this section the signal and system spaces considered in this dissertation are defined. Much of the material presented in this section is taken from the books [Wil71, DV75, FS82, FF90, SNF70].

2.3.1 Signals

In systems theory a signal space is usually defined to be a vector space of functions mapping a time interval of definition (for example $\mathbb{R}^+$ or $\mathbb{Z}^+$) to a Banach space. A system is then defined to be an operator mapping between signal spaces. Given a Banach space $\mathcal{B}$ and a time interval of definition $\mathbb{S}$, let $\mathscr{Y}_{\mathbb{S}}(\mathcal{B})$ denote the vector space of $\mathcal{B}$-valued functions defined on $\mathbb{S}$. That is, $\mathscr{Y}_{\mathbb{S}}(\mathcal{B}) :=\{f \ : \ f : \mathbb{S} \to \mathcal{B}\}$. Functions in $\mathscr{Y}_{\mathbb{Z}^+}(\mathcal{B})$ are often called discrete-time signals and functions in $\mathscr{Y}_{\mathbb{R}^+}(\mathcal{B})$ continuous-time signals. As it stands, $\mathscr{Y}_{\mathbb{S}}(\mathcal{B})$ has little structure and would be difficult to work with. So it is standard to introduce a norm on $\mathscr{Y}_{\mathbb{S}}(\mathcal{B})$, which gives rise to a normed (linear) subspace $\mathcal{W} :=\{f \in \mathscr{Y}_{\mathbb{S}}(\mathcal{B}) \ : \ \|f\|_{\mathcal{W}} < \infty\}$. Often, the norm is selected so that $\mathcal{W}$ is a Banach space or Hilbert space. Given such a $\mathcal{W}$, it is generally useful to define the extension of $\mathcal{W}$. Extended spaces play an important role in systems theory and are particularly important in the study of causal operators and the well-posedness of closed-loop systems [Wil71]. Let $\mathbf{P}_\tau$ denote the projection defined for $f \in \mathscr{Y}_{\mathbb{S}}(\mathcal{B})$ by

$$(\mathbf{P}_\tau f)(t) := \begin{cases} f(t) & \text{for } \ t \le \tau, \ t \in \mathbb{S} \\ 0 & \text{otherwise} \end{cases} .$$

Then for a Banach space $\mathcal{W} \subset \mathscr{Y}_{\mathbb{S}}(\mathcal{B})$, the extended space $\mathcal{W}^{\mathrm{e}}$ is defined to be

$$\mathcal{W}^{\mathrm{e}} :=\{f \in \mathscr{Y}_{\mathbb{S}}(\mathcal{B}) : \mathbf{P}_\tau f \in \mathcal{W} \ \text{ for all finite } \ \tau \in \mathbb{S}\}.$$

It is important to note that although $\mathcal{W}^{\mathrm{e}}$ is a vector space it is not normed. In this dissertation, attention is restricted to the *continuous-time* signal-space $\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{H})$ and the *discrete-time* signal-space $\ell^2_{\mathbb{Z}^+}(\mathcal{H})$ (and their respective extensions) defined below.

Let $\mathcal{H}$ be a Hilbert space. The Hilbert space of $\mathcal{H}$-valued functions $f \in \mathscr{Y}_{\mathbb{R}^+}(\mathcal{H})$ that satisfy

$$\|f\|^2_{\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{H})} := \int_0^\infty \langle f(t), f(t)\rangle_{\mathcal{H}}\, dt < \infty,$$

is denoted by $\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{H})$. The inner-product on $\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{H})$ is defined to be

$$\langle f, g\rangle_{\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{H})} := \int_0^\infty \langle f(t), g(t)\rangle_{\mathcal{H}}\, dt,$$

which induces the norm used in the definition. The symbol $\mathbf{U}_\tau$ denotes the unilateral shift by τ on $\mathscr{Y}_{\mathbb{R}^+}(\mathcal{H})$ and for all $f \in \mathscr{Y}_{\mathbb{R}^+}(\mathcal{H})$ is defined by

$$(\mathbf{U}_\tau f)(t) := \begin{cases} f(t-\tau) & \text{for } t \geq \tau \\ 0 & \text{for } 0 \leq t < \tau \end{cases},$$

which is an isometry on $\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{H})$. The Hilbert space $\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{H})$ is a *continuous-time* signal-space and corresponds to continuous-time signals with finite energy. $\mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{H})$ is used to denoted the associated extended space.

The Hilbert space $\ell^2_{\mathbb{Z}^+}(\mathcal{H})$ consists of all $\mathcal{H}$-valued functions $\mathsf{f} \in \mathscr{Y}_{\mathbb{Z}^+}(\mathcal{H})$ that satisfy

$$\|\mathsf{f}\|^2_{\ell^2_{\mathbb{Z}^+}(\mathcal{H})} := \sum_{k=0}^{\infty} \langle \mathsf{f}_k, \mathsf{f}_k\rangle_{\mathcal{H}} < \infty.$$

The inner-product on $\ell^2_{\mathbb{Z}^+}(\mathcal{H})$ is defined to be

$$\langle \mathsf{f}, \mathsf{g}\rangle_{\ell^2_{\mathbb{Z}^+}(\mathcal{H})} := \sum_{k=0}^{\infty} \langle \mathsf{f}_k, \mathsf{g}_k\rangle_{\mathcal{H}},$$

which induces the norm used in the definition. The symbol $\mathbf{S}$ denotes the unilateral shift on $\mathscr{Y}_{\mathbb{Z}^+}(\mathcal{H})$ and for all $\mathsf{f} \in \mathscr{Y}_{\mathbb{Z}^+}(\mathcal{H})$ is defined by

$$(\mathbf{S}\mathsf{f})_k := \begin{cases} \mathsf{f}_{(k-1)} & \text{for } k \geq 1 \\ 0 & \text{for } k = 0 \end{cases},$$

which is an isometry on $\ell^2_{\mathbb{Z}^+}(\mathcal{H})$. The Hilbert space $\ell^2_{\mathbb{Z}^+}(\mathcal{H})$ is a *discrete-time* signal-space and corresponds to discrete-time signals with finite energy. $\ell^{2,\mathrm{e}}_{\mathbb{Z}^+}(\mathcal{H})$ is used to denote the associated extended space. Lower-case Euler characters (for example f, x) are used throughout to denote discrete-time signals.

In SD control-systems the following elementary operators are used to map continuous-time signals to discrete-time (sampled) signals and vice-versa: given an arbitrary Hilbert space $\mathcal{H}$ and a real number $h > 0$, the ideal, periodic (period h) sampling-device (or analogue-to-digital converter) $\mathbf{A_D} : \mathscr{Y}_{\mathbb{R}^+}(\mathcal{H}) \to \mathscr{Y}_{\mathbb{Z}^+}(\mathcal{H})$, is defined by

$$\mathsf{f}_k := (\mathbf{A_D} f)_k := f(kh);$$

the zeroth-order hold-device (or digital-to-analogue converter) $\mathbf{D_A} : \mathscr{Y}_{\mathbb{Z}^+}(\mathcal{H}) \to \mathscr{Y}_{\mathbb{R}^+}(\mathcal{H})$, is defined by

$$f(t) := (\mathbf{D_A}\mathsf{f})(t) := \mathsf{f}_k \ \text{ for } \ kh \leq t < (k+1)h.$$

In systems theory it is often convenient to work with spaces that are isomorphic to the original time-domain signal-space concerned. To this end, let $\mathrm{L}^2_{\mathbb{T}}(\mathcal{H})$ denote the Hilbert space of functions $f : \mathbb{T} \to \mathcal{H}$ that satisfy

$$\|f\|^2_{\mathrm{L}^2_{\mathbb{T}}(\mathcal{H})} := \frac{1}{2\pi} \int_0^{2\pi} \langle f(e^{j\omega}), f(e^{j\omega}) \rangle_{\mathcal{H}}\, d\omega < \infty.$$

The inner-product that induces this norm is

$$\langle f, g \rangle_{\mathrm{L}^2_{\mathbb{T}}(\mathcal{H})} := \frac{1}{2\pi} \int_0^{2\pi} \langle f(e^{j\omega}), g(e^{j\omega}) \rangle_{\mathcal{H}}\, d\omega$$

and is it taken to be the inner-product on $\mathrm{L}^2_{\mathbb{T}}(\mathcal{H})$. Note that any $f \in \mathrm{L}^2_{\mathbb{T}}(\mathcal{H})$ has Fourier series expansion

$$f(e^{j\omega}) = \sum_{k=-\infty}^{\infty} e^{jk\omega} f_k,$$

where each $f_k \in \mathcal{H}$ and $\langle f, f \rangle_{\mathrm{L}^2_{\mathbb{T}}(\mathcal{H})} = \sum_{k=-\infty}^{\infty} \langle f_k, f_k \rangle_{\mathcal{H}}$, the set $\{f_k\}_{k=0}^{\infty}$ being the set of Fourier coefficients. An important subspace of $\mathrm{L}^2_{\mathbb{T}}(\mathcal{H})$ is denoted by $\mathrm{L}^{2+}_{\mathbb{T}}(\mathcal{H})$ and consists of all functions in $\mathrm{L}^2_{\mathbb{T}}(\mathcal{H})$ for which $f_k = 0$ when $k < 0$. Given any $f \in \mathrm{L}^{2+}_{\mathbb{T}}(\mathcal{H})$, $\sum_{k=0}^{\infty} \langle f_k, f_k \rangle_{\mathcal{H}} < \infty$ by definition and consequently, it is possible to associate with f the function $\varphi : \mathbb{D} \to \mathcal{H}$ given by

$$\varphi(\lambda) = \sum_{k=0}^{\infty} \lambda^k f_k. \tag{2.1}$$

Now, the Hardy space $\mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$ is defined to be the set of all $\mathcal{H}$-valued functions $\varphi : \mathbb{D} \to \mathcal{H}$ that are analytic in $\mathbb{D}$ and satisfy the following condition:

$$\int_0^{2\pi} \langle \varphi(re^{j\omega}), \varphi(re^{j\omega}) \rangle_{\mathcal{H}}\, d\omega \quad (0 \le r < 1) \tag{2.2}$$

has finite bound independent of r. Since each $\varphi \in \mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$ is analytic, it can be expressed as a power series of the form given in equation (2.1). Given such a representation, it follows that the condition associated with equation (2.2) implies $\sum_{k=0}^{\infty} \langle f_k, f_k \rangle_{\mathcal{H}} < \infty$. Hence, every function $\varphi(\lambda) \in \mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$ can be associated with a function $f(\omega) = \varphi(e^{j\omega}) \in \mathrm{L}^{2+}_{\mathbb{T}}(\mathcal{H})$ and in this way, $\mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$ inherits the Hilbert-space structure of $\mathrm{L}^{2+}_{\mathbb{T}}(\mathcal{H})$, embedding it in $\mathrm{L}^2_{\mathbb{T}}(\mathcal{H})$ as a subspace.

Proposition 2.1 [SNF70, pp. 184-185] *The Hilbert space $\ell^2_{\mathbb{Z}+}(\mathcal{H})$ is isomorphic to $\mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$ via the* $\mathbf{Z}$-*Transform isomorphism defined by*

$$\varphi(\lambda) = (\mathbf{Z}\mathrm{f})(\lambda) := \sum_{k=0}^{\infty} \lambda^k \mathrm{f}_k,$$

for all $\lambda \in \mathbb{D}$ and $\mathrm{f} := k \mapsto \mathrm{f}_k \in \ell^2_{\mathbb{Z}+}(\mathcal{H})$. $\mathbf{Z}^{-1}$ denotes the inverse $\mathbf{Z}$-*Transform isomorphism.*[4]

[4] $\lambda = z^{-1}$ in the usual engineering definition of the $\mathbf{Z}$-Transform.

By definition of the **Z**-Transform isomorphism, it follows that the unilateral shift acts as multiplication by λ on $\mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$. That is, $(\mathbf{S}\varphi)(\lambda) = \lambda\varphi(\lambda)$ for $\varphi \in \mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$.

Recall that given a real number $h > 0$, the symbol $\mathbb{H}$ denotes the interval $[0, h) \subset \mathbb{R}^+$. $\mathrm{L}^2_{\mathbb{H}}(\mathcal{H})$ is used to denoted the Hilbert space of $\mathcal{H}$-valued functions $f : \mathbb{H} \to \mathcal{H}$ that satisfy

$$\|f\|_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{H})} := \int_0^h \langle f(t), f(t)\rangle_{\mathcal{H}}\, dt < \infty.$$

The inner product on $\mathrm{L}^2_{\mathbb{H}}(\mathcal{H})$ is defined by

$$\langle f, g\rangle_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{H})} := \int_0^h \langle f(t), g(t)\rangle_{\mathcal{H}}\, dt,$$

which induces the norm used in the definition. $\mathrm{L}^2_{\mathbb{H}}(\mathcal{H})$ is really the truncation of $\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{H})$ to the interval $\mathbb{H}$ and henceforth, the symbol "$\multimap$" above a character is used to denote a vector in $\mathrm{L}^2_{\mathbb{H}}(\mathcal{H})$ or a sequence of such vectors.

The following operator plays a pivotal role throughout this dissertation: given a real number $h > 0$, the bijective operator $\mathbf{W} : \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{H}) \to \ell^{2,\mathrm{e}}_{\mathbb{Z}^+}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{H}))$ is defined by

$$\vec{\mathrm{f}}_k(\theta) := (\mathbf{W}f)_k = f(kh + \theta) \quad (\theta \in \mathbb{H}).$$

It acts as an isomorphism between the two Hilbert spaces $\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{H})$ and $\ell^2_{\mathbb{Z}^+}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{H}))$ and is referred to as the time-lifting isomorphism. Its inverse is denoted by $\mathbf{W}^{-1}$. The importance of this operator lies in the fact that it intertwines the continuous-time, unilateral shift by h and the discrete-time, unilateral shift. That is, $\mathbf{S}^k\mathbf{W} = \mathbf{W}\mathbf{U}_{kh}$ for all $k \in \mathbb{Z}^+$. In this way, a shift by h in $\mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{H})$ corresponds to a shift in $\ell^{2,\mathrm{e}}_{\mathbb{Z}^+}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{H}))$ by a single time-step.

Let $\mathcal{U}$ and $\mathcal{Y}$ be arbitrary Hilbert spaces. In the sequel, bounded operators that map the discrete-time signal space $\ell^2_{\mathbb{Z}^+}(\mathcal{U})$ to $\ell^2_{\mathbb{Z}^+}(\mathcal{Y})$ and commute with the unilateral shift, play a significant role. Related to these operators, is the Hardy space $\mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{U}\to\mathcal{Y}})$ of $\mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$-valued[5] functions $\Phi : \mathbb{D} \to \mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$ that are bounded and analytic in the open, unit disc $\mathbb{D}$. Given a function $\Phi \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{U}\to\mathcal{Y}})$, boundary values can be defined by application of Fatou's Theorem to $\Phi(\lambda)u$ for $u \in \mathcal{U}$ (cf. [FF90, pg. 234]). The resultant boundary values $\Phi(e^{j\omega})$ can be considered an extension of $\Phi(\lambda)$ to $\mathbb{T}$ and are essentially bounded on $\mathbb{T}$. $\|\cdot\|_\infty$ denotes the norm on $\mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{U}\to\mathcal{Y}})$, which given $\Phi \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{U}\to\mathcal{Y}})$ is defined by

$$\|\Phi\|_\infty := \sup_{\lambda\in\mathbb{D}} \|\Phi(\lambda)\| = \operatorname*{ess\,sup}_{\omega\in[0,2\pi)} \|\Phi(e^{j\omega})\|,$$

[5]Recall that $\mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$ denotes the Banach space of all bounded, linear operators mapping $\mathcal{U}$ to $\mathcal{Y}$.

where the second equality follows from the fact that $\|\Phi(\lambda)\|$ satisfies a maximum principle on $\mathbb{D}$ (cf. Sub-Section 2.2.3). An essentially-bounded, $\mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$-valued function $\Phi_\partial : \mathbb{T} \to \mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$ is said to be rigid if $\Phi_\partial(e^{j\omega})$ is an isometry almost everywhere on $\mathbb{T}$. A function $\Phi : \mathbb{D} \to \mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$ is said to be inner if $\Phi \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{U}\to\mathcal{Y}})$ and the function associated with its boundary values is rigid. Corresponding to each $\Phi \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{U}\to\mathcal{Y}})$ is a multiplication operator $\mathbf{M}_\Phi : \mathrm{H}^2_{\mathbb{D}}(\mathcal{U}) \to \mathrm{H}^2_{\mathbb{D}}(\mathcal{Y})$, defined by $(\mathbf{M}_\Phi\varphi)(\lambda) := \Phi(\lambda)\varphi(\lambda)$ for all $\varphi \in \mathrm{H}^2_{\mathbb{D}}(\mathcal{U})$ and $\lambda \in \mathbb{D}$.

Recall that $\ell^2_{\mathbb{Z}^+}(\mathcal{U})$ and $\mathrm{H}^2_{\mathbb{D}}(\mathcal{U})$ are isomorphic via the $\mathbf{Z}$-Transform isomorphism and hence, that a linear operator $\mathsf{F} : \ell^2_{\mathbb{Z}^+}(\mathcal{U}) \to \ell^2_{\mathbb{Z}^+}(\mathcal{Y})$ is *equivalent* to a linear operator $\mathbf{Z}\mathsf{F}\mathbf{Z}^{-1} : \mathrm{H}^2_{\mathbb{D}}(\mathcal{U}) \to \mathrm{H}^2_{\mathbb{D}}(\mathcal{U})$ (and vice-versa). Interestingly, if F is bounded and commutes with the unilateral shift, then $\mathbf{Z}\mathsf{F}\mathbf{Z}^{-1}$ can be expressed as a multiplication operator with symbol in $\mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{U}\to\mathcal{Y}})$, as is summarised in the following proposition:

Proposition 2.2 [FF90, pg. 235] *Given a bounded, linear operator* $\mathsf{F} : \ell^2_{\mathbb{Z}^+}(\mathcal{U}) \to \ell^2_{\mathbb{Z}^+}(\mathcal{Y})$ *that commutes with the unilateral shift, there exists a function* $\hat{\mathsf{F}} \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{U}\to\mathcal{Y}})$ *such that* $\mathsf{F} = \mathbf{Z}^{-1}\mathbf{M}_{\hat{\mathsf{F}}}\mathbf{Z}$ *and moreover,* F *is an isometry if and only if* $\hat{\mathsf{F}}$ *is inner. Furthermore, any multiplication operator* $\mathbf{M}_\Phi$ *with symbol* $\Phi \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{U}\to\mathcal{Y}})$ *is bounded and linear shift-invariant on* $\mathrm{H}^2_{\mathbb{D}}(\mathcal{U})$*, with* $\|\mathbf{M}_\Phi\| = \|\Phi\|_\infty$.

2.3.2 Systems

A system is defined to be an operator mapping between signal spaces. In this dissertation attention is restricted to systems defined on the continuous-time signal-space $\mathrm{L}^2_{\mathbb{R}^+}(\cdot)$ (or a subspace of this) and its extension $\mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\cdot)$, and the discrete-time signal-space $\ell^2_{\mathbb{Z}^+}(\cdot)$ (or a subspaces of this) and its extension $\ell^{2,\mathrm{e}}_{\mathbb{Z}^+}(\cdot)$. As a general rule, continuous-time systems and all associated operators (for example operators associated with a state-space realisation) are denoted throughout by italic, upper-case, Roman letters (P, F for example). Discrete-time systems and all associated operators (for example operators associated with a state-space realisation) are denoted by upper-case, Euler letters (P, F for example). In what follows, let $\mathcal{U}$ and $\mathcal{Y}$ be Hilbert spaces and for notational convenience define $\mathcal{V} := \mathcal{U} \oplus \mathcal{Y}$.

Two important concepts in systems theory concern the timing of input-output pairs. The first is causality, which is a fundamental property of physically realisable systems. An operator is said to be causal if the output does not depend on future values of the input. More formally, a continuous-time system $P : \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{U}) \to \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{Y})$ is said to be <u>causal</u> if given any $\tau \in \mathbb{R}^+$, $\mathbf{P}_\tau P u_1 = \mathbf{P}_\tau P u_2$ for all $u_1, u_2 \in \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{U})$ that satisfy $\mathbf{P}_\tau u_1 = \mathbf{P}_\tau u_2$. Further to this, P is said to be <u>strongly causal</u> if it is causal and given any $\tau \in \mathbb{R}^+$, $\epsilon > 0$ and $\tau_1 \leq \tau$ ($\tau_1 \in \mathbb{R}^+$), there exists a $\delta > 0$ such that for all

$u_1, u_2 \in \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{U})$ with $\mathbf{P}_{\tau_1} u_1 = \mathbf{P}_{\tau_1} u_2$,

$$\|\mathbf{P}_{(\tau_1+\delta)}(Pu_1 - Pu_2)\|_{\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{Y})} \leq \epsilon \|\mathbf{P}_{(\tau_1+\delta)}(u_1 - u_2)\|_{\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U})}.$$

Basically, a strongly causal system cannot respond instantaneously to inputs. For many causal systems encountered in systems theory, the way their behaviour varies with time imparts additional structure to the analysis at hand. Two types of variation are exploited in this dissertation; time-invariance and periodic time-variation. Specifically, a causal, continuous-time system $P : \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{U}) \to \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{Y})$ is said to be time-invariant if $P\mathbf{U}_\tau = \mathbf{U}_\tau P$ for all $\tau \geq 0$ and P is said to be periodically time-varying with period h, if $P\mathbf{U}_{kh} = \mathbf{U}_{kh} P$ for all $k \in \mathbb{Z}^+$. Note that as defined, a time-invariant system is also periodic. As for continuous-time systems, a discrete-time system $\mathsf{P} : \ell^{2,\mathrm{e}}_{\mathbb{Z}^+}(\mathcal{U}) \to \ell^{2,\mathrm{e}}_{\mathbb{Z}^+}(\mathcal{Y})$ is said to be causal if given any $i \in \mathbb{Z}^+$, $\mathbf{P}_i \mathsf{P} u_1 = \mathbf{P}_i \mathsf{P} u_2$ for all $u_1, u_2 \in \ell^{2,\mathrm{e}}_{\mathbb{Z}^+}(\mathcal{U})$ such that $\mathbf{P}_i u_1 = \mathbf{P}_i u_2$. Furthermore, a causal, discrete-time system is said to be shift-invariant if $\mathsf{P}\mathbf{S}^i = \mathbf{S}^i \mathsf{P}$ for all $i \in \mathbb{Z}^+$. That is, its behaviour does not vary with time.

Any system $P_e : \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{U}) \to \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{Y})$ induces a system

$$P : \mathcal{D}_P \subset \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U}) \to \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{Y})$$

by specifying $\mathcal{D}_P := \{u \in \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U}) : P_e u \in \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{Y})\}$ and $Pu := P_e u$ for all $u \in \mathcal{D}_P$. When two systems are related in this way, it is said that P_e induces P and that P_e is the extension of P. Note that any P_e uniquely induces a P. However, the converse is not true in general [GS93, FS82]. Before discussing a case for which the converse is true, it is necessary to clarify the notion of causality for systems not defined on extended space. A system $P : \mathcal{D}_P \subset \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U}) \to \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{Y})$ is said to be causal if

$$\mathcal{G}_P \cap \begin{bmatrix} (\mathrm{I} - \mathbf{P}_\tau)\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U}) \\ \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{Y}) \end{bmatrix} \subset (\mathrm{I} - \mathbf{P}_\tau)\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{V})$$

for all $\tau \in \mathbb{R}^+$. If P is linear, then this corresponds to $\mathbf{P}_\tau \mathcal{G}_P$ being the graph of a linear operator. Clearly, if P is induced by a causal system $P_e : \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{U}) \to \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{Y})$, then P is causal in the sense just defined. Now given a causal system $P : \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U}) \to \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{Y})$ with $\|P\| < \infty$, there is a one-to-one correspondence to a system on the extended space [FS82, pg. 179]. To see this, note that for such a P and given any $\tau \in \mathbb{R}^+$, the input can be chosen arbitrarily over the interval $[0, \tau]$. Then since P is causal, the output is uniquely determined over the same interval. In fact, for any causal P that satisfies

$$\mathbf{P}_\tau \mathcal{D}_P = \mathrm{L}^2_{[0,\tau]}(\mathcal{U}) \ \text{ for all finite real } \ \tau > 0, \tag{2.3}$$

where $\mathrm{L}^2_{[0,\tau]}(\mathcal{U}) := \mathbf{P}_\tau \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U})$, there is a one-to-one correspondence to a system on the relevant extended space [GS93]. With this in mind, a system P is called causally extendible if it is causal and satisfies condition (2.3). Note that all of this also holds in discrete-time with obvious notational modifications.

Further to the definition of causality for systems not defined on extended space, it is also convenient to clarify the notions of time-invariance and periodic time-variation for these systems. A causal system $P : \mathcal{D}_P \subset \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U}) \to \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{Y})$ is said to be time-invariant (respectively periodically time-varying) if $\mathbf{U}_\tau \mathcal{G}_P \subset \mathcal{G}_P$ (respectively $\mathbf{U}_{kh} \mathcal{G}_P \subset \mathcal{G}_P$) for all $\tau \in \mathbb{R}^+$ (respectively $k \in \mathbb{Z}^+$). Clearly, if P is induced by a time-invariant (respectively periodically time-varying) system P_e on the extended space, then P is time-invariant (respectively periodically time-varying) in the sense just defined. It goes without saying, that similar definitions hold in discrete-time. Now before proceeding to define precisely the systems considered in the sequel, it is necessary to make one more technical definition. A causal, system $P : \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{U}) \to \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{Y})$ is said to be (time) locally Lipschitz-continuous on $\mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{U})$ if and only if for all $\tau \in \mathbb{R}^+$,

$$\sup_{\substack{u_1, u_2 \in \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{U}) \\ \mathbf{P}_\tau u_1 \neq \mathbf{P}_\tau u_2}} \frac{\|\mathbf{P}_\tau (P u_1 - P u_2)\|_{\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{Y})}}{\|\mathbf{P}_\tau (u_1 - u_2)\|_{\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U})}} < \infty.$$

A similar definition (with obvious notational modifications) holds in discrete-time. Also note that although the concepts introduced above hold for a wide class of non-linear systems, only *linear* systems are considered in this dissertation.

Definition 2.3 Given two Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$, $\mathscr{P}_{\mathcal{U} \to \mathcal{Y}}$ is defined to be the set of *causal, linear, periodically-time-varying* (LPTV) (with period h), *continuous-time* systems

$$P : \mathcal{D}_P \subset \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U}) \to \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{Y}),$$

with *closed graphs*. Furthermore, $\mathscr{P}^{\mathrm{e}}_{\mathcal{U} \to \mathcal{Y}} \subset \mathscr{P}_{\mathcal{U} \to \mathcal{Y}}$ is defined to be the subset of *causally extendible* systems with *locally-Lipschitz-continuous* extension and $\mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U} \to \mathcal{Y}}$ the subset of *strongly-causal* systems in $\mathscr{P}^{\mathrm{e}}_{\mathcal{U} \to \mathcal{Y}}$. A system $P \in \mathscr{P}_{\mathcal{U} \to \mathcal{Y}}$ is said to be stable if $\mathcal{D}_P = \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U})$, which since it is assumed that P has closed graph implies that $\|P\| < \infty$ by the Closed Graph Theorem. In view of the discussion above, any stable system $P \in \mathscr{P}_{\mathcal{U} \to \mathcal{Y}}$ is an element of $\mathscr{P}^{\mathrm{e}}_{\mathcal{U} \to \mathcal{Y}}$. ♠

An important property of LPTV systems is that they are *equivalent*, via the time-lifting isomorphism ($\mathbf{W}$), to linear, shift-invariant (LSI), discrete-time systems, which by virtue of their shift-invariance are easier to work with analytically [BPFT91, BP92].

Proposition 2.4 *Given an LPTV system $P \in \mathscr{P}_{\mathcal{U} \to \mathcal{Y}}$, the equivalent system* $\mathsf{P} := \mathbf{W} P \mathbf{W}^{-1}$ *is causal and LSI.*

Proof : Note that the graph of P is simply the graph of P under the time-lifting isomorphism. That is, $\mathcal{G}_{\mathsf{P}} = \mathbf{W}\mathcal{G}_P$. So since $\mathcal{G}_P$ corresponds to a linear operator it follows that $\mathcal{G}_{\mathsf{P}}$ does too. That P is causal is immediate, because $\mathbf{P}_k\mathcal{G}_{\mathsf{P}} = \mathbf{W}\mathbf{P}_{kh}\mathcal{G}_P$ for all $k \in \mathbb{Z}^+$, which since $\mathbf{P}_{kh}\mathcal{G}_P$ corresponds to the graph of a linear operator (as P is causal), implies that P is causal. Similarly, since $\mathbf{S}^k\mathbf{W} = \mathbf{W}\mathbf{U}_{kh}$ for any $k \in \mathbb{Z}^+$, it follows that

$$\mathbf{S}^k\mathcal{G}_{\mathsf{P}} = \mathbf{W}\mathbf{U}_{kh}\mathcal{G}_P \subset \mathbf{W}\mathcal{G}_P = \mathcal{G}_{\mathsf{P}}$$

and hence, that P is also shift-invariant. ■

For notational convenience the set of causal, LSI operators is defined below:

Definition 2.5 Given two Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$, $\mathscr{D}_{\mathcal{U}\to\mathcal{Y}}$ is defined to be the set of causal, LSI, discrete-time systems

$$\mathsf{P} : \mathcal{D}_{\mathsf{P}} \subset \ell^2_{\mathbb{Z}^+}(\mathcal{U}) \to \ell^2_{\mathbb{Z}^+}(\mathcal{Y}).$$

A system $\mathsf{P} \in \mathscr{D}_{\mathcal{U}\to\mathcal{Y}}$ is called <u>stable</u> if $\mathcal{D}_{\mathsf{P}} = \ell^2_{\mathbb{Z}^+}(\mathcal{U})$ and $\|\mathsf{P}\| < \infty$. ♠

By representing each signal in $\ell^2_{\mathbb{Z}^+}(\cdot)$ as a column vector with k^{th} entry corresponding to the signal's value at time k (that is, with respect to the standard basis for $\ell^2_{\mathbb{Z}^+}(\cdot)$), it follows by causality and shift-invariance, that every system $\mathsf{P} \in \mathscr{D}_{\mathcal{U}\to\mathcal{Y}}$ has block, lower-triangular, Toeplitz structure

$$\begin{bmatrix} \mathsf{P}_{[0]} & 0 & \cdots & \cdots \\ \mathsf{P}_{[1]} & \mathsf{P}_{[0]} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

This Toeplitz representation can be uniquely identified with the sequence $\{\mathsf{P}_{[i]}\}_{i=0}^{\infty}$ and will be used extensively in the sequel. If $\mathsf{P} \in \mathscr{D}_{\mathcal{U}\to\mathcal{Y}}$ is causally extendible to a locally-Lipschitz-continuous system $\mathsf{P}_e : \ell^{2,e}_{\mathbb{Z}^+}(\mathcal{U}) \to \ell^{2,e}_{\mathbb{Z}^+}(\mathcal{Y})$, then each $\mathsf{P}_{[i]} \in \mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$.[6] Given such a system P, it is straightforward to show, using its Toeplitz representation, that P is equivalent via the **Z**-Transform isomorphism to a multiplication operator on (a subspace of) $\mathrm{H}^2_{\mathbb{D}}(\mathcal{U})$ with symbol

$$\hat{\mathsf{P}}(\lambda) := \sum_{i=0}^{\infty} \mathsf{P}_{[i]}\lambda^i,$$

analytically defined on some open subset of $\mathbb{D}$ (it is useful to note that $\mathsf{P}_{[0]} = \hat{\mathsf{P}}(0)$). If P is also stable, then it follows by Proposition 2.2, that $\hat{\mathsf{P}}(\lambda) \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{U}\to\mathcal{Y}})$. Henceforth, the symbol "^" is used to denote the frequency-domain symbol of the multiplication-operator representation of an LSI system.

[6] If $\mathcal{U}$ and $\mathcal{Y}$ are finite-dimensional Hilbert spaces, then all systems in $\mathscr{D}_{\mathcal{U}\to\mathcal{Y}}$ have locally-Lipschitz-continuous extension to the extend space $\ell^{2,e}_{\mathbb{Z}^+}(\cdot)$.

Remark 2.6 A system $\mathsf{P} \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}) \to \mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}$ is equivalent to a system $P \in \mathscr{P}_{\mathcal{U}\to\mathcal{Y}}$ (via $\mathbf{W}$) if and only if $\mathsf{P}_{[0]}$ is a causal map from $\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})$ to $\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})$. A system $\mathsf{P} \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}) \to \mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}$ not satisfying this condition is equivalent (via $\mathbf{W}$) to an LPTV, continuous-time system that is not locally causal in each $[kh, (k+1)h)$ interval of time ($k \in \mathbb{Z}^+$). ♣

Remark 2.7 A system $P \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ is equivalent (via $\mathbf{W}$) to a system in $\mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}) \to \mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}$ with locally-Lipschitz-continuous extension. ♣

2.3.3 Closed-Loop Systems

In this dissertation, the stability of closed-loop systems is of central concern. To define precisely what is meant by a stable closed-loop, a few preliminary, technical definitions and results are required. Let $\mathcal{U}$ and $\mathcal{Y}$ be Hilbert spaces and consider the system shown in Figure 2.1, where

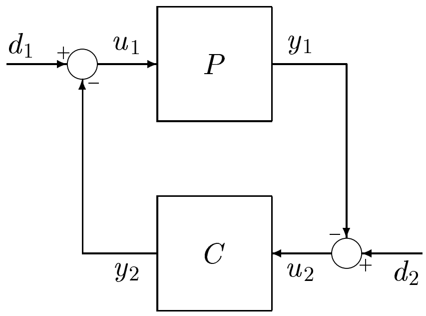

Figure 2.1 *Standard Feedback Configuration*

the systems $P : \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{U}) \to \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{Y})$ and $C : \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{Y}) \to \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{U})$ are causal and locally-Lipschitz-continuous. For notational convenience, define $\mathcal{V} := \mathcal{U} \oplus \mathcal{Y}$. The closed-loop system is denoted by $[P, C]$ and the functional equations describing it are

$$
\begin{aligned}
u_1 &= d_1 - y_2, \\
u_2 &= d_2 - y_1, \\
y_1 &= P u_1, \\
y_2 &= C u_2.
\end{aligned}
$$

An important concept in the study of closed-loop systems is well-posedness. Well-posedness is primarily concerned with the existence and uniqueness of solutions to the functional equations describing the closed-loop, causality of the closed-loop and adequacy of mathematical models to represent physical systems. For further details and discussion of the physical significance of well-posedness, the reader is referred to the seminal work of Willems [Wil71, Chap. 4].

Definition 2.8 Given causal, locally-Lipschitz-continuous systems P and C, the closed-loop $[P, C]$ is said to be well-posed [Wil71, pg. 90] if

(i) The system

$$\begin{bmatrix} I & C \\ P & I \end{bmatrix} : \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}+}(\mathcal{U}) \times \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}+}(\mathcal{Y}) \mapsto \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}+}(\mathcal{U}) \times \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}+}(\mathcal{Y}),$$

is bijective and hence, invertible on $\mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}+}(\mathcal{U}) \oplus \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}+}(\mathcal{Y})$. That is, for any inputs $d_1 \in \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}+}(\mathcal{U})$ and $d_2 \in \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}+}(\mathcal{Y})$ there exists a unique solution to the closed-loop equations.

(ii) The signals u_1, u_2, y_1 and y_2 depend causally on d_1 and d_2. Combined with (i), this requires that the following system be causal:

$$H_{\mathrm{e}}(P, C) := \begin{bmatrix} I & C \\ P & I \end{bmatrix}^{-1} : \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}+}(\mathcal{U}) \times \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}+}(\mathcal{Y}) \mapsto \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}+}(\mathcal{U}) \times \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}+}(\mathcal{Y}).$$

(iii) On finite intervals of time ($\mathbb{R}^+$), the signals u_1, u_2, y_1 and y_2 depend Lipschitz-continuously on d_1 and d_2. Combined with (*i*) and (*ii*), this requires that $H_{\mathrm{e}}(P, C)$ be causal and locally Lipschitz-continuous.

(iv) The solution to the closed-loop equations is insensitive to very-high-frequency modelling errors, such as small transmission delays. ♠

Proposition 2.9 [Wil71, Chap. 4] *A sufficient condition for the closed-loop $[P, C]$ to be well-posed is that either P or C be strongly causal.*

Remark 2.10 In a mathematical representation of a physical, closed-loop system, it is natural to assume that at least one of the mathematical models P and C is strongly causal. To see this, observe that in closed-loop P and C transmit information to each other and that in a physical system the transmission of information cannot occur instantaneously, which is in line with the definition of strong causality. ♣

In addition to well-posedness, it is desirable that the system $H(P, C) : \mathcal{D}_{H(P,C)} \to \mathrm{L}^2_{\mathbb{R}+}(\mathcal{V})$ induced by $H_{\mathrm{e}}(P, C)$, satisfies $\mathcal{D}_{H(P,C)} = \mathrm{L}^2_{\mathbb{R}+}(\mathcal{V})$. That is, an input $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in \mathrm{L}^2_{\mathbb{R}+}(\mathcal{V})$ to the closed-loop corresponds to closed-loop outputs $u_1, y_2 \in \mathrm{L}^2_{\mathbb{R}+}(\mathcal{U})$ and $y_1, u_2 \in \mathrm{L}^2_{\mathbb{R}+}(\mathcal{Y})$.

Definition 2.11 If for a given pair of causal, locally-Lipschitz-continuous systems P and C, the closed-loop $[P, C]$ is *well-posed* and $\mathcal{D}_{H(P,C)} = \mathrm{L}^2_{\mathbb{R}+}(\mathcal{V})$, then $[P, C]$ is said to be stable and P is said to be stabilised by C (and vice-versa). A stronger condition is that $[P, C]$ be *well-posed,*

$\mathcal{D}_{H(P,C)} = \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{V})$ and $\|H(P,C)\| < \infty$. In this case, $[P,C]$ is said to be stable with finite gain. If the two systems P and C are also *linear*, it can be shown using the Closed Graph Theorem that stable with finite gain and stable are equivalent [Wil71, pg. 117]. As such, throughout this dissertation only the term stable is used. ♠

It is now possible to characterise closed-loop stability for the systems considered in this dissertation. If in Figure 2.1, P is taken to be in $\mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ and C in $\mathscr{P}^{\mathrm{e,sc}}_{\mathcal{Y}\to\mathcal{U}}$, then by definition, P and C have causal extensions P_e and C_e that are locally-Lipschitz-continuous and since C_e is strongly causal, the closed-loop $[P,C]$ is well-posed (cf. Proposition 2.9). With the closed-loop being well-posed, it follows by condition (i) in Definition 2.8, that the system

$$\begin{bmatrix} I & C \\ P & I \end{bmatrix} : \mathcal{D}_C \oplus \mathcal{D}_P \to \mathcal{G}_P + \mathcal{G}_C^{'}$$

is injective onto $\mathcal{G}_P + \mathcal{G}_C^{'}$ and hence, that $\mathcal{G}_P \cap \mathcal{G}_C^{'} = \{0\}$. So on $\mathcal{D}_{H(P,C)} := \mathcal{G}_P + \mathcal{G}_C^{'}$ it has a well-defined (set theoretic) inverse

$$H(P,C) := \begin{bmatrix} I & C \\ P & I \end{bmatrix}^{-1} : \mathcal{D}_{H(P,C)} \subset \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{V}) \to \mathcal{D}_C \oplus \mathcal{D}_P.$$

By the definition of stability, it is then clear that $[P,C]$ is stable if and only if $\mathcal{D}_{H(P,C)} := \mathcal{G}_P + \mathcal{G}_C^{'} = \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{V})$ (with $\mathcal{G}_P \cap \mathcal{G}_C^{'} = \{0\}$). Furthermore, since all of the systems involved are linear, this is equivalent to stable with finite gain. This is summarised in the following proposition, which is reasonably standard and has also appeared in [OS91, FGS93, GS93], for example.

Proposition 2.12 *With $P \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ and $C \in \mathscr{P}^{\mathrm{e,sc}}_{\mathcal{Y}\to\mathcal{U}}$, the closed-loop $[P,C]$ is stable if and only if $\mathcal{G}_P$ and $\mathcal{G}_C^{'}$ induce a co-ordinatisation of $\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{V})$. That is, if and only if $\mathcal{G}_P \cap \mathcal{G}_C^{'} = \{0\}$ and $\mathcal{G}_P + \mathcal{G}_C^{'} = \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{V})$.*

Remark 2.13 Necessity of Proposition 2.12 holds for $P \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ and $C \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}}$, but sufficiency may not always hold in this case. To see why, note that while the condition $\{\mathcal{G}_P, \mathcal{G}_C^{'}\}$ *is a co-ordinatisation of* $\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{V})$ is necessary and sufficient for $\left[\begin{smallmatrix} I & C \\ P & I \end{smallmatrix}\right]: \mathcal{D}_P \oplus \mathcal{D}_C \to \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{V})$ to have bounded inverse on $\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{V})$, it does not guarantee that the closed-loop is well-posed. However, with C strongly causal the closed-loop is always well-posed (cf. Proposition 2.9) and for technical simplicity this was assumed. Although rather mild (cf. Remark 2.10), that C be strongly causal is not the weakest assumption required for the Proposition to hold.[7] In fact, Proposition 2.12 holds for any well-posed, plant/controller pair: for example, $P \in \mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U}\to\mathcal{Y}}$ and $C \in \mathscr{P}_{\mathcal{Y}\to\mathcal{U}}$ (cf. Proposition 2.9). ♣

[7] For example another sufficient condition for well-posedness is that the product of the instantaneous gains of P and C be strictly less than 1 [Wil71].

Remark 2.14 Given a stable closed-loop $[P,C]$,

$$
\begin{aligned}
H(P,C) &:= \begin{bmatrix} I & C \\ P & I \end{bmatrix}^{-1} = \begin{bmatrix} (I-CP)^{-1} & -C(I-PC)^{-1} \\ -P(I-CP)^{-1} & (I-PC)^{-1} \end{bmatrix}, \\
\mathbf{\Pi}_{\mathcal{G}_P \| \mathcal{G}'_C} &= \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} H(P,C) + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} \begin{bmatrix} (I-CP)^{-1} & -C(I-PC)^{-1} \end{bmatrix}
\end{aligned}
$$

and

$$
\mathbf{\Pi}_{\mathcal{G}'_C \| \mathcal{G}_P} = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} H(P,C) + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} C \\ I \end{bmatrix} \begin{bmatrix} -P(I-CP)^{-1} & (I-PC)^{-1} \end{bmatrix}.
$$

Since $H(P,C)$ is causal and bounded it is causally extendible to a locally-Lipschitz-continuous operator on $\mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{V})$ and it follows that $H(P,C) \in \mathscr{P}^{\mathrm{e}}_{\mathcal{V}\to\mathcal{V}}$. Similarly, the parallel projection operators $\mathbf{\Pi}_{\mathcal{G}_P \| \mathcal{G}'_C}$ and $\mathbf{\Pi}_{\mathcal{G}'_C \| \mathcal{G}_P}$ are also systems in $\mathscr{P}^{\mathrm{e}}_{\mathcal{V}\to\mathcal{V}}$. ♣

2.4 LSI State-Space Systems and Discrete-Time Synthesis

The purpose of this section is to assemble some basic properties of LSI, discrete-time, state-space systems and to present state-space solutions to two discrete-time, synthesis problems. All of the results presented are well-known for state-space systems that have *finite-dimensional* input, output and state spaces. They are perhaps less well-known however, for the case in which the state-space is importantly *finite-dimensional* but the input and output spaces are *infinite-dimensional*, which as will be seen in the sequel, arises frequently in the study of LPTV systems. Many of the ideas discussed here are fundamental to the computational procedures developed later.

Let $\mathcal{X}$ be an n-dimensional (n *finite*) Hilbert space and $\mathcal{U}$ and $\mathcal{Y}$ be (possibly *infinite-dimensional*) Hilbert spaces. Consider the discrete-time system of difference equations:

$$
\begin{aligned}
\mathsf{x}_{k+1} &= \mathsf{A}\mathsf{x}_k + \mathsf{B}\mathsf{u}_k; \quad \mathsf{x}_0 = x_o, && (2.4a) \\
\mathsf{y}_k &= \mathsf{C}\mathsf{x}_k + \mathsf{D}\mathsf{u}_k, && (2.4b)
\end{aligned}
$$

where for any fixed $k \in \mathbb{Z}^+$ the state $\mathsf{x}_k \in \mathcal{X}$, the input $\mathsf{u}_k \in \mathcal{U}$, the output $\mathsf{y}_k \in \mathcal{Y}$ and the initial state $x_o \in \mathcal{X}$. It is assumed that the bounded, linear operators $\mathsf{A} : \mathcal{X} \to \mathcal{X}$, $\mathsf{B} : \mathcal{U} \to \mathcal{X}$, $\mathsf{C} : \mathcal{X} \to \mathcal{Y}$ and $\mathsf{D} : \mathcal{U} \to \mathcal{Y}$ are all independent of time. With $x_0 = 0$, the system of difference equations (2.4) characterises a causal, LSI system $\mathsf{P}_e : \ell^{2,\mathrm{e}}_{\mathbb{Z}^+}(\mathcal{U}) \to \ell^{2,\mathrm{e}}_{\mathbb{Z}^+}(\mathcal{Y})$ defined by

$$
\mathsf{y}_k := (\mathsf{P}_e\mathsf{u})_k := \mathsf{C}\sum_{n=0}^{k-1} \mathsf{A}^{k-n-1}\mathsf{B}\mathsf{u}_n + \mathsf{D}\mathsf{u}_k. \tag{2.5}
$$

The system P_e induces (as described in Sub-Section 2.3.2) a system $\mathsf{P} : \mathcal{D}_{\mathsf{P}} \subset \ell^2_{\mathbb{Z}^+}(\mathcal{U}) \to \ell^2_{\mathbb{Z}^+}(\mathcal{Y})$ in $\mathscr{D}_{\mathcal{U}\to\mathcal{Y}}$ and the system of difference equations (2.4), is said to be a state-space realisation of P. It is often convenient to denote a state-space realisation such as that given in equations (2.4), by $\left[\begin{array}{c|c} \mathsf{A} & \mathsf{B} \\ \hline \mathsf{C} & \mathsf{D} \end{array}\right]$. Importantly, a state-space realisation of a system $\mathsf{P} \in \mathscr{D}_{\mathcal{U}\to\mathcal{Y}}$ (if it admits one) is not unique. Suppose that P admits the state-space realisation (2.4), then given any invertible basis-transformation $\mathsf{T} : \mathcal{X} \to \mathcal{X}$, $\left[\begin{array}{c|c} T^{-1}AT & T^{-1}B \\ \hline CT & D \end{array}\right]$ is also state-space realisation of P.

2.4.1 Reachability, Observability, Stabilisability and Detectability

In this section, the well understood concepts of reachability, observability, stabilisability and detectability, which are all properties of the state-space realisation chosen to represent a system, are introduced. For further details the reader is referred to [Bro70, Kai80, Won85, ZDG95].

Given an initial condition $x_o \in \mathcal{X}$ and input signal $\mathsf{u} =: k \mapsto \mathsf{u}_k \subset \ell^{2,\mathrm{e}}_{\mathbb{Z}^+}(\mathcal{U})$, the state of the system of difference equations (2.4) evolves as:

$$\mathsf{x}_k = \mathsf{A}^k x_o + \sum_{n=0}^{k-1} \mathsf{A}^{k-n-1} \mathsf{B} \mathsf{u}_n.$$

A terminal state $x_f \in \mathcal{X}$ is said to be reachable if there exists a time $N < \infty$ and input signal $\mathsf{u} \in \ell^{2,\mathrm{e}}_{\mathbb{Z}^+}(\mathcal{U})$ such that the solution to the state equation (2.4a), satisfies $\mathsf{x}_N = x_f$ when $x_o = 0$. The set of reachable states is a property of the pair (A, B) and can be characterised in terms of the sequence of subspaces $\mathcal{C}_0,\ \mathcal{C}_1,\ \mathcal{C}_2 \ldots \subset \mathcal{X}$, defined in terms of A and B as follows:

$$\mathcal{C}_p = \bigcup_{k=0}^{p} \mathcal{R}_{\mathsf{A}^k\mathsf{B}} := \mathcal{R}_{\left[\begin{array}{cccc} \mathsf{B} & \mathsf{AB} & \cdots & \mathsf{A}^p\mathsf{B} \end{array}\right]}, \quad (p \in \mathbb{Z}^+).$$

Clearly, $\mathcal{C}_0 \subset \mathcal{C}_1 \subset \mathcal{C}_2 \subset \ldots$ and it can be shown that the inclusions are strictly proper up to a certain index r (dependent on A and B), with equality holding thereafter. Since $\mathcal{C}_p \subset \mathcal{X}$ for all $p \in \mathbb{Z}^+$, it follows that $r < n$. $\mathcal{C}_{\mathsf{A},\mathsf{B}} := \mathcal{C}_r$ is called the reachable subspace and corresponds to the set of reachable states. It follows by definition, that $A\mathcal{C}_p \subset \mathcal{C}_{p+1}$ and in fact, that $\mathcal{C}_{\mathsf{A},\mathsf{B}}$ is the smallest A-invariant subspace containing $\mathcal{R}_{\mathsf{B}}$. The pair (A, B) is said to be completely reachable if $\mathcal{C}_{\mathsf{A},\mathsf{B}} = \mathcal{X}$.

Given two operators $\mathsf{B}_1, \mathsf{B}_2 : \mathcal{U} \to \mathcal{X}$ with $\mathcal{R}_{\mathsf{B}_1} = \mathcal{R}_{\mathsf{B}_2}$, it follows by definition, that $\mathcal{C}_{\mathsf{A},\mathsf{B}_1} = \mathcal{C}_{\mathsf{A},\mathsf{B}_2}$. In particular the pairs (A, B) and $(\mathsf{A}, \mathsf{B}\mathsf{B}^*)$ have the same reachable subspace. To see this, it is sufficient to show that $\mathcal{R}_{\mathsf{B}} = \mathcal{R}_{\mathsf{B}\mathsf{B}^*}$. To this end, note that for any bounded, linear operator X, $(\mathcal{R}_X)^\perp = \mathcal{K}_{X^*}$, but that it is *not* possible to write $\mathcal{R}_X = (\mathcal{K}_{X^*})^\perp$ unless the range of X is closed.

Now, since $\mathcal{R}_{\mathsf{B}} \subset \mathcal{X}$, which is finite-dimensional, it follows that B has closed range (since any finite-dimensional subspace is closed [Kre89b]). Hence, $\mathcal{R}_{\mathsf{B}} = (\mathcal{K}_{\mathsf{B}^*})^{\perp}$ and $\mathcal{R}_{\mathsf{BB}^*} = (\mathcal{K}_{\mathsf{BB}^*})^{\perp}$. So to obtain the desired result, it is sufficient to show that $\mathcal{K}_{\mathsf{B}^*} = \mathcal{K}_{\mathsf{BB}^*}$. That $\mathcal{K}_{\mathsf{B}^*} \subset \mathcal{K}_{\mathsf{BB}^*}$ is obvious. To see the opposite inclusion take any $x \in \mathcal{K}_{\mathsf{BB}^*}$, by which $\langle x, \mathsf{BB}^* x\rangle_{\mathcal{X}} = 0$ and thus, $\langle \mathsf{B}^* x, \mathsf{B}^* x\rangle_{\mathcal{U}}$. That is, $\mathsf{B}^* x = 0$ and the desired result follows as claimed. The importance of this equivalence can be seen by noting that BB^* is a finite-dimensional operator and hence, is more computationally tractable than B. For example, to test complete reachability of the pair (A, B), standard finite-dimensional tools can be used to test the equivalent condition of complete reachability of the (finite-dimensional) pair $(\mathsf{A}, \mathsf{BB}^*)$. The combination of infinite-dimensional but finite-rank, state-space operators in this way, arises throughout this dissertation and is fundamental to the development of numerical algorithms for automated analysis and synthesis of LPTV systems (cf. Chapters 5 and 6 for example).

Another sequence of subspaces of the state-space $\mathcal{X}$, useful in the study of the system of difference equations (2.4), is now introduced. Let $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2, \ldots \subset \mathcal{X}$ be defined in terms of A and C as follows:

$$\mathcal{N}_p = \bigcap_{k=0}^{p} \mathcal{K}_{\mathsf{CA}^k}, \quad (p \in \mathbb{Z}^+).$$

Observe that $\mathcal{N}_0 \supset \mathcal{N}_1 \supset \mathcal{N}_2 \supset \ldots$, and that the inclusions are strictly proper up to a certain index $r < n$ (dependent on A and C), with equality holding thereafter. $\mathcal{N}_{\mathsf{C},\mathsf{A}} :=\mathcal{N}_r$ is called the unobservable subspace and it corresponds to the the subspace of initial states $x_o \in \mathcal{X}$ that *cannot* be uniquely determined by observing $\{\mathsf{u}_k\}_{k=0}^N$ and $\{\mathsf{y}_k\}_{k=0}^N$ in equations (2.4), for some finite time $N \in \mathbb{Z}^+$. The pair (C, A) is said to be completely observable if $\mathcal{N}_{\mathsf{C},\mathsf{A}}^{\perp} = \mathcal{X}$. Since the kernel of a bounded, linear operator is always closed [Kre89b], it follows that

$$\left(\bigcap_{k=0}^{p} \mathcal{K}_{\mathsf{CA}^k}\right)^{\perp} = \bigcup_{k=0}^{p} \left(\mathcal{K}_{\mathsf{CA}^k}\right)^{\perp}, \quad (p \in \mathbb{Z}^+).$$

Furthermore, since $\mathcal{R}_{\mathsf{C}^*}$ and $\mathcal{R}_{\mathsf{A}^*}$ are subspaces of the finite-dimensional Hilbert space $\mathcal{X}$, it follows that $\mathcal{R}_{(\mathsf{A}^*)^k\mathsf{C}^*}$ is closed for all $k \in \mathbb{Z}^+$ and equal to $\left(\mathcal{K}_{\mathsf{CA}^k}\right)^{\perp}$. As such,

$$\mathcal{N}_p^{\perp} = \bigcup_{k=0}^{p} \left(\mathcal{K}_{\mathsf{CA}^k}\right)^{\perp} :=\mathcal{R}_{\left[\begin{array}{cccc} \mathsf{C}^* & \mathsf{A}^*\mathsf{C}^* & \ldots & (\mathsf{A}^*)^p\mathsf{C}^* \end{array}\right]} \quad (p \in \mathbb{Z}^+)$$

and hence, $\mathcal{N}_{\mathsf{C},\mathsf{A}}^{\perp} = \mathcal{C}_{A^*,C^*}$.

An important and very useful result is that complete reachability and observability of a state-space realisation $\left[\begin{array}{c|c} \mathsf{A} & \mathsf{B} \\ \hline \mathsf{C} & \mathsf{D} \end{array}\right]$, can be characterised in terms of the eigenvalues (or modes) of A and the range and kernel of B and C respectively. This is summarised in the following proposition.

Proposition 2.15 (cf. [ZDG95] for example).

I. The following are equivalent :

(i) The pair (A, B) *is completely reachable.*

(ii) $\mathcal{K}_{\mathsf{B}^*} \cap \mathcal{K}_{(\lambda \mathsf{I} - \mathsf{A})^*} = \{0\}$ *for all* $\lambda \in \mathbb{C}$.

II. The following are equivalent :

(i) The pair (C, A) *is completely observable.*

(ii) $\mathcal{K}_{\mathsf{C}} \cap \mathcal{K}_{(\lambda \mathsf{I} - \mathsf{A})} = \{0\}$ *for all* $\lambda \in \mathbb{C}$.

With this result in mind, all $\lambda \in \mathbb{C}$ such that $\mathcal{K}_{\mathsf{C}} \cap \mathcal{K}_{(\lambda \mathsf{I} - \mathsf{A})} \neq \{0\}$ are called unobservable modes of the pair (C, A). Similarly, all $\lambda \in \mathbb{C}$ such that $\mathcal{K}_{\mathsf{B}^*} \cap \mathcal{K}_{(\lambda \mathsf{I} - \mathsf{A})^*} \neq \{0\}$ are called unreachable modes of the pair (A, B).

Two concepts closely related to reachability and observability are stabilisability and detectability. The pair (A, B) is said to be stabilisable if there exist a bounded, linear operator $\mathsf{F} : \mathcal{X} \to \mathcal{U}$ such that the spectrum of $(\mathsf{A} + \mathsf{BF}) : \mathcal{X} \to \mathcal{X}$ lies in the open unit disc $\mathbb{D}$. This is often referred to as state-feedback stabilisability. Similarly, the pair (C, A) is said to be detectable if there exists a bounded, linear operator $\mathsf{L} : \mathcal{Y} \to \mathcal{X}$ such that the spectrum of $(\mathsf{A} + \mathsf{LC}) : \mathcal{X} \to \mathcal{X}$ lies in $\mathbb{D}$. The following proposition constitutes a useful characterisation of stabilisability and detectability.

Proposition 2.16 (cf. [ZDG95] for example)

I. The following are equivalent :

(i) The pair (A, B) *is stabilisable.*

(ii) $\mathcal{K}_{\mathsf{B}^*} \cap \mathcal{K}_{(\lambda \mathsf{I} - \mathsf{A})^*} = \{0\}$ *for all* $\lambda \in \mathbb{E} := \mathbb{C} - \mathbb{D}$.

II. The following are equivalent :

(i) The pair (C, A) *is detectable.*

(ii) $\mathcal{K}_{\mathsf{C}} \cap \mathcal{K}_{(\lambda \mathsf{I} - \mathsf{A})} = \{0\}$ *for all* $\lambda \in \mathbb{E}$.

In view of this result and Proposition 2.15, it can be said that if (A, B) is stabilisable then every A-invariant subspace of $\mathcal{X}$ corresponding to an eigenvalue (mode) of A in $\mathbb{E}$ is reachable. Similarly, if (C, A) is detectable then every A-invariant subspace of $\mathcal{X}$ corresponding to an eigenvalue (mode) of A in $\mathbb{E}$ is observable. As with reachability and observability, there is also a duality between stabilisability and detectability. Specifically, (C, A) is detectable if and only if $(\mathsf{A}^*, \mathsf{C}^*)$ is stabilisable.

Recall that a state-space realisation of a system is not unique. With this in mind, two useful canonical forms, due to Kalman, are now presented. Given a state-space realisation $\left[\begin{array}{c|c} \mathsf{A} & \mathsf{B} \\ \hline \mathsf{C} & \mathsf{D} \end{array}\right]$ of a system $\mathsf{P} \in \mathscr{D}_{\mathcal{U}\to\mathcal{Y}}$, suppose that (A,B) is not completely reachable (that is $\mathcal{C}_{\mathsf{A},\mathsf{B}}$ is not the whole of $\mathcal{X}$). Let $\{e_n\}_{n=0}^{r}$ be a basis for $\mathcal{C}_{\mathsf{A},\mathsf{B}}$ and extend this to the whole of $\mathcal{X}$. Since $\mathcal{C}_{\mathsf{A},\mathsf{B}}$ is A-invariant and $\mathcal{R}_{\mathsf{B}} \subset \mathcal{C}_{\mathsf{A},\mathsf{B}}$, the representations of A, B and C in this basis have the following form:

$$\begin{bmatrix} \mathsf{A}_1 & \mathsf{A}_2 \\ 0 & \mathsf{A}_3 \end{bmatrix}, \begin{bmatrix} \mathsf{B}_1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} \mathsf{C}_1 & \mathsf{C}_2 \end{bmatrix},$$

respectively. By the way the representations above are constructed, it is straightforward to see that the pair $(\mathsf{A}_1,\mathsf{B}_1)$ is completely reachable and that $\left[\begin{array}{c|c} \mathsf{A}_1 & \mathsf{B}_1 \\ \hline \mathsf{C}_1 & \mathsf{D} \end{array}\right]$ is a state-space realisation of P. Moreover, (A,B) is stabilisable if and only if $\operatorname{spec}(\mathsf{A}_3) \subset \mathbb{D}$. Similarly, if the pair $(\mathsf{C}_1,\mathsf{A}_1)$ is not completely observable then it is possible to construct a state-space, basis transformation T such that:

$$\mathsf{T}^{-1}\mathsf{A}_1\mathsf{T} = \begin{bmatrix} \mathsf{A}_{11} & \mathsf{A}_{12} \\ 0 & \mathsf{A}_{13} \end{bmatrix}, \mathsf{T}^{-1}\mathsf{B}_1 = \begin{bmatrix} \mathsf{B}_{11} \\ \mathsf{B}_{12} \end{bmatrix}, \mathsf{C}_1\mathsf{T} = \begin{bmatrix} \mathsf{C}_{11} & 0 \end{bmatrix};$$

$(\mathsf{C}_{11},\mathsf{A}_{11})$ is completely observable; and $\left[\begin{array}{c|c} \mathsf{A}_{11} & \mathsf{B}_{11} \\ \hline \mathsf{C}_{11} & \mathsf{D} \end{array}\right]$ is a state-space realisation of P. Furthermore, it follows that $(\mathsf{A}_1,\mathsf{C}_1)$ is detectable if and only if $\operatorname{spec}(\mathsf{A}_{13}) \subset \mathbb{D}$. The realisation $\left[\begin{array}{c|c} \mathsf{A}_{11} & \mathsf{B}_{11} \\ \hline \mathsf{C}_{11} & \mathsf{D} \end{array}\right]$, which is completely reachable and completely observable, is called a <u>minimal</u> realisation.

2.4.2 Stability

Consider the homogeneous component of the system of difference equations (2.4): $\mathsf{x}_{k+1} = \mathsf{A}\mathsf{x}_k$ with $\mathsf{x}_0 = x_o \in \mathcal{X}$. Then the state evolves as $\mathsf{x}_k = \mathsf{A}^k x_o$, which corresponds to a sequence in $\ell^2_{\mathbb{Z}^+}(\mathcal{X})$ if and only if $\operatorname{spec}(\mathsf{A}) \subset \mathbb{D}$. If this is the case, then the homogeneous system is said to be stable. Now given a state-space realisation $\left[\begin{array}{c|c} \mathsf{A} & \mathsf{B} \\ \hline \mathsf{C} & \mathsf{D} \end{array}\right]$ of a system $\mathsf{P} \in \mathscr{D}_{\mathcal{U}\to\mathcal{Y}}$, it can be shown [Vid78] (using the input-output description given in equation (2.5)) that if $\operatorname{spec}(\mathsf{A}) \subset \mathbb{D}$, then P is input-output stable in the sense that $\mathcal{D}_P = \ell^2_{\mathbb{Z}^+}(\mathcal{U})$ and $\|\mathsf{P}\| < \infty$. For the converse to hold however, additional assumptions are required on the state-space realisation considered. If the state-space realisation of P is completely reachable, then every A-invariant subspace (which covers the whole of $\mathcal{X}$) can be reached by an appropriate finite-energy input sequence. That is, every mode of A can be excited. If each of these subspaces is observable (that is if (C,A) is completely observable), then for P to be input-output stable it is necessary that all modes of A be stable (that is, $\operatorname{spec}(\mathsf{A}) \subset \mathbb{D}$). In other words if (A,B) is completely reachable and (C,A) is completely observable then input-output stability implies $\operatorname{spec}(\mathsf{A}) \subset \mathbb{D}$. In fact, in view of the Kalman canonical forms introduced in the previous subsection, the complete reachability and observability conditions can be relaxed to stabilisability and detectability, since under these relaxed conditions, A can be decomposed into a completely controllable and observable part and a part with spectrum contained in $\mathbb{D}$ (that is, a stable part).

An important tool in state-space system-theory is the Lyapunov equation. For discrete-time, state-space systems this is also known as the Stein equation and it takes the following form:

$$\mathsf{X} - \mathsf{A}^*\mathsf{X}\mathsf{A} = \mathsf{Q}.$$

It can be shown that the solution $\mathsf{X} : \mathcal{X} \to \mathcal{X}$ to this equation is unique provided $\lambda_i \overline{\lambda}_j \neq 1$ for every $\lambda_i, \lambda_j \in \operatorname{spec}(\mathsf{A})$ (cf. [ZDG95] for example). The following result is well-known:

Proposition 2.17 ([ZDG95]) *Let* X *be the unique solution to the Stein equation* $\mathsf{X} - \mathsf{A}^*\mathsf{X}\mathsf{A} = \mathsf{C}^*\mathsf{C}$. *Then* $\operatorname{spec}(\mathsf{A}) \subset \mathbb{D}$ *if and only if* $\mathsf{X} \geq 0$ *and* (C, A) *is detectable.*

Proof : ($\Leftarrow$) Suppose A has an eigenvalue $\lambda \in \mathbb{E}$ and let $x \in \mathcal{X}$ denote a corresponding eigenvector. Then

$$\langle x, \mathsf{X}x \rangle_{\mathcal{X}} - \langle x, \mathsf{A}^*\mathsf{X}\mathsf{A}x \rangle_{\mathcal{X}} = (1 - |\lambda|^2)\langle x, \mathsf{X}x \rangle_{\mathcal{X}} = \langle \mathsf{C}x, \mathsf{C}x \rangle_{\mathcal{Y}},$$

which since $\lambda \in \mathbb{E}$ and X is positive-semidefinite, implies that $\mathsf{C}x = 0$. That is, $\lambda \in \mathbb{E}$ must be an unobservable mode of (C, A), which contradicts detectability of (C, A). Hence, $\operatorname{spec}(\mathsf{A}) \subset \mathbb{D}$.

($\Rightarrow$) If $\operatorname{spec}(\mathsf{A}) \subset \mathbb{D}$ then a solution to the Stein equation is given by

$$\mathsf{X} = \sum_{k=0}^{\infty} (\mathsf{C}\mathsf{A}^k)^*\mathsf{C}\mathsf{A}^k \geq 0.$$

Furthermore, $\mathcal{K}_{(\lambda \mathsf{I} - \mathsf{A})} = \{0\}$ for all $\lambda \in \mathbb{E}$ and hence, (C, A) does not have any unobservable modes in $\mathbb{E}$. ∎

It follows by duality that if X is the solution to $\mathsf{X} - \mathsf{A}\mathsf{X}\mathsf{A}^* = \mathsf{B}\mathsf{B}^*$, then $\operatorname{spec}(A) \subset \mathbb{D}$ if and only if $\mathsf{X} \geq 0$ and (A, B) is stabilisable.

2.4.3 Quadratic-Regulator and Full-Information $\ell^2_{\mathbb{Z}^+}(\cdot)$-Induced-Norm Synthesis

In this section the quadratic-regulator and $\ell^2_{\mathbb{Z}^+}(\cdot)$-induced-norm, synthesis problems are considered. The results presented here are fundamental to a computational procedure developed in Chapter 5. Attention is restricted to control strategies having direct access to both the state and disturbance signals. That is, full-information control.[8] The system to be controlled is assumed to admit a state-space realisation with *finite-dimensional* state-space. Specifically,

$$\begin{aligned} x_{k+1} &= Ax_k + B_1 w_k + B_2 u_k; \quad x_0 = x_o, && \text{(2.6a)} \\ z_k &= Cx_k + D_1 w_k + D_2 u_k, && \text{(2.6b)} \end{aligned}$$

where for each fixed $k \in \mathbb{Z}^+$, the state $x_k \in \mathcal{X}$ (a finite-dimensional Hilbert space), the exogenous disturbance $w_k \in \mathcal{W}$, the control input $u_k \in \mathcal{U}$ and the controlled output $z_k \in \mathcal{Z}$ (with $\mathcal{U}, \mathcal{W}, \mathcal{Z}$ possibly infinite-dimensional, Hilbert spaces). The bounded, linear operators A, B_1, B_2, C, D_1 and D_2 are all assumed to be time independent.

In what follows, for $k \in \mathbb{Z}^+$, $x := k \mapsto x_k$, $u := k \mapsto u_k$, $w := k \mapsto w_k$ and $z := k \mapsto z_k$. Now, given an initial state $x_o \in \mathcal{X}$ and an identically-zero, exogenous disturbance signal w (that is, $w_k = 0$ for all $k \in \mathbb{Z}^+$), the objective of quadratic-regulator synthesis is to define a causal, LSI, stabilising control-law $u = Kx$, to minimise $\|z\|_{\ell^2_{\mathbb{Z}^+}(\mathcal{Z})}$.[9] On the other hand, given a zero initial state and some real number $\gamma > 0$, the objective of full-information, $\ell^2_{\mathbb{Z}^+}(\cdot)$-induced-norm synthesis is to define a causal, LSI, stabilising control-law $u = K\begin{bmatrix} x \\ w \end{bmatrix}$, such that with the control-law in place the $\ell^2_{\mathbb{Z}^+}(\cdot)$-induced norm of the system mapping w to z is bounded by γ. Clearly (A, B_2) stabilisable is a necessary condition for existence of the specific controllers defined for both problems and as such, it will be assumed throughout this sub-section.

Before going on to solve the synthesis problems specified above, some preliminary results concerning the discrete-time, algebraic, Riccati equation are required. The theory of discrete-time, algebraic, Riccati equations in optimal control originates in [DL70] and [Kuč72], with subsequent developments by many authors: notably, [PLS80, DGG86, BGPK85, IG91, GL95, IOW97]. The reader is referred to the books [LR95, BLW91] as general and quite comprehensive references.

Consider the following algebraic equation in X:

$$X = A^*XA + Q - (S + B^*XA)^*(R + B^*XB)^{-1}(S + B^*XA), \tag{2.7}$$

where $A \in \mathcal{B}_{\mathcal{X}\to\mathcal{X}}$, $Q^* = Q \in \mathcal{B}_{\mathcal{X}\to\mathcal{X}}$, $S \in \mathcal{B}_{\mathcal{X}\to\mathcal{U}}$, $B \in \mathcal{B}_{\mathcal{U}\to\mathcal{X}}$ and $R^* = R \in \mathcal{B}_{\mathcal{U}\to\mathcal{U}}$. This is known as the (discrete-time) algebraic, Riccati equation. By definition, a solution X must be a bounded, self-adjoint operator such that $R + B^*XB$ is boundedly invertible.

[8] For the quadratic-regulator problem there is no disturbance and hence, the controller only has access to the state.

[9] Here and below, stabilising means that with the control-law in place, $x \in \ell^2_{\mathbb{Z}^+}(\mathcal{X})$ and thus, that $\lim_{k\to\infty} x_k = 0$.

Proposition 2.18 *Let* R *be boundedly invertible and define*

$$\mathsf{A}_1 := \mathsf{A} - \mathsf{B}\mathsf{R}^{-1}\mathsf{S}, \text{ and } \mathsf{Q}_1 := \mathsf{Q} - \mathsf{S}^*\mathsf{R}^{-1}\mathsf{S}.$$

Then the following algebraic (Riccati) equations in $\mathsf{X} : \mathcal{X} \to \mathcal{X}$ *have the same solutions:*

$$\begin{aligned} \mathsf{X} &= \mathsf{A}^*\mathsf{X}\mathsf{A} + \mathsf{Q} - (\mathsf{S} + \mathsf{B}^*\mathsf{X}\mathsf{A})^*(\mathsf{R} + \mathsf{B}^*\mathsf{X}\mathsf{B})^{-1}(\mathsf{S} + \mathsf{B}^*\mathsf{X}\mathsf{A}), \\ \mathsf{X} &= \mathsf{A}_1^*\mathsf{X}\mathsf{A}_1^* + \mathsf{Q}_1 - \mathsf{A}_1^*\mathsf{X}\mathsf{B}(\mathsf{R} + \mathsf{B}^*\mathsf{X}\mathsf{B})^{-1}\mathsf{B}^*\mathsf{X}\mathsf{A}_1, \\ \mathsf{X} &= \mathsf{A}_1^*\mathsf{X}(\mathsf{I} + \mathsf{B}\mathsf{R}^{-1}\mathsf{B}^*\mathsf{X})^{-1}\mathsf{A}_1 + \mathsf{Q}_1. \end{aligned}$$

Note that with $\mathcal{X}$ finite-dimensional, the operators appearing in the last equation above can be grouped together so as to involve only finite-dimensional terms. That is, by defining the finite-dimensional operator $\mathsf{R}_1 := \mathsf{B}\mathsf{R}^{-1}\mathsf{B}^*$, the last equation can be expressed as the finite-dimensional, algebraic, Riccati equation

$$\mathsf{X} = \mathsf{A}_1^*\mathsf{X}(\mathsf{I} + \mathsf{R}_1^{-1}\mathsf{X})^{-1}\mathsf{A}_1 + \mathsf{Q}_1,$$

which given a method for computing $\mathsf{R}_1 : \mathcal{X} \to \mathcal{X}$, $\mathsf{Q}_1 : \mathcal{X} \to \mathcal{X}$ and $\mathsf{A}_1 : \mathcal{X} \to \mathcal{X}$, can be solved using standard finite-dimensional tools.

In the following proposition, what is called a stabilising solution to the algebraic, Riccati equation (2.7) is characterised. As part of this, it is necessary to assume that $\mathsf{R} > 0$. Under this assumption it is said that the algebraic, Riccati equation is sign definite (in fact, in this case positive definite).

Remark 2.19 Note that with $\mathsf{R} > 0$ there exists by definition an $\alpha > 0$ such that

$$\alpha^2\langle u, u\rangle_{\mathcal{U}}^2 \leq \langle u, \mathsf{R}u\rangle_{\mathcal{U}}^2 \leq \langle u, u\rangle_{\mathcal{U}}\langle \mathsf{R}u, \mathsf{R}u\rangle_{\mathcal{U}} \text{ for all } u \in \mathcal{U},$$

where the second inequality is called the Schwartz inequality. Hence, $\|\mathsf{R}\mathsf{u}\|_{\mathcal{U}} \geq \alpha^2\|\mathsf{u}\|_{\mathcal{U}}$ and since R is self-adjoint, it follows by [Kre89b, Theorem 9.1-2] that $0 \notin \mathrm{spec}(\mathsf{R})$. Consequently, R^{-1} is bounded and defined on a set dense in $\mathcal{U}$. In fact R^{-1} is defined on the whole of $\mathcal{U}$. Similarly, if R were negative definite, R^{-1} would be bounded and defined on the whole of $\mathcal{U}$. It should be noted that this is a non-trivial fact when $\mathcal{U}$ is infinite-dimensional. ♣

Proposition 2.20 *Suppose that* $\mathsf{R} > 0$ *and*

$$\begin{bmatrix} \mathsf{Q} & \mathsf{S}^* \\ \mathsf{S} & \mathsf{R} \end{bmatrix} \geq 0.$$

Then, there exist a self-adjoint, positive-semidefinite solution X *to the (discrete-time) algebraic, Riccati equation (2.7), such that with* $\mathsf{F} := (\mathsf{R} + \mathsf{B}^*\mathsf{X}\mathsf{B})^{-1}(\mathsf{S} + \mathsf{B}^*\mathsf{X}\mathsf{A})$, $\mathrm{spec}(\mathsf{A} - \mathsf{B}\mathsf{F}) \subset \mathbb{D}$ *if and only if* (A, B) *is stabilisable and* $(\mathsf{Q}_1, \mathsf{A}_1)$ *has no unobservable modes on* $\mathbb{T}$.

Proof : For a proof see Appendix A.1. ∎

This result is fundamental to the solution of both the quadratic-regulator and full-information, $\ell^2_{\mathbb{Z}^+}(\cdot)$-induced-norm, synthesis problems discussed below.

State-Feedback Quadratic-Regulator Synthesis

Consider the component of the state-space system (2.6), mapping u to z according to

$$\begin{aligned} \mathsf{x}_{k+1} &= \mathsf{A}\mathsf{x}_k + \mathsf{B}_2\mathsf{u}_k; \quad \mathsf{x}_0 = x_o, \\ \mathsf{z}_k &= \mathsf{C}\mathsf{x}_k + \mathsf{D}_2\mathsf{u}_k, \end{aligned}$$

under the standing assumption $\mathsf{D}_2^*\mathsf{D}_2 > 0$. Given an initial state $x_o \in \mathcal{X}$, recall that the objective of quadratic-regulator synthesis is to find a causal, LSI, stabilising control-law $\mathsf{u} = \mathsf{K}\mathsf{x}$, to minimise $\|\mathsf{z}\|_{\ell^2_{\mathbb{Z}^+}(\mathcal{Z})}$. Since the control-law is restricted to be causal and dependent only on the state, it follows that for each $k \in \mathbb{Z}^+$, $\|(\mathrm{I} - \mathbf{P}_{k-1})\mathsf{z}\|_{\ell^2_{\mathbb{Z}^+}(\mathcal{Z})}$ is dependent only on x_k when an admissible control-law is in place. With this in mind, the self-adjoint, positive-semidefinite operator $\mathsf{X}_2 : \mathcal{X} \to \mathcal{X}$ is defined to satisfy $\langle \mathsf{x}_k, \mathsf{X}_2\mathsf{x}_k\rangle_{\mathcal{X}} = \|(\mathrm{I}-\mathbf{P}_{k-1})\mathsf{z}^\star\|_{\ell^2_{\mathbb{Z}^+}(\mathcal{Z})}$, where $\mathsf{z}^\star$ denotes the output with the optimal control-law in place. In this way $\|\mathsf{z}^\star\|_{\ell^2_{\mathbb{Z}^+}(\mathcal{Z})} = \langle x_o, \mathsf{X}_2 x_o\rangle_{\mathcal{X}}$.

In dynamic programming terms, at each time $k \in \mathbb{Z}^+$, u_k is a decision variable and

$$\langle \mathsf{z}_k, \mathsf{z}_k\rangle_{\mathcal{Z}} + \langle \mathsf{x}_{k+1}, \mathsf{X}_2\mathsf{x}_{k+1}\rangle_{\mathcal{X}}$$

is the cost at stage k. By the principle of dynamic programming, which states that a sequence $\{\mathsf{u}_k\}_{k=0}^{\infty}$ of decisions is optimal for a problem defined over the horizon $[0,\infty)$ if and only if for all $N \in \mathbb{Z}^+$ the control sequence $\{\mathsf{u}_k\}_{k=N}^{\infty}$ is optimal for the problem considered over the sub-horizon $[N,\infty)$ [Bel62], it follows that

$$\langle \mathsf{x}_k, \mathsf{X}_2\mathsf{x}_k\rangle_{\mathcal{X}} = \min_{\mathsf{u}_k \in \mathcal{U}} \{\langle \mathsf{z}_k, \mathsf{z}_k\rangle_{\mathcal{Z}} + \langle \mathsf{x}_{k+1}, \mathsf{X}_2\mathsf{x}_{k+1}\rangle_{\mathcal{X}}\}. \tag{2.8}$$

This is known as Isaac's equation and is a special form of the Hamilton-Jacobi-Bellman equation. Now note that for all $k \in \mathbb{Z}^+$,

$$\begin{aligned} &\langle \mathsf{z}_k, \mathsf{z}_k\rangle_{\mathcal{Z}} + \langle \mathsf{x}_{k+1}, \mathsf{X}_2\mathsf{x}_{k+1}\rangle_{\mathcal{X}} \\ &= \left\langle \begin{bmatrix} \mathsf{x}_k \\ \mathsf{u}_k \end{bmatrix}, \begin{bmatrix} \mathsf{A}^* & \mathsf{C}^* \\ \mathsf{B}_2^* & \mathsf{D}_2^* \end{bmatrix} \begin{bmatrix} \mathsf{X}_2 & 0 \\ 0 & \mathrm{I} \end{bmatrix} \begin{bmatrix} \mathsf{A} & \mathsf{B}_2 \\ \mathsf{C} & \mathsf{D}_2 \end{bmatrix} \begin{bmatrix} \mathsf{x}_k \\ \mathsf{u}_k \end{bmatrix} \right\rangle_{\mathcal{X}\oplus\mathcal{U}} \\ &= \left\langle \begin{bmatrix} \mathsf{x}_k \\ \mathsf{u}_k \end{bmatrix}, \begin{bmatrix} \mathrm{I} & \mathsf{F}_2^* \\ 0 & \mathrm{I} \end{bmatrix} \begin{bmatrix} \mathsf{A}^*\mathsf{X}_2\mathsf{A} + \mathsf{C}^*\mathsf{C} - \mathsf{F}_2^*\mathsf{R}_2\mathsf{F}_2 & 0 \\ 0 & \mathsf{R}_2 \end{bmatrix} \begin{bmatrix} \mathrm{I} & 0 \\ \mathsf{F}_2 & \mathrm{I} \end{bmatrix} \begin{bmatrix} \mathsf{x}_k \\ \mathsf{u}_k \end{bmatrix} \right\rangle_{\mathcal{X}\oplus\mathcal{U}} \\ &= \left\langle \begin{bmatrix} \mathsf{x}_k \\ \mathsf{u}_k + \mathsf{F}_2\mathsf{x}_k \end{bmatrix}, \begin{bmatrix} \mathsf{A}^*\mathsf{X}_2\mathsf{A} + \mathsf{C}^*\mathsf{C} - \mathsf{F}_2^*\mathsf{R}_2\mathsf{F}_2 & 0 \\ 0 & \mathsf{R}_2 \end{bmatrix} \begin{bmatrix} \mathsf{x}_k \\ \mathsf{u}_k + \mathsf{F}_2\mathsf{x}_k \end{bmatrix} \right\rangle_{\mathcal{X}\oplus\mathcal{U}} \\ &= \langle \mathsf{x}_k, (\mathsf{A}^*\mathsf{X}_2\mathsf{A} + \mathsf{C}^*\mathsf{C} - \mathsf{F}_2^*\mathsf{R}_2\mathsf{F}_2)\mathsf{x}_k\rangle_{\mathcal{X}} + \langle \mathsf{u}_k + \mathsf{F}_2\mathsf{x}_k, \mathsf{R}_2(\mathsf{u}_k + \mathsf{F}_2\mathsf{x}_k)\rangle_{\mathcal{U}}, \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} \mathsf{F}_2 &:= \mathsf{R}_2^{-1}(\mathsf{D}_2^*\mathsf{C} + \mathsf{B}_2^*\mathsf{X}_2\mathsf{A}) : \mathcal{X} \to \mathcal{U}, \\ \mathsf{R}_2 &:= (\mathsf{D}_2^*\mathsf{D}_2 + \mathsf{B}_2^*\mathsf{X}_2\mathsf{B}_2) : \mathcal{U} \to \mathcal{U}. \end{aligned}$$

Then since $\mathsf{R}_2 > 0$, it can be seen that the minimum over $\mathsf{u}_k \in \mathcal{U}$ of equation (2.9) is achieved uniquely by $\mathsf{u}_k = -\mathsf{F}_2\mathsf{x}_k$. Furthermore, it follows from equation (2.8), that X_2 must satisfy

$$\mathsf{X}_2 = \mathsf{A}^*\mathsf{X}_2\mathsf{A} + \mathsf{C}^*\mathsf{C} - \mathsf{F}_2^*\mathsf{R}_2\mathsf{F}_2. \tag{2.10}$$

So if $\mathsf{u}_k = -\mathsf{F}_2\mathsf{x}_k$ is stabilising, the synthesis problem is solved.

Since $(\mathrm{I} - \mathsf{D}_2(\mathsf{D}_2^*\mathsf{D}_2)^{-1}\mathsf{D}_2)$ is self-adjoint and idempotent it is a projection and hence, $(\mathrm{I} - \mathsf{D}_2(\mathsf{D}_2^*\mathsf{D}_2)^{-1}\mathsf{D}_2) \geq 0$ (cf. [Kre89b]). It then follows that

$$\begin{bmatrix} \mathsf{C}^*\mathsf{C} & \mathsf{C}^*\mathsf{D}_2 \\ \mathsf{D}_2^*\mathsf{C} & \mathsf{D}_2^*\mathsf{D}_2 \end{bmatrix} \geq 0.$$

Thus, by Proposition 2.20, $(\mathsf{A}, \mathsf{B}_2)$ stabilisable and $(\mathsf{C}^*(\mathrm{I}-\mathsf{D}_2(\mathsf{D}_2^*\mathsf{D}_2)^{-1}\mathsf{D}_2^*)\mathsf{C}, \mathsf{A}-\mathsf{B}_2(\mathsf{D}_2^*\mathsf{D}_2)^{-1}\mathsf{D}_2^*\mathsf{C})$ having no unobservable modes on $\mathbb{T}$, is necessary and sufficient for the existence of a self-adjoint, positive-semidefinite solution X_2 to equation (2.10) such that $\mathrm{spec}(\mathsf{A} - \mathsf{B}\mathsf{F}_2) \subset \mathbb{D}$. That is, under these assumptions the control-law $\mathsf{u}_k = -\mathsf{F}_2\mathsf{x}_k$ is stabilising. This is summarised in the following theorem.

Proposition 2.21 *Assume that* $\mathsf{D}_2^*\mathsf{D}_2 > 0$, $(\mathsf{A}, \mathsf{B}_2)$ *is stabilisable and*

$$\mathcal{K}\begin{bmatrix} \mathsf{A} - e^{j\theta}\mathrm{I} & \mathsf{B}_2 \\ \mathsf{C} & \mathsf{D}_2 \end{bmatrix} = \{0\} \tag{2.11}$$

for all $\theta \in [0, 2\pi)$. *Then the unique, optimal, causal, LSI, stabilising control-law for the quadratic-regulator synthesis problem is* $\mathsf{u}_k = -\mathsf{F}_2\mathsf{x}_k$, *where*

$$\begin{aligned} \mathsf{F}_2 &:= \mathsf{R}_2^{-1}(\mathsf{D}_2^*\mathsf{C} + \mathsf{B}_2^*\mathsf{X}_2\mathsf{A}) : \mathcal{X} \to \mathcal{X}, \\ \mathsf{R}_2 &:= (\mathsf{D}_2^*\mathsf{D}_2 + \mathsf{B}_2^*\mathsf{X}_2\mathsf{B}_2) : \mathcal{U} \to \mathcal{U} \end{aligned}$$

and the self-adjoint, positive-semidefinite operator $\mathsf{X}_2 : \mathcal{X} \to \mathcal{X}$ *satisfies the (discrete-time) algebraic, Riccati equation (2.10).*

Proof : The proof follows from the discussion above by noting that the following are equivalent:

(i) $(\mathsf{C}^*(\mathrm{I} - \mathsf{D}_2(\mathsf{D}_2^*\mathsf{D}_2)^{-1}\mathsf{D}_2^*)\mathsf{C}, \mathsf{A} - \mathsf{B}_2(\mathsf{D}_2^*\mathsf{D}_2)^{-1}\mathsf{D}_2^*\mathsf{C})$ has no unobservable on $\mathbb{T}$;

(ii)

$$\mathcal{K}\begin{bmatrix} \mathsf{A} - e^{j\theta}\mathsf{I} & \mathsf{B}_2 \\ \mathsf{C} & \mathsf{D}_2 \end{bmatrix} = \{0\}$$

for all $\theta \in [0, 2\pi)$.

To see this suppose that (i) holds and that there exist a non-zero $\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{X} \oplus \mathcal{U}$ such that

$$\begin{bmatrix} \mathsf{A} - e^{j\theta} & \mathsf{B}_2 \\ \mathsf{C} & \mathsf{D}_2 \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} = 0,$$

for some $\theta \in [0, 2\pi)$. It then follows that $u = -(\mathsf{D}_2^*\mathsf{D}_2)^{-1}\mathsf{D}_2^*\mathsf{C}x$ and hence, that $\mathsf{A}x + \mathsf{B}_2 u = (\mathsf{A} - \mathsf{B}_2(\mathsf{D}_2^*\mathsf{D}_2)^{-1}\mathsf{D}_2^*\mathsf{C})x = e^{j\theta}x$ and $\mathsf{C}^*(\mathsf{I} - \mathsf{D}_2(\mathsf{D}_2^*\mathsf{D}_2)^{-1}\mathsf{D}_2^*)\mathsf{C}x = 0$. This contradicts (i) and hence, (ii) must hold.

To see the converse, suppose that (ii) holds and that there exists a non-zero $x \in \mathcal{X}$ and a $\theta \in [0, 2\pi)$ such that $(\mathsf{A} - \mathsf{B}_2(\mathsf{D}_2^*\mathsf{D}_2)^{-1}\mathsf{D}_2^*\mathsf{C})x = e^{j\theta}x$ and $\mathsf{C}^*(\mathsf{I} - \mathsf{D}_2(\mathsf{D}_2^*\mathsf{D}_2)^{-1}\mathsf{D}_2^*)\mathsf{C}x = 0$. Now note that $(\mathsf{I} - \mathsf{D}_2(\mathsf{D}_2^*\mathsf{D}_2)^{-1}\mathsf{D}_2^*)$ is self-adjoint and idempotent. Hence,

$$\mathsf{C}^*(\mathsf{I} - \mathsf{D}_2(\mathsf{D}_2^*\mathsf{D}_2)^{-1}\mathsf{D}_2^*)(\mathsf{I} - \mathsf{D}_2(\mathsf{D}_2^*\mathsf{D}_2)^{-1}\mathsf{D}_2^*)\mathsf{C}x = 0,$$

which implies that $(\mathsf{I} - \mathsf{D}_2(\mathsf{D}_2^*\mathsf{D}_2)^{-1}\mathsf{D}_2^*)\mathsf{C}x = 0$. It follows then, that

$$\begin{bmatrix} \mathsf{A} - e^{j\theta}\mathsf{I} & \mathsf{B}_2 \\ \mathsf{C} & \mathsf{D}_2 \end{bmatrix}\begin{bmatrix} \mathsf{I} & 0 \\ (\mathsf{D}_2^*\mathsf{D}_2)^{-1}\mathsf{D}_2^*\mathsf{C} & \mathsf{I} \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix} = 0,$$

which contradicts (ii) and hence, (i) must hold. This completes the proof ∎

Full-Information $\ell^2_{\mathbb{Z}+}(\cdot)$-Induced-Norm Synthesis

Consider the system of difference equations (2.6) re-written as

$$\begin{bmatrix} \mathsf{x}_{k+1} \\ \mathsf{z}_k \end{bmatrix} = \begin{bmatrix} \mathsf{A} & \mathsf{B}_1 & \mathsf{B}_2 \\ \mathsf{C} & \mathsf{D}_1 & \mathsf{D}_2 \end{bmatrix}\begin{bmatrix} \mathsf{x}_k \\ \mathsf{w}_k \\ \mathsf{u}_k \end{bmatrix}; \quad \mathsf{x}_0 = x_o,$$

under the standing assumption $\mathsf{D}_2^*\mathsf{D}_2 > 0$. Now recall that given a real number $\gamma > 0$, the objective of full-information $\ell^2_{\mathbb{Z}+}(\cdot)$-synthesis is to obtain a causal, LSI, stabilising control-law $\mathsf{u} = \mathsf{K}\begin{bmatrix} \mathsf{x} \\ \mathsf{w} \end{bmatrix}$, such that with it in place and with $x_o = 0$, the $\ell^2_{\mathbb{Z}+}(\cdot)$-induced norm of the operator mapping w to

z is bounded by γ. Note that when $x_o = 0$, the $\ell^2_{\mathbb{Z}+}(\cdot)$-induced norm of the mapping from w to z is strictly less than γ if and only if

$$\|z\|^2_{\ell^2_{\mathbb{Z}+}(\mathcal{Z})} - \gamma^2\|w\|^2_{\ell^2_{\mathbb{Z}+}(\mathcal{Z})} \leq -\epsilon\|w\|^2_{\ell^2_{\mathbb{Z}+}(\mathcal{W})}$$

for some $\epsilon > 0$ and all $w \in \ell^2_{\mathbb{Z}+}(\mathcal{W})$. As such, the following theorem characterises a solution to the full-information, $\ell^2_{\mathbb{Z}+}(\cdot)$-synthesis problem.

Proposition 2.22 *Let* $J := \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$ *and assume that* $D_2^* D_2 > 0$, (A, B_2) *is stabilisable and for all* $\theta \in [0, 2\pi)$

$$\mathcal{K}_{\begin{bmatrix} A - e^{j\theta}I & B_2 \\ C & D_2 \end{bmatrix}} = \{0\}.$$

Then there exists a causal, LSI, stabilising control-law of the form $u = K\begin{bmatrix} x \\ w \end{bmatrix}$*, such that given a real number* $\gamma > 0$,

$$\|z\|^2_{\ell^2_{\mathbb{Z}+}(\mathcal{Z})} - \gamma^2\|w\|^2_{\ell^2_{\mathbb{Z}+}(\mathcal{Z})} \leq -\epsilon\|w\|^2_{\ell^2_{\mathbb{Z}+}(\mathcal{Z})} \tag{2.12}$$

for all $w \in \ell^2_{\mathbb{Z}+}(\mathcal{W})$ *and some* $\epsilon > 0$ *when* $x_o = 0$*, if and only if the (discrete-time) algebraic, Riccati equation*

$$X_\infty = A^* X_\infty A + \tilde{C}^* J \tilde{C} - L^* Z^{-1} L, \tag{2.13}$$

has self-adjoint, positive-semidefinite solution $X_\infty : \mathcal{X} \to \mathcal{X}$ *such that* $\operatorname{spec}(A - BZ^{-1}L) \subset \mathbb{D}$ *and*

$$\nabla := Z_1 - Z_2^* Z_3^{-1} Z_2 < 0,$$

where

$$Z := \tilde{D}^* J \tilde{D} + B^* X_\infty B =: \begin{bmatrix} Z_1 & Z_2^* \\ Z_2 & Z_3 \end{bmatrix} : \mathcal{W} \oplus \mathcal{U} \to \mathcal{W} \oplus \mathcal{U}, \tag{2.14}$$

$$L := \tilde{D}^* J \tilde{C} + B^* X_\infty A =: \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} : \mathcal{X} \to \mathcal{W} \oplus \mathcal{U}, \tag{2.15}$$

$$\tilde{D} := \begin{bmatrix} D_1 & D_2 \\ I & 0 \end{bmatrix}, \qquad \tilde{C} := \begin{bmatrix} C \\ 0 \end{bmatrix} \qquad \text{and} \qquad B := \begin{bmatrix} B_1 & B_2 \end{bmatrix}.$$

In this case a causal, LSI, stabilising, full-information control-law achieving the objective is given by

$$u_k^\star = Z_3^{-1}\begin{bmatrix} L_2 & Z_2 \end{bmatrix}\begin{bmatrix} x_k \\ w_k \end{bmatrix}.$$

Proof : The proof of this result follows very closely that of Green and Limebeer [GL95, Appendix B.], with essentially only notational modifications to accommodate the possibility of infinite-dimensional input/output spaces. For completeness it is presented in Appendix A.2. ∎

To complete this section it is shown that for a solution to the full-information, $\ell^2_{\mathbb{Z}+}(\cdot)$-synthesis problem to exist it is necessary that $\tilde{\mathsf{D}}^*\mathsf{J}\tilde{\mathsf{D}} : \mathcal{W} \oplus \mathcal{U} \to \mathcal{W} \oplus \mathcal{U}$ be a boundedly invertible operator. In this way, it is possible to apply Proposition 2.18, by which the algebraic, Riccati equation (2.13) can be re-written in terms of finite-dimensional operators (in that they map the finite-dimensional space $\mathcal{X}$ to $\mathcal{X}$) only. That is, equation (2.13) can be re-written as the standard, finite-dimensional, algebraic, Riccati equation

$$\mathsf{X}_\infty = \grave{\mathsf{A}}^*\mathsf{X}_\infty(\mathsf{I} + \grave{\mathsf{R}}\mathsf{X}_\infty)^{-1}\grave{\mathsf{A}} + \grave{\mathsf{Q}},$$

where $\grave{\mathsf{A}} :=(\mathsf{A} - \mathsf{B}(\tilde{\mathsf{D}}^*\mathsf{J}\tilde{\mathsf{D}})^{-1}\tilde{\mathsf{D}}^*\mathsf{J}\tilde{\mathsf{C}}) : \mathcal{X} \to \mathcal{X}$,

$$\grave{\mathsf{R}} :=\mathsf{B}(\tilde{\mathsf{D}}^*\mathsf{J}\tilde{\mathsf{D}})^{-1}\mathsf{B}^* : \mathcal{X} \to \mathcal{X}$$

and

$$\grave{\mathsf{Q}} :=\tilde{\mathsf{C}}^*(\mathsf{I} - \tilde{\mathsf{D}}(\tilde{\mathsf{D}}^*\mathsf{J}\tilde{\mathsf{D}})^{-1}\tilde{\mathsf{D}}^*)\tilde{\mathsf{C}} : \mathcal{X} \to \mathcal{X}.$$

Lemma 2.23 *If $\mathsf{D}_2^*\mathsf{D}_2 > 0$ and there exists a causal, LSI, stabilising control-law of the form $u = \mathsf{K}\begin{bmatrix} x \\ w \end{bmatrix}$, such that given a $\gamma > 0$*

$$\|z\|^2_{\ell^2_{\mathbb{Z}+}(\mathcal{Z})} - \gamma^2\|w\|^2_{\ell^2_{\mathbb{Z}+}(\mathcal{Z})} \leq -\epsilon\|w\|^2_{\ell^2_{\mathbb{Z}+}(\mathcal{Z})}$$

for all $w \in \ell^2_{\mathbb{Z}+}(\mathcal{W})$ and some $\epsilon > 0$ when $x_o = 0$, then the operator $\tilde{\mathsf{D}}^*\mathsf{J}\tilde{\mathsf{D}} : \mathcal{W} \oplus \mathcal{U} \to \mathcal{W} \oplus \mathcal{U}$ is boundedly invertible.

Proof : Note that

$$\tilde{\mathsf{D}}^*\mathsf{J}\tilde{\mathsf{D}} = \begin{bmatrix} \mathsf{D}_1^*\mathsf{D}_1 - \gamma^2\mathsf{I} & \mathsf{D}_1^*\mathsf{D}_2 \\ \mathsf{D}_2^*\mathsf{D}_1 & \mathsf{D}_2^*\mathsf{D}_2 \end{bmatrix}.$$

Since $\mathsf{D}_2^*\mathsf{D}_2 > 0$, it follows as discussed in Remark 2.19, that $\mathsf{D}_2^*\mathsf{D}_2$ is boundedly invertible and as such, it is possible to write

$$\tilde{\mathsf{D}}^*\mathsf{J}\tilde{\mathsf{D}} = \begin{bmatrix} \mathsf{I} & \mathsf{D}_1^*\mathsf{D}_2(\mathsf{D}_2^*\mathsf{D}_2)^{-1} \\ 0 & \mathsf{I} \end{bmatrix}\begin{bmatrix} \Delta & 0 \\ 0 & \mathsf{D}_2^*\mathsf{D}_2 \end{bmatrix}\begin{bmatrix} \mathsf{I} & 0 \\ (\mathsf{D}_2^*\mathsf{D}_2)^{-1}\mathsf{D}_2^*\mathsf{D}_1 & \mathsf{I} \end{bmatrix}, \tag{2.16}$$

where $\Delta = \mathsf{D}_1^*(\mathsf{I} - \mathsf{D}_2(\mathsf{D}_2^*\mathsf{D}_2)^{-1}\mathsf{D}_2^*)\mathsf{D}_1 - \gamma^2\mathsf{I}$. Now suppose that a stabilising, full-information control-law that satisfies (2.12) is in place and set

$$w_k = \begin{cases} 0, & i \neq N \\ \tilde{w}, & i = N \end{cases}$$

for some $N \in \mathbb{Z}^+$ and $\tilde{w} \in \mathcal{W}$. Then since the control-law is causal and $x_o = 0$, $x_k = 0$ for $k \leq N$, $z_k = 0$ for $k < N$ and

$$\begin{aligned}
-\epsilon\langle \tilde{w}, \tilde{w}\rangle_{\mathcal{W}} &\geq \langle z_N, z_N\rangle_{\mathcal{Z}} - \gamma^2\langle w_N, w_N\rangle_{\mathcal{W}} + \sum_{k=N+1}^{\infty} \langle z_k, z_k\rangle_{\mathcal{Z}} \\
&\geq \langle z_N, z_N\rangle_{\mathcal{Z}} - \gamma^2\langle w_N, w_N\rangle_{\mathcal{W}} \\
&= \left\langle \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} x_N \\ \begin{bmatrix} w_N \\ u_N \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} x_N \\ \begin{bmatrix} w_N \\ u_N \end{bmatrix} \end{bmatrix} \right\rangle_{\mathcal{Z}\oplus\mathcal{W}} \\
&= \left\langle \begin{bmatrix} w_N \\ u_N \end{bmatrix}, \tilde{D}^*J\tilde{D} \begin{bmatrix} w_N \\ u_N \end{bmatrix} \right\rangle_{\mathcal{W}\oplus\mathcal{U}} \qquad \text{(since } x_N = 0\text{)} \\
&= \langle \tilde{w}, \Delta\tilde{w}\rangle_{\mathcal{W}} + \left\langle (u + (D_2^*D_2)^{-1}D_2^*D_1\tilde{w}), D_2^*D_2(u + (D_2^*D_2)^{-1}D_2^*D_1\tilde{w}) \right\rangle_{\mathcal{U}} \\
&\geq \langle \tilde{w}, \Delta\tilde{w}\rangle_{\mathcal{W}} \qquad \text{(since } D_2^*D_2 > 0.\text{)}
\end{aligned}$$

Since $\tilde{w} \in \mathcal{W}$ is arbitrary, it then follows that $\Delta < 0$ and hence, as discussed in Remark 2.19, that Δ is boundedly invertible. As such, it is possible to conclude from equation (2.16), that $\tilde{D}^*J\tilde{D}$ is boundedly invertible. ∎

2.5 SD Control-Systems and State-Space Time-Lifting

The purpose of this section is to illustrate the idea of time-lifting in the context of state-space systems and to collect some results concerning stability and induced-norm analysis of SD control-systems expressed in linear-fractional form. For further details the reader is referred to [FG88, CF90, CF91, BP92].

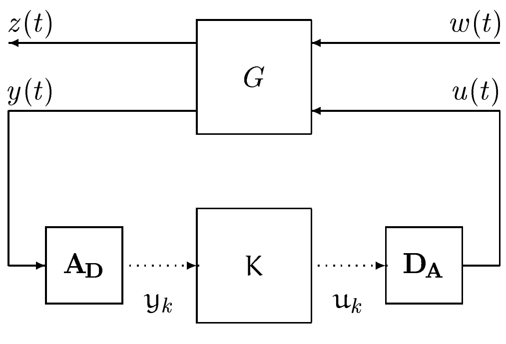

Figure 2.2 *SD Control System*

The lower, linear-fractional configuration shown in Figure 2.2, is a convenient setup for the synthesis of (SD) control-systems. $G =: \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ is called the generalised plant and it is typically constructed from a system that is to be controlled and parametric weights to reflect closed-loop objectives. For example, consider Figure 2.1 with the controller C removed. Then the system

$$\begin{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\ y_2 \end{bmatrix} \mapsto \begin{bmatrix} \begin{bmatrix} u_1 \\ y_1 \end{bmatrix} \\ u_2 \end{bmatrix},$$

is an example of a generalised plant that allows the standard closed-loop configuration to be expressed in lower, linear-fractional form. The closed-loop operator mapping w to z in Figure 2.2, is often denoted by the lower, linear-fractional transformation (LFT) $\mathbf{F}_\ell(G, K)$, defined by

$$\mathbf{F}_\ell(G, K) := G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21},$$

where $K := \mathbf{D_A} \mathsf{K} \mathbf{A_D}$. Typically, in an LFT framework, the objective of synthesis is to obtain a *stabilising* controller K such that some quantifiable criteria expressible in terms of $\mathbf{F}_\ell(G, K)$ is satisfied (for example, $\|\mathbf{F}_\ell(G, K)\| < \gamma$ for some $\gamma > 0$). Important issues concerning synthesis in this framework are presented in the rest of this section, where they are discussed in the context of SD control-systems (with an induced-norm synthesis criteria) and in terms of systems that admit state-space realisations.

Let $G =: \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ in Figure 2.2 be an LTI, continuous-time generalised-plant expressible in terms of the state-space system of differential equations

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t); \quad x(0) = 0, \tag{2.17a}$$

$$z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t), \tag{2.17b}$$

$$y(t) = C_2 x(t), \tag{2.17c}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^{m_1}$ is the disturbance input, $u(t) \in \mathbb{R}^{m_2}$ is the control input, $z(t) \in \mathbb{R}^{p_1}$ is the controlled output, $y(t) \in \mathbb{R}^{p_2}$ is the measured output and A, $B_\bullet$, $C_\bullet$ and $D_\bullet$ are all constant matrices, where "$\bullet$" denotes either 1 or 2 here. Since this state-space realisation of the generalised plant is (implicitly) used for synthesis in the sequel, the standard (cf. [ZDG95]) assumption that the pair (A, B_2) is stabilisable and (C_2, A) is detectable in the sense that $\mathcal{K}_{B_2^*} \cap \mathcal{K}_{(\lambda I - A)^*} = \{0\}$ and $\mathcal{K}_{C_2} \cap \mathcal{K}_{(\lambda I - A)} = \{0\}$ for all $\lambda \in \{(\sigma + j\omega) \in \mathbb{C} \, : \, \sigma \in \mathbb{R}^+, \omega \in \mathbb{R}\}$, is made. Let the controller K be a LSI, discrete-time system with state-space realisation, $\mathbf{A_D}$ denote an ideal, periodic (with period h) sampling-device and $\mathbf{D_A}$ a zeroth-order hold-device, which is assumed to be synchronised with $\mathbf{A_D}$. It is well-known [CF91], that the zero-order hold device $\mathbf{D_A}$ is bounded on $\ell^2_{\mathbb{Z}+}(\cdot)$ with unit norm, but that the ideal sampling device $\mathbf{A_D}$ is not bounded on the whole of $\mathrm{L}^2_{\mathbb{R}+}(\cdot)$. However, given any stable, low-pass, LTI filter F say, it can be shown that $\mathbf{A_D}F$ is bounded on $\mathrm{L}^2_{\mathbb{R}^+}(\cdot)$ [CF91]. As such, a sampling device is invariably preceded by a stable, strictly-causal, LTI filter, which is used to ensure boundedness of $\mathbf{A_D}$ and to limit frequency-domain aliasing of measured information. This is not shown in Figure 2.2, because it is assumed that such a filter has already been absorbed into the generalised plant, which explains why no direct feedthrough terms from the disturbance and control inputs to the measured output appear in the state-space realisation of the generalised plant given in equations (2.17).

Analysis and synthesis of SD control-systems (cf. Figure 2.2) is complicated by the fact that the measured-output and control-input signals as "seen" by K evolve in discrete-time, whereas the disturbance and controlled outputs, by which performance is often determined, evolve in continuous-time. Compounding this, since it is assumed that the sampling and hold devices are synchronised and periodic, the closed-loop system mapping the disturbance input w to the controlled output z is LPTV. These difficulties can be alleviated however, by characterising the evolution of the generalised-plant in terms of a system of LSI, difference equations, so that tools similar to those presented in Section 2.4 for LSI, state-space systems can be used. It is important that loss of continuous-time information does not occur in this process. As such, to capture all disturbance-input information and controlled-output information in discrete-time, it is necessary to use signals that take infinite-dimensional values at each point in time. To this end, the time-lifting isomorphism defined in Sub-Section 2.3.1 as

$$\vec{\mathsf{f}}_k(\theta) := (\mathbf{W}f)_k := f(kh+\theta) \quad (\theta \in \mathbb{H}),$$

is used to map the continuous-time signals to discrete-time signals. Using the time-lifting isomorphism also overcomes the difficulty imposed by periodic time-variation, since it intertwines the continuous-time and discrete-time unilateral shift as follows: $\mathbf{W}\mathbf{U}_{kh} = \mathbf{S}^k\mathbf{W}$ for all $k \in \mathbb{Z}^+$ (cf. Proposition 2.4). Now, absorbing the sampling and hold devices into G and expressing the evolution of this system in terms of the state at the sampling instants kh ($k \in \mathbb{Z}^+$), yields the LSI system of difference equations

$$\mathsf{x}_{k+1} = \mathsf{A}\mathsf{x}_k + \mathsf{B}_1\vec{\mathsf{w}}_k + \mathsf{B}_2\mathsf{u}_k; \qquad \mathsf{x}_0 = 0, \tag{2.18a}$$

$$\vec{\mathsf{z}}_k = \mathsf{C}_1\mathsf{x}_k + \mathsf{D}_{11}\vec{\mathsf{w}}_k + \mathsf{D}_{12}\mathsf{u}_k, \tag{2.18b}$$

$$\mathsf{y}_k = \mathsf{C}_2\mathsf{x}_k, \tag{2.18c}$$

where $\mathsf{x}_k = x(kh)$, $\vec{z}_k(\theta) = (\mathbf{W}z)_k(\theta) = z(kh+\theta)$, $\vec{w}_k(\theta) = (\mathbf{W}w)_k(\theta) = w(kh+\theta)$, $\mathsf{y}_k = (\mathbf{A_D}y)_k = y(kh)$ and for all $\vec{w} \in \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{m_1})$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^{m_2}$,

$$\mathsf{A} : \mathbb{R}^n \to \mathbb{R}^n, \qquad \mathsf{A}x := e^{hA}x, \tag{2.19}$$

$$\mathsf{B}_2 : \mathbb{R}^{m_2} \to \mathbb{R}^n, \qquad \mathsf{B}_2 u := \int_0^h e^{(h-\tau)A}\, d\tau\, B_2 u, \tag{2.20}$$

$$\mathsf{B}_1 : \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{m_1}) \to \mathbb{R}^n, \qquad \mathsf{B}_1 \vec{w} := \int_0^h e^{(h-\tau)A} B_1 \vec{w}(\tau)\, d\tau, \tag{2.21}$$

$$\mathsf{C}_1 : \mathbb{R}^n \to \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{p_1}), \qquad (\mathsf{C}_1 x)(\theta) := C_1 e^{\theta A} x, \tag{2.22}$$

$$\mathsf{C}_2 : \mathbb{R}^n \to \mathbb{R}^{p_2}, \qquad \mathsf{C}_2 x := C_2 x, \tag{2.23}$$

$$\mathsf{D}_{11} : \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{m_1}) \to \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{p_1}), \qquad (\mathsf{D}_{11}\vec{w})(\theta) := D_{11}\vec{w}(\theta) + \int_0^\theta C_1 e^{(\theta-\tau)A} B_1 \vec{w}(\tau)\, d\tau \tag{2.24}$$

and

$$\mathsf{D}_{12} : \mathbb{R}^{m_2} \to \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{p_1}), \qquad (\mathsf{D}_{12}u)(\theta) := D_{12}u + \int_0^\theta C_1 e^{(\theta-\tau)A}\, d\tau B_2 u. \tag{2.25}$$

This constitutes a state-space realisation of the LSI, discrete-time system

$$\mathsf{G} := \begin{bmatrix} \mathbf{W}G_{11}\mathbf{W}^{-1} & \mathbf{W}G_{12}\mathbf{D_A} \\ \mathbf{A_D}G_{21}\mathbf{W}^{-1} & \mathbf{A_D}G_{22}\mathbf{D_A} \end{bmatrix}.$$

Importantly, a sufficient condition for the 22-block of this state-space realisation to be stabilisable and detectable can be characterised in terms of the original continuous-time, state-space realisation of G. First though, the following technical definition is required: it is said that the sampling is not pathological if no two eigenvalues of A, which lie on a line parallel to the imaginary axis in the complex plane, are separated by an integer multiple of the sampling frequency $\left(\frac{2\pi}{h}\right)$. That is, if the sampling is not pathological, then no two continuous-time modes are *aliased* to the same discrete-time mode by sampling. It can be shown that if the sampling is not pathological, then (A, B_2) stabilisable implies $(\mathsf{A}, \mathsf{B}_2)$ stabilisable and (C_2, A) detectable implies $(\mathsf{C}_2, \mathsf{A})$ detectable [CF95]. As such, it is assumed henceforth that the sampling is not pathological and since (A, B_2, C_2) stabilisable and detectable is a standing assumption, it then follows that the 22-block of the state-space realisation of G given in equations (2.18) is both stabilisable and detectable.

Under the standing assumptions just mentioned, given a stabilisable and detectable state-space realisation

$$\left[\begin{array}{c|c} \mathsf{A}_\mathsf{K} & \mathsf{B}_\mathsf{K} \\ \hline \mathsf{C}_\mathsf{K} & \mathsf{D}_\mathsf{K} \end{array}\right],$$

of the LSI, discrete-time controller K, the closed-loop system $\mathbf{F}_\ell(\mathsf{G},\mathsf{K})$ is said to be internally stable if and only if the homogeneous component ($\vec{\hat{w}}$ set to zero) of the closed-loop, state-space realisation is stable. That is, if and only if

$$\operatorname{spec}\left(\mathsf{A}_{cl} := \begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{A}_\mathsf{K} \end{bmatrix} + \begin{bmatrix} \mathsf{B}_2 & 0 \\ 0 & \mathsf{B}_\mathsf{K} \end{bmatrix} \begin{bmatrix} \mathsf{I} & 0 \\ -\mathsf{D}_\mathsf{K} & \mathsf{I} \end{bmatrix}^{-1} \begin{bmatrix} 0 & \mathsf{C}_\mathsf{K} \\ \mathsf{C}_2 & 0 \end{bmatrix}\right) \subset \mathbb{D}.$$

(Note that for $\operatorname{spec}(\mathsf{A}_{cl})$ to lie inside the open unit disc it is *necessary* for both $(\mathsf{A},\mathsf{B}_2,\mathsf{C}_2)$ and $(\mathsf{A}_\mathsf{K},\mathsf{B}_\mathsf{K},\mathsf{C}_\mathsf{K})$ to be stabilisable and detectable.) The notion of internal stability just defined, corresponds to stability of the lower (closed) loop of the linear-fractional configuration in the sense discussed in Sub-Section 2.3.3 for standard closed-loop configurations, and closed-loop, asymptotic stability to zero of the states of G and K starting from any initial states (and initial time) when $\vec{\hat{w}} \equiv 0$. The latter condition (involving closed-loop, asymptotic stability to zero of the plant and controller states) is often used to define internal stability for LFT configurations of general state-space systems, as is the case for the continuous-time plant with sampled-data controller here. Now since x_k is always equal to $x(kh)$ and since for any time $t = kh + \theta$ ($\theta \in \mathbb{H}$), the state of the original, continuous-time generalised-plant when operating in closed-loop (with w set to zero) can be expressed as

$$x(kh+\theta) = \left(e^{\theta A} + \int_0^\theta e^{(\theta-\tau)A} B_2 \, d\tau \mathsf{D}_\mathsf{K} C_2\right) \mathsf{x}_k + \int_0^\theta e^{(\theta-\tau)A} B_2 \, d\tau \mathsf{C}_\mathsf{K} \mathsf{s}_k,$$

where s_k denotes the state of the controller, it follows that under the standing assumptions mentioned above, the LSI, discrete-time, closed-loop system $\mathbf{F}_\ell(\mathsf{G},\mathsf{K})$ is internally stable if and only if the LPTV, continuous-time, closed-loop system $\mathbf{F}_\ell(G,\mathbf{D_A}\mathsf{K}\mathbf{A_D})$ is internally stable. Furthermore, if the LFT configuration is internally stable then the system mapping w to z is bounded and moreover, since $\mathbf{F}_\ell(G,\mathbf{D_A}\mathsf{K}\mathbf{A_D}) = \mathbf{W}^{-1}\mathbf{F}_\ell(\mathsf{G},\mathsf{K})\mathbf{W}$ and $\mathbf{W}$ is an isomorphism, $\|\mathbf{F}_\ell(G,\mathbf{D_A}\mathsf{K}\mathbf{A_D})\| = \|\mathbf{F}_\ell(\mathsf{G},\mathsf{K})\|$ (cf. also [CF95, BP92]). As such, synthesis of K to achieve internal stability of $\mathbf{F}_\ell(G,\mathbf{D_A}\mathsf{K}\mathbf{A_D})$ and a specified bound on the closed-loop induced-norm $\|\mathbf{F}_\ell(G,\mathbf{D_A}\mathsf{K}\mathbf{A_D})\|$, is *equivalent* to synthesis of a K to achieve internal stability of $\mathbf{F}_\ell(\mathsf{G},\mathsf{K})$ and the same bound on $\|\mathbf{F}_\ell(\mathsf{G},\mathsf{K})\|$.

Synthesis of a controller K to achieve a specified bound on $\|\mathbf{F}_\ell(\mathsf{G},\mathsf{K})\|$ is complicated by the fact that the input and output spaces of the closed-loop are infinite-dimensional. Fortunately, there exist methods for transforming the problem into an equivalent finite-dimensional one [BP92, Toi92, Toi93, CF95]. This is possible because the state-space operators $\mathsf{B}_1 : \mathrm{L}^2_\mathbb{H}(\mathbb{R}^{m_1}) \to \mathbb{R}^n$, $\mathsf{C}_1 : \mathbb{R}^n \to \mathrm{L}^2_\mathbb{H}(\mathbb{R}^{p_1})$ and $\mathsf{D}_{12} : \mathbb{R}^{m_2} \to \mathrm{L}^2_\mathbb{H}(\mathbb{R}^{p_1})$ all have finite-rank and therefore, finite-dimensional matrix representations. However, $\mathsf{D}_{11} : \mathrm{L}^2_\mathbb{H}(\mathbb{R}^{m_1}) \to \mathrm{L}^2_\mathbb{H}(\mathbb{R}^{p_1})$ does not have finite-rank and as such, an additional transformation is required to handle this. The required transformation is an operator version of the loop-shifting principle (cf. [SLC90]) as summarised in the following lemma:

Proposition 2.24 [BP92] *Consider the LSI, discrete-time generalised-plant* $\tilde{\mathsf{G}}$ *with state-space realisation*

$$\left[\begin{array}{c|cc} \tilde{\mathsf{A}} & \tilde{\mathsf{B}}_1 & \tilde{\mathsf{B}}_2 \\ \hline \tilde{\mathsf{C}}_1 & 0 & \tilde{\mathsf{D}}_{12} \\ \mathsf{C}_2 & 0 & 0 \end{array}\right], \tag{2.26}$$

where

$$\begin{aligned} \tilde{\mathsf{A}} &:= \mathsf{A} + \gamma^{-2}\mathsf{B}_1\mathsf{D}_{11}^*\mathsf{S}^{-1}\mathsf{C}_1 \ : \ \mathbb{R}^n \to \mathbb{R}^n, & (2.27)\\ \tilde{\mathsf{B}}_1 &:= \gamma^{-1}\mathsf{B}_1\mathsf{R}^{-\frac{1}{2}} \ : \ \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{m_1}) \to \mathbb{R}^n, & (2.28)\\ \tilde{\mathsf{B}}_2 &:= \gamma^{-2}\mathsf{B}_1\mathsf{D}_{11}^*\mathsf{S}^{-1}\mathsf{D}_{12} + \mathsf{B}_2 \ : \ \mathbb{R}^{m_2} \to \mathbb{R}^n, & (2.29)\\ \tilde{\mathsf{C}}_1 &:= \gamma\mathsf{S}^{-\frac{1}{2}}\mathsf{C}_1 \ : \ \mathbb{R}^n \to \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{p_1}), & (2.30)\\ \tilde{\mathsf{D}}_{12} &:= \gamma\mathsf{S}^{-\frac{1}{2}}\mathsf{D}_{12} \ : \ \mathbb{R}^{m_2} \to \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{p_1}), & (2.31) \end{aligned}$$

$\mathsf{R} :=(I-\gamma^{-2}\mathsf{D}_{11}^*\mathsf{D}_{11})$ *and* $\mathsf{S} :=(I-\gamma^{-2}\mathsf{D}_{11}\mathsf{D}_{11}^*)$. *Provided* $\|\mathsf{D}_{11}\| < \gamma$ *and given an LSI, discrete-time controller* K, *the following are equivalent:*

(i) $\mathbf{F}_\ell(\mathsf{G},\mathsf{K})$ *is internally stable and* $\|\mathbf{F}_\ell(\mathsf{G},\mathsf{K})\| < \gamma$

(ii) $\mathbf{F}_\ell(\tilde{\mathsf{G}},\mathsf{K})$ *is internally stable and* $\|\mathbf{F}_\ell(\tilde{\mathsf{G}},\mathsf{K})\| < \gamma$.

Note that in analogy with the standard finite-dimensional H^∞ problem, the assumption $\gamma > \|\mathsf{D}_{11}\|$ is required. This ensures that R and S are well defined and as noted in [Toi93], is equivalent to the condition that for fixed $k \in \mathbb{Z}^+$, given any bounded $\mathsf{x}_k \in \mathbb{R}^n$ and $\mathsf{u}_k \in \mathbb{R}^{m_2}$,

$$\langle \vec{\mathring{z}}_k, \vec{\mathring{z}}_k\rangle_{\mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{p_1})} - \gamma^2\langle \vec{\mathring{w}}_k, \vec{\mathring{w}}_k\rangle_{\mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{m_1})}$$

has bounded supremum over $\vec{\mathring{w}}_k \in \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{m_1})$. To ensure that this assumption is not violated, a procedure for determining a bound on $\|\mathsf{D}_{11}\|$ is needed. Since $\mathsf{D}_{11} : \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{m_1}) \to \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{p_1})$ is not a compact operator, this is *not* simply a matter of bounding the maximum eigenvalue of $\mathsf{D}_{11}^*\mathsf{D}_{11}$ (cf. Sub-Section 2.2.2 or [Kre89b]). However, using the principle of loop-shifting again, a *compact* operator $\check{\mathsf{D}}_{11} : \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{m_1}) \to \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{p_1})$ can be constructed such that $\|\mathsf{D}_{11}\| < \gamma \Leftrightarrow \|\check{\mathsf{D}}_{11}\| < \gamma$. Given a $\vec{\mathring{v}} \in \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{m_1})$, the action of $\vec{\mathring{g}} = \check{\mathsf{D}}_{11}\vec{\mathring{v}}$ can be expressed in terms of the system of differential equations

$$\begin{aligned} \frac{d\vec{\mathring{a}}}{d\theta} &= \check{A}\,\vec{\mathring{a}}(\theta) + \check{B}\,\vec{\mathring{v}}(\theta); \quad \vec{\mathring{a}}(0) = 0, & (2.32a)\\ \vec{\mathring{g}}(\theta) &= \check{C}\,\vec{\mathring{a}}(\theta), & (2.32b) \end{aligned}$$

where $R :=(I-\gamma^{-2}D_{11}^*D_{11})$, $S :=(I-\gamma^{-2}D_{11}D_{11}^*)$,

$$\begin{aligned} \check{A} &:= A + \gamma^{-2}B_1R^{-1}D_{11}^*C_1 \ : \ \mathbb{R}^n \to \mathbb{R}^n,\\ \check{B} &:= \gamma^{-1}B_1R^{-\frac{1}{2}} \ : \ \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{p_1}) \to \mathbb{R}^n \end{aligned}$$

and

$$\check{C} := \gamma S^{-\frac{1}{2}} C_1 \quad : \quad \mathbb{R}^n \to \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{p_1}).$$

Note that provided $\overline{\sigma}\{D_{11}\} < \gamma$, $R^{-\frac{1}{2}}$ and $S^{-\frac{1}{2}}$ are well defined. Now since $\check{\mathrm{D}}_{11}$ is compact,[10] it follows that $\|\mathrm{D}_{11}\| < \gamma$ if and only if $\overline{\sigma}\{\check{\mathrm{D}}_{11}\} < \gamma$ (cf. Sub-Section 2.2.2). This is why the compact operator $\check{\mathrm{D}}_{11}$ is defined the in way it is, since it is possible to characterise in a computationally-tractable manner the non-zero singular values of a compact operator described by a system of differential equations such as that given in equation (2.32) (cf. Chapters 5 and 6 for details of such characterisations). Let

$$E := \begin{bmatrix} -A^* - \gamma^{-2} C_1^* D_{11} R^{-1} B_1^* & -C_1^* S^{-1} C_1 \\ \gamma^{-2} B_1 R^{-1} B_1^* & A + \gamma^{-2} B_1 R^{-1} D_{11}^* C_1 \end{bmatrix}, \tag{2.33}$$

and

$$Q(t) =: \begin{bmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{bmatrix} := e^{tE}. \tag{2.34}$$

Lemma 2.25 *Given a real number $\gamma > \overline{\sigma}\{D_{11}\}$, then γ is a singular value of $\check{\mathrm{D}}_{11}$ if and only if $\det(Q_{11}(h)) = 0$, where $\det(\cdot)$ denotes the determinant of a matrix (cf. [GVL89]).*

Proof : The proof is essentially the same as that of Theorem 13.5.1 in [CF95]. See also Sub-Section 5.4.3 for a similar result. ∎

Remark 2.26 Clearly, with $\|\mathrm{D}_{11}\| < \gamma$ the matrix $Q_{11}(h)$ is invertible. ♣

Although it has been suppressed for notational convenience, $\det(Q_{11}(h))$ is a function of γ. Define the function $d(\gamma) := \det(Q_{11}) : \mathbb{C} \to \mathbb{R}$. Then in view of Lemma 2.25 and since $\check{\mathrm{D}}_{11}^* \check{\mathrm{D}}_{11}$ is a self-adjoint, compact operator, the zeros of $d(\gamma)$ constitute a discrete, real set with zero being the only possible accumulation point (cf. Sub-Section 2.2.2). Furthermore, with $\gamma \mapsto \mu := 1/\gamma$, $d(\mu)$ is an analytic function in the whole of the complex plane and the zeros of $d(\mu)$ form a discrete, real set with no finite accumulation points. So with $\mathrm{wno}_r(d)$ defined to be the number of clock-wise encirclements of the origin made by $d(\mu)$ as μ traverses the standard Nyquist contour of radius r,[11] the following corollary is a simple consequence of Lemma 2.25 and the Principle of the Argument [MH87].[12]

Corollary 2.27 *Given $\gamma > \|D_{11}\|$,*

$$\|\mathrm{D}_{11}\| < \gamma \quad \Leftrightarrow \quad \|\check{\mathrm{D}}_{11}\| < \gamma \quad \Leftrightarrow \mathrm{wno}_{\frac{1}{\gamma}}(d) = 0.$$

[10] $\check{\mathrm{D}}_{11}$ is compact because it corresponds to an integral operator with no direct feed-through term [Kre89a].
[11] A semi-circle (centred at the origin) into the right-half plane of radius r.
[12] Prof. Bassam Bamieh must be acknowledged for first suggesting the use of winding numbers in this context.

Having established a method for determining a bound on $\|\mathsf{D}_{11}\|$, it is now appropriate to state the important result of [BP92], which implicitly characterises a finite-dimensional (in terms of input, output and state spaces) equivalent of $\tilde{\mathsf{G}}$.

Proposition 2.28 [BP92] *Given the LSI, discrete-time generalised-plant* $\tilde{\mathsf{G}}$ *with state-space realisation (2.26), obtain the symmetric matrix-factorisations*

$$\begin{aligned}\tilde{\mathsf{B}}_1\tilde{\mathsf{B}}_1^* &= \mathsf{T}_{\tilde{\mathsf{B}}_1}^*\begin{bmatrix}\Sigma_{\tilde{\mathsf{B}}_1} & 0\\ 0 & 0\end{bmatrix}\mathsf{T}_{\tilde{\mathsf{B}}_1},\\ \begin{bmatrix}\tilde{\mathsf{C}}_1^*\\ \tilde{\mathsf{D}}_{12}^*\end{bmatrix}\begin{bmatrix}\tilde{\mathsf{C}}_1 & \tilde{\mathsf{D}}_{12}\end{bmatrix} &= \mathsf{T}_{\tilde{\mathsf{C}}_1,\tilde{\mathsf{D}}_{12}}^*\begin{bmatrix}\Sigma_{\tilde{\mathsf{C}}_1,\tilde{\mathsf{D}}_{12}} & 0\\ 0 & 0\end{bmatrix}\mathsf{T}_{\tilde{\mathsf{C}}_1,\tilde{\mathsf{D}}_{12}}\end{aligned}$$

and define the (completely) <u>*finite-dimensional*</u>*, LSI, discrete-time generalised-plant* $\breve{\mathsf{G}}$ *to have state-space realisation*

$$\left[\begin{array}{c|cc}\tilde{\mathsf{A}} & \breve{\mathsf{B}}_1 & \tilde{\mathsf{B}}_2\\ \hline \breve{\mathsf{C}}_1 & 0 & \breve{\mathsf{D}}_{12}\\ \mathsf{C}_2 & 0 & 0\end{array}\right] \tag{2.35}$$

where

$$\breve{\mathsf{B}}_1 := \mathsf{T}_{\tilde{\mathsf{B}}_1}^*\begin{bmatrix}\Sigma_{\tilde{\mathsf{B}}_1}^{\frac{1}{2}}\\ 0\end{bmatrix} \tag{2.36}$$

and

$$\begin{bmatrix}\breve{\mathsf{C}}_1 & \breve{\mathsf{D}}_{12}\end{bmatrix} := \begin{bmatrix}\Sigma_{\tilde{\mathsf{C}}_1,\tilde{\mathsf{D}}_{12}}^{\frac{1}{2}} & 0\end{bmatrix}\mathsf{T}_{\tilde{\mathsf{C}}_1,\tilde{\mathsf{D}}_{12}}. \tag{2.37}$$

Provided $\gamma > \|\mathsf{D}_{11}\|$ *and given a LSI, discrete-time controller* K*, the following are equivalent:*

(i) $\mathbf{F}_\ell(\tilde{\mathsf{G}},\mathsf{K})$ *is internally stable and* $\|\mathbf{F}_\ell(\tilde{\mathsf{G}},\mathsf{K})\| < \gamma$

(ii) $\mathbf{F}_\ell(\breve{\mathsf{G}},\mathsf{K})$ *is internally stable and* $\|\mathbf{F}_\ell(\breve{\mathsf{G}},\mathsf{K})\| < \gamma$.

Remark 2.29 Recall that under the standing assumptions that the sampling is not pathological and that (A, B_2, C_2) is stabilisable and detectable,

$$\begin{gathered}\mathbf{F}_\ell(\mathsf{G},\mathsf{K}) \text{ is internally stable and } \|\mathbf{F}_\ell(\mathsf{G},\mathsf{K})\| < \gamma\\ \Updownarrow\\ \mathbf{F}_\ell(G,\mathbf{D_A}\mathsf{K}\mathbf{A_D}) \text{ is internally stable and } \|\mathbf{F}_\ell(G,\mathbf{D_A}\mathsf{K}\mathbf{A_D})\| < \gamma.\end{gathered}$$

So in view of Propositions 2.28 and 2.24, synthesis of an internally-stabilising, LSI, discrete-time controller K to achieve a specified bound on $\|\mathbf{F}_\ell(G,\mathbf{D_A}\mathsf{K}\mathbf{A_D})\|$ is equivalent to the synthesis of K

to achieve internal stability for $\mathbf{F}_\ell(\breve{\mathsf{G}}, \mathsf{K})$ and the same bound on $\|\mathbf{F}_\ell(\breve{\mathsf{G}}, \mathsf{K})\|$. Although singular in the sense that $\breve{\mathsf{D}}_{21} = 0$, the latter induced-norm synthesis problem can be solved using the LMI framework of [GA94] or often, by using a bilinear transformation to a continuous-time problem [KHI90]. As such, $\breve{\mathsf{G}}$ is referred to as the SD-equivalent generalised-plant. Computation of the state-space matrices for $\breve{\mathsf{G}}$ is discussed in detail in Chapter 6, including related numerical issues.
♣

3

Frequency-Domain Analysis and SD Control Design

3.1 Introduction

In this chapter a new frequency-domain, analysis tool is developed for studying the performance of LPTV systems. The motivation for this stems from the wide-spread use of frequency-domain, analysis tools in the important "design step" of control-system development. Development of control systems usually involves two major steps: first, a design step, in which a generalised plant is constructed from the system to be controlled and parametric weights that reflect closed-loop objectives; second, a synthesis step, in which a stabilising controller is synthesised by solving an optimisation problem involving the generalised plant obtained in the design step. These two main steps are often iterated in conjunction with an intermediate *analysis* step, in which tools such as those developed in Chapter 4 for example, are used. The H^∞-loopshaping procedure of McFarlane and Glover [MG90] for linear, time-invariant (LTI) systems, is an intuitive and reasonably well-understood example of such a control-system development procedure. It is based on classical loopshaping ideas in the design step, by which the singular values of the frequency-response matrix of the plant are shaped open-loop to reflect closed-loop objectives. A four-block generalised plant is then constructed from the shaped plant and an H^∞-optimisation is solved in the synthesis step. The ultimate goal of this chapter is to develop a similar design procedure for SD, control-system development. This first requires a suitable definition of frequency response for LPTV systems.

To date, most of the work on frequency-domain, analysis methods for LPTV systems has been presented in the context of SD systems [AI93, AHI93, YA94, GS94, YK96]. The standard definition of frequency response for LPTV systems is often expressed in terms of the induced-norm of a frequency-response operator, which characterises the asymptotic response of an LPTV system to an infinite series of sinusoidal signals of frequencies separated by integer multiples of the frequency of periodicity. Such a definition is useful for frequency-domain robustness-analysis of SD, closed-loop systems [Dul94]. However, from the perspective of designing for nominal performance, it is perhaps more meaningful and certainly more intuitive to characterise the asymptotic response of a system to sinusoidal inputs of a single frequency. For example, in disturbance rejection problems it is useful to know if the "gain" of the system is small for signals with power concentrated in a specific band of frequencies.

The robustness properties of a closed-loop system developed using any given procedure are to a large extent (but not completely) governed by the structure of the generalised plant defined in the design step and the type of optimisation problem solved in the synthesis step (cf. Chapter 4 for a suitable framework in which quantitative robustness result can be obtained). Bearing this in mind, a control-system development procedure in which design for performance is based on the response of performance-indicating, closed-loop operators to sinusoids of a single frequency, is sensible provided a suitable (from the perspective of robustness) generalised-plant structure and optimisation formulation are employed. Motivated by this, in the following section the average-power gain matrix is defined for LPTV systems. The singular values of this finite-dimensional matrix characterise the average-power gain (in input-directions corresponding to the singular vectors) of the asymptotic response of a system to sinusoidal inputs of a single frequency. In fact, when the system is LTI, the non-zero singular values of the average-power gain matrix correspond to the non-zero singular values of the standard, continuous-time, frequency-response matrix. This new frequency-domain, analysis tools is used to derive frequency-by-frequency bounds on performance-indicating, closed-loop operators, which facilitate the design of weights used to construct a generalised plant in an H^∞-loopshaping based design procedure proposed for SD, control-system development. State-space formulae for computing the average-power gain matrix are presented for the sampled-data case in Appendix B.2.

3.2 The Average-Power Gain Matrix

In this section, a new notion of frequency response is developed for LPTV systems. It is based on the average power of the asymptotic response to sinusoidal inputs of a single frequency and hence, differs from the standard definition of frequency response for LPTV systems [YA94, YK96].

3.2.1 Frequency Response

For notational convenience, let $\mathcal{U} := \mathbb{R}^m$ and $\mathcal{Y} := \mathbb{R}^p$. Now consider a stable system $P \in \mathscr{P}_{\mathcal{U}\to\mathcal{Y}}$ (cf. Definition 2.3) with period h and recall that P is equivalent via the time-lifting isomorphism $\mathbf{W}$

to a stable, LSI system

$$\mathsf{P} := \mathbf{W} P \mathbf{W}^{-1} \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}) \to \mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}.$$

Since P is a stable system, it has a locally Lipschitz-continuous extension and hence, its block-Toeplitz representation

$$\begin{bmatrix} \mathsf{P}_{[0]} & 0 & \cdots & \cdots \\ \mathsf{P}_{[1]} & \mathsf{P}_{[0]} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

can be uniquely identified with the sequence $\{\mathsf{P}_{[i]}\}_{i=0}^{\infty} \subset \mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}) \to \mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}$. Furthermore, by Proposition 2.2, P is equivalent via the $\mathbf{Z}$-Transform isomorphism to a multiplication operator with symbol

$$\hat{\mathsf{P}}(\lambda) := \sum_{i=0}^{\infty} \mathsf{P}_{[i]} \lambda^i \in \mathrm{H}^{\infty}_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}) \to \mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})})$$

and $\|P\| = \|\mathsf{P}\| = \|\hat{\mathsf{P}}\|_{\infty}$. In the sequel, $\hat{\mathsf{P}}(\lambda)$ is referred to as the transfer operator of P. For technical convenience it is assumed throughout this chapter only, that the sequence uniquely associated with the Toeplitz representation of the time-lifted equivalent of any stable system considered, for example $P \in \mathscr{P}_{\mathcal{U} \to \mathcal{Y}}$ here, satisfies $\sum_{i=0}^{\infty} \|\mathsf{P}_{[i]}\| < \infty$. In this way, the corresponding transfer operator $\hat{\mathsf{P}}(\lambda)$ will have continuous extension to $\mathbb{T}$. That is, $\hat{\mathsf{P}}(e^{j\omega h})$ is defined for all $\omega \in [0, \frac{2\pi}{h})$.

Central to the standard (cf. [YA94, YK96]) definition of frequency response for LPTV systems is the following lemma, which characterises the asymptotic response of P to a power signal.

Lemma 3.1 [Yam94] *Consider an input to P of the form*

$$u(kh + \theta) \;=\; \mu^k \, \vec{u}_d(\theta), \;\; \vec{u}_d \in \mathrm{L}^2_{\mathbb{H}}(\mathcal{U}), \;\; |\mu| \geq 1,$$

for $k \in \mathbb{Z}^+$ and $\theta \in \mathbb{H}$. Let $y(t) = (Pu)(t)$. Then $y(t)$ asymptotically approaches

$$y_{\mathrm{as}}(t) = y_{\mathrm{as}}(kh + \theta) = \mu^k (\hat{\mathsf{P}}(\mu^{-1}) \, \vec{u}_d)(\theta)$$

as $t \to \infty$.

Taking $\mu = e^{j\omega h}$ ($\omega \in [0, \frac{2\pi}{h})$), the signal

$$\mathfrak{u} \;:= \mathbf{W} u \;:= k \mapsto e^{j\omega hk} \, \vec{u}_d \quad (k \in \mathbb{Z}^+)$$

is simply a discrete-time sinusoid with direction $\vec{u}_d$ in the infinite-dimensional, Hilbert space $\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})$. It follows by Lemma 3.1, that the response of P to such a signal asymptotically approaches a discrete-time sinusoid of the same frequency, but new direction given by $\hat{\mathsf{P}}(e^{-j\omega h})\,\vec{\mathring{u}}_d$. This is by no means surprising, since P is a stable, LSI system. Note however, that unless $\vec{\mathring{u}}_d$ takes a particular form, the signal $\mathfrak{u}_k = e^{j\omega hk}\,\vec{\mathring{u}}_d$ does not correspond (via $\mathbf{W}$) to a sinusoidal signal of a single frequency in continuous-time. In fact, since for any fixed $\omega_0 \in [0, \frac{2\pi}{h})$ the sequence of functions

$$\left\{h^{-\frac{1}{2}} e^{j(\frac{2\pi n}{h}-\omega_0)\theta} u_{bi}\right\}_{n\in\mathbb{Z},\, i=1,\dots,m} \subset \mathrm{L}^2_{\mathbb{H}}(\mathcal{U}),$$

where $\{u_{bi}\}_{i=1,\dots,m}$ is any complete, orthonormal basis for the m-dimensional, Hilbert space $\mathcal{U}$, constitutes a complete orthonormal basis for $\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})$ [Dul94, YA94],[1] it is readily verifiable that in general, the signal $\mathfrak{u} := k \mapsto e^{j\omega hk}\,\vec{\mathring{u}}_d$ ($k \in \mathbb{Z}^+$) corresponds to an infinite series of continuous-time sinusoids with frequencies separated by integer multiples of the frequency of periodicity ($\frac{2\pi}{h}$). [2]

The standard definition of frequency response for LPTV systems is the $\mathrm{L}^2_{\mathbb{H}}$-induced norm of $\hat{\mathsf{P}}(e^{j\omega h})$ ($\omega \in [0, \frac{2\pi}{h})$). This characterises the maximal gain over all directions in $\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})$ and as such corresponds to characterising the maximal, asymptotic gain of the system to an infinite series of continuous-time sinusoids, each separated by integer multiples of the frequency of periodicity. Although this definition is useful for studying the robustness of closed-loop stability and performance to LTI uncertainty for example [Dul94], all input-directional information is lost in the definition and its usefulness from the perspective of performance is not immediately clear. With this in mind, a new definition of frequency response is developed in what follows, in terms of the average power of the asymptotic response to sinusoidal inputs of single frequencies.

Observe that a continuous-time, sinusoidal signal

$$u(t) = e^{j\omega t} u_d \qquad (u_d \in \mathcal{U} := \mathbb{R}^m)$$

of frequency $\omega \in \mathbb{R}$, corresponds (via $\mathbf{W}$) to a discrete-time sinusoid

$$\mathfrak{u}_k = e^{j\omega_0 hk}\,\vec{\mathring{u}}_d,$$

where

$$\vec{\mathring{u}}_d(\theta) = e^{j\omega\theta} u_d \in \mathrm{L}^2_{\mathbb{H}}(\mathcal{U}),$$

[1] To see this simply take the Fourier series expansion of the periodic extension of any function in $\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})$.

[2] Correspondingly, by Lemma 3.1, the asymptotic response to such a signal is in general an infinite-series of sinusoids with frequencies separated by integer multiples of the frequency of periodicity. By this, it can be said that such signals are *asymptotically invariant* for LPTV systems in the same way that sinusoids of a single frequency are *asymptotically invariant* for LTI systems.

$\omega_0 = \omega -]\frac{h\omega}{2\pi}[\ (\frac{2\pi}{h})$ and $]\cdot[$ denotes the integer part of a real number. As such, it follows by Lemma 3.1, that the asymptotic response of an LPTV system to a continuous-time, sinusoidal input of frequency ω is given by

$$y^{\omega}_{\mathrm{as}}(t) = y^{\omega}_{\mathrm{as}}(kh+\theta) = (e^{j\omega_0 h})^k(\hat{\mathsf{P}}(e^{-j\omega_0 h})\,\mathring{\vec{u}}_d)(\theta). \tag{3.1}$$

Again, note that this corresponds to a discrete-time sinusoid with direction $\hat{\mathsf{P}}(e^{-j\omega_0 h})\,\mathring{\vec{u}}_d \in \mathrm{L}^2_{\mathbb{H}}(\mathcal{U})$, but that in general, $y^{\omega}_{\mathrm{as}}(t)$ is not simply a continuous-time sinusoid of a single frequency. Despite this, it is possible to characterise the average power of the asymptotic response as follows:

Lemma 3.2 *Let $y^{\omega}_{\mathrm{as}}(t)$ be as defined in (3.1). Then,*

$$\begin{aligned}\|y^{\omega}_{\mathrm{as}}\|_{\mathrm{pow}} &:= \sqrt{\lim_{T\to\infty}\frac{1}{T}\int_0^T \langle y^{\omega}_{\mathrm{as}}(t), y^{\omega}_{\mathrm{as}}(t)\rangle_{\mathcal{Y}}\,dt}\\ &= \frac{1}{\sqrt{h}}\|\hat{\mathsf{P}}(e^{-j\omega_0 h})\,\mathring{\vec{u}}_d\|_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}.\end{aligned}$$

Proof : Let $\mathring{\vec{y}} := \hat{\mathsf{P}}(e^{-j\omega_0 h})\,\mathring{\vec{u}}_d \in \mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})$. Then,

$$\begin{aligned}\|y^{\omega}_{\mathrm{as}}\|^2_{\mathrm{pow}} &:= \lim_{T\to\infty}\frac{1}{T}\int_0^T \langle y^{\omega}_{\mathrm{as}}(t), y^{\omega}_{\mathrm{as}}(t)\rangle_{\mathcal{Y}}\,dt\\ &= \lim_{N\to\infty}\frac{1}{Nh}\sum_{k=0}^{N-1}\int_{kh}^{(k+1)h}\langle y^{\omega}_{\mathrm{as}}(t), y^{\omega}_{\mathrm{as}}(t)\rangle_{\mathcal{Y}}\,dt\\ &= \lim_{N\to\infty}\frac{1}{Nh}\sum_{k=0}^{N-1}\langle\,\mathring{\vec{y}},\,\mathring{\vec{y}}\rangle_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}\\ &= \frac{1}{h}\|\,\mathring{\vec{y}}\|^2_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}.\end{aligned}$$

This completes the proof. ∎

For each $u_d \in \mathcal{U}$, let the $\mathcal{B}_{\mathcal{U}\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}$-valued function $\mathbf{\Gamma}_P(j\omega) : j\mathbb{R} \to \mathcal{B}_{\mathcal{U}\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}$ be defined by

$$\mathring{\vec{y}} := \mathbf{\Gamma}_P(j\omega)u_d := \hat{\mathsf{P}}(e^{-j\omega_0 h})\Omega(j\omega)u_d \in \mathrm{L}^2_{\mathbb{H}}(\mathcal{Y}), \tag{3.2}$$

where $\Omega(j\omega) : j\mathbb{R} \to \mathcal{B}_{\mathcal{U}\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})}$ is the $\mathcal{B}_{\mathcal{U}\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})}$-valued function defined by

$$\mathring{\vec{u}}_d := \Omega(j\omega)u_d := e^{j\omega\theta}u_d \in \mathrm{L}^2_{\mathbb{H}}(\mathcal{U}) \tag{3.3}$$

and $\omega_0 := \omega - \,]\,\frac{h\omega}{2\pi}\,[\,\frac{2\pi}{h}$. Then by Lemma 3.2,

$$\|y^{\omega}_{\mathrm{as}}\|_{\mathrm{pow}} = \frac{1}{\sqrt{h}}\sqrt{\langle\mathbf{\Gamma}_P(j\omega)u_d, \mathbf{\Gamma}_P(j\omega)u_d\rangle_{\mathcal{Y}}} \;=\; \frac{1}{\sqrt{h}}\sqrt{\langle(\mathbf{\Gamma}_P(j\omega))^*\mathbf{\Gamma}_P(j\omega)u_d, u_d\rangle_{\mathcal{U}}}. \tag{3.4}$$

For fixed $\omega \in \mathbb{R}$, $(\mathbf{\Gamma}_P(j\omega))^*\mathbf{\Gamma}_P(j\omega)$ is a square, symmetric, positive-semidefinite matrix and hence, its square root exists. With this in mind, define

$$\mathbf{\Theta}_P(j\omega) := \frac{1}{\sqrt{h}}\{(\mathbf{\Gamma}_P(j\omega))^*\mathbf{\Gamma}_P(j\omega)\}^{\frac{1}{2}}, \tag{3.5}$$

noting from equation (3.4), that the average power of the asymptotic response can be characterised in terms of the singular values of this matrix function of frequency. Accordingly the following definition is made:

Definition 3.3 The frequency-dependent $m \times m$ matrix

$$\mathbf{\Theta}_P(j\omega) := \frac{1}{\sqrt{h}}\{(\mathbf{\Gamma}_P(j\omega))^*\mathbf{\Gamma}_P(j\omega)\}^{\frac{1}{2}},$$

where $\mathbf{\Gamma}_P(j\omega)$ is defined in equation (3.2), is defined to be the the average-power gain matrix of the LPTV system P. ♠

Remark 3.4 Note that any LTI system F is LPTV and as such, if F is also stable, that $\mathbf{\Theta}_F(j\omega)$ is well defined. In fact, by the way the average-power gain matrix has been defined, it follows that $\mathbf{\Theta}_F(j\omega) = (\hat{F}(j\omega)^*\hat{F}(j\omega))^{\frac{1}{2}}$, where $\hat{F}(j\omega)$ denotes the standard, continuous-time, frequency-response matrix. That is, the non-zero singular values of $\hat{F}(j\omega)$ correspond to the non-zero singular values of $\mathbf{\Theta}_F(j\omega)$. In forming $\mathbf{\Theta}_F(j\omega)$ the phase information of $\hat{F}(j\omega)$ is lost but the gain information is retained. ♣

3.2.2 Properties of the Average-Power Gain Matrix

In this sub-section a few technical results are presented, which characterise some important properties of the average-power gain matix used in the sequel.

Lemma 3.5 *Given a stable, LTI system $F \in \mathscr{P}_{\mathfrak{U}\to\mathfrak{Y}}$ with frequency-domain transfer-matrix $\hat{F}(j\omega)$, for any $\omega_0 \in [0, \frac{2\pi}{h})$*

$$\|\hat{\mathsf{F}}(e^{-j\omega_0 h})\| = \overline{\sigma}^{\,\omega_0}_{\max}\{\hat{F}\},$$

where $\hat{\mathsf{F}}$ is the transfer operator of the time-lifted, equivalent system $\mathsf{F} = \mathbf{W}F\mathbf{W}^{-1}$,

$$\overline{\sigma}^{\,\omega_0}_{\max}\{\hat{F}\} := \max_{n\in\mathbb{Z}} \overline{\sigma}\{\hat{F}(j\omega_n)\}, \tag{3.6}$$

and $\omega_n := (\omega_0 + n\frac{2\pi}{h})$.

Proof : Consider any $\vec{u}_d \in \mathrm{L}^2_{\mathbb{H}}(\mathcal{U})$ and note that by taking a Fourier series expansion of the periodic extension $u_d(t) = u_d(kh+\theta) := \vec{u}_d(\theta)$, for any $\omega_0 \in [0, 2\pi)$ the function $\vec{u}_d(\theta)$ can be expressed as (cf. [YA94, Lemma 3.3])

$$\vec{u}_d(\theta) = \sum_{n\in\mathbb{Z}} e^{j\omega_n\theta}\alpha_n,$$

where $\omega_n = \omega_0 + n\frac{2\pi}{h}$ and

$$\alpha_n = \frac{1}{h}\int_0^h e^{-j\omega_n\theta}\,\vec{u}_d(\theta)\;d\theta \in \mathcal{U}.$$

Furthermore,

$$\|\,\vec{u}_d\|^2_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})} = h\sum_{n\in\mathbb{Z}} \langle\alpha_n, \alpha_n\rangle_{\mathcal{U}}. \tag{3.7}$$

Now given an LTI system F, denote by $\hat{\mathsf{F}}$ the transfer operator of $\mathsf{F} := \mathbf{W}F\mathbf{W}^{-1}$. Then since $\hat{\mathsf{F}}$ is linear, it is possible to write

$$(\hat{\mathsf{F}}(e^{-j\omega_0 h})\,\vec{u}_d)(\theta) = \sum_{n\in\mathbb{Z}}(\hat{\mathsf{F}}(e^{-j\omega_0 h})e^{j\omega_n\tau}\alpha_n)(\theta) \tag{3.8}$$

Also note that the asymptotic response of F to a sinusoidal input $u(t) = e^{j\omega_n t}\alpha_n$ is given by

$$y^{\omega_n}_{\mathrm{as}}(t) = e^{j\omega_n t}\hat{F}(j\omega_n)\alpha_n,$$

which via $\mathbf{W}$ corresponds to the discrete-time sinusoidal signal

$$\vec{\mathcal{Y}}^{\omega_n}_{\mathrm{as},k}(\theta) := y^{\omega_n}_{\mathrm{as}}(kh+\theta) = (e^{j\hat{\omega}h})^k e^{j\omega_n\theta}\hat{F}(j\omega_n)\alpha_n. \tag{3.9}$$

By Lemma 3.1,

$$\vec{\mathcal{Y}}^{\omega_n}_{\mathrm{as},k}(\theta) := y^{\omega_n}_{\mathrm{as}}(kh+\theta) = (e^{j\hat{\omega}h})^k(\hat{\mathsf{F}}(e^{-j\hat{\omega}h})e^{j\omega_n\tau}\alpha_n)(\theta),$$

which combined with equation (3.9) implies

$$(\hat{\mathsf{F}}(e^{-j\omega_0 h})e^{j\omega_n\tau}\alpha_n)(\theta) = e^{j\omega_n\theta}\hat{F}(j\omega_n)\alpha_n.$$

So from equation (3.8), it follows that

$$(\hat{\mathsf{F}}(e^{-j\omega_0 h})\,\vec{u}_d)(\theta) = \sum_{n\in\mathbb{Z}} e^{j\omega_n\theta}\hat{F}(j\omega_n)\alpha_n$$

and consequently, that

$$\begin{aligned}
\|\hat{\mathsf{F}}(e^{-j\omega_0 h})\,\vec{u}_d\|^2_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})} &= h\sum_{n\in\mathbb{Z}}\left\langle \hat{F}(j\omega_n)\alpha_n, \hat{F}(j\omega_n)\alpha_n\right\rangle_{\mathcal{Y}} \\
&\le h\max_{n\in\mathbb{Z}}\overline{\sigma}\{\hat{F}(j\omega_n)\}^2\sum_{n\in\mathbb{Z}}\langle\alpha_n,\alpha_n\rangle_{\mathcal{U}} \\
&= \max_{n\in\mathbb{Z}}\overline{\sigma}\{\hat{F}(j\omega_n)\}^2\;\|\,\vec{u}_d\|^2_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})},
\end{aligned}$$

where the last equality follows from equation (3.7). That is,

$$\frac{\|\hat{\mathsf{F}}(e^{-j\omega_0 h})\,\vec{u}_d\|_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}}{\|\,\vec{u}_d\|_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})}} \le \max_{n\in\mathbb{Z}} \overline{\sigma}\{\hat{F}(j\omega_n)\}. \tag{3.10}$$

In fact by taking $\vec{u}_d(\theta) = \frac{1}{\sqrt{h}} e^{j\omega_{n_{\max}}\theta} u_{d\max}$, where $n_{\max}$:=arg $\max_{n\in\mathbb{Z}} \overline{\sigma}\{\hat{F}(j\omega_n)\}$ and $u_{d\max}$ is the singular-vector corresponding to $\overline{\sigma}\{\hat{F}(j\omega_{n_{\max}})\}$, equality holds in equation (3.10) and therefore,

$$\begin{aligned} \|\hat{\mathsf{F}}(e^{-j\hat{\omega}h})\| \quad &:= \sup_{\substack{\vec{u}_d\in \mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\\ \vec{u}_d\neq 0}} \frac{\|\hat{\mathsf{F}}(e^{-j\hat{\omega}h})\,\vec{u}_d\|_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}}{\|\,\vec{u}_d\|_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})}} \\ &= \max_{n\in\mathbb{Z}} \overline{\sigma}\{\hat{F}(j\omega_n)\}, \end{aligned}$$

as claimed. ∎

Lemma 3.6 *Given a stable, LTI system $F \in \mathscr{P}_{\mathcal{U}\to\mathcal{X}}$ and a stable, LPTV system $P \in \mathscr{P}_{\mathcal{X}\to\mathcal{Y}}$, where $\mathcal{X}$ is some finite-dimensional, Hilbert space, the system $X := PF \in \mathscr{P}_{\mathcal{U}\to\mathcal{Y}}$ is stable and*

$$\overline{\sigma}\{\mathbf{\Theta}_X(j\omega)\} \le \overline{\sigma}\{\mathbf{\Theta}_P(j\omega)\}\overline{\sigma}\{\hat{F}(j\omega)\},$$

where $\mathbf{\Theta}.(j\omega)$ is the average-power gain matrix defined in Definition 3.3 and $\hat{F}$ is the standard, frequency-domain transfer-matrix of F.

Proof : Since F is LTI, its asymptotic response to an input $u(t) = e^{j\omega t}u_d$ is given by $y^{\omega}_{\text{as}}(t) = e^{j\omega t}\hat{F}(j\omega)u_d$. It then follows by the definition of $\mathbf{\Theta}.(\omega)$, that the average power of the asymptotic response of $X := PF$ to the sinusoidal input $u(t)$ is given by

$$\begin{aligned} \|y^{\omega}_{\text{as}}\|_{\text{pow}} &= \|\mathbf{\Theta}_X(j\omega)u_d\|_{\mathcal{U}} \\ &= \|\mathbf{\Theta}_P(j\omega)\hat{F}(j\omega)u_d\|_{\mathcal{X}} \\ &\le \overline{\sigma}\{\mathbf{\Theta}_P(j\omega)\}\overline{\sigma}\{\hat{F}(j\omega)\}\|u_d\|_{\mathcal{U}}, \end{aligned}$$

Then since u_d is an arbitrary vector in $\mathcal{U}$,

$$\overline{\sigma}\{\mathbf{\Theta}_X(j\omega)\} = \sup_{\substack{u_d\in\mathcal{U}\\ u_d\neq 0}} \frac{\|\mathbf{\Theta}_X(j\omega)u_d\|_{\mathcal{U}}}{\|u_d\|_{\mathcal{U}}} \le \overline{\sigma}\{\mathbf{\Theta}_P(j\omega)\}\overline{\sigma}\{\hat{F}(j\omega)\},$$

as required. ∎

Lemma 3.7 *Given a stable, LPTV system $P \in \mathscr{P}_{\mathcal{U}\to\mathcal{X}}$ and a stable, LTI system $F \in \mathscr{P}_{\mathcal{X}\to\mathcal{Y}}$, where $\mathcal{X}$ is some finite-dimensional Hilbert space, $X := FP$ is a stable system in $\mathscr{P}_{\mathcal{U}\to\mathcal{Y}}$ and*

$$\overline{\sigma}\{\mathbf{\Theta}_X(j\omega)\} \leq \overline{\sigma}^{\omega_0}_{\max}\{\hat{F}\}\overline{\sigma}\{\mathbf{\Theta}_P(j\omega)\}.$$

where $\mathbf{\Theta}_{\cdot}(j\omega)$ is the average-power gain matrix defined in Definition 3.3, $\hat{F}$ is the standard, frequency-domain transfer-matrix of F, $\overline{\sigma}^{\omega_0}_{\max}\{\cdot\}$ is defined in equation (3.6) and $\omega_0 := \omega - \left]\frac{h\omega}{2\pi}\right[\frac{2\pi}{h}$.

Proof : By Lemma 3.1, the asymptotic response of P to a sinusoidal input $u(t) = e^{j\omega t}u_d$ ($u_d \in \mathcal{U}$) is given by

$$x^{\omega}_{\text{as}}(t) = x^{\omega}_{\text{as}}(kh+\theta) = (e^{j\omega_0 h})^k\, \mathring{x}_d(\theta), \tag{3.11}$$

where $\mathring{x}_d(\theta) = (\hat{\mathsf{P}}(e^{-j\omega_0 h})e^{j\omega\tau}u_d)(\theta)$, $\omega_0 = \omega - \left]\frac{h\omega}{2\pi}\right[\frac{2\pi}{h}$ and $\hat{\mathsf{P}}$ is the transfer operator of the time-lifted, equivalent system $\mathsf{P} := \mathbf{W}P\mathbf{W}^{-1}$. Moreover, by Lemma 3.2 and the definition of $\mathbf{\Theta}_P(j\omega)$,

$$\|x^{\omega}_{\text{as}}\|_{\text{pow}} = \frac{1}{\sqrt{h}}\|\,\mathring{x}_d\|_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{X})} = \|\mathbf{\Theta}_P(j\omega)u_d\|_{\mathcal{U}}. \tag{3.12}$$

Note from equation (3.11), that $x^{\omega}_{\text{as}}(t)$ is of the form considered in Lemma 3.1. Then since the asymptotic response of X to $u(t)$ is the same as the asymptotic response of F to $x^{\omega}_{\text{as}}(t)$, it follows by Lemma 3.2 and the definition of $\mathbf{\Theta}_X(j\omega)$, that

$$\|y^{\omega}_{\text{as}}\|_{\text{pow}} = \|\mathbf{\Theta}_X u_d\|_{\mathcal{U}} = \frac{1}{\sqrt{h}}\|\hat{\mathsf{F}}(e^{-j\omega_0 h})\,\mathring{x}_d\|_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}, \tag{3.13}$$

where $\hat{\mathsf{F}}$ is the transfer operator of the time-lifted, equivalent system $\mathsf{F} := \mathbf{W}F\mathbf{W}^{-1}$. Then, from equations (3.12) and (3.13),

$$\begin{aligned}
\|y^{\omega}_{\text{as}}\|_{\text{pow}} &= \|\mathbf{\Theta}_X u_d\|_{\mathcal{U}} \\
&\leq \frac{1}{\sqrt{h}}\|\hat{\mathsf{F}}(e^{j\omega_0 h})\| \cdot \|\,\mathring{x}_d\|_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{X})} \\
&\leq \|\hat{\mathsf{F}}(e^{j\omega_0 h})\|\, \overline{\sigma}\{\mathbf{\Theta}_P(j\omega)\}\|u_d\|_{\mathcal{U}}.
\end{aligned}$$

Since u_d is an arbitrary vector in $\mathcal{U}$, it follows that

$$\begin{aligned}
\overline{\sigma}\{\mathbf{\Theta}_X(j\omega)\} = \sup_{\substack{u_d\in\mathcal{U}\\ u_d\neq 0}} \frac{\|\mathbf{\Theta}_X u_d\|_{\mathcal{U}}}{\|u_d\|_{\mathcal{U}}} &\leq \|\hat{\mathsf{F}}(e^{j\omega_0 h})\|\, \overline{\sigma}\{\mathbf{\Theta}_P(j\omega)\} \\
&= \overline{\sigma}^{\omega_0}_{\max}\{\hat{F}\}\, \overline{\sigma}\{\mathbf{\Theta}_P(j\omega)\}
\end{aligned}$$

where $\overline{\sigma}^{\omega_0}_{\max}\{\hat{F}\}$ is defined in equation (3.6). This completes the proof. ∎

3.3 A Framework for SD Closed-Loop Design

As mentioned previously, an intuitive procedure for the development of LTI, closed-loop systems is that of McFarlane and Glover [MG90]. It is based on classical loopshaping in the design step and an H^∞ optimisation in the synthesis step. By virtue of the structure of the generalised plant constructed and the optimisation problem solved, robustness to coprime-factor (gap) uncertainty and the deviation from desired performance can be readily quantified for closed-loop systems designed using this framework. A brief introduction to the H^∞-loopshaping procedure of McFarlane and Glover is given in Sub-Section 3.3.1. The main purpose of this section is to propose a H^∞-loopshaping based design procedure for SD control-system development. Details of the proposed procedure are presented in Sub-Section 3.3.2, in which the new notion of frequency response introduced for LPTV systems in Section 3.2, plays an important role.

3.3.1 H^∞-Loopshaping for LTI Systems

The design step of McFarlane and Glover's H^∞-loopshaping, design procedure is based on the notions of classical loopshaping. The basic idea in classical loopshaping is to trade off desired closed-loop performance and robustness objectives by shaping open-loop gains in the frequency-domain [DFT92, ZDG95]. Consider for example, the typical closed-loop configuration shown below, where P and C are causal, LTI systems. Simple algebraic manipulation leads to the following functional

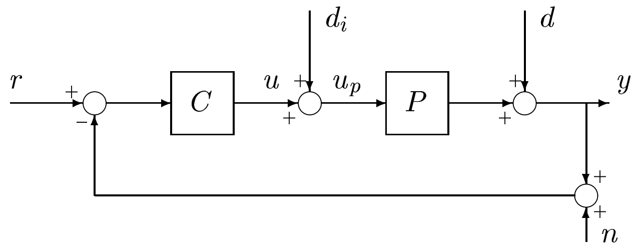

Figure 3.1 *Typical Closed-Loop Configuration*

characterisation of the closed-loop:

$$y = T_o(r-n) + S_oPd_i + S_od, \tag{3.14a}$$

$$r-y = S_o(r-d) + T_on - S_oPd_i, \tag{3.14b}$$

$$u = CS_o(r-n) - CS_od - T_id_i, \tag{3.14c}$$

$$u_p = CS_o(r-n) - CS_od + S_id_i, \tag{3.14d}$$

where $L_o := PC$, $L_i := CP$, $S_\bullet = (I + L_\bullet)^{-1}$ and $T_\bullet = I - S_\bullet$. Recall that the symbol "^" denotes transfer operator (matrix) of a system. Now from equation (3.14a), it can be seen that disturbance rejection at the plant output y can be achieved by making $\overline{\sigma}\{\hat{S}_o(j\omega)\} = \frac{1}{\underline{\sigma}\{I+\hat{L}_o(j\omega)\}}$ small in frequency ranges where d has significant power and $\overline{\sigma}\{\hat{S}_o\hat{P}(j\omega)\} = \overline{\sigma}\{\hat{P}\hat{S}_i(j\omega)\}$ small in frequency ranges where d_i has significant power. Similarly, from equation (3.14d), it is clear that disturbance rejection at the plant input u_p can be achieved by making $\overline{\sigma}\{\hat{S}_i(j\omega)\} = \frac{1}{\underline{\sigma}\{I+\hat{L}_i(j\omega)\}}$ small in frequency ranges where d_i has significant power and $\overline{\sigma}\{\hat{S}_i\hat{C}(j\omega)\} = \overline{\sigma}\{\hat{C}\hat{S}_o(j\omega)\}$ small in frequency ranges where d has significant power. Now since

$$\underline{\sigma}\{\hat{L}_o(j\omega)\} - 1 \leq \underline{\sigma}\{I + \hat{L}_o(j\omega)\} \leq 1 + \underline{\sigma}\{\hat{L}_o(j\omega)\}$$

and

$$\underline{\sigma}\{\hat{L}_i(j\omega)\} - 1 \leq \underline{\sigma}\{I + \hat{L}_i(j\omega)\} \leq 1 + \underline{\sigma}\{\hat{L}_i(j\omega)\},$$

it follows that if $\underline{\sigma}\{\hat{L}_o(j\omega)\} > 1$ and $\underline{\sigma}\{\hat{L}_i(j\omega)\} > 1$,

$$\frac{1}{1 + \underline{\sigma}\{\hat{L}_o(j\omega)\}} \leq \overline{\sigma}\{\hat{S}_o(j\omega)\} \leq \frac{1}{\underline{\sigma}\{\hat{L}_o(j\omega)\} - 1}$$

and

$$\frac{1}{1 + \underline{\sigma}\{\hat{L}_i(j\omega)\}} \leq \overline{\sigma}\{\hat{S}_i(j\omega)\} \leq \frac{1}{\underline{\sigma}\{\hat{L}_i(j\omega)\} - 1}.$$

Thus, disturbance rejection of d at y and d_i at u_p can be achieved by selecting a stabilising C to shape the minimal, open-loop gains $\underline{\sigma}\{\hat{L}_o(j\omega)\}$ and $\underline{\sigma}\{\hat{L}_i(j\omega)\}$ to be large in the frequency ranges where the disturbances d and d_i have significant power respectively. This should also be done in frequency regions (typically low-frequency) where good tracking performance is desired (cf. equation (3.14b)).

Similarly, it is desirable to design the stabilising C to achieve small maximal, open-loop gains $\overline{\sigma}\{\hat{L}_o(j\omega)\}$ and $\overline{\sigma}\{\hat{L}_i(j\omega)\}$ in frequency ranges where the sensor noise[3] n is significant and where high-levels of control activity are not acceptable (cf. equations (3.14)). A typical loop-shape is shown in Figure 3.2. In the H^∞-loopshaping procedure of McFarlane and Glover the plant P is shaped open-loop and a stabilising controller synthesised to solve an H^∞ optimisation problem. Briefly, the procedure can be summarised as follows:

(i) Using the notions of classical loopshaping, shape the plant's frequency-response matrix $\hat{P}(j\omega)$ open-loop with stable, pre- and post-compensators $\hat{W}_1(j\omega)$ and $\hat{W}_2(j\omega)$ according to the closed-loop objectives. There should be no unstable pole/zero cancellations between P and the weights. For notational convenience the shaped plant is denoted by $P_s := W_2PW_1$.

[3]This can also be used to characterise unmodelled, high-frequency dynamics.

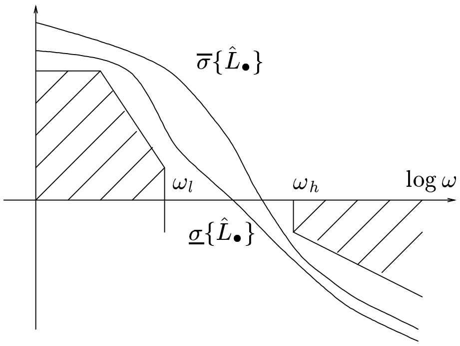

Figure 3.2 *Typical Loop-Shape*

(ii) Synthesise a stabilising controller C_∞ to achieve some specified bound on the induced-norm of the operator mapping the disturbances w_1 and w_2 to z_1 and z_2 of the closed-loop $[P_s, C_\infty]$ shown in Figure 3.3;

$$\left\| \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \mapsto \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} \hat{C}_\infty \\ I \end{bmatrix} (I - \hat{P}_s \hat{C}_\infty)^{-1} \begin{bmatrix} \hat{P}_s & I \end{bmatrix} \right\|_\infty =: b^{-1}_{P_s, C_\infty}.$$

If b_{P_s,C_∞} is sufficiently large (that is, > 0.25 but preferably 0.35) then continue to implement the controller. If b_{P_s,C_∞} is too small then redesign the weights.

(iii) The final controller C is then constructed by absorbing the weights W_1 and W_2 into the controller: $C := W_1 C_\infty W_2$.

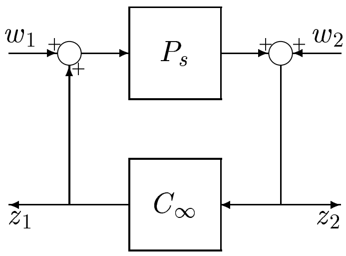

Figure 3.3 *Standard Loopshaping Design Configuration*

An advantage of using the McFarlane-Glover, loopshaping, design procedure over classical loop-shaping, is that the designer does not have to explicitly consider closed-loop stability. Of course, ignoring completely the fundamental objective of closed-loop stability (by not shaping the phase characteristic appropriately at cross-over for example) however, can lead to poor robustness as would be indicated by a small b_{P_s,C_∞}.

Note that $b^{-1}_{P_s,C_\infty}$ is a bound on all six common, performance-indicating, closed-loop operators, although tighter bounds are possible since the the closed-loop operators are not independent [Vin93b]. Furthermore, b_{P_s,C_∞} can be used to indicate the level of degradation of the desired loop-shape P_s and bounds on performance-indicating, closed-loop operators can be expressed in terms of P_s, W_1, W_2 and b_{P_s,C_∞} only [MG90]. For example, the input and output sensitivity of the closed-loop $[P, C] = [P, W_1 C_\infty W_2]$ are bounded by

$$\overline{\sigma}\{(I + \hat{P}\hat{C}(j\omega))^{-1}\} \leq \min\left\{b_{P_s,C_\infty}\overline{\sigma}\{\hat{M}_{l,s}(j\omega)\}\kappa\{\hat{W}_2(j\omega)\}, 1 + b_{P_s,C_\infty}\overline{\sigma}\{\hat{N}_{r,s}(j\omega)\}\kappa\{\hat{W}_2(j\omega)\}\right\} \quad (3.15a)$$

and

$$\overline{\sigma}\{(I + \hat{C}\hat{P}(j\omega))^{-1}\} \leq \min\left\{1 + b_{P_s,C_\infty}\overline{\sigma}\{\hat{N}_{l,s}(j\omega)\}\kappa\{\hat{W}_1(j\omega)\}, b_{P_s,C_\infty}\overline{\sigma}\{\hat{M}_{r,s}(j\omega)\}\kappa\{\hat{W}_1(j\omega)\}\right\}, \quad (3.15b)$$

where $P_s = N_{r,s}M^{-1}_{r,s} = M^{-1}_{l,s}N_{l,s}$ are normalised, right- and left-coprime factorisations[4] respectively and $\kappa\{\cdot\}$ denotes the condition number of a matrix. Since frequency-by-frequency

$$\overline{\sigma}\{\hat{M}_{l,s}\} = \overline{\sigma}\{\hat{M}_{r,s}\} = \left(\frac{1}{1 + \underline{\sigma}^2\{\hat{P}_s\}}\right)^{1/2} \quad \text{and} \quad \overline{\sigma}\{\hat{N}_{l,s}\} = \overline{\sigma}\{\hat{N}_{l,s}\} = \left(\frac{\overline{\sigma}^2\{\hat{P}_s\}}{1 + \overline{\sigma}^2\{\hat{P}_s\}}\right)^{1/2},$$

bounds such as those given in equations (3.15) give the designer a feel for how the weights selected in the design step affect the final closed-loop.

The H$^\infty$-loopshaping procedure of McFarlane-Glover is also appealing from the perspective of robust stability. By virtue of the generalised plant's structure and the optimisation problem solved in the synthesis step, it is possible to quantify the level of robustness of the resultant closed-loop system to coprime-factor uncertainty [MG90], which is equivalent to gap-metric uncertainty, as shown in [GS90]. The tightest bounds on degradation in performance and the strongest possible robustness results for the H$^\infty$-loopshaping framework are given in [Vin93b] in terms of the ν-gap metric, which induces the same topology on the class of LTI systems as the gap metric. For further information on H$^\infty$-loopshaping for LTI systems and its application in practice, the reader is referred to [Vin98, Pap98, MG90, Hyd95].

3.3.2 H$^\infty$-Loopshaping Based Design for SD Control-System Development

In this sub-section, a new procedure is proposed for the development of SD control-systems. The proposed procedure is related to the H$^\infty$-loopshaping procedure of McFarlane and Glover [MG90]

[4]For LTI systems, a right-coprime factorisation $N_{r,s}M^{-1}_{r,s}$ is said to be normalised if the frequency-domain symbol $\hat{G}_{r,s} = \begin{bmatrix} \hat{M}_{r,s} \\ \hat{N}_{r,s} \end{bmatrix}$ is inner (that is, if $\hat{G}_{r,s}(-s)^T\hat{G}_{r,s}(s) = I$) and a left-coprime factorisation $M^{-1}_{l,s}N_{l,s}$ is said to be normalised if frequency-domain symbol $\hat{G}_{l,s} = \begin{bmatrix} -\hat{M}_{l,s} & \hat{N}_{l,s} \end{bmatrix}$ is co-inner (that is, if $\hat{G}_{l,s}(s)\hat{G}_{l,s}(-s)^T = I$), where the superscript T denotes matrix transpose (without conjugation) (cf. [ZDG95] for example).

described above, with the main difference being a slight modification of the structure of the generalised plant constructed in the design step. Recall that in the final step of the McFarlane-Glover procedure, the weights used to shape the plant open-loop are absorbed into the synthesised controller. This is not possible if the controller is constrained to be SD, because the weights cannot be absorbed directly across the sampling and hold devices. The alternative of leaving the weights on the plant side of the sampling and hold devices defeats the purpose of using SD control, although using W_2 to shape a simple, stable, low-pass, LTI filter used to ensure boundedness of $\mathbf{A_D}$ for synthesis, is sensible (which is the strategy adopted in the formulation presented here). Importantly though, with $C_{\infty o} := W_1 C_\infty$ and $P_{so} := W_2 P$,

$$\left\| \begin{bmatrix} C_\infty \\ I \end{bmatrix} (I - P_s C_\infty)^{-1} \begin{bmatrix} I & P_s \end{bmatrix} \right\| = \left\| \begin{bmatrix} W_1^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} C_{\infty o} \\ I \end{bmatrix} (I - P_{so} C_{\infty o})^{-1} \begin{bmatrix} P_{so} & I \end{bmatrix} \begin{bmatrix} W_1 & 0 \\ 0 & I \end{bmatrix} \right\|.$$

Using this identity it follows that if W_1 is stably invertible, then synthesising a stabilising C_∞ to bound the first expression and absorbing W_1 at the output and W_2 at the input of this controller, is equivalent to synthesising a stabilising $C_{\infty o}$ to bound the second expression and absorbing the weight W_2 at the input of the resultant controller. In the context of SD control, the latter can be viewed schematically as shown in Figure 3.4, where $\mathbf{A_D}$ is an ideal, periodic (period h) sampling-device, $\mathbf{D_A}$ is a zeroth-order hold-device synchronised with $\mathbf{A_D}$ and F_b is a simple, stable, low-pass, LTI filter used to ensure bounded operation of the sampling device and to limit frequency-domain aliasing. The new design procedure for SD, control-system development is summarised below:

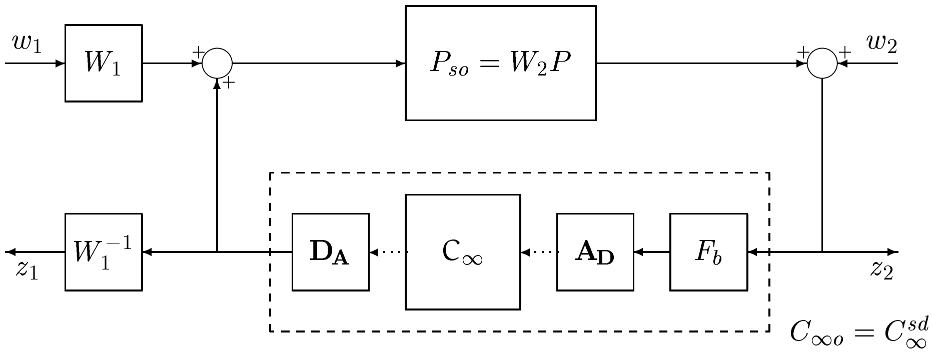

Figure 3.4 *SD Closed-Loop Design Configuration*

(i) Given an LTI plant $P \in \mathscr{P}^{\mathrm{e}}_{\mathbb{U}\to\mathbb{Y}}$, select the LTI weights W_1 and W_2 shown in Figure 3.4 to reflect closed-loop objectives, subject to the constraints that both W_1 and W_1^{-1} are stable and that there should be no unstable pole/zero cancellations between W_2 and P. Closed-loop performance bounds similar to those given in equation (3.15) for the LTI case, are obtained in Theorem 3.8 to facilitate the design of W_1 and W_2.

(ii) Synthesise a stabilising, LSI, discrete-time controller C_∞ to achieve some specified bound on the induced-norm of the closed-loop operator mapping the disturbances w_1 and w_2 to z_1 and

z_2 of the closed-loop configuration shown in Figure 3.4;

$$\left\| \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \mapsto \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} W_1^{-1} C_\infty^{sd} \\ I \end{bmatrix} (I - P_{so} C_\infty^{sd})^{-1} \begin{bmatrix} P_{so} W_1 & I \end{bmatrix} \right\| =: b^{-1}_{P_s, W_1^{-1} C_\infty^{sd}}, \quad (3.16)$$

where $P_{so} := W_2 P$, $P_s = P_{so} W_1$ and $C_\infty^{sd} := \mathbf{D_A} \mathsf{C}_\infty \mathbf{A_D} F_b$.

(iii) Shape the simple, stable, low-pass, LTI filter F_b with the weight W_2 to form the final controller $C := C_\infty^{sd} W_2$. Since by design there are no hidden unstable modes in $W_2 P$, moving W_2 around the loop in this manner does not affect closed-loop stability.

The synthesis step can be carried out as described in Section 2.5, by noting that the closed-loop operator mapping w_1 and w_2 to z_1 and z_2 in Figure 3.4 can be expressed in the linear-fractional form

$$\mathbf{F}_\ell(G, \mathbf{D_A} \mathsf{C}_\infty \mathbf{A_D}) = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \mapsto \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},$$

where

$$G := \begin{bmatrix} 0 & 0 & W_1^{-1} \\ P_{so} W_1 & I & P_{so} \\ F_b P_{so} W_1 & F_b & F_b P_{so} \end{bmatrix}.$$

Explicit formulae for H^∞, SD synthesis and related numerical issues are discussed in Chapter 6.

As in the LTI case, it is possible to quantify the robustness of SD control-systems designed using the proposed procedure. More specifically, in [CG96] it is shown that if $\beta \le b_{P_s, W_1^{-1} C_\infty^{sd}}$, all LTI plants in the weighted coprime-factor ball

$$\mathcal{N}_{NLCF}(P_s, \beta) := \left\{ \grave{P} = (M_{l,s} W_2 + \Delta_{M_l} W_2)^{-1} (N_{l,s} W_1^{-1} + \Delta_{N_l} W_1^{-1}) : \left\| \begin{bmatrix} \hat{\Delta}_{M_l} & \hat{\Delta}_{N_l} \end{bmatrix} \right\|_\infty < \beta \right\},$$

where $P_s = M_{l,s}^{-1} N_{l,s}$ is a normalised, left-coprime factorisation, are stabilised by the controller $C := C_\infty^{sd} W_2$. Furthermore, using results developed in Chapter 4, it can be shown (cf. Corollary 4.28) that all LPTV plants in the weighted gap-ball

$$\mathcal{N}^{\mathrm{e}}_{\delta_g, W_1, W_2}(P_s, \beta) := \{ \grave{P} \in \mathscr{P}^{\mathrm{e}}_{\mathfrak{U} \to \mathcal{Y}} \ : \ \delta_g(P_s, W_2 \grave{P} W_1) < \beta \}$$

are stabilised by $C := C_\infty^{sd} W_2$ if and only if $\beta \le b_{P_s, W_1^{-1} C_\infty^{sd}}$.

Since (cf. [DGS93] and Lemma 4.21)

$$\left\| \begin{bmatrix} W^{-1}C_{\infty}^{sd} \\ I \end{bmatrix} (I - P_{so}C_{\infty}^{sd})^{-1} \begin{bmatrix} P_{so}W_1 & I \end{bmatrix} \right\| = \left\| \begin{bmatrix} P_{so}W_1 \\ I \end{bmatrix} (I - C_{\infty}^{sd}P_{so})^{-1} \begin{bmatrix} W_1^{-1}C_{\infty}^{sd} & I \end{bmatrix} \right\|, \quad (3.17)$$

it is possible to bound the behaviour of all six common performance-indicating, closed-loop operators. Moreover, as for the LTI case (cf. equations (3.15)), it is possible to express bounds on the the average-power gain of a subset of these operators to sinusoidal inputs of a single frequency in terms of $P_s := W_2PW_1$, W_1, W_2 and $b_{P_s, W_1^{-1}C_{\infty}^{sd}}$ only. These bounds aid in the design of W_1 and W_2 in the first step of the proposed procedure and are summarised in the following theorem.[5]

Theorem 3.8 *Let*

$$\begin{aligned} W_2 S_o &:= W_2(I - PC)^{-1}, \\ S_i &:= (I - CP)^{-1} \end{aligned}$$

and

$$S_i C := (I - CP)^{-1}C,$$

where $C := C_{\infty}^{sd}W_2 = \mathbf{D_A}\mathsf{C}_{\infty}\mathbf{A_D}F_bW_2$. *Recall that these operators characterise the (weighted) tracking error, the plant-input disturbance to plant-output error and the sensor noise to plant-input error (cf. equations (3.14)). Then, the maximal, average-power gain of these performance-indicating, closed-loop operators in response to sinusoidal inputs of a single frequency can be bounded as follows:*

$$\overline{\sigma}\{\mathbf{\Theta}_{W_2S_o}(j\omega)\} \leq b^{-1}_{P_s, W_1^{-1}C_{\infty}^{sd}}\, \overline{\sigma}\{\hat{M}_s^l(j\omega)\}\overline{\sigma}\{\hat{W}_2(j\omega)\}, \quad (3.18a)$$

$$\overline{\sigma}\{\mathbf{\Theta}_{S_i}(j\omega)\} \leq b^{-1}_{P_s, W_1^{-1}C_{\infty}^{sd}}\, \frac{\overline{\sigma}^{\omega_0}_{\max}\{\hat{W}_1\hat{M}_{r,s}\}}{\underline{\sigma}\{\hat{W}_1(j\omega)\}} \quad (3.18b)$$

$$(3.18c)$$

and

$$\overline{\sigma}\{\mathbf{\Theta}_{S_iC}(j\omega)\} \leq b^{-1}_{P_s, W_1^{-1}C_{\infty}^{sd}}\, \overline{\sigma}^{\omega_0}_{\max}\{\hat{W}_1\hat{M}_{r,s}\}\overline{\sigma}\{\hat{W}_2(j\omega)\}, \quad (3.18d)$$

where $\mathbf{\Theta}$. *denotes the average-power gain matrix of a system (cf. Definition 3.3),* $\overline{\sigma}^{\omega_0}_{\max}\{\cdot\}$ *is defined in (3.6),* $P_s := W_2PW_1 = N_{r,s}(M_{r,s})^{-1} = (M_{l,s})^{-1}N_{l,s}$ *are normalised, right- and left-coprime factorisations and* $\omega_0 := \omega - \left]\frac{h\omega}{2\pi}\right[\frac{2\pi}{h}$.

[5]The operators for which bounds are given have been chosen according to their significance from a performance perspective. Such frequency-by-frequency bounds for closed-loop operators that reflect robustness are meaningless when the closed-loop is LPTV. To study robustness properties of LPTV systems in the frequency-domain, it is necessary to consider the asymptotic response of the system to infinite series of sinusoids with frequencies separated by integer multiples of the sampling frequency [Dul94]. Recall that such signals are asymptotically invariant for LPTV systems in the same way that sinusoids of single frequencies are asymptotically invariant for LTI systems.

Before proving this, the following technical result is required:

Lemma 3.9 *Let $\mathcal{W}$, $\mathcal{U}$ and $\mathcal{Y}$ be finite-dimensional Hilbert spaces. Given a stable, LPTV system $X \in \mathscr{P}_{\mathcal{W}\to\mathcal{Y}}$ and a stable LTI system G equivalent, via the Laplace-Transform isomorphism, to a multiplication operator $\mathbf{M}_{\hat{G}(s)}$ with symbol $\hat{G}(s) \in \mathrm{H}^{\infty}_{\mathbb{C}_+}(\mathcal{B}_{\mathcal{U}\to\mathcal{W}})$ that satisfies $\hat{G}(s)\hat{G}(-s)^T = I$ (i.e. $\hat{G}$ is co-inner), where the superscript T denotes matrix transpose, then $\|XG\| = \|X\|$. Similarly, if $\hat{G}(-s)^T\hat{G}(s) = I$ (i.e. $\hat{G}$ is inner) then $\|GX\| = \|X\|$.*

Proof : The proof is deferred to Appendix B.1. ∎

Proof : (of Theorem 3.8) Only the bound on $\overline{\sigma}\{\mathbf{\Theta}_{S_iC}(j\omega)\}$ is proved, since the other bounds follow in a similar manner. First note that by Lemma 3.9 and from equations (3.17) and (3.16),

$$\begin{aligned}&\left\|\begin{bmatrix} W_1^{-1}C_{\infty}^{sd} \\ I\end{bmatrix}(I - P_sW_1^{-1}C_{\infty}^{sd})^{-1}M_{l,s}^{-1}\right\| = \\ &\quad\left\|M_{r,s}^{-1}(I - W_1^{-1}C_{sd}^{\infty}P_s)^{-1}\begin{bmatrix} W_1^{-1}C_{sd}^{\infty} & I\end{bmatrix}\right\| = b^{-1}_{P_s,W_1^{-1}C_{\infty}^{sd}},\end{aligned} \tag{3.19}$$

where $P_s := W_2PW_1 = N_{r,s}(M_{r,s})^{-1} = (M_{l,s})^{-1}N_{l,s}$ are normalised, right- and left-coprime factorisations respectively. Now let

$$Q \quad := \quad M_{r,s}^{-1}(I - W_1^{-1}C_{\infty}^{sd}P_s)^{-1}W_1^{-1}C_{\infty}^{sd}$$

and $u := Qy$ with $y(t) = e^{j\omega t}y_d$ ($y_d \in \mathcal{Y}$). Defining $\vec{y}_d(\theta) = e^{j\omega\theta}y_d$, it follows by Lemma 3.2 and the definition of $\Theta_Q(j\omega)$, that

$$\begin{aligned}\|\mathbf{\Theta}_Q(j\omega)y_d\|_{\mathcal{Y}} &= \frac{1}{\sqrt{h}}\|\hat{\mathrm{Q}}(e^{-j\omega_0 h})\,\vec{y}_d\|_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})} \\ &\leq \|\hat{\mathrm{Q}}(e^{-j\omega_0 h})\|\|y_d\|_{\mathcal{Y}},\end{aligned} \tag{3.20}$$

where $\omega_o := \omega - \left]\frac{\omega h}{2\pi}\right[\frac{2\pi}{h}$ and $\hat{\mathrm{Q}}$ is the transfer operator of the LSI equivalent system $\mathrm{Q} := \mathbf{W}Q\mathbf{W}^{-1}$. Then since

$$\|\hat{\mathrm{Q}}(e^{-j\omega_0 h})\| \leq \|Q\| = \operatorname*{ess\,sup}_{\omega\in[0,2\pi)}\|\hat{\mathrm{Q}}(e^{-j\omega h})\|,$$

it follows from equation (3.19) that

$$\|\hat{\mathrm{Q}}(e^{-j\omega_0 h})\| \leq b^{-1}_{P_s,W_1^{-1}C_{\infty}^{sd}}.$$

So using equation (3.20),

$$\overline{\sigma}\{\mathbf{\Theta}_Q(j\omega)\} := \sup_{\substack{y_d\in\mathcal{Y}\\ y_d\neq 0}}\frac{\|\mathbf{\Theta}_Q(j\omega)y_d\|_{\mathcal{Y}}}{\|y_d\|_{\mathcal{Y}}} \leq \|\hat{\mathrm{Q}}(e^{-j\omega_0 h})\| \leq b^{-1}_{P_s,W_1^{-1}C_{\infty}^{sd}}.$$

Now note that $S_iC = W_1 M_{r,s} Q W_2$ and hence, by Lemmas 3.6 and 3.7, that

$$\begin{aligned}\overline{\sigma}\{\boldsymbol{\Theta}_{S_iC}(j\omega)\} &\leq \overline{\sigma}^{\omega_0}_{\max}\{\hat{W}_1\hat{M}_{r,s}\}\overline{\sigma}\{\boldsymbol{\Theta}_{Q}(j\omega)\}\overline{\sigma}\{\hat{W}_2(j\omega)\} \\ &\leq b^{-1}_{P_s,W_1^{-1}C^{sd}_\infty}\,\overline{\sigma}^{\omega_0}_{\max}\{\hat{W}_1\hat{M}_{r,s}\}\overline{\sigma}\{\hat{W}_2(j\omega)\},\end{aligned}$$

as claimed. This completes the proof. ∎

Remark 3.10 The bounds presented in Theorem 3.8 are similar to those given in [MG90] for the LTI case (cf. equations (3.15) for example). Note that the "tracking-error" bound is weighted by W_2. This was necessary to obtain a reasonable bound and does not pose a significant problem, since provided $\hat{W}_2(j\omega)$ is designed to roll-off at a frequency sufficiently greater than the bandwidth of the un-weighted open-loop P, the bound given in equation (3.18a) divided by $\underline{\sigma}\{W_2(j\omega)\}$ will not be too far from that of the actual maximal, tracking-error gain in frequency ranges where tracking is important. Furthermore, since frequency-by-frequency

$$\overline{\sigma}\{\hat{M}_{l,s}\} = \overline{\sigma}\{\hat{M}_{r,s}\} = \left(\frac{1}{1+\underline{\sigma}^2\{\hat{P}_s\}}\right)^{1/2} \quad\text{and}\quad \overline{\sigma}\{\hat{N}_{l,s}\} = \overline{\sigma}\{\hat{N}_{l,s}\} = \left(\frac{\overline{\sigma}^2\{\hat{P}_s\}}{1+\overline{\sigma}^2\{\hat{P}_s\}}\right)^{1/2},$$

it is possible to design $\hat{W}_1(j\omega)$ so that

$$\sup_\omega \overline{\sigma}\{\hat{W}_1\hat{M}_{r,s}(j\omega)\} \approx 1.$$

As such, the bounds derived on input-disturbance and sensor-noise rejection are sensible in the frequency ranges where they are important.

It is suggested that as a first step, the weights W_1 and W_2 be selected to be sensible in the context of standard H^∞ loopshaping for LTI control-system design, accounting for the simple, stable, low-pass, LTI filter used to ensure boundedness of $\mathbf{A_D}$ in the SD synthesis step. These can then be adjusted (using the bounds given in Theorem 3.8 as a guide) according to the sampling-period chosen or perhaps more simply, a sampling-period can be chosen so that these weights are sensible in the context of SD control. ♣

3.4 Summary

In this section a H^∞-loopshaping based design procedure has been established for SD, control-system development. To this end, a new notion of frequency response has been defined for LPTV systems. Specifically, a frequency-domain, analysis tool called the average-power, gain matrix has been defined, which as its name suggests, characterises the average-power gain of an LPTV system to sinusoidal inputs of a single frequency. This matrix is finite-dimensional (can be computed as described in Appendix B.2) and retains input-directional information. By the way it is defined the new, frequency-domain, analysis tool has clear performance-indicating value, although it should be pointed out that it would not be suitable for robustness analysis in the frequency-domain.

4

Robustness Analysis of LPTV Closed-Loop Systems

4.1 Introduction

Motivation for this chapter stems from a desire to characterise the robustness properties of SD control-systems developed using the design procedure proposed in Chapter 3. The approach taken is to study a general class of LPTV, closed-loop systems, of which the SD control-systems considered in Chapter 3 are a special case.

One of the important properties of closed-loop systems is that they can be desensitised to uncertainty. This property is often referred to as robustness. This is particularly important from the perspective of model-based design, since any mathematical model used to describe a system's behaviour is uncertain to some degree. The question is: how to qualify and quantify the uncertainty a closed-loop system can tolerate? The answer is to define a suitable topology on the set of systems concerned and a metric to induce this topology. An appropriate topology must capture all perturbations to closed-loop components that do not cause instability, thus qualifying permissible uncertainty. Furthermore, since performance is also an important objective when designing closed-loop systems, suitable indicators of performance must vary continuously in the topology selected.

For stable systems, a topology suitable for studying robustness is the induced-norm topology, which lays the foundation for H^∞ control-theory as developed in the seminal paper [Zam81]. A short-coming of the induced-norm topology is the inability to consider explicitly, unstable components of a given closed-loop. This led to the development of the graph topology for robustness analysis [VSF83, Vid84, ZES80]. The foundations for robustness analysis in the graph topology were established in [VSF83], where a topology deemed suitable for this was proposed in the following way. Consider the closed-loop system shown in Figure 4.1, which is said to be stable if the operator $H(P,C) := \begin{bmatrix} I & C \\ P & I \end{bmatrix}^{-1}$ mapping $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ to $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is causal and bounded. A topology on a set of plants (or controllers) suitable for robustness analysis is one that satisfies the following properties: given any plant P in the set of interest and a stabilising controller C, there exists a neighbourhood of P (in the topology) such that all plants in this neighbourhood are also stabilised by C and the mapping $P \mapsto H(P,C)$ is continuous at P (with respect to the topology on the set of plants concerned and the induced norm topology on the closed-loop system set). That is, for any sequence $\{P_i\}$ converging to P (with respect to a topology with these properties), all but a finite number of the plants in that sequence must be stabilised by C and $H(P_i,C) \to H(P,C)$ in the induced norm topology.

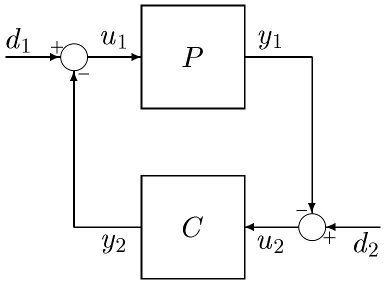

Figure 4.1 *Standard Closed-Loop Configuration*

For certain classes of system, a topology that satisfies the properties described above can be characterised explicitly in terms of the graph representation of a system. Hence, such a topology is often called the graph topology [Vid85]. For systems known to admit fractional representations that are coprime (such as LTI systems [Vid85, Smi89] for example), a basis for the graph topology can be constructed by additively perturbing a representation of the graph derived from coprime factorisations [Vid84]. Qualitatively, two systems are "close" in the graph topology if their graphs are "close". It is shown in [Vid84], that for LTI systems the graph topology (defined in terms of the basis just described) is the weakest with respect to which closed-loop performance varies continuously and closed-loop stability is a robust property.

The topological aspects of robustness analysis described above qualify the types of uncertainty to which closed-loop systems can be desensitised. To quantify a level of robustness requires a metric to induce the graph topology. For LTI systems a number of metrics have this property, including: the graph metric [Vid84]; the gap metric [ZES80, ES85, Geo88, GS90, OS91]; and the ν-gap metric [Vin93a]. In this chapter attention is restricted to the gap metric, which is defined below. Given two operators defined on the same ambient Hilbert spaces, the distance between them can be defined in terms of the aperture between their graphs [Kat66]. More specifically, the gap between two (closed) linear operators P_0 and P_1 is defined to be

$$\delta_g(P_0, P_1) := \|\mathbf{\Pi}_{\mathcal{G}_{P_0}} - \mathbf{\Pi}_{\mathcal{G}_{P_1}}\|,$$

where $\mathbf{\Pi}_{\mathcal{G}_{P_i}}$ denotes the orthogonal projection onto $\mathcal{G}_{P_i}$. This is a metric and as for any two (closed) subspaces of a Hilbert space, can be expressed as the maximum of two directed gaps $\vec{\delta}_g(P_i, P_j) := \|(\mathrm{I} - \mathbf{\Pi}_{\mathcal{G}_{P_i}})\mathbf{\Pi}_{\mathcal{G}_{P_j}}\|$ [KVZ+72].

All of the robustness results derived in this chapter, are obtained in terms of the gap metric. To this end, a formula is derived for the directed gap between two LPTV systems, which is expressed in terms of a particular representation of the graph and an H^∞ optimisation. The formula obtained is essentially a generalisation of that given in [Geo88] for LTI systems and is fundamental to the framework used in the sequel to quantify and qualify robustness of LPTV, closed-loop systems. The main quantitative robust stability results presented concern robustness to perturbations measured in the gap metric and are to some extent analogous to the results of [GS90] for LTI systems and those of [FGS93] for linear, time-varying (LTV) systems, although they do not follow from either directly. In particular, problems concerning causality are not addressed in [FGS93] and even though the periodic nature of the time-variation considered here imparts a high degree of structure to the analysis, it is somewhat more involved than in the time-invariant case. Qualitatively, it is shown that the gap metric induces on a general class of LPTV systems, the weakest topology with respect to which closed-loop performance varies continuously and closed-loop stability is a robust property. All of the results obtained accommodate possibly infinite-dimensional input and output spaces and importantly, can be applied to LPTV, SD control-systems as a special case.

The existence of particular representations of the graph of a class of LPTV systems is fundamental to the robustness results obtained. Specifically, is it shown that the graph of a stabilisable, LPTV system can be expressed as the range and kernel of stable, LPTV systems that are respectively, left and right invertible by stable, LPTV systems. These so-called strong-right and strong-left representations resemble the coprime-factor representations known to exist for LTI systems [Vid85]. Given a well-posed plant/controller pair $[P, C]$, a useful characterisation of closed-loop stability is obtained in terms of the strong-right and strong-left representations of $\mathcal{G}_P$ and $\mathcal{G}_C$. In turn, this leads to a Youla-style parameterisation of stabilising controllers.

Briefly, the chapter is structured as follows: Section 4.2 is devoted to presenting results concerning the representation of LPTV systems and closed-loop stability; in Section 4.3, the gap metric is introduced and a formula for the directed gap between two LPTV systems is derived; in Section 4.4, quantitative robustness results are obtained in terms of the gap metric and in Section 4.5, it is shown that the gap metric induces a topology on a class of LPTV systems with the properties of the graph topology describe above; in Section 4.6, results obtained in preceding sections are applied to the special case of a sampled-data-controlled, closed-loop systems; finally, a brief summary is given in Section 4.7.

4.2 Representations of LPTV Systems

In this section, various representations of LPTV systems are developed. Fundamental to the results presented, is the equivalence, via the time-lifting isomorphism, of LPTV, continuous-time systems to LSI, discrete-time systems (cf. Proposition 2.4 and [BPFT91, BP92]).

4.2.1 LPTV Systems and LSI Equivalents

Let $\mathcal{U}$ and $\mathcal{Y}$ be (possibly infinite-dimensional) Hilbert spaces and for notational convenience let $\mathcal{V} := \mathcal{U} \oplus \mathcal{Y}$. Recall that the graph of an operator $P : \mathcal{D}_P \subset \mathcal{U} \to \mathcal{Y}$ is defined to be the set of all bounded input-output pairs

$$\mathcal{G}_P := \begin{bmatrix} I \\ P \end{bmatrix} \mathcal{D}_P \subset \mathcal{V},$$

and that for notational convenience, the inverse graph is defined to be

$$\mathcal{G}_P' := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \mathcal{G}_P = \begin{bmatrix} P \\ I \end{bmatrix} \mathcal{D}_P \subset \mathcal{Y} \oplus \mathcal{U}.$$

Furthermore, recall that a linear subspace $\mathcal{G} \subset \mathcal{V}$ is the graph of a linear operator if and only if

$$\begin{bmatrix} 0 \\ y \end{bmatrix} \in \mathcal{G} \quad \Rightarrow \quad y = 0.$$

Note that the bounded behaviour of an operator is completely characterised by its graph and as such, graphs prove to be a useful representation of systems, particularly when studying the stability of closed-loops (cf. Sub-Section 2.3.3).

In this chapter, attention is restricted to LPTV systems. In particular, the class of systems considered is defined in Definition 2.3 of Sub-Section 2.3.2, which for convenience is re-stated below:

Definition 4.1 Let $\mathcal{U}$ and $\mathcal{Y}$ be (possibly infinite-dimensional) Hilbert spaces. Then $\mathscr{P}_{\mathcal{U}\to\mathcal{Y}}$ is defined to be the set of causal, LPTV (with period h), continuous-time systems

$$P : \mathcal{D}_P \subset \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U}) \to \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{Y}),$$

with closed graphs. Furthermore, $\mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}} \subset \mathscr{P}_{\mathcal{U}\to\mathcal{Y}}$ is defined to be the subset of causally-extendible systems with locally-Lipschitz-continuous extension and $\mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U}\to\mathcal{Y}}$ the subset of strongly-causal systems in $\mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$. A system in $\mathscr{P}_{\mathcal{U}\to\mathcal{Y}}$ is said to be stable if and only if $\mathcal{D}_P = \mathrm{L}^2_{\mathbb{R}+}(\mathcal{U})$ and $\|P\| < \infty$. Note that any stable system $P \in \mathscr{P}_{\mathcal{U}\to\mathcal{Y}}$ is an element of $\mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$. ♠

Recall the important property of LPTV systems, by which they are equivalent via the time-lifting isomorphism to LSI systems (cf. Proposition 2.4 and [BP92, BPFT91]). More precisely, each $P \in \mathscr{P}_{\mathcal{U}\to\mathcal{Y}}$ is equivalent (via $\mathbf{W}$) to a system in the set $\mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}$ defined in Definition 2.5 of Sub-Section 2.3.2. This equivalence is used extensively throughout this chapter.

Remark 4.2 Recall that a system $\mathsf{P} \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}$ is equivalent to a system $P \in \mathscr{P}_{\mathcal{U}\to\mathcal{Y}}$ (via $\mathbf{W}$) if and only if $\mathsf{P}_{[0]}$ is a causal map from $\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})$ to $\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})$, where $\mathsf{P}_{[0]}$ is the first element of the sequence uniquely identifiable with the Toeplitz representation of P. A system $\mathsf{P} \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}$ not satisfying this condition is equivalent (via $\mathbf{W}$) to a continuous-time, LPTV system that is not locally causal in each $[kh, (k+1)h)$ interval of time ($k \in \mathbb{Z}^+$). ♣

As previously mentioned, the graph is a useful representation of a system. Since an LPTV system is equivalent (via $\mathbf{W}$) to an LSI system, its graph is isomorphic to an LSI subspace. Motivated by this, the following sub-section is concerned with the characterisation of LSI subspaces. To this end, the Beurling-Lax-Halmos Theorem [Lax59, RR85, FF90] is used, which briefly, states that any LSI subspace of $\mathrm{H}^2_{\mathbb{D}}$ can be expressed as the range of a multiplication operator with inner symbol. This result is extended to show that if the LSI subspace is LSI co-ordinatisable, then it can be expressed as the kernel of a multiplication operator with symbol in $\mathrm{H}^\infty_{\mathbb{D}}$. Furthermore, these range and kernel representations are respectively right and left invertible by multiplication operators with symbols in $\mathrm{H}^\infty_{\mathbb{D}}$. Using this, range and kernel representations (with (causal) stable, left and right inverses respectively) of the graphs of stabilisable systems in $\mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ can be constructed.

4.2.2 LSI Subspaces

Let $\mathcal{G} \subset \mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$ be closed and LSI, where $\mathcal{H}$ is an arbitrary Hilbert space. Then $\mathcal{G}$ is said to be LSI co-ordinatisable if there exists a closed, LSI subspace $\mathcal{F} \subset \mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$ such that $\{\mathcal{G}, \mathcal{F}\}$ is a co-ordinatisation of $\mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$. That is, if $\mathcal{G} \cap \mathcal{F} = \{0\}$ and $\mathcal{G} + \mathcal{F} = \mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$. Recall that if $\{\mathcal{G}, \mathcal{F}\}$ is a co-ordinatisation of $\mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$, then any $\eta \in \mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$ can be uniquely decomposed as the sum $\eta = \gamma + \varphi$, where $\gamma \in \mathcal{G}$ and $\varphi \in \mathcal{F}$ (cf. [DGS93]). Furthermore, recall that the operator $\mathbf{\Pi}_{\mathcal{G}\|\mathcal{F}} := \eta \mapsto \gamma$ is called the parallel projection onto $\mathcal{G}$ along $\mathcal{F}$ and similarly, that $\mathbf{\Pi}_{\mathcal{F}\|\mathcal{G}} := \eta \mapsto \varphi$ is called the parallel projection onto $\mathcal{F}$ along $\mathcal{G}$. Since $\mathcal{G}$ and $\mathcal{F}$ are closed, linear subspaces it follows that the graphs of the parallel projection operators are linear and closed. Hence, by the Closed Graph Theorem

[Bol90, pg. 80], the parallel projection operators are bounded and clearly linear. That they are also shift-invariant, is immediate because $\mathbf{S}\mathcal{G} \subset \mathcal{G}$ and $\mathbf{S}\mathcal{F} \subset \mathcal{F}$.

Proposition 4.3 (Beurling-Lax-Halmos Theorem) [Lax59, RR85, FF90] *Given an LSI subspace $\mathcal{S}$ of $\mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$ there exists an inner function $\Sigma_r \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{H}_1 \to \mathcal{H}})$ such that $\mathcal{S} = \Sigma_r \mathrm{H}^2_{\mathbb{D}}(\mathcal{H}_1)$, where $\mathcal{H}_1$ can be any Hilbert space isomorphic to $\mathcal{S} \ominus \mathbf{S}\mathcal{S}$.*

So corresponding to the LSI subspaces $\mathcal{G}$ and $\mathcal{F}$ discussed above, there exist inner functions $\Gamma_r \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{H}_1 \to \mathcal{H}})$ and $\Phi_r \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{H}_2 \to \mathcal{H}})$ such that $\mathcal{G} = \Gamma_r \mathrm{H}^2_{\mathbb{D}}(\mathcal{H}_1)$ and $\mathcal{F} = \Phi_r \mathrm{H}^2_{\mathbb{D}}(\mathcal{H}_2)$, where $\mathcal{H}_1$ is some Hilbert space isomorphic to $\mathcal{G} \ominus \mathbf{S}\mathcal{G}$ and $\mathcal{H}_2$ any Hilbert space isomorphic to $\mathcal{F} \ominus \mathbf{S}\mathcal{F}$. Note that since $\mathbf{M}_{\Gamma_r}$ and $\mathbf{M}_{\Phi_r}$ are isometries (cf. Proposition 2.2), the orthogonal projections $\mathbf{\Pi}_{\mathcal{G}}$ onto $\mathcal{G}$ and $\mathbf{\Pi}_{\mathcal{F}}$ onto $\mathcal{F}$ can be expressed as $\mathbf{M}_{\Gamma_r}(\mathbf{M}_{\Gamma_r})^*$ and $\mathbf{M}_{\Phi_r}(\mathbf{M}_{\Phi_r})^*$ respectively. This fact is used in the proof of the following technical result, which in turn, is fundamental to the main Theorem of this sub-section.

Lemma 4.4 *Given two closed, LSI subspaces $\mathcal{G}$ and $\mathcal{F}$ that induce a co-ordinatisation of $\mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$ and inner functions $\Gamma_r \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{H}_1 \to \mathcal{H}})$ and $\Phi_r \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{H}_2 \to \mathcal{H}})$ such that $\mathcal{G} = \Gamma_r \mathrm{H}^2_{\mathbb{D}}(\mathcal{H}_1)$ and $\mathcal{F} = \Phi_r \mathrm{H}^2_{\mathbb{D}}(\mathcal{H}_2)$, the operators $(\mathbf{M}_{\Gamma_r})^* \mathbf{\Pi}_{\mathcal{G}\|\mathcal{F}}$ and $(\mathbf{M}_{\Phi_r})^* \mathbf{\Pi}_{\mathcal{F}\|\mathcal{G}}$ are bounded and LSI.*

Proof : It is shown that $(\mathbf{M}_{\Gamma_r})^* \mathbf{\Pi}_{\mathcal{G}\|\mathcal{F}}$ is bounded and LSI, the proof that $(\mathbf{M}_{\Phi_r})^* \mathbf{\Pi}_{\mathcal{F}\|\mathcal{G}}$ is bounded and LSI follows similarly. $(\mathbf{M}_{\Gamma_r})^* \mathbf{\Pi}_{\mathcal{G}\|\mathcal{F}}$ is clearly linear and bounded on $\mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$ since it is the product of two bounded, linear operators. To see that it is also shift invariant, note that any $\eta \in \mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$ is uniquely expressible as $\eta = \gamma + \varphi$, where $\gamma = \mathbf{\Pi}_{\mathcal{G}\|\mathcal{F}}\eta \in \mathcal{G}$ and $\varphi = \mathbf{\Pi}_{\mathcal{F}\|\mathcal{G}}\eta \in \mathcal{F}$. Let ζ be the unique element in $\mathrm{H}^2_{\mathbb{D}}(\mathcal{H}_1)$ corresponding to $\gamma = \mathbf{M}_{\Gamma_r}\zeta$. Then since $\mathbf{M}_{\Gamma_r}$ is an isometry it follows that $(\mathbf{M}_{\Gamma_r})^* \mathbf{\Pi}_{\mathcal{G}\|\mathcal{F}}\eta = \zeta$. Furthermore, since $\mathbf{S}\eta = \mathbf{S}\gamma + \mathbf{S}\varphi$, $\mathbf{S}\gamma = \mathbf{M}_{\Gamma_r}\mathbf{S}\zeta \in \mathcal{G}$ and $\mathbf{S}\varphi \in \mathcal{F}$ it is clear that

$$(\mathbf{M}_{\Gamma_r})^* \mathbf{\Pi}_{\mathcal{G}\|\mathcal{F}} \mathbf{S}\eta = (\mathbf{M}_{\Gamma_r})^* \mathbf{S}\gamma = \mathbf{S}\zeta = \mathbf{S}(\mathbf{M}_{\Gamma_r})^* \mathbf{\Pi}_{\mathcal{G}\|\mathcal{F}}\eta.$$

That is, $(\mathbf{M}_{\Gamma_r})^* \mathbf{\Pi}_{\mathcal{G}\|\mathcal{F}}$ is also shift invariant. ∎

Theorem 4.5 *Given a closed, LSI subspace $\mathcal{G} \subset \mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$ that is LSI co-ordinatisable, there exist functions $\Gamma_r \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{H}_1 \to \mathcal{H}})$, $\Gamma_l \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{H} \to \mathcal{H}_2})$, $\Phi_r \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{H}_2 \to \mathcal{H}})$ and $\Phi_l \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{H} \to \mathcal{H}_1})$ such that*

$$\mathcal{G} = \mathcal{R}_{\mathbf{M}_{\Gamma_r}} = \mathcal{K}_{\mathbf{M}_{\Gamma_l}},$$

$$\begin{bmatrix} \mathbf{M}_{\Gamma_r} & \mathbf{M}_{\Phi_r} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{\Phi_l} \\ \mathbf{M}_{\Gamma_l} \end{bmatrix} = \mathrm{I} \quad \textit{and} \quad \begin{bmatrix} \mathbf{M}_{\Phi_l} \\ \mathbf{M}_{\Gamma_l} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{\Gamma_r} & \mathbf{M}_{\Phi_r} \end{bmatrix} = \begin{bmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{bmatrix},$$

where $\mathcal{H}_1$ can be any Hilbert space isomorphic to $\mathcal{G} \ominus \mathbf{S}\mathcal{G}$, $\mathcal{H}_2$ any Hilbert space isomorphic to $\mathcal{F} \ominus \mathbf{S}\mathcal{F}$ and $\mathcal{F}$ is any closed, LSI subspace such that $\{\mathcal{G}, \mathcal{F}\}$ is a co-ordinatisation of $\mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$.

Proof : Let $\{\mathcal{G}, \mathcal{F}\}$ be a co-ordinatisation of $\mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$ and the operators $\mathbf{\Pi}_{\mathcal{G}||\mathcal{F}}$, $\mathbf{\Pi}_{\mathcal{F}||\mathcal{G}}$, $\mathbf{M}_{\Gamma_r}$ and $\mathbf{M}_{\Phi_r}$ be defined as above, recalling that $\mathcal{G} = \mathcal{R}_{\mathbf{M}_{\Gamma_r}}$ and $\mathcal{F} = \mathcal{R}_{\mathbf{M}_{\Phi_r}}$. Furthermore, note that

$$\mathbf{\Pi}_{\mathcal{G}||\mathcal{F}} = \mathbf{\Pi}_{\mathcal{G}}\mathbf{\Pi}_{\mathcal{G}||\mathcal{F}} = \mathbf{M}_{\Gamma_r}(\mathbf{M}_{\Gamma_r})^*\mathbf{\Pi}_{\mathcal{G}||\mathcal{F}} \tag{4.1}$$

and

$$\mathbf{\Pi}_{\mathcal{F}||\mathcal{G}} = \mathbf{\Pi}_{\mathcal{F}}\mathbf{\Pi}_{\mathcal{F}||\mathcal{G}} = \mathbf{M}_{\Phi_r}(\mathbf{M}_{\Phi_r})^*\mathbf{\Pi}_{\mathcal{F}||\mathcal{G}}. \tag{4.2}$$

Now by Lemma 4.4, $(\mathbf{M}_{\Gamma_r})^*\mathbf{\Pi}_{\mathcal{G}||\mathcal{F}}$ and $(\mathbf{M}_{\Phi_r})^*\mathbf{\Pi}_{\mathcal{F}||\mathcal{G}}$ are bounded, LSI operators on $\mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$ and hence, can be expressed as multiplication operators with symbols $\Phi_l \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{H}\to\mathcal{H}_1})$ and $\Gamma_l \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{H}\to\mathcal{H}_2})$ respectively (cf. Proposition 2.2). That is,

$$\mathbf{M}_{\Phi_l} = (\mathbf{M}_{\Gamma_r})^*\mathbf{\Pi}_{\mathcal{G}||\mathcal{F}} \quad \text{and} \quad \mathbf{M}_{\Gamma_l} = (\mathbf{M}_{\Phi_r})^*\mathbf{\Pi}_{\mathcal{F}||\mathcal{G}}. \tag{4.3}$$

Then since $\mathcal{F} = \mathcal{R}_{\mathbf{M}_{\Phi_r}}$, it follows that $\mathcal{R}_{\mathbf{\Pi}_{\mathcal{F}||\mathcal{G}}}$ is orthogonal to $\mathcal{K}_{(\mathbf{M}_{\Phi_r})^*}$ and consequently, that $\mathcal{K}_{\mathbf{M}_{\Gamma_l}} = \mathcal{K}_{\mathbf{\Pi}_{\mathcal{F}||\mathcal{G}}} = \mathcal{G}$. Similarly, $\mathcal{F} = \mathcal{K}_{\mathbf{M}_{\Phi_l}}$. So it remains to show that

$$\begin{bmatrix} \mathbf{M}_{\Gamma_r} & \mathbf{M}_{\Phi_r} \end{bmatrix}\begin{bmatrix} \mathbf{M}_{\Phi_l} \\ \mathbf{M}_{\Gamma_l} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{\Phi_l} \\ \mathbf{M}_{\Gamma_l} \end{bmatrix}\begin{bmatrix} \mathbf{M}_{\Gamma_r} & \mathbf{M}_{\Phi_r} \end{bmatrix} = \begin{bmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{bmatrix}.$$

It follows from the relationship $\mathbf{\Pi}_{\mathcal{G}||\mathcal{F}} + \mathbf{\Pi}_{\mathcal{F}||\mathcal{G}} = \mathrm{I}$ and equations (4.1-4.3), that

$$\begin{bmatrix} \mathbf{M}_{\Gamma_r} & \mathbf{M}_{\Phi_r} \end{bmatrix}\begin{bmatrix} \mathbf{M}_{\Phi_l} \\ \mathbf{M}_{\Gamma_l} \end{bmatrix} = \mathrm{I}. \tag{4.4}$$

Now since $\mathcal{G} = \mathcal{R}_{\mathbf{M}_{\Gamma_r}} = \mathcal{K}_{\mathbf{M}_{\Gamma_l}}$, $\mathcal{F} = \mathcal{R}_{\mathbf{M}_{\Phi_r}} = \mathcal{K}_{\mathbf{M}_{\Phi_l}}$ and $\{\mathcal{G}, \mathcal{F}\}$ is a co-ordinatisation of $\mathrm{H}^2_{\mathbb{D}}(\mathcal{H})$, both $\begin{bmatrix} \mathbf{M}_{\Gamma_r} & \mathbf{M}_{\Phi_r} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{M}_{\Phi_l} \\ \mathbf{M}_{\Gamma_l} \end{bmatrix}$ have zero kernel. Thus, they are the inverse of each other (cf. equation (4.4)) and

$$\begin{bmatrix} \mathbf{M}_{\Phi_l} \\ \mathbf{M}_{\Gamma_l} \end{bmatrix}\begin{bmatrix} \mathbf{M}_{\Gamma_r} & \mathbf{M}_{\Phi_r} \end{bmatrix} = \begin{bmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{bmatrix},$$

which completes the proof. ∎

4.2.3 Strong-Right/Left Representations and Closed-Loop Stability

Results derived in the previous sub-section are now used to develop range and kernel representations of the graphs of stabilisable systems in $\mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$, that are themselves (causal) stable, LPTV systems with (causal) stable, LPTV, left and right inverses respectively. All notions of well-posedness and closed-loop stability used in this sub-section are defined in Sub-Section 2.3.2. Recall the standard, closed-loop configuration, denoted $[P, C]$, shown in Figure 4.2. The following result characterises the graph of a stabilisable, LPTV, continuous-time system in terms the range and kernel of stable, LPTV systems that are respectively, left and right invertible by stable, LPTV systems. These representations resemble the coprime-factor representations known to exist for LTI systems (cf. [Vid84, Smi89]) and linear, time-varying, discrete-time systems with finite-dimensional input-output spaces (cf. [DS93]).

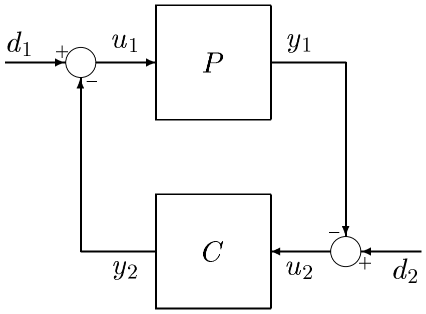

Figure 4.2 *Standard Feedback Configuration*

Theorem 4.6 *If $P \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ can be stabilised by some $C \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}}$, then there exist <u>stable</u> systems $G_r \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{V}}$,[1] $G_l \in \mathscr{P}^{\mathrm{e}}_{\mathcal{V}\to\mathcal{Y}}$, $K_l \in \mathscr{P}^{\mathrm{e}}_{\mathcal{V}\to\mathcal{U}}$ and $K_r \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{V}}$ such that*

$$\mathcal{G}_P = \mathcal{R}_{G_r} = \mathcal{K}_{G_l},$$

and

$$\begin{bmatrix} G_r & K_r \end{bmatrix}\begin{bmatrix} K_l \\ G_l \end{bmatrix} = \begin{bmatrix} K_l \\ G_l \end{bmatrix}\begin{bmatrix} G_r & K_r \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Proof : By assumption, P is stabilised by some $C \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$. So by Proposition 2.12 and Remark 2.13, it follows that $\mathcal{G}_P$ and $\mathcal{G}_C$ induce a co-ordinatisation of $\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{V})$. Let $\mathsf{P} := \mathbf{W}P\mathbf{W}^{-1}$ and $\mathsf{C} := \mathbf{W}C\mathbf{W}^{-1}$, noting that that $\mathcal{G}_{\mathsf{P}}$ and $\mathcal{G}'_{\mathsf{C}}$ are LSI subspaces of $\ell^2_{\mathbb{Z}^+}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{V}))$. These are respectively, isomorphic (via $\mathbf{Z}$) to the LSI subspaces $\hat{\mathcal{G}}_{\mathsf{P}} := \mathbf{Z}\mathcal{G}_{\mathsf{P}}$ and $\hat{\mathcal{G}}'_{\mathsf{C}} := \mathbf{Z}\mathcal{G}'_{\mathsf{C}}$ of $\mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{V}))$, which since $\mathcal{G}_P$ and $\mathcal{G}_C$ induce a co-ordinatisation of $\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{V})$, induce a co-ordinatisation of $\mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{V}))$. So by Theorem 4.5, there exist functions $\hat{\mathsf{G}}_r \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{H}_1\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})})$, $\hat{\mathsf{G}}_l \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})\to\mathcal{H}_2})$, $\hat{\mathsf{K}}_l \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})\to\mathcal{H}_1})$ and $\hat{\mathsf{K}}_r \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathcal{H}_2\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})})$, such that

$$\hat{\mathcal{G}}_{\mathsf{P}} = \mathcal{R}_{\mathbf{M}_{\hat{\mathsf{G}}_r}} = \mathcal{K}_{\mathbf{M}_{\hat{\mathsf{G}}_l}}$$

[1]Recall that $\mathcal{V} := \mathcal{U} \oplus \mathcal{Y}$.

and

$$\begin{bmatrix} \mathbf{M}_{\hat{\mathsf{G}}_r} & \mathbf{M}_{\hat{\mathsf{K}}_r} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{\hat{\mathsf{K}}_l} \\ \mathbf{M}_{\hat{\mathsf{G}}_l} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{\hat{\mathsf{K}}_l} \\ \mathbf{M}_{\hat{\mathsf{G}}_l} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{\hat{\mathsf{G}}_r} & \mathbf{M}_{\hat{\mathsf{K}}_r} \end{bmatrix} = \begin{bmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{bmatrix},$$

where $\mathcal{H}_1$ is a Hilbert space isomorphic to $\hat{\mathcal{G}}_{\mathsf{P}} \ominus \mathbf{S}\hat{\mathcal{G}}_{\mathsf{P}}$ and $\mathcal{H}_2$ is a Hilbert space isomorphic to $\hat{\mathcal{G}}'_{\mathsf{C}} \ominus \mathbf{S}\hat{\mathcal{G}}'_{\mathsf{C}}$. In fact, it is possible to take $\mathcal{H}_1$ to be $\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})$ and $\mathcal{H}_2$ to be $\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})$. To see this, note that since $P \in \mathscr{P}_{\mathcal{U}\to\mathcal{Y}}$ is causally extendible,

$$\mathbf{P}_0\mathcal{G}_{\mathsf{P}} = \mathsf{G}_{r[0]}\mathcal{H}_1 = \begin{bmatrix} \mathrm{I} \\ \mathsf{P}_{[0]} \end{bmatrix} \mathrm{L}^2_{\mathbb{H}}(\mathcal{U}), \tag{4.5}$$

where $\mathsf{G}_{r[0]} = \hat{\mathsf{G}}_r(0)$ and $\mathsf{P}_{[0]}$ denotes the first element of the sequence uniquely identifiable with the Toeplitz representation of P. Also, since $\mathbf{M}_{\hat{\mathsf{K}}_l}\mathbf{M}_{\hat{\mathsf{G}}_r} = \mathrm{I}$, it follows that $\mathsf{K}_{l[0]} = \hat{\mathsf{K}}_l(0)$ is a bounded left-inverse of $\mathsf{G}_{r[0]}$.[2] Define

$$\mathsf{Q} := \mathsf{K}_{l[0]} \begin{bmatrix} \mathrm{I} \\ \mathsf{P}_{[0]} \end{bmatrix} : \mathrm{L}^2_{\mathbb{H}}(\mathcal{U}) \to \mathcal{H}_1.$$

Then since $\begin{bmatrix} \mathrm{I} \\ \mathsf{P}_{[0]} \end{bmatrix}$ has zero kernel and $\mathsf{K}_{l[0]}\mathsf{G}_{r[0]} = \mathrm{I}$, it follows from equation (4.5) that Q is a bijective mapping. Hence, $\mathsf{Q}(\mathsf{Q}^*\mathsf{Q})^{-1}$ is an isomorphism between $\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})$ and $\mathcal{H}_1$, and correspondingly, $\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})$ is isomorphic to $\hat{\mathcal{G}}_{\mathsf{P}} \ominus \mathbf{S}\hat{\mathcal{G}}_{\mathsf{P}} \sim \mathcal{H}_1$. It can be shown in a similar manner that $\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})$ is isomorphic to $\hat{\mathcal{G}}'_{\mathsf{C}} \ominus \mathbf{S}\hat{\mathcal{G}}'_{\mathsf{C}}$.

Now the multiplication operators $\mathbf{M}_{\hat{\mathsf{G}}_r}$, $\mathbf{M}_{\hat{\mathsf{G}}_l}$, $\mathbf{M}_{\hat{\mathsf{K}}_l}$ and $\mathbf{M}_{\hat{\mathsf{K}}_r}$ are equivalent (via $\mathbf{Z}$) to stable systems $\mathsf{G}_r \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})}$, $\mathsf{G}_l \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}$, $\mathsf{K}_l \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})}$ and $\mathsf{K}_r \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})}$ respectively, with $\mathcal{G}_{\mathsf{P}} = \mathcal{R}_{\mathsf{G}_r} = \mathcal{K}_{\mathsf{G}_l}$ and

$$\begin{bmatrix} \mathsf{G}_r & \mathsf{K}_r \end{bmatrix} \begin{bmatrix} \mathsf{K}_l \\ \mathsf{G}_l \end{bmatrix} = \begin{bmatrix} \mathsf{K}_l \\ \mathsf{G}_l \end{bmatrix} \begin{bmatrix} \mathsf{G}_r & \mathsf{K}_r \end{bmatrix} = \begin{bmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{bmatrix}.$$

Furthermore, observe that given any stable system $\mathsf{Q}_0 \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}$ and stable systems $\mathsf{Q}_1 \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})}$ and $\mathsf{Q}_2 \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}$ with stable inverses,

$$\mathcal{G}_{\mathsf{P}} = \mathcal{R}_{\mathsf{G}_r\mathsf{Q}_1} = \mathcal{K}_{\mathsf{Q}_2^{-1}\mathsf{G}_l}$$

[2] That $\mathsf{G}_{r[0]}$ has a bounded left-inverse also follows from the fact that $\hat{\mathsf{G}}_r$ may be taken to be inner (see the proof of Theorem 4.5).

and

$$\begin{aligned}
\begin{bmatrix} \mathsf{G}_{r\star} & \mathsf{K}_{r\star} \end{bmatrix}\begin{bmatrix} \mathsf{K}_{l\star} \\ \mathsf{G}_{l\star} \end{bmatrix} &:= \begin{bmatrix} \mathsf{G}_r\mathsf{Q}_1 & (\mathsf{K}_r - \mathsf{G}_r\mathsf{Q}_0)\mathsf{Q}_2 \end{bmatrix}\begin{bmatrix} \mathsf{Q}_1^{-1}(\mathsf{K}_l + \mathsf{Q}_0\mathsf{G}_l) \\ \mathsf{Q}_2^{-1}\mathsf{G}_l \end{bmatrix} \\
&= \begin{bmatrix} \mathsf{Q}_1^{-1}(\mathsf{K}_l + \mathsf{Q}_0\mathsf{G}_l) \\ \mathsf{Q}_2^{-1}\mathsf{G}_l \end{bmatrix}\begin{bmatrix} \mathsf{G}_r\mathsf{Q}_1 & (\mathsf{K}_r - \mathsf{G}_r\mathsf{Q}_0)\mathsf{Q}_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathsf{Q}_1^{-1} & \mathsf{Q}_1^{-1}\mathsf{Q}_0 \\ 0 & \mathsf{Q}_2^{-1} \end{bmatrix}\begin{bmatrix} \mathsf{K}_l \\ \mathsf{G}_l \end{bmatrix}\begin{bmatrix} \mathsf{G}_r & \mathsf{K}_r \end{bmatrix}\begin{bmatrix} \mathsf{Q}_1 & -\mathsf{Q}_0\mathsf{Q}_2 \\ 0 & \mathsf{Q}_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{bmatrix}.
\end{aligned} \tag{4.6}$$

So if Q_0, Q_1 and Q_2 can be constructed to make $(\mathsf{G}_r\mathsf{Q}_1)_{[0]}$, $(\mathsf{Q}_2^{-1}\mathsf{G}_l)_{[0]}$, $(\mathsf{Q}_1^{-1}(\mathsf{K}_l + \mathsf{Q}_0\mathsf{G}_l))_{[0]}$ and $((\mathsf{K}_r + \mathsf{G}_r\mathsf{Q}_0)\mathsf{Q}_2)_{[0]}$ causal mappings on the finite horizon $\mathbb{H}$, it would follow by Remark 4.2, that the stable systems $\mathsf{G}_{r\star} \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})}$, $\mathsf{G}_{l\star} \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}$, $\mathsf{K}_{l\star} \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})}$ and $\mathsf{K}_{r\star} \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})}$ are equivalent (via $\mathbf{W}$) to stable systems $G_{r\star} \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{V}}$, $G_{l\star} \in \mathscr{P}^{\mathrm{e}}_{\mathcal{V}\to\mathcal{Y}}$, $K_{l\star} \in \mathscr{P}^{\mathrm{e}}_{\mathcal{V}\to\mathcal{U}}$ and $K_{r\star} \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{V}}$ respectively,[2] that satisfy

$$\mathcal{G}_P = \mathcal{R}_{G_{r\star}} = \mathcal{K}_{G_{l\star}}$$

and

$$\begin{bmatrix} G_{r\star} & K_{r\star} \end{bmatrix}\begin{bmatrix} K_{l\star} \\ G_{l\star} \end{bmatrix} = \begin{bmatrix} K_{l\star} \\ G_{l\star} \end{bmatrix}\begin{bmatrix} G_{r\star} & K_{r\star} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Accordingly, the rest of the proof is dedicated to showing that this is possible.

Let

$$\begin{bmatrix} \mathsf{M}_{r[0]} \\ \mathsf{N}_{r[0]} \end{bmatrix} := \mathsf{G}_{r[0]} \quad \text{and} \quad \begin{bmatrix} -\mathsf{N}_{l[0]} & \mathsf{M}_{l[0]} \end{bmatrix} := \mathsf{G}_{l[0]},$$

where the partitioning is conformal with that of $\mathcal{G}_\mathsf{P}$. Then since $P \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ is causally extendible with locally-Lipschitz-continuous extension, it follows that both $\mathsf{M}_{r[0]} \in \mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})}$ and $\mathsf{M}_{l[0]} \in \mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}$ are bijective and therefore, boundedly invertible by the Open Mapping Theorem [Bol90, pg. 79]. To see this, first note that since $P \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$,

$$\mathcal{R}_{\mathsf{M}_{r[0]}} = \mathbf{P}_0\mathcal{D}_\mathsf{P} = \mathbf{P}_h\mathcal{D}_P = \mathrm{L}^2_{\mathbb{H}}(\mathcal{U})$$

[2] Since these systems are all stable, they have locally-Lipschitz-continuous, causal extensions.

and hence, that $\mathsf{M}_{r[0]}$ is surjective.[3] Suppose now, that there exists some non-zero $\vec{q} \in \mathrm{L}^2_{\mathbb{H}}(\mathcal{U})$ such that $\mathsf{M}_{r[0]}\vec{q} = 0$. Then by causality of P, it follows that $\mathsf{N}_{r[0]}\vec{q}$ must also be zero. Since $\mathsf{K}_{l[0]}\mathsf{G}_{r[0]} = \mathrm{I}$, this implies that

$$0 = \mathsf{K}_{l[0]} \begin{bmatrix} \mathsf{M}_{r[0]} \\ \mathsf{N}_{r[0]} \end{bmatrix} \vec{q} = \vec{q},$$

which is a contradiction. Therefore, the linear operator $\mathsf{M}_{r[0]}$ has zero kernel and is by definition, injective. This combined with the fact that it is also surjective, implies that $\mathsf{M}_{r[0]}$ is bijective as claimed. Now consider $\mathsf{M}_{l[0]}$ and suppose there exists a non-zero $\vec{q} \in \mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})$ such that $\mathsf{M}_{l[0]}\vec{q} = 0$. Then since

$$\mathcal{K}_{\begin{bmatrix} -\mathsf{N}_{l[0]} & \mathsf{M}_{l[0]} \end{bmatrix}} = \mathbf{P}_0\mathcal{G}_{\mathsf{P}} = \mathbf{P}_h\mathcal{G}_P,$$

it follows that $\begin{bmatrix} 0 \\ \vec{q} \end{bmatrix} \in \mathbf{P}_h\mathcal{G}_P$. But this contradicts the causality of P and hence, $\vec{q}$ must be 0. Consequently, $\mathsf{M}_{l[0]}$ has zero kernel and is thereby injective. Moreover, since $\mathsf{G}_{l[0]}\mathsf{K}_{r[0]} = \mathrm{I}$,

$$\mathcal{R}_{\begin{bmatrix} -\mathsf{N}_{l[0]} & \mathsf{M}_{l[0]} \end{bmatrix}} = \mathrm{L}^2_{\mathbb{H}}(\mathcal{Y}).$$

So for all $\vec{e} \in \mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})$, there exists a $\vec{g} = \begin{bmatrix} \vec{g}_1 \\ \vec{g}_2 \end{bmatrix} \in \mathrm{L}^2_{\mathbb{H}}(\mathcal{V})$ such that

$$\vec{e} = \begin{bmatrix} -\mathsf{N}_{l[0]} & \mathsf{M}_{l[0]} \end{bmatrix} \begin{bmatrix} \vec{g}_1 \\ \vec{g}_2 \end{bmatrix},$$

In fact,

$$\vec{e} = \begin{bmatrix} -\mathsf{N}_{l[0]} & \mathsf{M}_{l[0]} \end{bmatrix} \begin{bmatrix} \vec{g}_1 + \vec{u} \\ \vec{g}_2 + \mathsf{P}_{[0]}\vec{u} \end{bmatrix}$$

for all $\vec{u} \in \mathbf{P}_h\mathcal{D}_P$. Since $P \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$, $\mathbf{P}_h\mathcal{D}_P = \mathrm{L}^2_{\mathbb{H}}(\mathcal{U})$ and consequently, $\vec{u}$ can be set to $-\vec{g}_1$. Doing this, gives $\vec{e} = \mathsf{M}_{l[0]}(\vec{g}_2 - \mathsf{P}_{[0]}\vec{g}_1)$, from which it is evident that $\mathcal{R}_{\mathsf{M}_{l[0]}} = \mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})$. Therefore, $\mathsf{M}_{l[0]}$ is bijective as claimed.

Having shown that $\mathsf{M}_{r[0]}$ and $\mathsf{M}_{l[0]}$ are boundedly invertible, it is possible to construct the required Q_0, Q_1 and Q_2. Let Q_1 satisfy $\mathrm{Q}_{1[0]} = \mathsf{M}^{-1}_{r[0]}$ and $\mathrm{Q}_{1[i]} = 0$ for $i \neq 0$, where $\{\mathrm{Q}_{1[i]}\}_{i=0}^{\infty}$ denotes the sequence uniquely identifiable with the Toeplitz representation of Q_1. Then

$$(\mathsf{G}_r\mathrm{Q}_1)_{[0]} = \begin{bmatrix} \mathrm{I} \\ \mathsf{N}_{r[0]}\mathsf{M}^{-1}_{r[0]} \end{bmatrix},$$

[3]Note that in the equation above $\mathbf{P}_0$ corresponds to a discrete-time truncation and $\mathbf{P}_h$ to truncation in a continuous-time signal-space.

which by causality of P is causal on $\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})$. Now let $\begin{bmatrix} \mathsf{Y}_{l[0]} & \mathsf{X}_{l[0]} \end{bmatrix} := \mathsf{K}_{l[0]}$, where the partition is conformal with $\mathcal{G}_{\mathsf{P}}$ and construct Q_0 so that $\mathsf{Q}_{0[0]} = -\mathsf{X}_{l[0]}\mathsf{M}_{l[0]}^{-1}$ and $\mathsf{Q}_{0[i]} = 0$ for $i \neq 0$. Then since $\mathsf{N}_{r[0]}\mathsf{M}_{r[0]}^{-1} = \mathsf{M}_{l[0]}^{-1}\mathsf{N}_{l[0]} = \mathsf{P}_{[0]}$ and $\mathsf{Y}_{l[0]}\mathsf{M}_{r[0]} + \mathsf{X}_{l[0]}\mathsf{N}_{r[0]} = \mathrm{I}$, it follows that

$$\begin{aligned}
(\mathsf{Q}_1^{-1}(\mathsf{K}_l + \mathsf{Q}_0\mathsf{G}_l))_{[0]} &= (\mathsf{Q}_1^{-1})_{[0]}\left(\mathsf{K}_{l[0]} + \mathsf{Q}_{0[0]}\begin{bmatrix} -\mathsf{N}_{l[0]} & \mathsf{M}_{l[0]} \end{bmatrix}\right) \\
&= \mathsf{M}_{r[0]}\begin{bmatrix} \mathsf{Y}_{l[0]} + \mathsf{X}_{l[0]}\mathsf{M}_{l[0]}^{-1}\mathsf{N}_{l[0]} & 0 \end{bmatrix} \\
&= \mathsf{M}_{r[0]}\begin{bmatrix} \mathsf{M}_{r[0]}^{-1} & 0 \end{bmatrix} = \begin{bmatrix} \mathrm{I} & 0 \end{bmatrix},
\end{aligned}$$

which is obviously causal on $\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})$ as required. Finally, let $\begin{bmatrix} -\mathsf{X}_{r[0]} \\ \mathsf{Y}_{r[0]} \end{bmatrix} := \mathsf{K}_{r[0]}$. Then from equation (4.6), it follows that $\mathsf{M}_{r[0]}\mathsf{X}_{l[0]} - \mathsf{X}_{r[0]}\mathsf{M}_{l[0]} = 0$ and hence, that $\mathsf{Q}_{0[0]} = -\mathsf{X}_{l[0]}\mathsf{M}_{l[0]}^{-1} = \mathsf{M}_{r[0]}^{-1}\mathsf{X}_{r[0]}$. Defining Q_2 to satisfy $\mathsf{Q}_{2[0]} = \mathsf{M}_{l[0]}$ and $\mathsf{Q}_{2[i]} = 0$ for $i \neq 0$, gives $(\mathsf{Q}_2^{-1}\mathsf{G}_l)_{[0]} = \begin{bmatrix} -\mathsf{M}_{l[0]}^{-1}\mathsf{N}_{l[0]} & \mathrm{I} \end{bmatrix}$ and $((\mathsf{K}_r + \mathsf{G}_r\mathsf{Q}_0)\mathsf{Q}_2)_{[0]} = \begin{bmatrix} 0 \\ \mathrm{I} \end{bmatrix}$. As required, both are causal on $\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})$ and $\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})$ respectively. This completes the proof. ∎

Remark 4.7 Since G_r and G_l in Theorem 4.6 are respectively, left and right invertible by stable, LPTV systems, they are called <u>strong-right</u> and respectively <u>strong-left</u> representations of $\mathcal{G}_P$. Note that they are only unique up to invertible factors. That is, for any stable systems $Q_1, Q_1^{-1} \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{U}}$ and $Q_2, Q_2^{-1} \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{Y}}$, the systems G_rQ_1 and $Q_2^{-1}G_l$ are also strong-right and strong-left representations of $\mathcal{G}_P$ respectively. ♣

Given a well-posed, plant/controller pair $[P, C]$, the following lemma constitutes a useful characterisation of closed-loop stability in terms of range and kernel representations of $\mathcal{G}_P$ and $\mathcal{G}'_C$.

Lemma 4.8 *Given a plant $P \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ and controller $C \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}}$ such that $[P, C]$ is well-posed, let $G_r \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{V}}$ be any stable system with zero kernel such that $\mathcal{G}_P = \mathcal{R}_{G_r}$ and $K_l \in \mathscr{P}^{\mathrm{e}}_{\mathcal{V}\to\mathcal{U}}$ be any stable system such that $\mathcal{R}_{K_l} = \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U})$ and $\mathcal{G}'_C = \mathcal{K}_{K_l}$. Then the following are equivalent:*

(i) The closed-loop system $[P, C]$ is stable;

(ii) The stable, LPTV system K_lG_r is bijective (and hence boundedly invertible, although the inverse may not be causal).

Furthermore, if $[P, C]$ is stable and G_r and K_l are <u>strong-right</u> and <u>strong-left</u> representations of $\mathcal{G}_P$ and $\mathcal{G}'_C$ respectively, then $(K_lG_r)^{-1}$ is a stable system in $\mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{U}}$. That is the inverse is also causal.

Proof : Since the closed-loop is well-posed by assumption, Proposition 2.12 holds (cf. Remark 2.13), and using the identities $\mathcal{G}_P = \mathcal{R}_{G_r}$ and $\mathcal{G}_C' = \mathcal{K}_{K_l}$, it follows that

$$[P, C] \text{ is stable } \Leftrightarrow \mathcal{R}_{G_r} + \mathcal{K}_{K_l} = \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{V}) \text{ and } \mathcal{R}_{G_r} \cap \mathcal{K}_{K_l} = \{0\}.$$

Then since $\mathcal{R}_{K_l} = \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U})$ and $\mathcal{K}_{G_r} = \{0\}$, it is clear that

$$\begin{gathered} \mathcal{R}_{G_r} + \mathcal{K}_{K_l} = \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{V}) \quad \text{and} \quad \mathcal{R}_{G_r} \cap \mathcal{K}_{K_l} = \{0\} \\ \Downarrow \\ \mathcal{R}_{K_l G_r} = \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U}) \quad \text{and} \quad \mathcal{K}_{K_l G_r} = \{0\}. \end{gathered}$$

To see that the converse is also true, let $\mathcal{R}_{K_l G_r} = \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U})$ and $\mathcal{K}_{K_l G_r} = \{0\}$. That $\mathcal{R}_{G_r} \cap \mathcal{K}_{K_l} = \{0\}$ is immediate. Now suppose there exists a $v \in \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{V})$ such that $v \notin \mathcal{R}_{G_r} + \mathcal{K}_{K_l}$ and define $e := K_l v$. Then since $\mathcal{R}_{K_l G_r} = \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U})$ and $\mathcal{K}_{K_l G_r} = \{0\}$, there exist a unique $q \in \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U})$ such that $e = K_l G_r q = K_l g$, where $g := G_r q$. Consequently, $K_l(g - v) = 0$, which implies that either $v = g$ or $(g - v) \in \mathcal{K}_{K_l}$. Since $g \in \mathcal{R}_{G_r}$, this contradicts $v \notin \mathcal{R}_{G_r} + \mathcal{K}_{K_l}$ and it follows that $[P, C]$ is stable if and only if the bounded linear operator $K_l G_r$ is bijective (and hence, boundedly invertible by the Open Mapping Theorem [Bol90, pg. 79]). Note that $(K_l G_r)^{-1}$ is periodically time-varying, but may not be locally causal in $\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})$.

It is now shown that if $[P, C]$ is stable and G_r and K_l are strong-right and strong-left representations of $\mathcal{G}_P$ and $\mathcal{G}_C'$ respectively, then $(K_l G_r)^{-1} \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U} \to \mathcal{U}}$. First note that for any K_l and G_r that satisfy the original conditions of the lemma,

$$\mathbf{\Pi}_{\mathcal{G}_P \| \mathcal{G}_C'} = G_r (K_l G_r)^{-1} K_l. \tag{4.7}$$

To see this, note that since $[P, C]$ is stable, $\{\mathcal{G}_P, \mathcal{G}_C'\}$ induces a co-ordinatisation of $\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{V})$. Therefore, any $g \in \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{V})$ can be written uniquely as the sum $g = p + c$, where $p \in \mathcal{G}_P$ and $c \in \mathcal{G}_C'$. Furthermore, since $\mathcal{K}_{G_r} = \{0\}$ and $\mathcal{R}_{G_r} = \mathcal{G}_P$, there exists a unique $q \in \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U})$ such that $p = G_r q$. Consequently,

$$\begin{aligned} (\mathbf{\Pi}_{\mathcal{G}_P \| \mathcal{G}_C'} - G_r (K_l G_r)^{-1} K_l) g &= p - G_r (K_l G_r)^{-1} K_l p \\ &= p - G_r (K_l G_r)^{-1} K_l G_r q \\ &= p - G_r q = 0, \end{aligned}$$

for all $g \in \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U})$, which in turn, implies that $\mathbf{\Pi}_{\mathcal{G}_P \| \mathcal{G}_C'} = G_r (K_l G_r)^{-1} K_l$. Now, take G_r and K_l to be strong-right and strong-left representations. Then by definition, G_r and K_l satisfy the original conditions of the lemma and there exist stable systems $K_{l\star} \in \mathscr{P}^{\mathrm{e}}_{\mathcal{V} \to \mathcal{U}}$ and $G_{r\star} \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U} \to \mathcal{V}}$ such that $K_{l\star} G_r = I$ and $K_l G_{r\star} = I$. Furthermore, from equation (4.7),

$$(K_l G_r)^{-1} = K_{l\star} \mathbf{\Pi}_{\mathcal{G}_P \| \mathcal{G}_C'} G_{r\star}.$$

Then recalling that $\mathbf{\Pi}_{\mathcal{G}_P \| \mathcal{G}'_C} \in \mathscr{P}^{\mathrm{e}}_{\mathcal{V}\to\mathcal{V}}$ (cf. Remark 2.14), it follows that $(K_l G_r)^{-1} \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{U}}$. ∎

Remark 4.9 Note that by interchanging the roles of P and C in Lemma 4.8, given any stable system $G_l \in \mathscr{P}^{\mathrm{e}}_{\mathcal{V}\to\mathcal{Y}}$ such that $\mathcal{R}_{G_l} = \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U})$ and $\mathcal{G}_P = \mathcal{K}_{G_l}$ and any stable system $K_r \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{V}}$ with zero kernel such that $\mathcal{G}'_C = \mathcal{R}_{K_r}$, the following are equivalent :

(i) The closed-loop system $[P, C]$ is stable;

(ii) The stable, LPTV system $G_l K_r$ is bijective (and hence boundedly invertible, although the inverse may not be causal).

Furthermore, if G_l and K_r are <u>strong-left</u> and <u>strong-right</u> representations of $\mathcal{G}_P$ and $\mathcal{G}'_C$ respectively and $[P, C]$ is stable, then $(G_l K_r)^{-1}$ is a stable system $\mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{Y}}$. ♣

Remark 4.10 Given a well-posed, plant/controller pair $[P, C]$ where $P \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ and $C \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}}$, let $\mathsf{P} := \mathbf{W} P \mathbf{W}^{-1}$ and $\mathsf{C} := \mathbf{W} C \mathbf{W}^{-1}$. Furthermore, let $\mathsf{G}_r \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}$ and $\mathsf{K}_l \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})}$ be stable systems that satisfy $\mathcal{K}_{\mathsf{G}_r} = \{0\}$, $\mathcal{G}_{\mathsf{P}} = \mathcal{R}_{\mathsf{G}_r}$, $\mathcal{R}_{\mathsf{K}_l} = \ell^2_{\mathbb{Z}^+}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}))$ and $\mathcal{G}'_{\mathsf{C}} = \mathcal{K}_{\mathsf{K}_l}$. Then the closed-loop $[P, C]$ is stable if and only if the bounded linear operator $\mathsf{K}_l \mathsf{G}_r$ is boundedly invertible. In this case $(\mathsf{K}_l \mathsf{G}_r)^{-1}$ is a stable system in $\mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})}$ and hence, equivalent (via $\mathbf{Z}$) to a multiplication operator with symbol in $\mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})})$ (cf. Proposition 2.2). Moreover, $\mathbf{\Pi}_{\mathcal{G}_{\mathsf{P}} \| \mathcal{G}'_{\mathsf{C}}} = \mathsf{G}_r (\mathsf{K}_l \mathsf{G}_r)^{-1} \mathsf{K}_l$. ♣

It is interesting to note that given a system $P \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ and a stabilising $C \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}}$, strong-right and strong-left representations of $\mathcal{G}'_C$ can be constructed from strong-right and strong-left representations of $\mathcal{G}_P$ and the respective stable, left and right inverses. This is analogous to necessity of the well-known Youla parameterisation of stabilising controllers for certain classes of systems [YJB76, DLJS80, DG84, DS93] and is summarised in the following lemma.

Lemma 4.11 *Given a stabilisable system $P \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ and stable systems $G_r \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{V}}$, $G_l \in \mathscr{P}^{\mathrm{e}}_{\mathcal{V}\to\mathcal{Y}}$, $K_l \in \mathscr{P}^{\mathrm{e}}_{\mathcal{V}\to\mathcal{U}}$ and $K_r \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{V}}$ such that*

$$\mathcal{G}_P = \mathcal{R}_{G_r} = \mathcal{K}_{G_l}$$

and

$$\begin{bmatrix} G_r & K_r \end{bmatrix} \begin{bmatrix} K_l \\ G_l \end{bmatrix} = \begin{bmatrix} K_l \\ G_l \end{bmatrix} \begin{bmatrix} G_r & K_r \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

a system $C \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}}$ stabilises P only if there exists a stable $Q \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}}$ such that

$$K_r - G_r Q$$

is a strong-right representation of $\mathcal{G}_C'$ *and*

$$K_l + QG_l$$

is a strong-left representation of $\mathcal{G}_C'$.

Proof : Let $C \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{Y}}$ denote a stabilising controller for P. Then it follows by Theorem 4.6, that there exist stable systems $G_r \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{V}}$, $G_l \in \mathscr{P}^{\mathrm{e}}_{\mathcal{V}\to\mathcal{Y}}$, $K_l \in \mathscr{P}^{\mathrm{e}}_{\mathcal{V}\to\mathcal{U}}$ and $K_r \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{V}}$ such that

$$\mathcal{G}_P = \mathcal{R}_{G_r} = \mathcal{K}_{G_l}$$

and

$$\begin{bmatrix} G_r & K_r \end{bmatrix}\begin{bmatrix} K_l \\ G_l \end{bmatrix} = \begin{bmatrix} K_l \\ G_l \end{bmatrix}\begin{bmatrix} G_r & K_r \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{4.8}$$

Furthermore, for any stable system $Q \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}}$,

$$\begin{bmatrix} G_r & (K_r - G_rQ) \end{bmatrix}\begin{bmatrix} (K_l + QG_l) \\ G_l \end{bmatrix} = \begin{bmatrix} (K_l + QG_l) \\ G_l \end{bmatrix}\begin{bmatrix} G_r & (K_r - G_rQ) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{4.9}$$

Now from equation (4.9),

$$(K_l + QG_l)(K_r - G_rQ) = 0,$$

which implies $\mathcal{R}_{(K_r - G_rQ)} \subset \mathcal{K}_{(K_l+QG_l)}$. To see that the opposite inclusion also holds, note from equation (4.9), that

$$(K_l + QG_l)\begin{bmatrix} G_r & (K_r - G_rQ) \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix}. \tag{4.10}$$

Now suppose there exists a non-zero $x \in \mathcal{K}_{(K_l+QG_l)}$ that is not in $\mathcal{R}_{(K_r-G_rQ)}$. Then since

$$\mathcal{K}_{\begin{bmatrix} G_r & (K_r - G_rQ) \end{bmatrix}} = \{0\} \quad \text{and} \quad \mathcal{R}_{\begin{bmatrix} G_r & (K_r - G_rQ) \end{bmatrix}} = \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{V}),$$

there exists a unique $u =: \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{V})$ such that

$$x = \begin{bmatrix} G_r & (K_r - G_rQ) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Furthermore, $0 = (K_l + QG_l)x = u_1$, which using equation (4.10), implies that $x \in \mathcal{R}_{(K_r-G_rQ)}$. This is a contradiction and therefore, $\mathcal{K}_{(K_l+QG_l)} \subset \mathcal{R}_{(K_r-G_rQ)}$.

Since $[P, C]$ is stable (or equivalently C is stabilised P), it follows by Theorem 4.6, that there exist strong-right and strong-left representations of $\mathcal{G}_C$. Noting that $\mathcal{G}_C^{'} := \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]\mathcal{G}_C$, these give rise to strong-right and strong-left representations of $\mathcal{G}_C^{'}$, denoted by $K_{r\star} \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{V}}$ and $K_{l\star} \in \mathscr{P}^{\mathrm{e}}_{\mathcal{V}\to\mathcal{U}}$ respectively. It follows by Lemma 4.8, that $K_{l\star}G_r$ is boundedly invertible in $\mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{U}}$. Now define

$$Q := -\left(K_{l\star}G_r\right)^{-1}K_{l\star}K_r$$

and note that Q is a stable system in $\mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}}$. Then using equation (4.8),

$$\begin{aligned} K_l - QG_l &= K_l + \left(K_{l\star}G_r\right)^{-1}K_{l\star}K_rG_l \\ &= K_l + \left(K_{l\star}G_r\right)^{-1}K_{l\star}(I - G_rK_l) \\ &= K_l - K_l + \left(K_{l\star}G_r\right)^{-1}K_{l\star} = \left(K_{l\star}G_r\right)^{-1}K_{l\star}, \end{aligned}$$

which implies $\mathcal{K}_{(K_l - QG_l)} = \mathcal{K}_{K_{l\star}}$. Since $\mathcal{R}_{(K_r + G_rQ)} = \mathcal{K}_{(K_l - QG_l)}$ and $\mathcal{R}_{K_{r\star}} = \mathcal{K}_{K_{l\star}}$, it also follows that $\mathcal{R}_{(K_r + G_rQ)} = \mathcal{R}_{K_{r\star}}$. That is, the strong-right and strong-left representations of $\mathcal{G}_C^{'}$ are in the form required. ∎

Remark 4.12 If in Lemma 4.11, the choice of $Q \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ is restricted to those for which $\mathcal{R}_{(K_r - G_rQ)} = \mathcal{K}_{(K_l + QG_l)}$ correspond to the inverse graph of a system $C \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ such that the closed-loop $[P, C]$ is *well-posed*, then C stabilises P. That is, the Lemma would also be sufficient. To see this, note from equation (4.9) that for such a Q, the system $(K_l + QG_l)$ is a *strong-left* representation of $\mathcal{G}_C^{'}$. So by arguments similar to those in the proof of Lemma 4.8 and the fact that $(K_l + QG_l)G_r = I$ is obviously invertible, it follows that $[P, C]$ is stable. Again from equation (4.9) and since $\mathcal{R}_{(K_r - G_rQ)} = \mathcal{K}_{(K_l + QG_l)} = \mathcal{G}_C^{'}$, it is also clear that $(K_r - G_rQ)$ is a strong-right representation of the inverse graph of a stabilising controller. ♣

4.3 The Gap Metric

In this section, the gap metric is formally introduced. The main result of the section is a formula for the directed gap between two LPTV systems. This formula is essentially a generalisation of Georgiou's [Geo88] for LTI systems. It is fundamental to the main robustness result of this chapter and hence, to the proof that the topology induced by the gap metric on a class of LPTV systems, is the weakest with respect to which closed-loop performance varies continuously and closed-loop stability is a robust property. For additional information on the gap metric the reader is referred to [FGS93, SO93, GS90, Geo88, ZES80, KVZ+72, Kat66].

The gap (or aperture) between two (closed) subspaces $\mathcal{H}_0$ and $\mathcal{H}_1$ of a Hilbert space $\mathcal{H}$, is defined to be

$$\delta(\mathcal{H}_0, \mathcal{H}_1) := \|\mathbf{\Pi}_{\mathcal{H}_0} - \mathbf{\Pi}_{\mathcal{H}_1}\|$$

and it can be shown (cf. [KVZ+72, §15 Chap. 4]) that

$$\delta(\mathcal{H}_0,\mathcal{H}_1) = \max\{\vec{\delta}(\mathcal{H}_0,\mathcal{H}_1), \vec{\delta}(\mathcal{H}_1,\mathcal{H}_0)\},$$

where

$$\vec{\delta}(\mathcal{H}_i,\mathcal{H}_j) = \|\mathbf{\Pi}_{\mathcal{H}_j^{\perp}}\mathbf{\Pi}_{\mathcal{H}_i}\| \quad i,j = 0,1,$$

is the directed gap. Note that $0 \leq \delta(\mathcal{H}_0,\mathcal{H}_1) \leq 1$ and that it defines a metric on $\mathcal{H}$.

Proposition 4.13 [KVZ+72]. *For $i = 0,1$, let $\mathcal{H}_i$ be a closed subspace of a Hilbert space $\mathcal{H}$. Then $\mathbf{\Pi}_{\mathcal{H}_1}|_{\mathcal{H}_0}$ is a bijective mapping from $\mathcal{H}_0$ to $\mathcal{H}_1$ if and only if $\delta(\mathcal{H}_0,\mathcal{H}_1) < 1$. Moreover, if $\delta(\mathcal{H}_0,\mathcal{H}_1) < 1$ then*

$$\delta(\mathcal{H}_0,\mathcal{H}_1) = \vec{\delta}(\mathcal{H}_0,\mathcal{H}_1) = \vec{\delta}(\mathcal{H}_1,\mathcal{H}_0).$$

The gap between two (closed) operators P_0 and P_1 is defined to be the gap between their respective graphs,

$$\delta_g(P_0,P_1) := \delta(\mathcal{G}_{P_0},\mathcal{G}_{P_1}),$$

with the directed gap being defined similarly [Kat66]. This is a metric and can be thought of as a measure of the distance between two systems. In the next section the gap metric is used to quantify the robust stability properties of LPTV, closed-loop systems.

Consider a system $P_0 \in \mathscr{P}_{\mathcal{U}\to\mathcal{Y}}$ and recall that $\mathcal{G}_{P_0}$ is isomorphic to both $\mathcal{G}_{\mathsf{P}_0}$ and $\hat{\mathcal{G}}_{\mathsf{P}_0} := \mathbf{Z}\mathcal{G}_{\mathsf{P}_0}$, where $\mathsf{P}_0 := \mathbf{W}P_0\mathbf{W}^{-1}$. Given another system $P_1 \in \mathscr{P}_{\mathcal{U}\to\mathcal{Y}}$, the following hold :

$$\delta_g(P_0,P_1) = \delta(\mathcal{G}_{P_0},\mathcal{G}_{P_1}) = \delta(\hat{\mathcal{G}}_{\mathsf{P}_0},\hat{\mathcal{G}}_{\mathsf{P}_1}),$$

for the gap and

$$\vec{\delta}_g(P_i,P_j) = \vec{\delta}(\mathcal{G}_{P_i},\mathcal{G}_{P_j}) = \vec{\delta}(\hat{\mathcal{G}}_{\mathsf{P}_i},\hat{\mathcal{G}}_{\mathsf{P}_j}), \quad (i,j = 0,1)$$

for the directed gap. The following corollary of Proposition 4.13 constitutes a useful characterisation of when the gap between two systems is strictly less than one. It is used in the proof of the main robustness result in the next section. A similar result appeared in [SO93] and [Vin93b] for LTI systems.

Corollary 4.14 *Given two systems P_0 and P_1 in $\mathscr{P}_{\mathcal{U}\to\mathcal{Y}}$, the following are equivalent :*

(i) $\delta_g(P_0,P_1) = \delta(\hat{\mathcal{G}}_{\mathsf{P}_0},\hat{\mathcal{G}}_{\mathsf{P}_1}) < 1$;

(ii) The bounded linear operator $(\mathbf{M}_{\hat{\mathrm{G}}_{r1}})^*\mathbf{M}_{\hat{\mathrm{G}}_{r0}}$ *is bijective (and hence boundedly invertible), where* $\hat{\mathrm{G}}_{ri}$ *is any inner function in* $\mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})})$ *such that* $\hat{\mathcal{G}}_{\mathrm{P}_i} := \mathbf{ZW}\mathcal{G}_{P_i} = \hat{\mathrm{G}}_{ri}\mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}))$ *for* $i = 0, 1$.

Proof : Since $\mathbf{\Pi}_{\hat{\mathcal{G}}_{\mathrm{P}_i}} = \mathbf{M}_{\hat{\mathrm{G}}_{ri}}(\mathbf{M}_{\hat{\mathrm{G}}_{ri}})^*$, $\hat{\mathcal{G}}_{\mathrm{P}_i} = \mathcal{R}_{\mathbf{M}_{\hat{\mathrm{G}}_{ri}}}$ and $\mathcal{K}_{\mathbf{M}_{\hat{\mathrm{G}}_{ri}}} = \{0\}$, it follows by Proposition 4.13 (cf. a similar result in Nikol'skii [Nik86, pg. 201] concerning closed subspaces), that $\delta(\hat{\mathcal{G}}_{\mathrm{P}_0}, \hat{\mathcal{G}}_{\mathrm{P}_1}) < 1$ if and only if $\mathbf{M}_{\hat{\mathrm{G}}_{r1}}(\mathbf{M}_{\hat{\mathrm{G}}_{r1}})^*\mathbf{M}_{\hat{\mathrm{G}}_{r0}}$ is a bijective mapping from $\mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}))$ to $\hat{\mathcal{G}}_{\mathrm{P}_1}$. Again using the fact that $\hat{\mathcal{G}}_{\mathrm{P}_i} = \mathcal{R}_{\mathbf{M}_{\hat{\mathrm{G}}_{ri}}}$ and $\mathcal{K}_{\mathbf{M}_{\hat{\mathrm{G}}_{ri}}} = \{0\}$, this is equivalent to

$$\mathcal{R}_{(\mathbf{M}_{\hat{\mathrm{G}}_{r1}})^*\mathbf{M}_{\hat{\mathrm{G}}_{r0}}} = \mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})) \text{ and } \mathcal{K}_{(\mathbf{M}_{\hat{\mathrm{G}}_{r1}})^*\mathbf{M}_{\hat{\mathrm{G}}_{r0}}} = \{0\}.$$

That is, $\delta(\hat{\mathcal{G}}_{\mathrm{P}_0}, \hat{\mathcal{G}}_{\mathrm{P}_1}) < 1$ if and only if $(\mathbf{M}_{\hat{\mathrm{G}}_{r1}})^*\mathbf{M}_{\hat{\mathrm{G}}_{r0}}$ is bijective (and hence, boundedly invertible by the open mapping theorem [Bol90]). ■

An important theorem, characterising the directed gap between LPTV systems in terms of an optimisation problem over $\mathrm{H}^\infty_{\mathbb{D}}$, is now presented. It is essentially a generalisation of Georgiou's result for LTI systems [Geo88] (cf. also [You86, Theorem 1] and [SO93, Corollary 4.4]). The relevant formula plays a crucial role in the proof of the main robustness result in the following section.

Theorem 4.15 *For* $i, j = 0, 1$,

$$\vec{\delta}_g(P_i, P_j) = \vec{\delta}(\hat{\mathcal{G}}_{\mathrm{P}_i}, \hat{\mathcal{G}}_{\mathrm{P}_j}) = \inf_{\hat{\mathrm{Q}}\in\mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})})} \|\hat{\mathrm{G}}_{ri} - \hat{\mathrm{G}}_{rj}\hat{\mathrm{Q}}\|_\infty,$$

where $\hat{\mathrm{G}}_{ri}$ *is any inner function in* $\mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})})$ *such that* $\hat{\mathcal{G}}_{\mathrm{P}_i} = \hat{\mathrm{G}}_{ri}\mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}))$.

Proof : The proof follows that of an H^∞ optimisation result in [FF90, pg. 248]. Note that (for $i = 0, 1$) $\hat{\mathcal{G}}_{\mathrm{P}_i}$ is an LSI subspace of $\mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{V}))$. As before, it follows by the Beurling-Lax-Halmos Theorem, that there exists an inner function $\hat{\mathrm{G}}_{ri} \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})})$ such that $\hat{\mathcal{G}}_{\mathrm{P}_i} = \hat{\mathrm{G}}_{ri}\mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}))$. Since $\mathbf{M}_{\hat{\mathrm{G}}_{ri}}$ is an isometry it can be shown that $\mathbf{\Pi}_{\hat{\mathcal{G}}_{\mathrm{P}_i}} = \mathbf{M}_{\hat{\mathrm{G}}_{ri}}(\mathbf{M}_{\hat{\mathrm{G}}_{ri}})^*$ and hence, that

$$\vec{\delta}(\hat{\mathcal{G}}_{\mathrm{P}_i}, \hat{\mathcal{G}}_{\mathrm{P}_j}) = \|\mathbf{\Pi}_{\hat{\mathcal{G}}^\perp_{\mathrm{P}_j}}\mathbf{M}_{\hat{\mathrm{G}}_{ri}}(\mathbf{M}_{\hat{\mathrm{G}}_{ri}})^*\| = \|\mathbf{\Pi}_{\hat{\mathcal{G}}^\perp_{\mathrm{P}_j}}\mathbf{M}_{\hat{\mathrm{G}}_{ri}}\|,$$

where the last equality follows from the fact that $\mathbf{M}_{\hat{\mathrm{G}}_{ri}}$ is an isometry. Now recall that $\hat{\mathcal{G}}^\perp_{\mathrm{P}_j} := \mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})) \ominus \hat{\mathcal{G}}_{\mathrm{P}_j}$ and take any $\eta \in \mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{V}))$, expressing it as $\eta = \gamma + \varphi$, where $\gamma = \mathbf{\Pi}_{\hat{\mathcal{G}}_{\mathrm{P}_j}}\eta$ and $\varphi = \mathbf{\Pi}_{\hat{\mathcal{G}}^\perp_{\mathrm{P}_j}}\eta$. Then $\mathbf{S}\eta = \mathbf{S}\gamma + \mathbf{S}\varphi$ and

$$\begin{aligned} \mathbf{\Pi}_{\hat{\mathcal{G}}^\perp_{\mathrm{P}_j}}\mathbf{S}\eta &= \mathbf{\Pi}_{\hat{\mathcal{G}}^\perp_{\mathrm{P}_j}}\mathbf{S}\gamma + \mathbf{\Pi}_{\hat{\mathcal{G}}^\perp_{\mathrm{P}_j}}\mathbf{S}\varphi \\ &= \mathbf{\Pi}_{\hat{\mathcal{G}}^\perp_{\mathrm{P}_j}}\mathbf{S}\varphi, \end{aligned}$$

where the second equality follows from the fact that $\mathbf{S}\hat{\mathcal{G}}_{\mathsf{P}_j} \subset \hat{\mathcal{G}}_{\mathsf{P}_j}$. That is,

$$\mathbf{\Pi}_{\hat{\mathcal{G}}_{\mathsf{P}_j}^{\perp}} \mathbf{S}\eta = \mathbf{\Pi}_{\hat{\mathcal{G}}_{\mathsf{P}_j}^{\perp}} \mathbf{S}\mathbf{\Pi}_{\hat{\mathcal{G}}_{\mathsf{P}_j}^{\perp}} \eta = \mathbf{S}_{\star}\mathbf{\Pi}_{\hat{\mathcal{G}}_{\mathsf{P}_j}^{\perp}} \eta \tag{4.11}$$

where

$$\mathbf{S}_{\star} := \mathbf{\Pi}_{\hat{\mathcal{G}}_{\mathsf{P}_j}^{\perp}} \mathbf{S}|_{\hat{\mathcal{G}}_{\mathsf{P}_j}^{\perp}} \ : \ \hat{\mathcal{G}}_{\mathsf{P}_j}^{\perp} \to \hat{\mathcal{G}}_{\mathsf{P}_j}^{\perp}.$$

The unilateral shift $\mathbf{S}$ is an isometric dilation (lifting) of $\mathbf{S}_{\star}$ in that $\mathbf{S}_{\star}^{n} = \mathbf{\Pi}_{\hat{\mathcal{G}}_{\mathsf{P}_j}^{\perp}} \mathbf{S}^{n}|_{\hat{\mathcal{G}}_{\mathsf{P}_j}^{\perp}}$ for all $n \geq 0$. In fact, by the way $\hat{\mathcal{G}}_{\mathsf{P}_j}^{\perp}$ is defined and since $\hat{\mathcal{G}}_{\mathsf{P}_j}$ is shift-invariant, it follows that $\mathbf{S}$ is the minimal, isometric dilatation of $\mathbf{S}_{\star}$ in the sense that $\mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{V}))$ is the smallest shift-invariant space that contains $\hat{\mathcal{G}}_{\mathsf{P}_j}^{\perp}$.[4] Furthermore, from equation (4.11), it follows that

$$\mathbf{S}_{\star}\mathbf{\Pi}_{\hat{\mathcal{G}}_{\mathsf{P}_j}^{\perp}} = \mathbf{\Pi}_{\hat{\mathcal{G}}_{\mathsf{P}_j}^{\perp}} \mathbf{S}$$

and consequently, with the generalised, Hankel operator $\mathbf{H}_{\hat{\mathsf{G}}_{ri}}$ defined to be

$$\mathbf{H}_{\hat{\mathsf{G}}_{ri}} := \mathbf{\Pi}_{\hat{\mathcal{G}}_{\mathsf{P}_j}^{\perp}} \mathbf{M}_{\hat{\mathsf{G}}_{ri}}, \tag{4.12}$$

that

$$\mathbf{S}_{\star}\mathbf{H}_{\hat{\mathsf{G}}_{ri}} = \mathbf{\Pi}_{\hat{\mathcal{G}}_{\mathsf{P}_j}^{\perp}} \mathbf{S}\mathbf{M}_{\hat{\mathsf{G}}_{ri}} = \mathbf{H}_{\hat{\mathsf{G}}_{ri}}\mathbf{S}.$$

Since $\mathbf{S}$ is a minimal, isometric dilation of $\mathbf{S}_{\star}$, it follows by the Commutant Lifting Theorem (cf. [FF90, Chapter VII] and [SNF70]), that there exists an LSI operator $\mathsf{H} : \mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})) \to \mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{V}))$ that satisfies $\mathbf{\Pi}_{\hat{\mathcal{G}}_{\mathsf{P}_j}^{\perp}} \mathsf{H} = \mathbf{H}_{\hat{\mathsf{G}}_{ri}}$ and

$$\|\mathsf{H}\| = \|\mathbf{H}_{\hat{\mathsf{G}}_{ri}}\|. \tag{4.13}$$

Furthermore, by Proposition 2.2, the operator H can be expressed as a multiplication operator with symbol $\hat{\mathsf{H}} \in \mathrm{H}^{\infty}_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}) \to \mathrm{L}^2_{\mathbb{H}}(\mathcal{V})})$ and

$$\|\mathsf{H}\| = \|\hat{\mathsf{H}}\|_{\infty}. \tag{4.14}$$

Thus,

$$\mathbf{\Pi}_{\hat{\mathcal{G}}_{\mathsf{P}_j}^{\perp}} \mathbf{M}_{\hat{\mathsf{H}}} = \mathbf{H}_{\hat{\mathsf{G}}_{ri}} = \mathbf{\Pi}_{\hat{\mathcal{G}}_{\mathsf{P}_j}^{\perp}} \mathbf{M}_{\hat{\mathsf{G}}_{ri}},$$

which implies that $\mathbf{\Pi}_{\hat{\mathcal{G}}_{\mathsf{P}_j}^{\perp}} (\mathbf{M}_{\hat{\mathsf{H}}} - \mathbf{M}_{\hat{\mathsf{G}}_{ir}}) = 0$ or equivalently (since $\hat{\mathcal{G}}_{\mathsf{P}_j} = \hat{\mathsf{G}}_{rj}\mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}))$) that

$$(\hat{\mathsf{H}} - \hat{\mathsf{G}}_{ri})\mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})) \subset \hat{\mathsf{G}}_{rj}\mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})).$$

[4] cf. [FF90, SNF70] for a precise definition and treatment of minimal dilations.

Therefore, since $\hat{\mathsf{G}}_{rj}$ is inner and by Corollary IX.2.2 in [FF90, pp. 239-240] (a corollary of the Beurling-Lax-Halmos Theorem), there exists a $\hat{\mathsf{Q}}_\star \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})})$ that satisfies

$$\hat{\mathsf{H}} - \hat{\mathsf{G}}_{ri} = -\hat{\mathsf{G}}_{rj}\hat{\mathsf{Q}}_\star.$$

Hence, $\hat{\mathsf{H}} = \hat{\mathsf{G}}_{ri} - \hat{\mathsf{G}}_{rj}\hat{\mathsf{Q}}_\star$ and

$$\|\hat{\mathsf{G}}_{ri} - \hat{\mathsf{G}}_{rj}\hat{\mathsf{Q}}_\star\|_\infty = \|\mathbf{H}_{\hat{\mathsf{G}}_{ri}}\| = \|\mathbf{H}_{\hat{\mathsf{G}}_{ri}-\hat{\mathsf{G}}_{rj}\hat{\mathsf{Q}}}\| \leq \inf_{\hat{\mathsf{Q}}\in\mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})})} \|\hat{\mathsf{G}}_{ri} - \hat{\mathsf{G}}_{rj}\hat{\mathsf{Q}}\|_\infty,$$

where the first equality follows from equations (4.13) and (4.14), the second by the fact that $\mathbf{H}_{\hat{\mathsf{G}}_{rj}\hat{\mathsf{Q}}} = \mathbf{\Pi}_{\hat{\mathcal{G}}^\perp_{\mathsf{P}_j}}\mathbf{M}_{\hat{\mathsf{G}}_{rj}\hat{\mathsf{Q}}} = 0$ for all $\hat{\mathsf{Q}}$ and finally, the third inequality by the fact that $\|\mathbf{\Pi}_{\hat{\mathcal{G}}^\perp_{\mathsf{P}_j}}\| \leq 1$. Clearly, equality holds when $\hat{\mathsf{Q}} = \hat{\mathsf{Q}}_\star$. In conclusion,

$$\|\mathbf{\Pi}_{\hat{\mathcal{G}}^\perp_{\mathsf{P}_j}}\mathbf{\Pi}_{\hat{\mathcal{G}}_{\mathsf{P}_i}}\| = \|\mathbf{\Pi}_{\hat{\mathcal{G}}^\perp_{\mathsf{P}_j}}\mathbf{M}_{\hat{\mathsf{G}}_{ri}}\| = \|\mathbf{H}_{\hat{\mathsf{G}}_{ri}}\| = \inf_{\hat{\mathsf{Q}}\in\mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})})} \|\hat{\mathsf{G}}_{ri} - \hat{\mathsf{G}}_{rj}\hat{\mathsf{Q}}\|_\infty,$$

which completes the proof. ∎

Interestingly, the difference between two stable closed-loops with the same controller but different plants can be bounded (above and below) in terms of the gap between the two plants. As will be seen later, these bounds lead to a particularly simple proof that the gap metric induces what is commonly known as the graph topology (the weakest with respect to which closed-loop performance varies continuously and closed-loop stability is a robust property). Since the parallel projection onto the graph of a plant along the inverse graph of a stabilising controller completely characterises a closed-loop (cf. Remark 2.14), in the following lemma the difference between the closed-loops concerned is quantified by the induced norm of the difference between the respective parallel-projection operators. A similar result has appeared in [Vin93b] for LTI systems. As an aside, note that for the closed-loop configuration specified in the H^∞-loopshaping, design procedure proposed in Sub-Section 3.3.2, the pertinent closed-loop operator is simply a weighted, parallel projection (compare equations in Remark 2.14 and equation (3.16)). As such, the following result also characterises explicitly the degradation in performance (in the context of the proposed design procedure) that may be observed if the plant deviates from its nominal value.

Lemma 4.16 *For any $P_0, P_1 \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ and $C \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}}$ such that $[P_0, C]$ and $[P_1, C]$ are stable,*

$$\delta_g(P_0, P_1) \leq \|\mathbf{\Pi}_{\mathcal{G}_{P_0}\|\mathcal{G}'_C} - \mathbf{\Pi}_{\mathcal{G}_{P_1}\|\mathcal{G}'_C}\| \leq \frac{\delta_g(P_0, P_1)}{b_{P_0,C}\, b_{P_1,C}},$$

where $b_{P_i,C} := \|\mathbf{\Pi}_{\mathcal{G}_{P_i}\|\mathcal{G}'_C}\|^{-1}$, $i = 0, 1$.

Proof : Given any stable plant/controller pair $[P_i, C]$ with $P_i \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ $(i = 0, 1)$ and $C \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}}$, recall that $\mathcal{G}_{P_i}$ and $\mathcal{G}'_C$ induce a co-ordinatisation of $\mathrm{L}^2_{\mathbb{R}+}(\mathcal{V})$ [5] (cf. Proposition 2.12 and Remark 2.13) and that the parallel projection $\mathbf{\Pi}_{\mathcal{G}_{P_i}\|\mathcal{G}'_C}$ is a bounded, LPTV operator. Now (for $i, j = 0, 1$) define

$$\Delta \;:=\; \mathbf{\Pi}_{\mathcal{G}_{P_i}\|\mathcal{G}'_C} - \mathbf{\Pi}_{\mathcal{G}_{P_j}\|\mathcal{G}'_C}$$

and note that

$$(\mathbf{\Pi}_{\mathcal{G}_{P_i}\|\mathcal{G}'_C} - \mathbf{\Pi}_{\mathcal{G}_{P_j}\|\mathcal{G}'_C}) = \mathbf{\Pi}_{\mathcal{G}'_C\|\mathcal{G}_{P_i}}(\mathbf{\Pi}_{\mathcal{G}_{P_i}\|\mathcal{G}'_C} - \mathbf{\Pi}_{\mathcal{G}_{P_j}\|\mathcal{G}'_C}) = -\mathbf{\Pi}_{\mathcal{G}'_C\|\mathcal{G}_{P_i}}\mathbf{\Pi}_{\mathcal{G}_{P_j}\|\mathcal{G}'_C}. \tag{4.15}$$

To see this, take any $e \in \mathrm{L}^2_{\mathbb{R}+}(\mathcal{V})$, which since $\mathcal{G}_{P_i}$ $(i = 0, 1)$ and $\mathcal{G}'_C$ induce a co-ordinatisation of $\mathrm{L}^2_{\mathbb{R}+}(\mathcal{V})$ can be uniquely decomposed as

$$e = p_i + c_i = p_j + c_j, \tag{4.16}$$

where $p_i \in \mathcal{G}_{P_i} \subset \mathrm{L}^2_{\mathbb{R}+}(\mathcal{V})$, $p_j \in \mathcal{G}_{P_j} \subset \mathrm{L}^2_{\mathbb{R}+}(\mathcal{V})$ and $c_i, c_j \in \mathcal{G}'_C \subset \mathrm{L}^2_{\mathbb{R}+}(\mathcal{V})$. Then from equation (4.16), it follows that

$$(\mathbf{\Pi}_{\mathcal{G}_{P_i}\|\mathcal{G}'_C} - \mathbf{\Pi}_{\mathcal{G}_{P_j}\|\mathcal{G}'_C})e = (p_i - p_j) = (c_j - c_i) \in \mathcal{G}'_C,$$

by which equation (4.15) is immediate.

Using equation (4.15) and the identity

$$\mathbf{\Pi}_{\mathcal{G}_{P_j}\|\mathcal{G}'_C} = (\mathbf{\Pi}_{\mathcal{G}_{P_i}} + \mathbf{\Pi}_{\mathcal{G}^{\perp}_{P_i}})\mathbf{\Pi}_{\mathcal{G}_{P_j}}\mathbf{\Pi}_{\mathcal{G}_{P_j}\|\mathcal{G}'_C},$$

it follows that

$$\begin{aligned}
\Delta &= -\mathbf{\Pi}_{\mathcal{G}'_C\|\mathcal{G}_{P_i}}\mathbf{\Pi}_{\mathcal{G}_{P_j}\|\mathcal{G}'_C} \\
&= -\mathbf{\Pi}_{\mathcal{G}'_C\|\mathcal{G}_{P_i}}(\mathbf{\Pi}_{\mathcal{G}_{P_i}} + \mathbf{\Pi}_{\mathcal{G}^{\perp}_{P_i}})\mathbf{\Pi}_{\mathcal{G}_{P_j}}\mathbf{\Pi}_{\mathcal{G}_{P_j}\|\mathcal{G}'_C} \\
&= -\mathbf{\Pi}_{\mathcal{G}'_C\|\mathcal{G}_{P_i}}\mathbf{\Pi}_{\mathcal{G}^{\perp}_{P_i}}\mathbf{\Pi}_{\mathcal{G}_{P_j}}\mathbf{\Pi}_{\mathcal{G}_{P_j}\|\mathcal{G}'_C},
\end{aligned}$$

where the last equality holds since $\mathbf{\Pi}_{\mathcal{G}'_C\|\mathcal{G}_{P_i}}\mathbf{\Pi}_{\mathcal{G}_{P_i}} = 0$. Hence,

$$\|\Delta\| \leq \|\mathbf{\Pi}_{\mathcal{G}'_C\|\mathcal{G}_{P_i}}\| \cdot \|\mathbf{\Pi}_{\mathcal{G}^{\perp}_{P_i}}\mathbf{\Pi}_{\mathcal{G}_{P_j}}\| \cdot \|\mathbf{\Pi}_{\mathcal{G}_{P_j}\|\mathcal{G}'_C}\|,$$

which by definition of $b_{P_i,C}$, the directed gap and its relationship to the gap, implies

$$\|\Delta\| \leq \frac{\delta_g(P_0, P_1)}{b_{P_0,C}\,b_{P_1,C}}.$$

This is precisely the upper bound stated in the lemma.

[5] Also recall that $\mathcal{V} := \mathcal{U} \oplus \mathcal{Y}$.

To establish the lower bound, consider

$$\begin{aligned}\mathbf{\Pi}_{\mathcal{G}_{P_j}^{\perp}}\Delta\mathbf{\Pi}_{\mathcal{G}_{P_i}} &= \mathbf{\Pi}_{\mathcal{G}_{P_j}^{\perp}}(\mathbf{\Pi}_{\mathcal{G}_{P_i}\|\mathcal{G}_C'} - \mathbf{\Pi}_{\mathcal{G}_{P_j}\|\mathcal{G}_C'})\mathbf{\Pi}_{\mathcal{G}_{P_i}} \\ &= \mathbf{\Pi}_{\mathcal{G}_{P_j}^{\perp}}\mathbf{\Pi}_{\mathcal{G}_{P_i}},\end{aligned}$$

where the second equality follows from the identities $\mathbf{\Pi}_{\mathcal{G}_{P_j}^{\perp}}\mathbf{\Pi}_{\mathcal{G}_{P_j}\|\mathcal{G}_C'} = 0$ and $\mathbf{\Pi}_{\mathcal{G}_{P_i}\|\mathcal{G}_C'}\mathbf{\Pi}_{\mathcal{G}_{P_i}} = \mathbf{\Pi}_{\mathcal{G}_{P_i}}$. Hence, for $i, j = 0, 1$

$$\|\mathbf{\Pi}_{\mathcal{G}_{P_j}^{\perp}}\mathbf{\Pi}_{\mathcal{G}_{P_i}}\| = \|\mathbf{\Pi}_{\mathcal{G}_{P_j}^{\perp}}\Delta\mathbf{\Pi}_{\mathcal{G}_{P_i}}\| \leq \|\Delta\|,$$

from which it follows that

$$\delta_g(P_0, P_1) \leq \|\Delta\|,$$

as claimed. ∎

4.4 Quantitative Robustness Analysis

A stable, closed-loop system $[P, C]$ is said to be robustly stable if closed-loop stability is maintained when P and C undergo reasonable perturbations. An important property of closed-loop systems is that stability can be desensitised to such perturbations and in the sub-sections to follow, this property is quantitatively analysed in terms of the gap metric for LPTV, closed-loop systems.

4.4.1 Plant Perturbations

In this sub-section, the robustness of closed-loop stability is studied within the context of plant perturbations quantified using the gap metric. Before going on, it is instructive to recall the following definition (cf. Lemma 4.16):

Definition 4.17 Given a stable closed-loop $[P, C]$ with $P \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ and $C \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}}$, define

$$b_{P,C}^{-1} := \|\mathbf{\Pi}_{\mathcal{G}_P\|\mathcal{G}_C'}\| = \|\mathbf{\Pi}_{\mathcal{G}_{\mathsf{P}}\|\mathcal{G}_{\mathsf{C}}'}\| = \|\mathsf{G}_r(\mathsf{K}_l\mathsf{G}_r)^{-1}\mathsf{K}_l\|, \tag{4.17}$$

where $\mathsf{P} := \mathbf{W}P\mathbf{W}^{-1}$, $\mathsf{C} := \mathbf{W}C\mathbf{W}^{-1}$ and $\mathsf{G}_r \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}$ and $\mathsf{K}_l \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})}$ are any stable systems that satisfy $\mathcal{K}_{\mathsf{G}_r} = \{0\}$, $\mathcal{G}_{\mathsf{P}} = \mathcal{R}_{\mathsf{G}_r}$, $\mathcal{R}_{\mathsf{K}_l} = \ell^2_{\mathbb{Z}^+}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}))$ and $\mathcal{G}_{\mathsf{C}}' = \mathcal{K}_{\mathsf{K}_l}$ (cf. Remark 4.10). For reasons that will soon become apparent, $b_{P,C}$ is called the robust-stability margin. ♠

Using results obtained in the previous sections, it is now possible to state and prove the main robust stability result, which identifies the largest gap-ball of LPTV plants, centred at a nominal plant, that a nominal, stabilising, LPTV controller is guaranteed to stabilise. To a certain extent, the result is analogous to the LTI result of [GS90] and the general LTV result of [FGS93]. Note however, that it does not follow directly from either of these. In particular, considerable effort is required to prove necessity, where the approach taken is to construct a *causal* plant P_1 of appropriate gap distance from the nominal plant so that $[P_1, C]$ is well-posed but not stable.

Theorem 4.18 *Let $[P_0, C]$ be a stable, closed-loop system, where $P_0 \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ and $C \in \mathscr{P}^{\mathrm{e,sc}}_{\mathcal{Y}\to\mathcal{U}}$. Then, the following are equivalent :*

(i) $b_{P_0,C} \geq \beta$;

(ii) $[P_1, C]$ is stable for all $P_1 \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ that satisfy $\delta_g(P_0, P_1) < \beta$.

Proof : **(**$(i) \Rightarrow (ii)$**)** By assumption, $\delta_g(P_0, P_1) = \delta(\mathcal{G}_{\mathsf{P}_0}, \mathcal{G}_{\mathsf{P}_1}) < \beta$ and $b^{-1}_{P_0,C} \leq \beta^{-1}$. For $i = 0, 1$, let $\mathsf{G}_{ri} \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})}$ be an isometry such that $\mathcal{G}_{\mathsf{P}_i} = \mathcal{R}_{\mathsf{G}_{ri}}$, where $\mathsf{P}_i := \mathbf{W}P_i\mathbf{W}^{-1}$.[6] Then $\mathbf{\Pi}_{\mathcal{G}_{\mathsf{P}_i}} = \mathsf{G}_{ri}(\mathsf{G}_{ri})^*$ and

$$\delta(\mathcal{G}_{\mathsf{P}_0}, \mathcal{G}_{\mathsf{P}_1}) = \|\mathbf{\Pi}_{\mathcal{G}_{\mathsf{P}_0}} - \mathbf{\Pi}_{\mathcal{G}_{\mathsf{P}_1}}\| = \|\mathsf{G}_{r0}\mathsf{G}^*_{r0} - \mathsf{G}_{r1}\mathsf{G}^*_{r1}\| < \beta. \tag{4.18}$$

Moreover, since G_{r0} is an isometry,

$$b^{-1}_{P_0,C} = \|(\mathsf{K}_l\mathsf{G}_{r0})^{-1}\mathsf{K}_l\| \leq \beta^{-1}, \tag{4.19}$$

where $\mathsf{K}_l \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})}$ is any stable system that satisfies $\mathcal{K}_{\mathsf{K}_l} = \mathcal{G}'_{\mathsf{C}}$ and $\mathcal{R}_{\mathsf{K}_l} = \ell^2_{\mathbb{Z}^+}(\mathcal{U})$.[7] Combining equations (4.18) and (4.19) yields

$$\|(\mathsf{K}_l\mathsf{G}_{r0})^{-1}\mathsf{K}_l(\mathsf{G}_{r0}\mathsf{G}^*_{r0} - \mathsf{G}_{r1}\mathsf{G}^*_{r1})\| < 1,$$

which implies

$$\|\mathrm{I} - (\mathsf{K}_l\mathsf{G}_{r0})^{-1}\mathsf{K}_l\mathsf{G}_{r1}\mathsf{G}^*_{r1}\mathsf{G}_{r0}\| \leq \|\mathsf{G}^*_{r0} - (\mathsf{K}_l\mathsf{G}_{r0})^{-1}\mathsf{K}_l\mathsf{G}_{r1}\mathsf{G}^*_{r1}\|\|\mathsf{G}_{r0}\| < 1.$$

Hence, it follows that $(\mathsf{K}_l\mathsf{G}_{r0})^{-1}\mathsf{K}_l\mathsf{G}_{r1}(\mathsf{G}^*_{r1}\mathsf{G}_{r0})$ has bounded inverse. Since $\delta_g(P_0, P_1) < 1$, note that the operator $\mathsf{G}^*_{r1}\mathsf{G}_{r0}$ also has bounded inverse (cf. Corollary 4.14) and therefore, that $\mathsf{K}_l\mathsf{G}_{r1}$ is boundedly-invertible, which by Lemma 4.8, implies stability of the closed-loop $[P_1, C]$.

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[6] $\mathcal{G}_{\mathsf{P}_0}$ and $\mathcal{G}_{\mathsf{P}_1}$ are LSI subspaces and hence, the required G_{r0} and G_{r1} exist by the Beurling-Lax-Halmos Theorem.
[7] Since $[P_0, C]$ is stable, such a K_l exists by Theorem 4.6.

($(i) \Leftarrow (ii)$) Suppose that (ii) holds for some $1 \geq \beta > b_{P_0,C}$.[8] Under this hypothesis it is sufficient to construct a system $P_1 \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ such that $\delta_g(P_0, P_1) < \beta$ and $[P_1, C]$ is well-posed, but not stable. Let G_{r0} and K_l be defined as in the proof of $(i) \Rightarrow (ii)$. Then by Proposition 2.2, G_{r0} and K_l are equivalent (via $\mathbf{Z}$) to multiplication operators with symbols $\hat{\mathsf{G}}_{r0} \in \mathrm{H}^{\infty}_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})})$ and $\hat{\mathsf{K}}_l \in \mathrm{H}^{\infty}_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})})$ respectively. Furthermore, it follows by Lemma 4.8 and Remark 4.10, that $(\hat{\mathsf{K}}_l\hat{\mathsf{G}}_{r0})^{-1} \in \mathrm{H}^{\infty}_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})})$. Hence,

$$\hat{\mathsf{K}}_{l\star} := (\hat{\mathsf{G}}_{r0}\hat{\mathsf{K}}_l)^{-1}\hat{\mathsf{K}}_l \in \mathrm{H}^{\infty}_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})})$$

and $\mathsf{K}_{l\star} := \mathbf{Z}^{-1}\mathbf{M}_{\hat{\mathsf{K}}_{l\star}}\mathbf{Z}$ is a stable system in $\mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})}$ that satisfies $\mathcal{K}_{\mathsf{K}_{l\star}} = \mathcal{G}'_{\mathsf{C}}$ and $\mathcal{R}_{\mathsf{K}_{l\star}} = \ell^2_{\mathbb{Z}+}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}))$. In fact, by definition,

$$b^{-1}_{P_0,C} := \|\mathsf{G}_{r0}\mathsf{K}_{l\star}\| = \|\mathsf{K}_{l\star}\| = \|\hat{\mathsf{K}}_{l\star}\|_{\infty} = \sup_{\lambda\in\mathbb{D}} \|\hat{\mathsf{K}}_{l\star}(\lambda)\|, \tag{4.20}$$

where the second equality holds because G_{r0} is an isometry. Since the function $\hat{\mathsf{K}}_{l\star}(\lambda)$ is analytic in $\mathbb{D}$, $\|\hat{\mathsf{K}}_{l\star}(\lambda)\|$ satisfies a maximum principle on any connected, open subset of $\mathbb{D}$ (cf. Sub-Section 2.2.3). Using this property and equation (4.20), it follows that for any $\epsilon_0 > 0$ and $\epsilon_1 > 0$, there exists a $\lambda_{0,\epsilon_0} \in \mathbb{A}_{\epsilon_0} := \{\lambda : (1-\epsilon_0) \leq |\lambda| < 1\}$ such that

$$(b_{P_0,C} + \epsilon_1)^{-1} \leq \|\hat{\mathsf{K}}_{l\star}(\lambda_{0,\epsilon_0})\| \leq b^{-1}_{P_0,C}.$$

Now for any $\epsilon_2 > \epsilon_1$ it is possible to construct an operator $\Delta_{0,\epsilon_2} \in \mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})}$ with

$$\|\Delta_{0,\epsilon_2}\| \leq (b_{P_0,C} + \epsilon_2), \tag{4.21}$$

such that $(\mathrm{I} + \hat{\mathsf{K}}_{l\star}(\lambda_{0,\epsilon_0})\Delta_{0,\epsilon_2})$ is not invertible in $\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})}$.[9] Defining

$$\hat{\Delta}_{\epsilon_2}(\lambda) := \frac{\lambda}{\lambda_{0,\epsilon_0}}\Delta_{0,\epsilon_2} \in \mathrm{H}^{\infty}_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})}), \tag{4.22}$$

[8] $\delta_g(P_0, P_1) \leq 1$ for all $P_1 \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$. So it is without loss of generality that β is taken to be less than or equal to 1, otherwise there is an obvious contradiction.

[9] To see this, let A denote a bounded, linear operator in $\mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$. Then for any $\epsilon > 0$, it follows by definition of the induced norm that there exists a $u \in \mathcal{U}$ (which is taken to have unit norm) so that with $y := Au$,

$$\|A\| \geq \|y\|_{\mathcal{Y}} \geq \|A\| - \epsilon.$$

Defining $\Delta : \mathcal{Y} \to \mathcal{U}$ to be the operator that maps αy to $-\alpha u$ for all $\alpha \in \mathbb{C}$ and every other direction orthogonal to this in $\mathcal{Y}$ to 0, it follows that

$$\|\Delta\| \leq \frac{1}{\|A\| - \epsilon}.$$

Clearly then $y \in \mathcal{K}_{(I+A\Delta)}$ and hence, $(I + A\Delta)$ is not invertible.

it follows that

$$\|(\mathrm{I} + \hat{\mathsf{K}}_{l\star}(\lambda)\hat{\Delta}_{\epsilon_2}(\lambda))^{-1}\| \to \infty$$

as $\lambda \to \lambda_{0,\epsilon_0}$, since $(\mathrm{I} + \hat{\mathsf{K}}_{l\star}\hat{\Delta})$ is continuous on $\mathbb{D}$ and $(\mathrm{I} + \hat{\mathsf{K}}_{l\star}(\lambda_{0,\epsilon_0})\Delta_{0,\epsilon_2})$ is not invertible.[10] That is, $(\mathrm{I} + \hat{\mathsf{K}}_{l\star}\hat{\Delta}_{\epsilon_2})$ in not invertible in $\mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})})$.

Let

$$\hat{\mathsf{G}}_{r1} := (\hat{\mathsf{G}}_{r0} + \hat{\Delta}_{\epsilon_2})\hat{\mathsf{Q}}_1,$$

for some $\hat{\mathsf{Q}}_1, \hat{\mathsf{Q}}_1^{-1} \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})})$ and note that

$$\hat{\mathsf{K}}_{l\star}\hat{\mathsf{G}}_{r1} = (\mathrm{I} + \hat{\mathsf{K}}_{l\star}\hat{\Delta}_{\epsilon_2})\mathrm{Q}_1.$$

Clearly, $\hat{\mathsf{K}}_{l\star}\hat{\mathsf{G}}_{r1}$ is not invertible in $\mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})})$. Consequently, with $\mathsf{G}_{r1} := \mathbf{Z}^{-1}\mathbf{M}_{\hat{\mathsf{G}}_{r1}}\mathbf{Z}$, the stable system $\mathsf{K}_{l\star}\mathsf{G}_{r1} \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})}$ is not boundedly invertible. So provided

(a) $\mathcal{K}_{\mathsf{G}_{r1}} = \{0\}$;

(b) $\mathcal{R}_{\mathsf{G}_{r1}}$ is isomorphic (via $\mathbf{W}$) to the graph of a system $P_1 \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$;

(c) and $\delta_g(P_0, P_1) < \beta$,

a contradiction to the initial hypothesis that (ii) holds for some $1 \geq \beta > b_{P_0,C}$ can be established using Lemma 4.8 and Remark 4.10. Accordingly, the remainder of the proof is devoted to showing that by appropriate choice of ϵ_0, ϵ_2 and $\hat{\mathsf{Q}}_1$, conditions (a), (b) and (c) can be satisfied.

First note that $\mathsf{G}_{r1} = (\mathsf{G}_{r0} + \Delta_{\epsilon_2})\mathsf{Q}_1$, where

$$\mathsf{G}_{r0} := \mathbf{Z}^{-1}\mathbf{M}_{\hat{\mathsf{G}}_{r0}}\mathbf{Z},\ \ \Delta_{\epsilon_2} := \mathbf{Z}^{-1}\mathbf{M}_{\hat{\Delta}_{\epsilon_2}}\mathbf{Z},\ \text{ and }\ \mathsf{Q}_1 := \mathbf{Z}^{-1}\mathbf{M}_{\hat{\mathsf{Q}}_1}\mathbf{Z}.$$

Furthermore, note that $\hat{\Delta}_{\epsilon_2}(\lambda)$ has continuous extension to the closed, unit disc and hence, that $\|\hat{\Delta}_{\epsilon_2}\|_\infty = \max_{\omega\in[0,2\pi)} \|\hat{\Delta}_{\epsilon_2}(e^{j\omega})\|$. It then follows by Proposition 2.2 and from equations (4.21) and (4.22), that

$$\|\Delta_{\epsilon_2}\| = \|\hat{\Delta}_{\epsilon_2}\|_\infty = \frac{1}{|\lambda_{0,\epsilon_0}|}\|\Delta_{0,\epsilon_2}\| \leq \frac{b_{P_0,C} + \epsilon_2}{1 - \epsilon_0}.$$

By hypothesis $(\beta - b_{P_0,C}) > 0$ and setting $\epsilon_2 = (\beta - b_{P_0,C})/2 > 0$, $\epsilon_1 < \epsilon_2$ and $\epsilon_0 < \epsilon_2/2\beta$, gives

$$\|\Delta_{\epsilon_2}\| < \beta \leq 1. \tag{4.23}$$

It is now immediate that G_{r1} has zero kernel, since G_{r0} is an isometry. That is, condition (a) is satisfied.

[10]See [Dul94, Lemma 4.3 (only if part)] for a similar result.

$\hat{\mathsf{Q}}_1 \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})})$ is now constructed so that $\mathcal{R}_{\mathsf{G}_{r1}}$ is isomorphic (via $\mathbf{W}$) to the graph of a system $P_1 \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$. First let

$$\begin{bmatrix} \mathsf{M}_{r1} \\ \mathsf{N}_{r1} \end{bmatrix} := \mathsf{G}_{r1} \text{ and } \begin{bmatrix} \mathsf{M}_{r0} \\ \mathsf{N}_{r0} \end{bmatrix} := \mathsf{G}_{r0},$$

where the partitioning is conformal with $\mathcal{G}_{\mathsf{P}_0}$. By construction $\Delta_{[0]} = 0$, where the subscript $[0]$ denotes the first term in the sequence uniquely identifiable with the Toeplitz representation. Thus, $\mathsf{G}_{r1[0]} = (\mathsf{G}_{r0}\mathsf{Q}_1)_{[0]}$ and

$$\begin{bmatrix} \mathsf{M}_{r1[0]} \\ \mathsf{N}_{r1[0]} \end{bmatrix} = \left(\begin{bmatrix} \mathsf{M}_{r0} \\ \mathsf{N}_{r0} \end{bmatrix} \mathsf{Q}_1 \right)_{[0]} = \begin{bmatrix} \mathsf{M}_{r0[0]} \\ \mathsf{N}_{r0[0]} \end{bmatrix} \mathsf{Q}_{1[0]}. \tag{4.24}$$

Since $\mathcal{R}_{\mathsf{G}_{r0}}$ is isomorphic to $\mathcal{G}_{P_0}$ and $P_0 \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$, it follows (as is detailed in the proof of Theorem 4.6) that $\mathsf{M}_{r0[0]} \in \mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})}$ is boundedly invertible. So by the assumed invertibility of Q_1 and equation (4.24), $\mathsf{M}_{r1[0]}$ is also invertible in $\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})}$. Consequently, M_{r1} has zero kernel, which confirms that $\mathcal{R}_{\mathsf{G}_{r1}}$ is the graph of a linear operator. In fact, it is isomorphic to the graph of an LPTV system P_1 defined on a subspace of $\mathrm{L}^2_{\mathbb{R}+}(\mathcal{U})$. It is sufficient therefore, to select $\hat{\mathsf{Q}}_1$ to ensure that P_1 is causal and has locally-Lipschitz-continuous extension. To this end, set $\hat{\mathsf{Q}}_1 = \mathsf{M}^{-1}_{r0[0]}$, which is clearly an invertible element in $\mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})})$, so that $\mathsf{Q}_{1[0]} = \mathsf{M}^{-1}_{r0[0]}$ and

$$\mathsf{G}_{r1[0]} = \begin{bmatrix} \mathsf{M}_{r1[0]} \\ \mathsf{N}_{r1[0]} \end{bmatrix} = (\mathsf{G}_{r0}\mathsf{Q}_1)_{[0]} = \begin{bmatrix} \mathrm{I} \\ \mathsf{N}_{r0[0]}\mathsf{M}^{-1}_{r0[0]} \end{bmatrix}.$$

Since $\mathsf{P}_{0[0]} = \mathsf{N}_{r0[0]}\mathsf{M}^{-1}_{r0[0]}$ is causal on $\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})$, this is also a causal map on $\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})$. It then follows by Remark 2.6, that $\begin{bmatrix} M_{r1} \\ N_{r1} \end{bmatrix} := G_{r1} = \mathbf{W}^{-1}\mathsf{G}_{r1}\mathbf{W}$ is a stable system in $\mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{V}}$, which in turn implies that P_1 is causal. To see this, suppose that P_1 is not causal. Then by definition, there exists a point $\begin{bmatrix} u \\ y \end{bmatrix} \in \mathcal{G}_{P_1} = \mathcal{R}_{G_{r1}}$ and a $\tau_1 \in \mathbb{R}^+$ such that

$$\mathbf{P}_{\tau_1}\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{y} \neq 0 \end{bmatrix}.$$

That is, there exists a $q \in \mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U})$ such that $\tilde{y} = \mathbf{P}_{\tau_1} N_{r1} q$ and $0 = \mathbf{P}_{\tau_1} M_{r1} q$. Since M_{r1} is causal,

$$
\begin{aligned}
\mathbf{P}_{\tau_1} M_{r1} q &= \mathbf{P}_{\tau_1} \mathbf{W}^{-1} \mathbf{P}_n \mathsf{M}_{r1} \mathbf{W} q \\
&= \mathbf{P}_{\tau_1} \mathbf{W}^{-1} \begin{bmatrix} \underbrace{\begin{bmatrix} \mathrm{I} & 0 & \cdots & 0 \\ \mathsf{M}_{r1[1]} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \mathsf{M}_{r1[n]} & \cdots & \mathsf{M}_{r1[1]} & \mathrm{I} \end{bmatrix}}_{:=\tilde{\mathsf{M}}_n} \begin{bmatrix} \vec{\mathsf{q}}_0 \\ \vec{\mathsf{q}}_1 \\ \vdots \\ \vdots \\ \vec{\mathsf{q}}_n \end{bmatrix} \\ 0 \\ \vdots \end{bmatrix},
\end{aligned} \tag{4.25}
$$

where $\mathsf{q}(=: k \mapsto \vec{\mathsf{q}}_k) := \mathbf{W} q$, $n = \left]\frac{\tau_1}{h}\right[+ 1$ and $]\cdot[$ denotes the integer part of a real number. Now note that $\tilde{\mathsf{M}}_i$ is boundedly invertible for all finite $i \in \mathbb{Z}^+$ and therefore, has zero kernel and full range.[11] So if $0 = \mathbf{P}_{\tau_1} M_{r1} q$, it follows from equation (4.25), that $\vec{\mathsf{q}}_0 = \vec{\mathsf{q}}_1 = \ldots = \vec{\mathsf{q}}_{n-1} = 0$ and $\mathbf{P}_{(nh+1-\tau_1)} \vec{\mathsf{q}}_n = 0$ or equivalently, $\mathbf{P}_{\tau_1} q = 0$. Then since N_{r1} is causal, this implies that $\tilde{y} = \mathbf{P}_{\tau_1} N_{r1} q = 0$, which is a contradiction and consequently, P_1 is causal. This line of reasoning also leads to the conclusion that P_1 is causally extendible to a locally-Lipschitz-continuous operator on $\mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{U})$. More specifically, since $\tilde{\mathsf{M}}_i$ has full range for all finite $i \in \mathbb{Z}^+$,

$$
\mathbf{P}_\tau \mathcal{D}_{P_1} = \mathbf{P}_\tau \mathcal{R}_{M_{r1}} = \mathbf{P}_\tau \mathbf{W}^{-1} \begin{bmatrix} \mathcal{R}_{\tilde{\mathsf{M}}_n} \\ 0 \end{bmatrix} = \mathbf{P}_\tau \mathrm{L}^2_{[0,nh]}(\mathcal{U}) = \mathrm{L}^2_{[0,\tau]}(\mathcal{U})
$$

for all τ, where $n = \left]\frac{\tau}{h}\right[+ 1$. That is, P_1 is causally extendible (by definition). Now denote the extension of P_1 by P_{1e}. Then,

$$
\sup_{\substack{u_1, u_2 \in \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{U}) \\ \mathbf{P}_\tau u_1 \neq \mathbf{P}_\tau u_2}} \frac{\|\mathbf{P}_\tau (P_{1e} u_1 - P_{1e} u_2)\|_{\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{Y})}}{\|\mathbf{P}_\tau (u_1 - u_2)\|_{\mathrm{L}^2_{\mathbb{R}^+}(\mathcal{U})}} \leq \sup_{\substack{\mathsf{u} \in \mathbf{P}_n \ell^2_{\mathbb{Z}^+}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})) \\ \mathsf{u} \neq 0}} \frac{\|\tilde{\mathsf{N}}_n \tilde{\mathsf{M}}_n^{-1} \mathsf{u}\|_{\mathbf{P}_n \ell^2_{\mathbb{Z}^+}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y}))}}{\|\mathsf{u}\|_{\mathbf{P}_n \ell^2_{\mathbb{Z}^+}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}))}},
$$

where

$$
\tilde{\mathsf{N}}_n := \begin{bmatrix} \mathsf{N}_{r1[0]} & 0 & \cdots & 0 \\ \mathsf{N}_{r1[1]} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \mathsf{N}_{r1[n]} & \cdots & \mathsf{N}_{r1[1]} & \mathsf{N}_{r1[0]} \end{bmatrix}.
$$

Since $\tilde{\mathsf{N}}_i$ and $\tilde{\mathsf{M}}_i^{-1}$ are both bounded for all finite $i \in \mathbb{Z}^+$, it is then clear (by definition) that P_{1e} is locally Lipschitz-continuous as claimed.

[11]In fact $\tilde{\mathsf{M}}_i^{-1}$ is block lower-triangular with identities down the diagonal.

It remains to show that $\delta_g(P_0, P_1) < \beta$. First it is shown that $\vec{\delta}_g(P_0, P_1) < \beta$ and then, that $\delta_g(P_0, P_1) = \vec{\delta}_g(P_0, P_1) < \beta$. Note that $\hat{\mathcal{G}}_{\mathsf{P}_1} := \mathbf{ZW}\mathcal{G}_{\mathsf{P}_1}$ is an LSI subspace of $\mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{V}))$. Therefore, by the Beurling-Lax-Halmos Theorem, there exists an inner function $\hat{\mathsf{G}}_{r\star} \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{V})})$ such that $\hat{\mathcal{G}}_{\mathsf{P}_1} = \hat{\mathsf{G}}_{r1\star}\mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}))$. Moreover, since $\mathcal{R}_{\mathbf{M}_{\hat{\mathsf{G}}_{r1}}} = \mathcal{R}_{\mathbf{M}_{\hat{\mathsf{G}}_{r1\star}}}$ and $\mathcal{K}_{\mathbf{M}_{\hat{\mathsf{G}}_{r1}}} = \{0\}$, there exists a $\hat{\mathsf{Q}}_\star, \hat{\mathsf{Q}}_\star^{-1} \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})})$ such that $\hat{\mathsf{G}}_{r1\star}\hat{\mathsf{Q}}_\star = \hat{\mathsf{G}}_{r1} = (\hat{\mathsf{G}}_{r0} + \hat{\Delta})\hat{\mathsf{Q}}_1$ (cf. Corollary IX.2.2 of the Beurling-Lax-Halmos Theorem in [FF90]). Then by Theorem 4.15,

$$\begin{aligned}\vec{\delta}_g(P_0, P_1) = \vec{\delta}(\hat{\mathcal{G}}_{\mathsf{P}_0}, \hat{\mathcal{G}}_{\mathsf{P}_1}) &= \inf_{\hat{\mathsf{Q}}\in\mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})})} \|\hat{\mathsf{G}}_{r0} - \hat{\mathsf{G}}_{r1\star}\hat{\mathsf{Q}}\|_\infty \\ &= \inf_{\hat{\mathsf{Q}}\in\mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})})} \|\hat{\mathsf{G}}_{r0} - (\hat{\mathsf{G}}_{r0} + \hat{\Delta}_{\epsilon_2})\hat{\mathsf{Q}}_1\hat{\mathsf{Q}}_\star^{-1}\hat{\mathsf{Q}}\|_\infty \\ &\leq \|\hat{\Delta}_{\epsilon_2}\|_\infty < \beta,\end{aligned}$$

where the third inequality follows from the fact that $\hat{\mathsf{Q}}_\star\hat{\mathsf{Q}}_1^{-1} \in \mathrm{H}^\infty_{\mathbb{D}}(\mathcal{B}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})})$ and the last inequality from equation (4.23). Now for any $\hat{\mathsf{q}} \in \mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}))$ consider the projection of $\mathbf{M}_{\hat{\mathsf{G}}_{r1}}\hat{\mathsf{q}} \in \hat{\mathcal{G}}_{\mathsf{P}_1}$ onto $\hat{\mathcal{G}}_{\mathsf{P}_0}$. Since $\mathbf{M}_{\hat{\mathsf{G}}_{r0}}$ is an isometry, $\mathbf{\Pi}_{\hat{\mathcal{G}}_{\mathsf{P}_0}} = \mathbf{M}_{\hat{\mathsf{G}}_{r0}}(\mathbf{M}_{\hat{\mathsf{G}}_{r0}})^*$ and the projection described above can be expressed as

$$\mathbf{M}_{\hat{\mathsf{G}}_{r0}}(\mathbf{M}_{\hat{\mathsf{G}}_{r0}})^*(\mathbf{M}_{\hat{\mathsf{G}}_{r0}} + \mathbf{M}_{\hat{\Delta}_{\epsilon_2}})\mathbf{M}_{\hat{\mathsf{Q}}_1}\hat{\mathsf{q}} = \mathbf{M}_{\hat{\mathsf{G}}_{r0}}\hat{\mathsf{g}},$$

where

$$\hat{\mathsf{g}} := (\mathrm{I} + (\mathbf{M}_{\hat{\mathsf{G}}_{r0}})^*\mathbf{M}_{\hat{\Delta}_{\epsilon_2}})\mathbf{M}_{\hat{\mathsf{Q}}_1}\hat{\mathsf{q}}.$$

Furthermore,

$$\|(\mathbf{M}_{\hat{\mathsf{G}}_{r0}})^*\mathbf{M}_{\hat{\Delta}_{\epsilon_2}}\| \leq \|\mathbf{M}_{\hat{\mathsf{G}}_{r0}}\| \cdot \|\mathbf{M}_{\hat{\Delta}_{\epsilon_2}}\| < \beta \leq 1.$$

Thus, $(\mathrm{I} + (\mathbf{M}_{\hat{\mathsf{G}}_{r0}})^*\mathbf{M}_{\hat{\Delta}_\epsilon})$ is a *one-to-one* mapping *onto* $\mathrm{H}^2_{\mathbb{D}}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}))$ and since $\mathbf{M}_{\hat{\mathsf{Q}}_1}$ is bijective, it follows that the projection described above is also bijective. So by Proposition 4.13,

$$\vec{\delta}(\hat{\mathcal{G}}_{\mathsf{P}_0}, \hat{\mathcal{G}}_{\mathsf{P}_1}) = \vec{\delta}(\hat{\mathcal{G}}_{\mathsf{P}_1}, \hat{\mathcal{G}}_{\mathsf{P}_0}) = \delta(\hat{\mathcal{G}}_{\mathsf{P}_0}, \hat{\mathcal{G}}_{\mathsf{P}_1})$$

and hence, $\delta_g(P_0, P_1) = \delta(\hat{\mathcal{G}}_{\mathsf{P}_0}, \hat{\mathcal{G}}_{\mathsf{P}_1}) = \vec{\delta}(\hat{\mathcal{G}}_{\mathsf{P}_0}, \hat{\mathcal{G}}_{\mathsf{P}_1}) < \beta$ as required. This completes the proof. ■

Consider a closed-loop $[P_0, C]$, with $P_0 \in \mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U}\to\mathcal{Y}}$ and $C \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}}$. Then by Proposition 2.9, $[P_0, C]$ is well-posed and if all permissible perturbations are restricted to $\mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U}\to\mathcal{Y}}$, the closed-loop is always well-posed. Furthermore, by the way the destabilising plant P_1 is constructed in the proof above, $\mathsf{P}_{0[0]} = \mathsf{P}_{1[0]}$. That is, the instantaneous behaviour of the nominal plant and the plant constructed to destabilise the closed-loop is the same. As such, if $P_0 \in \mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U}\to\mathcal{Y}}$, then $P_1 \in \mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U}\to\mathcal{Y}}$. Consequently, $(ii) \Rightarrow (i)$ would still hold if permissible perturbations were restricted to only strongly causal ones.

Corollary 4.19 *Given a stable closed-loop* $[P_0, C]$*, where* $P_0 \in \mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U}\to\mathcal{Y}}$ *and* $C \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}}$*, the following are equivalent:*

(i) $b_{P_0,C} \geq \beta$*;*

(ii) $[P_1, C]$ *is stable for all* $P_1 \in \mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U}\to\mathcal{Y}}$ *such that* $\delta_g(P_0, P_1) < \beta$*.*

Similarly, although it is very difficult to characterise analytically, restricting the set of permissible perturbations to those which do not cause the nominal closed-loop to become ill-posed,[12] leads to the following restatement of Theorem 4.18.

Corollary 4.20 *Given a stable closed-loop* $[P_0, C]$*, where* $P_0 \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ *and* $C \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}}$*, the following are equivalent:*

(i) $b_{P_0,C} \geq \beta$*;*

(ii) $[P_1, C]$ *is stable for all* $P_1 \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ *such that* $\delta_g(P_0, P_1) < \beta$ *and* $[P_1, C]$ *is well-posed.*

4.4.2 Simultaneous Plant and Controller Perturbations

In this sub-section, the maximum combined perturbation to both P and C that does not cause instability is quantified in terms of the gap metric. As a corollary of Theorem 4.18, it is shown that a given C stabilises a gap-ball centred at P if and only if P is stabilised by all controllers in a gap-ball of the same radius and centred at C. Then, the maximum combined perturbation (as a sum) to P and C that the closed-loop can tolerate is considered. The results presented are analogous to the LTI results of [GS90, §VI] and the LTV results of [FGS93, §4]. First a technical result.

Lemma 4.21 [DGS93] *Let* $\{\mathcal{F}, \mathcal{G}\}$ *be a co-ordinatisation of* $\mathcal{V}$*, where* $\mathcal{F}$*,* $\mathcal{G}$ *and* $\mathcal{V}$ *are (closed) Hilbert spaces. If* $\mathcal{F}$ *is a linear subspace then*

$$\|\mathbf{\Pi}_{\mathcal{F}\|\mathcal{G}}\| = \|\mathbf{\Pi}_{\mathcal{G}\|\mathcal{F}}\|.$$

Recall the definition of $b_{P,C}$ (cf. Definition 4.17). Then since $\mathcal{G}'_C$ is a linear subspace, the following corollary of Lemma 4.21 is immediate.

Corollary 4.22

$$b_{P,C} = b_{C,P}.$$

[12]This is necessary for $(i) \Rightarrow (ii)$ to hold.

Proof : Note that

$$b_{C,P} := \|\mathbf{\Pi}_{\mathcal{G}_C \| \mathcal{G}'_P}\| = \left\| \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \mathbf{\Pi}_{\mathcal{G}'_C \| \mathcal{G}_P} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \right\| = \|\mathbf{\Pi}_{\mathcal{G}'_C \| \mathcal{G}_P}\|.$$

Since $[P, C]$ is stable, $\{\mathcal{G}_P, \mathcal{G}'_C\}$ induces a co-ordinatisation of $\mathrm{L}^2_{\mathbb{R}+}(\mathcal{V})$ (cf. Proposition 2.12) and so by Lemma 4.21, $b_{C,P} = \|\mathbf{\Pi}_{\mathcal{G}'_C \| \mathcal{G}_P}\| = \|\mathbf{\Pi}_{\mathcal{G}_P \| \mathcal{G}'_C}\| = b_{P,C}$. ∎

Similarly, the following corollary, which characterises robust stability to perturbations of the controller, is an immediate consequence of Corollary 4.22 and Theorem 4.18.

Corollary 4.23 *Let $[P, C]$ be a stable closed-loop system, where $P \in \mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U}\to\mathcal{Y}}$ and $C \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}}$. Then the following are equivalent :*

(i) $[P_1, C]$ is stable for all $P_1 \in \mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U}\to\mathcal{Y}}$ such that $\delta_g(P, P_1) < \beta$;

(ii) $[P, C_1]$ is stable for all $C_1 \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}}$ such that $\delta_g(C, C_1) < \beta$.

Remark 4.24 As with Corollaries 4.19 and 4.20, Corollary 4.23 can be restated with various different constraints on the strength of causality of the plants and the controllers. ♣

Now, with this established, it is possible to quantitatively characterise robustness to perturbations of both P and its stabilising controller C.

Theorem 4.25 *Consider a stable closed-loop system $[P, C]$, where $P \in \mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U}\to\mathcal{Y}}$ and $C \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}}$. The following are equivalent:*

(i) $\beta \le b_{P,C}$;

(ii) $[P_1, C_1]$ is stable for all $P_1 \in \mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U}\to\mathcal{Y}}$ and $C_1 \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}}$ such that

$$\delta_g(P, P_1) + \delta_g(C, C_1) < \beta.$$

Proof : The proof follows directly from Corollary 4.19 of Theorem 4.18 and Corollary 4.23, in exactly the same way as the corresponding result in [GS90, Theorem 7]. Assume that (i) holds so that by Corollaries 4.19 and 4.23, $[P, C_1]$ is stable for all $C_1 \in \mathscr{B}^{\mathrm{e}}_{\delta_g}(C, \beta)$, where

$$\mathscr{B}^{\mathrm{e}}_{\delta_g}(F, \epsilon) := \{\grave{F} \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}} \ : \ \delta_g(F, \grave{F}) < \epsilon\}.$$

Consider a controller $C_1 \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}}$, such that $\tau := \delta_g(C, C_1) < \beta$. Now by the metric property of the gap, $\mathscr{B}^{\mathrm{e}}_{\delta_g}(C_1, \beta - \tau) \subset \mathscr{B}^{\mathrm{e}}_{\delta_g}(C, \beta)$ and hence, $[P, C_2]$ is stable for all $C_2 \in \mathscr{B}^{\mathrm{e}}_{\delta_g}(C_1, \beta - \tau)$. Using

Corollary 4.23, it is then possible to conclude that $[P_1, C_1]$ is stable for all $P_1 \in \mathscr{B}^{\mathrm{e,sc}}_{\delta_g}(P, \beta - \tau)$, where

$$\mathscr{B}^{\mathrm{e,sc}}_{\delta_g}(F, \epsilon) := \{\grave{F} \in \mathscr{P}^{\mathrm{e,sc}}_{\mathcal{Y}\to\mathcal{U}} \ : \ \delta_g(F, \grave{F}) < \beta\}.$$

Since this holds for any $C_1 \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ such that $\delta_g(C, C_1) < \beta$, it is clear that (ii) holds. That $(ii) \Rightarrow (i)$ is immediate, since (ii) here implies (i) in Corollary 4.19, which is $\beta \leq b_{P,C}$. ∎

4.5 The Graph (Gap) Topology

Combining the quantitative robustness results presented in Section 4.4 with the upper and lower bounds of Lemma 4.16, leads to simple proofs of the following results, which concern the topology induced by the gap metric on the set of stabilisable systems in $\mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U}\to\mathcal{Y}}$. This topology will be referred to as the graph topology because its properties are the same as those of the graph topology defined on the set of LTI systems (cf. [Vid85] and recall that the graph topology on LTI systems is induced by a number of metrics, of which the gap metric is one [Geo88, Vin93b]).

The following identity (cf. Remark 2.14) is useful in the proof of the next result, which characterises convergent sequences in the topology induced by the gap metric:

$$H(P, C) = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \mathbf{\Pi}_{\mathcal{G}_P \| \mathcal{G}'_C} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}.$$

Lemma 4.26 *Given a stabilisable $P \in \mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U}\to\mathcal{Y}}$ and an infinite sequence $\{P_i\}$ in $\mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U}\to\mathcal{Y}}$, the following are equivalent:*

(i) $\delta_g(P, P_i) \to 0$;

(ii) There exists a $C \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ such that $[P, C]$ is stable and $H(P_i, C) \to H(P, C)$ in the induced-norm topology.

(iii) For any $C \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ such that $[P, C]$ is stable, $H(P_i, C) \to H(P, C)$ in the induced-norm topology.

Proof : The proof is essentially identical to that of Corollary IV.6 in [Vin93b]. Recall that

$$\mathscr{B}^{\mathrm{e,sc}}_{\delta_g}(P, \epsilon) := \{\grave{P} \in \mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U}\to\mathcal{Y}} \ : \ \delta_g(P, \grave{P}) < \epsilon\}.$$

Now given any $C \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}}$ such that $[P, C]$ is stable, it follows by Corollary 4.19 of Theorem 4.18, that there is a corresponding open ball $\mathscr{B}^{\mathrm{e,sc}}_{\delta_g}(P, b_{P,C})$ such that $[\grave{P}, C]$ is stable for all $\grave{P} \in \mathscr{B}^{\mathrm{e,sc}}_{\delta_g}(P, b_{P,C})$. Now assume that (i) holds, so that all but a finite number of the terms in the sequence $\{P_i\}$ lie in $\mathscr{B}^{\mathrm{e,sc}}_{\delta_g}(P, b_{P,C})$ and are therefore stabilised by C. By the upper bound in Lemma 4.16 it is then immediate that

$$\mathbf{\Pi}_{\mathcal{G}_{P_i}\|\mathcal{G}'_C} \to \mathbf{\Pi}_{\mathcal{G}_P\|\mathcal{G}'_C}$$

in the induced-norm topology. Consequently, since

$$\|H(P,C) - H(P_i,C)\| = \|\mathbf{\Pi}_{\mathcal{G}_P\|\mathcal{G}'_C} - \mathbf{\Pi}_{\mathcal{G}_{P_i}\|\mathcal{G}'_C}\|,$$

$H(P_i, C) \to H(P, C)$ in the induced-norm topology. That is, $(i) \Rightarrow (iii)$.

Now suppose that (ii) holds. Then for all but a finite number of terms in the sequence $\{P_i\}$, the closed-loop $[P_i, C]$ is stable and

$$\|\mathbf{\Pi}_{\mathcal{G}_P\|\mathcal{G}'_C} - \mathbf{\Pi}_{\mathcal{G}_{P_i}\|\mathcal{G}'_C}\| \to 0.$$

It is then immediate from the lower bound in Lemma 4.16 that $\delta_g(P, P_i) \to 0$, which is (i). That $(iii) \Rightarrow (ii)$ is obvious. ∎

The next corollary also appears in [Vin93b] and [Vid85] for LTI systems. Define

$$\mathscr{S}_C := \{P \in \mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U}\to\mathcal{Y}} \ : \ [P,C] \ \text{is stable}\}$$

and

$$\mathcal{B}^{\mathrm{e}}_{\mathcal{V}\to\mathcal{V}} := \{H \in \mathscr{P}^{\mathrm{e}}_{\mathcal{V}\to\mathcal{V}} \ : \ H \ \text{is stable}\}.$$

Corollary 4.27 *For any $C \in \mathscr{P}^{\mathrm{e}}_{\mathcal{Y}\to\mathcal{U}}$, the map $P \mapsto H(P,C)$ from $\mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U}\to\mathcal{Y}}$ to $\mathcal{B}^{\mathrm{e}}_{\mathcal{V}\to\mathcal{V}}$ is continuous at all points $P \in \mathscr{S}_C$ with respect to the graph topology (induced by the gap metric) on $\mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U}\to\mathcal{Y}}$ and the induced norm topology on $\mathcal{B}^{\mathrm{e}}_{\mathcal{V}\to\mathcal{V}}$. Furthermore, the graph topology is the weakest topology on $\mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U}\to\mathcal{Y}}$ for which this holds.*

Proof : The proof is identical to that of Corollary IV.9 in [Vin93b], but is included for completeness. To show that the map $P \mapsto H(P,C)$ is continuous at all points in $\mathscr{S}_C$, it is necessary to show that for any $P \in \mathscr{S}_C$ and any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\delta_g(P, P_i) < \delta \text{ with } P_i \in \mathscr{P}^{\mathrm{e,sc}}_{\mathcal{U}\to\mathcal{Y}} \ \Rightarrow \ H(P_i, C) \in \mathcal{B}^{\mathrm{e}}_{\mathcal{V}\to\mathcal{V}} \text{ and } \|H(P,C) - H(P_i,C)\| < \epsilon.$$

Since $\|H(P,C) - H(P_i,C)\| = \|\mathbf{\Pi}_{\mathcal{G}_P\|\mathcal{G}'_C} - \mathbf{\Pi}_{\mathcal{G}_{P_i}\|\mathcal{G}'_C}\|$, the existence of such a $\delta > 0$ is guaranteed by the lower bound in Lemma 4.16.

To see that the graph topology is the weakest on $\mathscr{S}_C$ for which the result holds, note that for any other topology $\boldsymbol{\tau}$ on $\mathscr{S}_C$ to have this property it is necessary for there to exist a neighbourhood $\mathscr{N}_\epsilon(P)$[13] for any given $P \in \mathscr{S}_C$ and $\epsilon > 0$, such that

$$H(P_i, C) \in \mathcal{B}^{\mathrm{e}}_{\mathcal{V}\to\mathcal{V}} \text{ and } \|H(P,C) - H(P_i,C)\| < \epsilon \text{ for all } P_i \in \mathscr{N}_\epsilon(P).$$

If this holds, it follows by the lower bound in Lemma 4.16, that

$$\delta_g(P, P_i) < \epsilon \text{ for all } P_i \in \mathscr{N}_\epsilon(P).$$

That is, for any $P \in \mathscr{S}_C$ and $\epsilon > 0$ there exists an open set in $\boldsymbol{\tau}$ that is a strict subset of $\mathscr{B}^{\mathrm{e,sc}}_{\delta_g}(P, \epsilon)$. For this to be possible, the graph topology (induced by the gap metric) must be a sub-collection of $\boldsymbol{\tau}$. ∎

4.6 Robustness of SD Closed-Loop Systems

In this section, results of the last two sections are specialised to SD, closed-loop systems. Set $\mathcal{U} = \mathbb{R}^m$ and $\mathcal{Y} = \mathbb{R}^p$ and consider the feedback configuration shown in Figure 4.3, where $P \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$, $\mathsf{C} \in \mathscr{D}_{\mathcal{Y}\to\mathcal{U}}$ is an LSI, discrete-time controller, $\mathbf{A_D}$ is an ideal, periodic (period h) sampling-device, $\mathbf{D_A}$ is a zeroth-order hold-device synchronised with $\mathbf{A_D}$ and F is a stable, low-pass, LTI filter required to ensure boundedness of the sampling device. Note that $F \in \mathscr{P}^{\mathrm{e,sc}}_{\mathcal{Y}\to\mathcal{Y}}$ and hence, that $C^{sd} := \mathbf{D_A}\mathsf{C}\mathbf{A_D}F \in \mathscr{P}^{\mathrm{e,sc}}_{\mathcal{Y}\to\mathcal{U}}$. As such, Theorem 4.18 applies as summarised below.

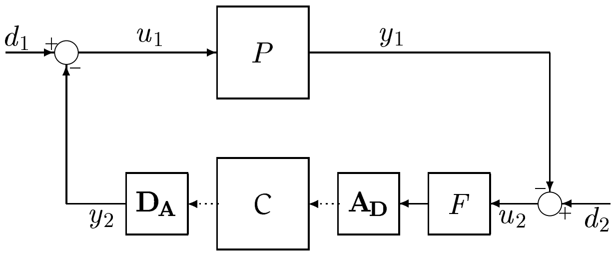

Figure 4.3 *Sampled-Data Control System*

Corollary 4.28 *Consider a stable, closed-loop system $[P_0, C^{sd}]$. The following are equivalent :*

(i) $b_{P,C^{sd}} \geq \beta$.

(ii) $[P_1, C^{sd}]$ *is stable for all* $P_1 \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ *such that* $\delta_g(P_0, P_1) < \beta$.

Remark 4.29 The results presented in Sub-Section 4.4.2 concerning simultaneous perturbations to the plant and controller also apply. One possible application of such a result is in the area of approximating LTI, continuous-time controllers by SD controllers. ♣

[13]Note the ϵ-dependence of the neighbourhood.

4.7 Summary

The primary motivation of this Chapter stems from a desire to characterise the robustness properties of SD control-systems developed using the design framework proposed in Chapter 3 (cf. Section 4.6). To this end, a more general result concerning the robustness of closed-loop stability to perturbations measured in the gap metric is obtained for (potentially infinite-dimensional) LPTV, closed-loop systems. Specifically, the largest gap-ball of LPTV plants, centred at a nominal plant, that a nominal, stabilising controller is guaranteed to stabilise, is identified. A similar result is obtained for simultaneous plant and controller perturbations and it is shown that the gap metric induces the graph topology on a quite general class of LPTV systems.

Along the way, various results of independent interest are derived. Specifically, it is shown that the graph of a stabilisable LPTV system can be expressed as the range and kernel of stable, LPTV systems, which are respectively, left and right invertible by stable, LPTV systems (cf. Theorem 4.6). The existence of these so-called strong-right and strong-left representations plays a crucial role throughout the chapter. In particular, given a well-posed plant/controller pair, it is shown that closed-loop stability can be readily characterised in terms of strong-right and strong-left representations of the plant and controller graphs. In turn, this leads to a Youla-style parameterisation of stabilising controllers.

In Section 4.3, a formula is derived for the directed gap between two LPTV systems. The formula obtained is essentially a generalisation of the formula derived in [Geo88] for LTI systems. It is fundamental to the framework used to establish robustness results and can be used to compute the gap metric as is detailed in Chapter 5.

The results presented in this chapter may find application in the important area of approximating LTI, continuous-time controllers with SD controllers and may be seen as a step towards characterising the quality of such approximations.

5

Computing the Gap

5.1 Introduction

In this chapter a procedure is developed for computing (to any desired accuracy) the gap between LPTV systems that admit stabilisable and detectable, state-space realisations. Using a formula derived in Section 4.3, it follows that bounding the directed gap between two LPTV systems is equivalent to solving a LSI, full-information, $\ell^2_{\mathbb{Z}^+}(\cdot)$-synthesis problem.[1] More specifically, it is shown that the directed gap between two LPTV, continuous-time systems P_a and P_b say, is less than some real number $\gamma > 0$ if and only if there exists a stabilising, full-information control-law that achieves a closed-loop, $\ell^2_{\mathbb{Z}^+}(\cdot)$-induced norm of less than γ, when applied to a particular generalised plant. The particular generalised plant is LSI and can be expressed in terms of normalised, right-coprime factorisations of the equivalent, LSI systems $\mathsf{P}_a := \mathbf{W} P_a \mathbf{W}^{-1}$ and $\mathsf{P}_b := \mathbf{W} P_b \mathbf{W}^{-1}$. The existence of a solution to the equivalent, LSI, full-information, $\ell^2_{\mathbb{Z}^+}(\cdot)$-synthesis problem can be determined using the state-space, necessary and sufficient condition of Proposition 2.22. Given a computationally-tractable characterisation of this necessary and sufficient condition and a method for constructing state-space realisations of the required normalised, right-coprime factors, it follows that a bisection-search can be used to compute (to any desired accuracy) the directed gap (and hence gap) between the two LPTV systems concerned.

Much of this chapter is devoted to constructing state-space realisations of normalised, right-coprime factors in the time-lifted domain and obtaining computationally-tractable conditions that characterise the existence of solutions to the equivalent, full-information, $\ell^2_{\mathbb{Z}^+}(\cdot)$-synthesis problem described above. First, given a stabilisable and detectable, state-space realisation of an LPTV system P_a say, it is shown how to construct a state-space realisation of a normalised, right-coprime

[1] cf. Sub-Section 2.4.3 for the definition of the full-information, $\ell^2_{\mathbb{Z}^+}(\cdot)$-synthesis problem

factorisation of $\mathsf{P}_a := \mathbf{W} P_a \mathbf{W}^{-1}$. This is done explicitly in terms of the original state-space description of P_a and involves solving a finite-dimensional and hence, computationally-tractable, algebraic, Riccati equation. Using this, a computationally-tractable, necessary and sufficient condition which characterises the existence of a solution to the LSI, full-information, $\ell^2_{\mathbb{Z}^+}(\cdot)$-synthesis problem associated with bounding the directed gap, is obtained. This characterisation is expressible explicitly in terms of the original stabilisable and detectable, state-space realisations of the two LPTV systems concerned and involves computing the state-transition operator of a periodic, matrix-valued function of time at only one point. The latter step can be achieved by integrating the relevant homogeneous system of differential equations forward in time.

5.2 The Gap Metric and Full-Information Synthesis

Recall (cf. Section 4.3) that the gap between two operators P_a and P_b is defined to be the gap between their respective graphs as follows:

$$\begin{aligned}\delta_g(P_a, P_b) &:= \delta(\mathcal{G}_{P_a}, \mathcal{G}_{P_b}) := \|\mathbf{\Pi}_{\mathcal{G}_{P_a}} - \mathbf{\Pi}_{\mathcal{G}_{P_b}}\| \\ &= \max\{\vec{\delta}_g(P_a, P_b), \vec{\delta}_g(P_b, P_a)\},\end{aligned}$$

where

$$\vec{\delta}_g(P_a, P_b) := \|\mathbf{\Pi}_{\mathcal{G}^{\perp}_{P_b}} \mathbf{\Pi}_{\mathcal{G}_{P_a}}\| \text{ and } \vec{\delta}_g(P_b, P_a) := \|\mathbf{\Pi}_{\mathcal{G}^{\perp}_{P_a}} \mathbf{\Pi}_{\mathcal{G}_{P_b}}\|$$

are the directed gaps. Furthermore, recall the formula derived in Section 4.3 for the directed gap between two LPTV systems P_a and P_b in $\mathscr{P}_{\mathcal{U}\to\mathcal{Y}}$:

$$\vec{\delta}_g(P_a, P_b) = \inf_{\text{stable } \mathsf{Q}\in\mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}} \|\mathsf{G}_a - \mathsf{G}_b\mathsf{Q}\|, \tag{5.1}$$

where $\mathcal{G}_{\mathsf{P}_\bullet} := \mathbf{W}\mathcal{G}_{P_\bullet}$ and $\mathsf{G}_\bullet$ is any isometry in $\mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}\oplus\mathcal{Y})}$ such that $\mathcal{G}_{\mathsf{P}_\bullet} = \mathcal{R}_{\mathsf{G}_\bullet}$ (the subscript "$\bullet$" here denotes a or b). Existence of the required $\mathsf{G}_\bullet$ is discussed in Section 4.2 (cf. Theorem 4.6). An important result used in the sequel is that given a state-space realisation of $P_\bullet$ (which must satisfy some mild stabilisability and detectability conditions), it is possible to construct an isometric, strong-right representation $\mathsf{G}_\bullet$ of $\mathcal{G}_{\mathsf{P}_\bullet}$. This is done by first constructing a normalised, right-coprime pair $(\mathsf{M}_\bullet, \mathsf{N}_\bullet)$, such that

$$\mathsf{P}_\bullet := \mathbf{W} P_\bullet \mathbf{W}^{-1} = \mathsf{N}_\bullet \mathsf{M}_\bullet^{-1}$$

and then taking

$$\mathsf{G}_\bullet = \begin{bmatrix} \mathsf{M}_\bullet \\ \mathsf{N}_\bullet \end{bmatrix}.$$

This is described in more detail in Section 5.3.

Remark 5.1 Note that although any system $P \in \mathscr{P}_{\mathcal{U}\to\mathcal{Y}}$ is equivalent to a system $\mathsf{P} := \mathbf{W}P\mathbf{W}^{-1}$ in $\mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}$, the converse is not true in general (cf. Remark 2.6). As such, the formula for the directed gap (cf. equation (5.1)) is only valid in the time-lifted domain. ♣

Interestingly, the right-hand side of equation (5.1) can be expressed in the following lower, linear-fractional form:

$$\mathsf{G}_a - \mathsf{G}_b\mathsf{Q} = \mathbf{F}_\ell(\mathsf{H}, \mathsf{Q}), \tag{5.2}$$

where

$$\mathsf{H} := \begin{bmatrix} \mathsf{G}_a & \mathsf{G}_b \\ \mathrm{I} & 0 \end{bmatrix} : \ell^2_{\mathbb{Z}+}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})) \oplus \ell^2_{\mathbb{Z}+}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})) \to \ell^2_{\mathbb{Z}+}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}\oplus\mathcal{Y})) \oplus \ell^2_{\mathbb{Z}+}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})). \tag{5.3}$$

The stable system H can be thought of as generalised plant mapping an exogenous disturbance and control signal $\begin{bmatrix} \mathsf{w} \\ \mathsf{u} \end{bmatrix}$, to a controlled and measured output $\begin{bmatrix} \mathsf{z} \\ \mathsf{y} \end{bmatrix}$. Now given state-space realisations of isometric, strong-right representations $\mathsf{G}_\bullet$ of $\mathcal{G}_{\mathsf{P}_\bullet}$, the following equivalence holds.

Lemma 5.2 *Given a state-space realisation of the generalised plant* H *defined in equation (5.3), let* x *denote the state. Then* $\vec{\delta}_g(P_a, P_b) < \gamma$ *if and only if there exists a causal, LSI, stabilising, disturbance-feedforward control-law* $\mathsf{u} = \mathsf{K}_{df}\mathsf{w}$*, such that* $\|\mathbf{F}_\ell(\mathsf{H}, \mathsf{K}_{df})\| < \gamma$*. Furthermore, this is equivalent to the existence of a causal, LSI, stabilising, full-information control-law* $\mathsf{u} = \mathsf{K}_{fi}\begin{bmatrix} \mathsf{x} \\ \mathsf{w} \end{bmatrix}$*, such that with it in place, the* $\ell^2_{\mathbb{Z}+}$*-induced norm of the mapping from* w *to* z *is strictly less than* γ*.*

Proof : Since G_b has stable left-inverse in $\mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}\oplus\mathcal{Y})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})}$, it follows from equation (5.2), that $\mathbf{F}_\ell(\mathsf{H}, \mathsf{Q})$ is stable if and only if Q is stable. That is, the existence of a stable $\mathsf{Q} \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})}$ such that $\|\mathsf{G}_a - \mathsf{G}_b\mathsf{Q}\| < \gamma$, is equivalent to the existence of a causal, LSI, stabilising control-law K_{df} acting on w alone, such that $\|\mathbf{F}_\ell(\mathsf{H}, \mathsf{K})\| < \gamma$. Furthermore, since the generalised plant H is a stable system, given a state-space realisation of H and the disturbance w, it is possible to re-construct the state. Thus, without loss of generality, it can be assumed that the control-law has access to the state x in addition to the exogenous disturbance w. That is, the directed gap is less than some number γ if and only if there exists a causal, LSI, stabilising, full-information control-law $u = \mathsf{K}_{fi}\begin{bmatrix} \mathsf{x} \\ \mathsf{w} \end{bmatrix}$ that achieves a closed-loop, $\ell^2_{\mathbb{Z}+}$-induced norm strictly less than γ. ∎

5.3 Normalised Right-Coprime Factorisations

A pair of stable systems (M, N), with $\mathsf{M} \in \mathscr{D}_{\mathcal{U}\to\mathcal{U}}$ and $\mathsf{N} \in \mathscr{D}_{\mathcal{U}\to\mathcal{Y}}$ ($\mathcal{U}$ and $\mathcal{Y}$ possibly infinite-dimensional, Hilbert spaces), is said to be a right-coprime pair if there exist stable systems $\mathsf{X} \in$

$\mathscr{D}_{\mathcal{U}\to\mathcal{U}}$ and $\mathsf{Y} \in \mathscr{D}_{\mathcal{Y}\to\mathcal{U}}$ such that $\mathsf{XM}+\mathsf{YN} = \mathrm{I}$. Further to this, the pair is said to be normalised if

$$\begin{bmatrix} \mathsf{M}^* & \mathsf{N}^* \end{bmatrix}\begin{bmatrix} \mathsf{M} \\ \mathsf{N} \end{bmatrix} = \mathrm{I}.$$

For a system $\mathsf{P} \in \mathscr{D}_{\mathcal{U}\to\mathcal{Y}}$ a (normalised) right-coprime pair (M,N) is said to constitute a (normalised) right-coprime factorisation of P if $\mathsf{P} = \mathsf{NM}^{-1}$. Importantly, the graph of such a P can be characterised in terms of any right-coprime factorisation of it.

Lemma 5.3 *If* $\mathsf{P} = \mathsf{NM}^{-1} \in \mathscr{D}_{\mathcal{U}\to\mathcal{Y}}$ *with* (M,N) *a right-coprime pair, then*

$$\mathcal{G}_{\mathsf{P}} := \begin{bmatrix} \mathrm{I} \\ \mathsf{P} \end{bmatrix}\mathcal{D}_{\mathsf{P}} = \mathcal{R}_{\mathsf{G}},$$

where

$$\mathsf{G} := \begin{bmatrix} \mathsf{M} \\ \mathsf{N} \end{bmatrix} \in \mathscr{D}_{\mathcal{U}\to\mathcal{U}\oplus\mathcal{Y}}.$$

In fact, since such a G *is stably left-invertible it is a strong-right representation of* $\mathcal{G}_{\mathsf{P}}$.

Proof : Suppose that $\begin{bmatrix} \mathsf{u} \\ \mathsf{y} \end{bmatrix} \in \mathcal{R}_{\mathsf{G}} \subset \ell^2_{\mathbb{Z}+}(\mathcal{U}\oplus\mathcal{Y})$. That is, for some $\mathsf{q} \in \ell^2_{\mathbb{Z}+}(\mathcal{U})$, $\mathsf{u} = \mathsf{Mq}$ and $\mathsf{y} = \mathsf{Nq}$. Then, $\mathsf{y} = \mathsf{NM}^{-1}\mathsf{u}$ and it follows that $\begin{bmatrix} \mathsf{u} \\ \mathsf{y} \end{bmatrix} \in \mathcal{G}_{\mathsf{P}}$ (or equivalently $\mathcal{R}_{\mathsf{G}} \subset \mathcal{G}_{\mathsf{P}}$). Now to see the opposite inclusion, suppose that $\begin{bmatrix} \mathsf{u} \\ \mathsf{y} \end{bmatrix} \in \mathcal{G}_{\mathsf{P}}$. Then $\mathsf{y} = \mathsf{NM}^{-1}\mathsf{u}$ and defining $\mathsf{q} := \mathsf{M}^{-1}\mathsf{u}$, it follows that $\begin{bmatrix} \mathsf{u} \\ \mathsf{y} \end{bmatrix} = \mathsf{Gq}$. Since (M,N) is right coprime, there exists a stable pair (Y,X) such that $\begin{bmatrix} \mathsf{X} & \mathsf{Y} \end{bmatrix}\mathsf{G} = \mathrm{I}$. Hence, $\mathsf{q} = \begin{bmatrix} \mathsf{X} & \mathsf{Y} \end{bmatrix}\begin{bmatrix} \mathsf{u} \\ \mathsf{y} \end{bmatrix} \in \ell^2_{\mathbb{Z}+}(\mathcal{U})$, which implies $\mathcal{G}_{\mathsf{P}} \subset \mathcal{R}_{\mathsf{G}}$. ■

Given an LPTV system $P \in \mathscr{P}_{\mathcal{U}\to\mathcal{Y}}$ described by a state-space system of differential equations satisfying some mild stabilisability and detectability conditions, it is shown in the next sub-section how to construct a state-space realisation of a normalised, right-coprime factorisation of the equivalent system $\mathsf{P} := \mathbf{W}P\mathbf{W}^{-1} \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}$. It then follows by Lemma 5.3, that this is a state-space realisation of an isometric, strong-right representation of $\mathcal{G}_{\mathsf{P}}$, which can in turn be used for directed gap computation (cf. Lemma 5.2). First though, it is shown how to construct a state-space realisation of a normalised, right-coprime factorisation of any LSI, state-space system that satisfies a stabilisability and detectability condition.

5.3.1 LSI State-Space Systems

Consider the system $\mathsf{P} \in \mathscr{D}_{\mathcal{U}\to\mathcal{Y}}$ (with $\mathcal{U}$ and $\mathcal{Y}$ possibly infinite-dimensional, Hilbert spaces) described by the LSI, state-space system of difference equations

$$x_{k+1} = Ax_k + Bu_k; \quad x_0 = 0, \tag{5.4a}$$
$$y_k = Cx_k + Du_k, \tag{5.4b}$$

where it is assumed that the state x_k evolves in a finite-dimensional, Hilbert space $\mathcal{X}$. In this subsection, a state-space realisation of a normalised, right-coprime factorisation (M, N) of $P = NM^{-1}$ is constructed in terms of the state-space operators A, B, C and D. Throughout, it is assumed that the pair (A, B) is stabilisable and (C, A) is detectable. That is, there exists an operator $F : \mathcal{X} \to \mathcal{U}$ such that $\mathrm{spec}(A + BF) \subset \mathbb{D}$ and an operator $L : \mathcal{Y} \to \mathcal{X}$ such that $\mathrm{spec}(A + LC) \subset \mathbb{D}$. Define $G := \begin{bmatrix} M \\ N \end{bmatrix}$ to be the stable system in $\mathscr{D}_{\mathcal{U}\to\mathcal{U}\oplus\mathcal{Y}}$ described by the LSI system of difference equations

$$\begin{aligned} s_{k+1} &= \check{A}s_k + \check{B}q_k \\ &:= (A + BF)s_k + BVq_k; \quad s_0 = 0, \end{aligned} \tag{5.5a}$$
$$\begin{aligned} \begin{bmatrix} u_k \\ y_k \end{bmatrix} &= \check{C}s_k + \check{D}q_k \\ &:= \begin{bmatrix} F \\ C + DF \end{bmatrix} s_k + \begin{bmatrix} V \\ DV \end{bmatrix} q_k, \end{aligned} \tag{5.5b}$$

where V is any operator in $\mathcal{B}_{\mathcal{U}\to\mathcal{U}}$ with bounded inverse. Then it is straight-forward to verify that the pair (M, N) is a right-coprime factorisation of $P = NM^{-1}$ [NJB84]. In fact, a stable left-inverse of G is characterised by the LSI system of difference equations

$$\begin{aligned} \nu_{k+1} &= (A + LC)\nu_k - (B + LD)u_k + Ly_k; \quad \nu_0 = 0, \\ q_k &= V^{-1}F\nu_k + V^{-1}u_k. \end{aligned}$$

The first step towards characterising normalised, right-coprime factorisations of P is to determine conditions under which the state-space realisation of G given in equations (5.5), is isometric. It follows by causality and shift-invariance, that G can be expressed in terms of the block, lower-triangular, Toeplitz operator

$$\begin{bmatrix} \check{D} & 0 & 0 & 0 & \cdots \\ \check{C}\check{B} & \check{D} & 0 & 0 & \cdots \\ \check{C}\check{A}\check{B} & \check{C}\check{B} & \check{D} & 0 & \cdots \\ \check{C}\check{A}^2\check{B} & \check{C}\check{A}\check{B} & \check{C}\check{B} & \check{D} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Since $\mathrm{spec}(\check{\mathsf{A}}) \subset \mathbb{D}$, the system of difference equations describes an $\ell^2_{\mathbb{Z}+}(\cdot)$-bounded operator and hence, its adjoint is well-defined. In fact, the adjoint can be expressed in terms of the block, upper-triangular, Toeplitz operator

$$\begin{bmatrix} \check{\mathsf{D}}^* & \check{\mathsf{B}}^*\check{\mathsf{C}}^* & \check{\mathsf{B}}^*\check{\mathsf{A}}^*\check{\mathsf{C}}^* & \check{\mathsf{B}}^*\check{\mathsf{A}}^{2*}\check{\mathsf{C}}^* & \cdots \\ 0 & \check{\mathsf{D}}^* & \check{\mathsf{B}}^*\check{\mathsf{C}}^* & \check{\mathsf{B}}^*\check{\mathsf{A}}^*\check{\mathsf{C}}^* & \cdots \\ 0 & 0 & \check{\mathsf{D}}^* & \check{\mathsf{B}}^*\check{\mathsf{C}}^* & \cdots \\ 0 & 0 & 0 & \check{\mathsf{D}}^* & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Now let X be a self-adjoint, positive-semidefinite operator satisfying the Stein equation $\mathsf{X} - \check{\mathsf{A}}^*\mathsf{X}\check{\mathsf{A}} = \check{\mathsf{C}}^*\check{\mathsf{C}}$. Then since $\lim_{k\to\infty} \check{\mathsf{A}}^k = 0$, the blocks of the Toeplitz representation of $\mathsf{G}^*\mathsf{G}$ satisfy

$$(\mathsf{G}^*\mathsf{G})_{[i,j]} = \begin{cases} \check{\mathsf{D}}^*\check{\mathsf{D}} + \check{\mathsf{B}}^*\mathsf{X}\check{\mathsf{B}} & i = j \\ \check{\mathsf{B}}(\check{\mathsf{A}}^*)^{i-j}(\check{\mathsf{C}}^*\check{\mathsf{D}} + \check{\mathsf{A}}^*\mathsf{X}\check{\mathsf{B}}) & i > j \\ (\check{\mathsf{D}}^*\check{\mathsf{C}} + \check{\mathsf{B}}^*\mathsf{X}\check{\mathsf{A}})\check{\mathsf{A}}^{j-i}\check{\mathsf{B}} & i < j \end{cases}.$$

Hence, if there exists a positive-semidefinite, self-adjoint operator X such that

$$\begin{aligned} &(i) && \mathsf{X} - \check{\mathsf{A}}^*\mathsf{X}\check{\mathsf{A}} = \check{\mathsf{C}}^*\check{\mathsf{C}}, \\ &(ii) && \check{\mathsf{D}}^*\check{\mathsf{C}} + \check{\mathsf{B}}^*\mathsf{X}\check{\mathsf{A}} = 0 \end{aligned}$$

and

$$(iii) \qquad \check{\mathsf{D}}^*\check{\mathsf{D}} + \check{\mathsf{B}}^*\mathsf{X}\check{\mathsf{B}} = \mathrm{I},$$

it follows that G is an isometry. In fact, these conditions are also necessary if F and V are such that $(\check{\mathsf{A}}, \check{\mathsf{B}})$ is completely reachable. Using these conditions, it is possible to select F and V in equation (5.5), to ensure that G corresponds to a normalised (isometric), right-coprime factorisation.

Lemma 5.4 *Consider the system* P *defined by the system of difference equations (5.4). Suppose that* (A, B) *is stabilisable and* (C, A) *is detectable. Now define*

$$\mathsf{R} := \mathrm{I} + \mathsf{D}^*\mathsf{D} > 0, \quad \tilde{\mathsf{R}} := \mathrm{I} + \mathsf{D}\mathsf{D}^* > 0$$

and $\begin{bmatrix} \mathsf{M} \\ \mathsf{N} \end{bmatrix}$ *by the system of difference equations*

$$\mathsf{s}_{k+1} = (\mathsf{A} + \mathsf{B}\mathsf{F})\mathsf{s}_k + \mathsf{B}\mathsf{V}\mathsf{q}_k; \quad \mathsf{s}_0 = 0, \tag{5.6a}$$

$$\begin{bmatrix} \mathsf{u}_k \\ \mathsf{y}_k \end{bmatrix} = \begin{bmatrix} \mathsf{F} \\ \mathsf{C} + \mathsf{D}\mathsf{F} \end{bmatrix}\mathsf{s}_k + \begin{bmatrix} \mathsf{V} \\ \mathsf{D}\mathsf{V} \end{bmatrix}\mathsf{q}_k, \tag{5.6b}$$

where

$$\mathsf{V} := (\mathsf{R}+\mathsf{B}^*\mathsf{X}\mathsf{B})^{-\frac{1}{2}}, \tag{5.7}$$

$$\mathsf{F} := -(\mathsf{R}+\mathsf{B}^*\mathsf{X}\mathsf{B})^{-1}(\mathsf{B}^*\mathsf{X}\mathsf{A}+\mathsf{D}^*\mathsf{C}) \tag{5.8}$$

and X *is a self-adjoint, positive-semidefinite solution to the algebraic, Riccati equation*

$$\mathsf{X} = \mathsf{A}^*\mathsf{X}\mathsf{A}+\mathsf{C}^*\mathsf{C}-(\mathsf{D}^*\mathsf{C}+\mathsf{B}^*\mathsf{X}\mathsf{A})(\mathsf{R}+\mathsf{B}^*\mathsf{X}\mathsf{B})^{-1}(\mathsf{C}^*\mathsf{D}+\mathsf{B}^*\mathsf{X}\mathsf{A}), \tag{5.9}$$

that satisfies $\operatorname{spec}(\mathsf{A}+\mathsf{B}\mathsf{F}) \subset \mathbb{D}$. *Then the pair* (M,N) *is a* <u>*normalised, right-coprime factorisation*</u> *of* $\mathsf{P} = \mathsf{N}\mathsf{M}^{-1}$.

Proof : Clearly the pair (M,N) is a right-coprime factorisation of P. So it sufficient to show that it is also normalised. That is, $\begin{bmatrix} \mathsf{M} \\ \mathsf{N} \end{bmatrix}$ defined by the system of difference equation (5.6) is an isometry. First note that under the assumption that (A,B) is stabilisable and (C,A) is detectable, there exists a self-adjoint, positive-semidefinite solution X to the algebraic, Riccati equation (5.9), that satisfies $\operatorname{spec}(\mathsf{A}+\mathsf{B}\mathsf{F}) \subset \mathbb{D}$. To see this, note that $\lambda \in \mathbb{C}$ is an unobservable mode of (C,A) if and only if it is an unobservable mode of $(\mathsf{C}^*\mathsf{C}-\mathsf{C}^*\mathsf{D}\mathsf{R}^{-1}\mathsf{D}^*\mathsf{C}, \mathsf{A}-\mathsf{B}\mathsf{R}^{-1}\mathsf{D}^*\mathsf{C}) = (\mathsf{C}^*\tilde{\mathsf{R}}^{-1}\mathsf{C}, \mathsf{A}-\mathsf{B}\mathsf{R}^{-1}\mathsf{D}^*\mathsf{C})$. Then since

$$\begin{bmatrix} \mathsf{C}^*\mathsf{C} & \mathsf{C}^*\mathsf{D} \\ \mathsf{D}^*\mathsf{C} & \mathsf{R} \end{bmatrix} = \begin{bmatrix} I & \mathsf{C}^*\mathsf{D}\mathsf{R}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathsf{C}^*\tilde{\mathsf{R}}^{-1}\mathsf{C} & 0 \\ 0 & \mathsf{R} \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathsf{R}^{-1}\mathsf{D}^*\mathsf{C} & I \end{bmatrix} \geq 0$$

and (C,A) has no unobservable modes on $\mathbb{T}$, it follows by Proposition 2.20, that the algebraic, Riccati equation (5.9), admits a self-adjoint, positive-semidefinite solution X so that $\operatorname{spec}(\mathsf{A}+\mathsf{B}\mathsf{F}) \subset \mathbb{D}$.

Now, it follows from equation (5.8) that

$$\begin{bmatrix} \mathsf{V}^* & \mathsf{V}^*\mathsf{D}^* \end{bmatrix} \begin{bmatrix} \mathsf{F} \\ \mathsf{C}+\mathsf{D}\mathsf{F} \end{bmatrix} + \mathsf{V}^*\mathsf{B}^*\mathsf{X}(\mathsf{A}+\mathsf{B}\mathsf{F}) = 0. \tag{5.10}$$

Moreover, from equation (5.7),

$$\begin{bmatrix} \mathsf{V}^* & \mathsf{V}^*\mathsf{D}^* \end{bmatrix} \begin{bmatrix} \mathsf{V} \\ \mathsf{D}\mathsf{V} \end{bmatrix} + \mathsf{V}^*\mathsf{B}^*\mathsf{X}\mathsf{B}\mathsf{V} = I. \tag{5.11}$$

Note also that the algebraic, Riccati equation (5.9), can be re-written as the Stein equation

$$\mathsf{X}-(\mathsf{A}+\mathsf{B}\mathsf{F})^*\mathsf{X}(\mathsf{A}+\mathsf{B}\mathsf{F}) = \begin{bmatrix} \mathsf{F}^* & (\mathsf{C}^*+\mathsf{F}^*\mathsf{D}^*) \end{bmatrix} \begin{bmatrix} \mathsf{F} \\ \mathsf{C}+\mathsf{D}\mathsf{F} \end{bmatrix}. \tag{5.12}$$

So from the discussion above concerning conditions for a state-space system to be isometric, it follows from equations (5.10), (5.11) and (5.12), that $\begin{bmatrix} \mathsf{M} \\ \mathsf{N} \end{bmatrix}$ is an isometry as claimed. ∎

5.3.2 LPTV State-Space Systems

Consider the system of differential equations

$$\frac{dx_a}{dt} = A_a(t)x_a(t) + B_a(t)u(t); \quad x_a(0) = 0, \tag{5.13a}$$

$$y(t) = C_a(t)x_a(t) + D_a(t)u(t), \tag{5.13b}$$

where for fixed $t \in \mathbb{R}^+$ the input $u(t) \in \mathcal{U}$, the output $y(t) \in \mathcal{Y}$, the state $x_a(t) \in \mathcal{X}_a$, and $\mathcal{U}$, $\mathcal{Y}$ and $\mathcal{X}_a$ are all *finite-dimensional*, Hilbert spaces. It is assumed that $A_a(t)$, $B_a(t)$, $C_a(t)$ and $D_a(t)$ are all piecewise-continuous functions of t, so that with the state-transition operator $\mathbf{\Phi}_{A_a}(t,\tau) : \mathbb{R} \times \mathbb{R} \to \mathcal{B}_{\mathcal{X}_a \to \mathcal{X}_a}$ defined to be the unique, continuous solution to the homogeneous, differential equation

$$\frac{d}{dt}\mathbf{\Phi}_{A_a}(t,\tau) = A_a(t)\mathbf{\Phi}_{A_a}(t,\tau); \quad \mathbf{\Phi}_{A_a}(\tau,\tau) = I,$$

the state-space system of differential equations (5.13), characterises the causal, linear, continuous-time system $P_a^{\mathrm{e}} : \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{Y}) \to \mathrm{L}^{2,\mathrm{e}}_{\mathbb{R}^+}(\mathcal{Y})$ defined by

$$y(t) :=(P_a^{\mathrm{e}}u)(t) := \int_0^t C_a(t)\mathbf{\Phi}_{A_a}(t,\tau)B_a(\tau)\, d\tau + D_a(t)u(t) \tag{5.14}$$

[Che84, CD91, GL95]. If for some real number $h > 0$ and all $k \in \mathbb{Z}^+$, $A_a(t+kh) = A_a(t)$, $B_a(t+kh) = B_a(t)$, $C_a(t+kh) = C_a(t)$ and $D_a(t+kh) = D_a(t)$, then $\mathbf{\Phi}_{A_a}(t+kh, \tau+kh) = \mathbf{\Phi}_{A_a}(t,\tau)$ and it follows from equation (5.14), that P_a^{e} is periodically time-varying in the sense described in Sub-Section 2.3.2. As such, P_a^{e} induces a system $P_a \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$, which in turn, is equivalent to an LSI system $\mathsf{P}_a :=\mathbf{W}P_a\mathbf{W}^{-1} \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathcal{U})\to\mathrm{L}^2_{\mathbb{H}}(\mathcal{Y})}$. Equations (5.13) are called a state-space realisation of P_a and such a systems is said to admit a state-space realisation. A state-space realisation of the equivalent, LSI system P_a can be obtained as follows, by characterising the evolution of the state in equation (5.13) in terms of its values at the discrete points in time $\{kh\}_{k\in\mathbb{Z}^+}$:

$$\mathsf{x}_{a,k+1} = \mathsf{A}_a\mathsf{x}_{a,k} + \mathsf{B}_a\vec{\mathsf{u}}_k; \quad \mathsf{x}_{a,0} = 0, \tag{5.15a}$$

$$\vec{\mathsf{y}}_k = \mathsf{C}_a\mathsf{x}_{a,k} + \mathsf{D}_a\vec{\mathsf{u}}_k, \tag{5.15b}$$

where $\mathsf{x}_{a,k} = x_a(kh)$, $\vec{\mathsf{u}}_k(\theta) = u(kh+\theta)$, $\vec{\mathsf{y}}_k(\theta) = y(kh+\theta)$ and for all $x \in \mathcal{X}_a$ and $\vec{u} \in \mathrm{L}^2_{\mathbb{H}}(\mathcal{U})$

$$\mathsf{A}_a : \mathcal{X}_a \to \mathcal{X}_a, \qquad \mathsf{A}_a x :=\mathbf{\Phi}_{A_a}(h,0)x, \tag{5.16a}$$

$$\mathsf{B}_a : \mathrm{L}^2_{\mathbb{H}}(\mathcal{U}) \to \mathcal{X}_a, \qquad \mathsf{B}_a\vec{u} := \int_0^h \mathbf{\Phi}_{A_a}(h,\tau)B_a(\tau)\vec{u}(\tau)d\tau, \tag{5.16b}$$

$$\mathsf{C}_a : \mathcal{X}_a \to \mathrm{L}^2_{\mathbb{H}}(\mathcal{Y}), \qquad (\mathsf{C}_a x)(\theta) :=C_a(\theta)\mathbf{\Phi}_{A_a}(\theta,0)x, \tag{5.16c}$$

$$\mathsf{D}_a : \mathrm{L}^2_{\mathbb{H}}(\mathcal{U}) \to \mathrm{L}^2_{\mathbb{H}}(\mathcal{Y}), \qquad (\mathsf{D}_a\vec{u})(\theta) :=D_a\vec{u}(\theta) + \int_0^\theta C_a(\theta)\mathbf{\Phi}_{A_a}(\theta,\tau)B_a(\tau)\vec{u}(\tau)d\tau. \tag{5.16d}$$

If the state-space realisation of P_a given in equations (5.13) is stabilisable and detectable (in a sense to be defined shortly), then the state-space realisation of P_a given in (5.15) is also stabilisable and detectable in the sense defined in Sub-Section 2.4.1, for LSI, state-space systems. The h-periodic pair $(A_a(t), B_a(t))$ is said to be stabilisable if there exists a piecewise-continuous, h-periodic function $F_a(t) : \mathbb{R}^+ \to \mathcal{B}_{\mathcal{X}_a \to \mathcal{U}}$ such that

$$\|\mathbf{\Phi}_{A_a+B_aF_a}(t_1,t_2)\| \leq c_1 e^{-c_2(t_1-t_2)}, \tag{5.17}$$

for some positive constants c_1 and c_2 and all $t_1, t_2 \in \mathbb{R}$ [RPK92]. Now consider the state-space system

$$\begin{aligned} \frac{dx}{dt} &= A_a(t)x(t) + B_a(t)u(t); \quad x(0) = \tilde{x}, & \text{(5.18a)} \\ y(t) &= x(t), & \text{(5.18b)} \end{aligned}$$

with state-feedback $u(t) := F_a(t)y(t) = F_a(t)x(t)$, where the ($h$-periodic) operator $F_a(t)$ is chosen so that equation (5.17) holds for some positive constants c_1 and c_2. The system described by equations (5.18) is equivalent (in an input-output sense via $\mathbf{W}$) to

$$\begin{aligned} \mathsf{x}_{k+1} &= \mathsf{A}_a\mathsf{x}_k + \mathsf{B}_a\vec{\mathtt{u}}_k; \quad \mathsf{x}_0 = \tilde{x}, \\ \vec{\mathtt{y}}_k(\theta) &= \mathbf{\Phi}_{A_a}(\theta,0)\mathsf{x}_k + \int_0^\theta \mathbf{\Phi}_{A_a}(\theta,\tau)B_a(\tau)\vec{\mathtt{u}}_k(\tau)\, d\tau, \end{aligned}$$

where $\vec{\mathtt{u}}_k(\theta) := (\mathbf{W}u)_k = u(kh+\theta)$, $\vec{\mathtt{y}}_k(\theta) := (\mathbf{W}y)_k = y(kh+\theta)$ and $\mathsf{x}_k := x(kh)$. However, with the state-feedback $u(t) = F_a(t)y(t) = F_a(t)x(t)$ in place, $\vec{\mathtt{y}}_k(\theta) = \mathbf{\Phi}_{A_a+B_aF_a}(\theta,0)\mathsf{x}_k$ and hence,

$$\vec{\mathtt{u}}_k(\theta) := F(\theta)\vec{\mathtt{y}}_k(\theta) = F_a(\theta)\mathbf{\Phi}_{A_a+B_aF_a}(\theta,0)\mathsf{x}_k.$$

So with the bounded, linear operator $\mathsf{F}_a : \mathcal{X}_a \to \mathrm{L}^2_{\mathbb{H}}(\mathcal{U})$ defined by

$$(\mathsf{F}_a x)(\theta) := F_a(\theta)\mathbf{\Phi}_{A_a+B_aF_a}(\theta,0)x$$

for all $x \in \mathcal{X}_a$, it follows that $\mathsf{A}_a + \mathsf{B}_a\mathsf{F}_a = \mathbf{\Phi}_{A_a+B_aF_a}(h,0)$ and since $\mathbf{\Phi}_{A_a+B_aF_a}((k+1)h, kh) = \mathbf{\Phi}_{A_a+B_aF_a}(h,0)$ for all $k \in \mathbb{Z}^+$, that $(\mathsf{A}_a + \mathsf{B}_a\mathsf{F}_a)^k = \mathbf{\Phi}_{A_a+B_aF_a}(kh,0)$. Thus, from equation (5.17), $\|(\mathsf{A}_a + \mathsf{B}_a\mathsf{F}_a)^k\| \leq c_1e^{-c_2kh} \to 0$ as $k \to \infty$. This implies that $\mathrm{spec}(\mathsf{A}_a + \mathsf{B}_a\mathsf{F}_a) \subset \mathbb{D}$ and therefore, that $(\mathsf{A}_a, \mathsf{B}_a)$ is stabilisable. In conclusion, if the pair $(A_a(t), B_a(t))$ is stabilisable then $(\mathsf{A}_a, \mathsf{B}_a)$ is stabilisable. Similarly, by considering the estimation-error dynamics that result from estimating the state of

$$\begin{aligned} \frac{dx}{dt} &= A_a(t)x(t); \quad x(0) = \tilde{x}, \\ y(t) &= C_a(t)x(t), \end{aligned}$$

with a simple Luenberger observer

$$
\begin{aligned}
\frac{dx_e}{dt} &= A_a(t)x_e(t) + L_a(t)(y_e(t) - y(t)); \quad x_e(0) = \tilde{x}_e, \\
y_e(t) &= C_a(t)x_e(t),
\end{aligned}
$$

it can be shown that if there exist a h-periodic function $L_a(t) : \mathbb{R} \to \mathcal{B}_{\mathcal{Y}\to\mathcal{X}_a}$ such that

$$
\|\mathbf{\Phi}_{A_a+L_aC_a}(t_1,t_2)\| \leq c_1 e^{-c_2(t_1-t_2)}, \tag{5.19}
$$

for some positive constants c_1 and c_2 and all $t_1, t_2 \in \mathbb{R}$,[2] then there exists a bounded, linear operator $\mathsf{L}_a : \mathsf{L}^2_{\mathbb{H}}(\mathcal{Y}) \to \mathcal{X}_a$ such that $\operatorname{spec}(\mathsf{A}_a + \mathsf{L}\mathsf{C}_a) \subset \mathbb{D}$. That is, if the pair $(C_a(t), A_a(t))$ is detectable in the sense just described, then the pair $(\mathsf{C}_a, \mathsf{A}_a)$ is detectable in the sense defined in Sub-Section 2.4.1, for LSI, state-space systems. So under the assumption that $(A_a(t), B_a(t))$ is stabilisable and $(C_a(t), A_a(t))$ is detectable, the following corollary follows directly by Lemma 5.4 and Proposition 2.18:

Corollary 5.5 *Let* $\mathsf{G}_a := \begin{bmatrix} \mathsf{M}_a \\ \mathsf{N}_a \end{bmatrix}$ *be defined by the state-space system of difference equations*

$$
\nu_{a,k+1} = (\mathsf{A}_a + \mathsf{B}_a\mathsf{F}_a)\nu_{a,k} + \mathsf{B}_a\mathsf{V}_a\vec{\mathring{\mathsf{q}}}_{a,k}; \quad \nu_{a,0} = 0, \tag{5.20a}
$$

$$
\begin{bmatrix} \vec{\mathring{\mathsf{u}}}_k \\ \vec{\mathring{\mathsf{y}}}_k \end{bmatrix} = \begin{bmatrix} \mathsf{F}_a \\ \mathsf{C}_a + \mathsf{D}_a\mathsf{F}_a \end{bmatrix}\nu_{a,k} + \begin{bmatrix} \mathsf{V}_a \\ \mathsf{D}_a\mathsf{V}_a \end{bmatrix}\vec{\mathring{\mathsf{q}}}_{a,k}, \tag{5.20b}
$$

where

$$
\begin{aligned}
\mathsf{V}_a &:= (\mathsf{R}_a + \mathsf{B}_a^*\mathsf{X}_a\mathsf{B}_a)^{-\frac{1}{2}}, && (5.21a)\\
\mathsf{F}_a &:= -(\mathsf{R}_a + \mathsf{B}_a^*\mathsf{X}_a\mathsf{B}_a)^{-1}(\mathsf{B}_a^*\mathsf{X}_a\mathsf{A}_a + \mathsf{D}_a^*\mathsf{C}_a), && (5.21b)\\
\mathsf{R}_a &:= \mathsf{I} + \mathsf{D}_a^*\mathsf{D}_a, && (5.21c)\\
\tilde{\mathsf{R}}_a &:= \mathsf{I} + \mathsf{D}_a\mathsf{D}_a^*, && (5.21d)\\
\grave{\mathsf{A}}_a &:= \mathsf{A}_a - \mathsf{B}_a\mathsf{R}_a^{-1}\mathsf{D}_a^*\mathsf{C}_a && (5.21e)
\end{aligned}
$$

and X_a *is a self-adjoint, positive-semidefinite, stabilising solution to the (discrete-time) algebraic, Riccati equation*

$$
\mathsf{X}_a = \grave{\mathsf{A}}_a^*\mathsf{X}_a(\mathsf{I} + \mathsf{B}_a\mathsf{R}_a^{-1}\mathsf{B}_a^*\mathsf{X}_a)^{-1}\grave{\mathsf{A}}_a + \mathsf{C}_a^*\tilde{\mathsf{R}}_a^{-1}\mathsf{C}_a. \tag{5.22}
$$

Then the pair $(\mathsf{M}_a, \mathsf{N}_a)$ *is a normalised, right-coprime factorisation of* $\mathsf{P}_a := \mathbf{W}P_a\mathbf{W}^{-1}$, *where* $P_a \in \mathscr{P}^{\mathrm{e}}_{\mathcal{U}\to\mathcal{Y}}$ *is characterised by the state-space system of differential equations (5.13).*

[2]This is the definition of $(C_a(t), A_a(t))$ detectable [RPK92].

5.4 Computing the Directed Gap

Given a real number $\gamma > 0$ and two systems P_a and P_b in $\mathscr{P}_{\mathcal{U}\to\mathcal{Y}}$ that admit stabilisable and detectable, state-space realisations, a necessary and sufficient condition for the directed gap between the two systems to be strictly less than γ is obtained in this section. The necessary and sufficient condition ultimately involves only finite-dimensional operators and is therefore, computationally tractable. Using this result, a bisection-search can be employed to compute the directed gap to any desired accuracy.

5.4.1 A Necessary and Sufficient Condition for Bounding the Directed Gap

Consider two LPTV systems P_a and P_b in $\mathscr{P}_{\mathcal{U}\to\mathcal{Y}}$ (with $\mathcal{U}$ and $\mathcal{Y}$ *finite-dimensional*, Hilbert spaces) that admit state-space realisations

$$\begin{aligned} \frac{dx_a}{dt} &= A_a(t)x_a(t) + B_a(t)u(t); \quad x_a(0) = 0, &(5.23a)\\ y(t) &= C_a(t)x_a(t) + D_a(t)u(t) &(5.23b) \end{aligned}$$

and

$$\begin{aligned} \frac{dx_b}{dt} &= A_b(t)x_b(t) + B_b(t)u(t); \quad x_b(0) = 0, &(5.24a)\\ y(t) &= C_b(t)x_b(t) + D_b(t)u(t) &(5.24b) \end{aligned}$$

respectively, where the states $x_a(t)$ and $x_b(t)$ evolve in the finite-dimensional, Hilbert spaces $\mathcal{X}_a$ and $\mathcal{X}_b$ respectively, and the state-space operators $A_\bullet(t)$, $B_\bullet(t)$, $C_\bullet(t)$ and $D_\bullet(t)$ are all piecewise-continuous, h-periodic functions of t (the subscript "$\bullet$" denotes a or b). It is assumed that $(A_\bullet(t), B_\bullet(t))$ is stabilisable and $(C_\bullet(t), A_\bullet(t))$ is detectable, so that by Corollary 5.5 and Lemma 5.3, the system $\mathsf{G}_a := \begin{bmatrix} \mathsf{M}_a \\ \mathsf{N}_a \end{bmatrix}$ (respectively $\mathsf{G}_b := \begin{bmatrix} \mathsf{M}_b \\ \mathsf{N}_b \end{bmatrix}$) described by the state-space system of difference equations

$$\begin{aligned} \mathsf{s}_{a,k+1} &= (\mathsf{A}_a + \mathsf{B}_a\mathsf{F}_a)\mathsf{s}_{a,k} + \mathsf{B}_a\mathsf{V}_a\vec{\mathsf{q}}_{a,k}; \quad \mathsf{s}_{a,0} = 0, &(5.25a)\\ \begin{bmatrix} \vec{\mathsf{u}}_{a,k} \\ \vec{\mathsf{y}}_{a,k} \end{bmatrix} &= \begin{bmatrix} \mathsf{F}_a \\ \mathsf{C}_a + \mathsf{D}_a\mathsf{F}_a \end{bmatrix}\mathsf{s}_{a,k} + \begin{bmatrix} \mathsf{V}_a \\ \mathsf{D}_a\mathsf{V}_a \end{bmatrix}\vec{\mathsf{q}}_{a,k}, &(5.25b) \end{aligned}$$

where A_a, B_a, C_a and D_a are defined in equations (5.16), V_a and F_a are defined in equations (5.21) and X_a is a self-adjoint, positive-semidefinite, stabilising solution to the algebraic, Riccati equation (5.22) (respectively described by equations (5.25) and the related equations (5.16), (5.21) and (5.22) with the subscript a replaced by b), constitutes a strong-right representation of the graph of the equivalent, LSI system $\mathsf{P}_a := \mathbf{W}P_a\mathbf{W}^{-1}$ (respectively $\mathsf{P}_b := \mathbf{W}P_b\mathbf{W}^{-1}$).

Let $\mathsf{s}_k := \begin{bmatrix} \mathsf{s}_{a,k} \\ \mathsf{s}_{b,k} \end{bmatrix}$ and rename $\vec{\mathsf{q}}_{a,k}$ by $\vec{\mathsf{w}}_k$ and $\vec{\mathsf{q}}_{b,k}$ by $\vec{\mathsf{u}}_k$. Then the system of difference equations

$$\mathsf{s}_{k+1} = \begin{bmatrix} (\mathsf{A}_a + \mathsf{B}_a\mathsf{F}_a) & 0 \\ 0 & (\mathsf{A}_b + \mathsf{B}_b\mathsf{F}_b) \end{bmatrix} \mathsf{s}_k + \begin{bmatrix} \mathsf{B}_a\mathsf{V}_a \\ 0 \end{bmatrix} \vec{\mathsf{w}}_k + \begin{bmatrix} 0 \\ \mathsf{B}_b\mathsf{V}_b \end{bmatrix} \vec{\mathsf{u}}_k; \quad \mathsf{s}_0 = 0, \tag{5.26a}$$

$$\vec{\mathsf{z}}_k = \begin{bmatrix} \mathsf{F}_a & \mathsf{F}_b \\ (\mathsf{C}_a + \mathsf{D}_a\mathsf{F}_a) & (\mathsf{C}_b + \mathsf{D}_b\mathsf{F}_b) \end{bmatrix} \mathsf{s}_k + \begin{bmatrix} \mathsf{V}_a \\ \mathsf{D}_a\mathsf{V}_a \end{bmatrix} \vec{\mathsf{w}}_k + \begin{bmatrix} \mathsf{V}_b \\ \mathsf{D}_b\mathsf{V}_b \end{bmatrix} \vec{\mathsf{u}}_k, \tag{5.26b}$$

is a state-space realisation of the system $\begin{bmatrix} \mathsf{G}_a & \mathsf{G}_b \end{bmatrix} : \ell^2_{\mathbb{Z}^+}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U} \oplus \mathcal{U})) \to \ell^2_{\mathbb{Z}^+}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{V}))$, where $\mathcal{V} := \mathcal{U} \oplus \mathcal{Y}$. It now follows by Lemma 5.2, that given a real number $\gamma > 0$, the directed gap $\vec{\delta}_g(P_a, P_b) < \gamma$ if and only if there exists a causal, LSI, stabilising control-law $\vec{\mathsf{u}} = \mathsf{K}\begin{bmatrix} \mathsf{s} \\ \vec{\mathsf{w}} \end{bmatrix}$ such that

$$\|\vec{\mathsf{z}}\|^2_{\ell^2_{\mathbb{Z}^+}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{V}))} - \gamma^2 \|\vec{\mathsf{w}}\|_{\ell^2_{\mathbb{Z}^+}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}))} \le -\epsilon \|\vec{\mathsf{w}}\|_{\ell^2_{\mathbb{Z}^+}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}))}$$

for some $\epsilon > 0$ and all $\vec{\mathsf{w}} \in \ell^2_{\mathbb{Z}^+}(\mathrm{L}^2_{\mathbb{H}}(\mathcal{U}))$, where $\vec{\mathsf{w}} := k \mapsto \vec{\mathsf{w}}_k$, $\vec{\mathsf{u}} := k \mapsto \vec{\mathsf{u}}_k$, $\vec{\mathsf{z}} := k \mapsto \vec{\mathsf{z}}_k$ and $\mathsf{s} := k \mapsto \mathsf{s}_k$ denote the signals associated with state-space system (5.26). That is, if and only if there exist a solution to the full-information, $\ell^2_{\mathbb{Z}^+}(\cdot)$-synthesis problem (cf. Proposition 2.22).

Now note that since both $\mathrm{spec}(\mathsf{A}_a + \mathsf{B}_a\mathsf{F}_a)$ and $\mathrm{spec}(\mathsf{A}_b + \mathsf{B}_b\mathsf{F}_b)$ are subsets of $\mathbb{D}$, the pair

$$\left(\begin{bmatrix} (\mathsf{A}_a + \mathsf{B}_a\mathsf{F}_a) & 0 \\ 0 & (\mathsf{A}_b + \mathsf{B}_b\mathsf{F}_b) \end{bmatrix}, \begin{bmatrix} 0 \\ \mathsf{B}_b\mathsf{V}_b \end{bmatrix} \right)$$

is stabilisable. Furthermore, since $\mathsf{V}_b = \mathsf{V}_b^*$ is boundedly invertible, it follows that

$$\begin{bmatrix} \mathsf{V}_b^* & \mathsf{V}_b^*\mathsf{D}_b^* \end{bmatrix} \begin{bmatrix} \mathsf{V}_b \\ \mathsf{D}_b\mathsf{V}_b \end{bmatrix} > 0.$$

Moreover, since $(\mathsf{C}_b, \mathsf{A}_b)$ is detectable by assumption, it can be shown that

$$\mathcal{K} \begin{bmatrix} \begin{bmatrix} (\mathsf{A}_a + \mathsf{B}_a\mathsf{F}_a) & 0 \\ 0 & (\mathsf{A}_b + \mathsf{B}_b\mathsf{F}_b) \end{bmatrix} - e^{j\theta}\mathrm{I} & \begin{bmatrix} 0 \\ \mathsf{B}_b\mathsf{V}_b \end{bmatrix} \\ \begin{bmatrix} \mathsf{F}_a & \mathsf{F}_b \\ (\mathsf{C}_a + \mathsf{D}_a\mathsf{F}_a) & (\mathsf{C}_b + \mathsf{D}_b\mathsf{F}_b) \end{bmatrix} & \begin{bmatrix} \mathsf{V}_b \\ \mathsf{D}_b\mathsf{V}_b \end{bmatrix} \end{bmatrix} = \{0\}$$

for all $\theta \in [0, 2\pi)$. That is, all of the assumption required for Proposition 2.22 to hold are satisfied. The following theorem then follows directly by Lemma 5.2 and Proposition 2.22.

Theorem 5.6 *Consider two LPTV systems P_a and P_b that admit the state-space realisations given in (5.23) and (5.24) respectively, where $(A_a(t), B_a(t))$ and $(A_b(t), B_b(t))$ are stabilisable and $(C_a(t), A_a(t))$ and $(C_b(t), A_b(t))$ are detectable. Then given a real number $\gamma > 0$, the directed gap $\vec{\delta}_g(P_a, P_b) < \gamma$ if and only if there exists a self-adjoint, positive-semidefinite operator $\mathsf{X}_\infty : \mathcal{X}_a \oplus \mathcal{X}_b \to \mathcal{X}_a \oplus \mathcal{X}_b$ that satisfies :*

(i) the finite-dimensional, (discrete-time) algebraic, Riccati equation

$$\mathsf{X}_\infty = \grave{\mathsf{A}}^*\mathsf{X}_\infty(\mathrm{I} + \grave{\mathsf{R}}\mathsf{X}_\infty)^{-1}\grave{\mathsf{A}} + \grave{\mathsf{Q}}; \tag{5.27}$$

(ii) $\operatorname{spec}(\mathsf{A} - \mathsf{B}\mathsf{Z}^{-1}\mathsf{L}) = \operatorname{spec}((\mathrm{I} + \grave{\mathsf{R}}\mathsf{X}_\infty)^{-1}\grave{\mathsf{A}}) \subset \mathbb{D}$*;*

(iii) $\mathsf{Z}_1 - \mathsf{Z}_2^*\mathsf{Z}_3^{-1}\mathsf{Z}_2 < 0$,

where

$$\mathsf{J} := \begin{bmatrix} \mathrm{I} & 0 \\ 0 & -\gamma^2\mathrm{I} \end{bmatrix}, \qquad \grave{\mathsf{A}} := \mathsf{A} - \mathsf{B}(\tilde{\mathsf{D}}^*\mathsf{J}\tilde{\mathsf{D}})^{-1}\tilde{\mathsf{D}}^*\mathsf{J}\tilde{\mathsf{C}},$$

$$\grave{\mathsf{R}} := \mathsf{B}(\tilde{\mathsf{D}}^*\mathsf{J}\tilde{\mathsf{D}})^{-1}\mathsf{B}^*, \qquad \grave{\mathsf{Q}} := \tilde{\mathsf{C}}^*(\mathrm{I} - \tilde{\mathsf{D}}(\tilde{\mathsf{D}}^*\mathsf{J}\tilde{\mathsf{D}})^{-1}\tilde{\mathsf{D}}^*)\tilde{\mathsf{C}},$$

$$\mathsf{A} := \begin{bmatrix} (\mathsf{A}_a + \mathsf{B}_a\mathsf{F}_a) & 0 \\ 0 & (\mathsf{A}_b + \mathsf{B}_b\mathsf{F}_b) \end{bmatrix}, \qquad \mathsf{B} := \begin{bmatrix} \mathsf{B}_a\mathsf{V}_a & 0 \\ 0 & \mathsf{B}_b\mathsf{V}_b \end{bmatrix},$$

$$\tilde{\mathsf{C}} := \begin{bmatrix} \begin{bmatrix} \mathsf{F}_a & \mathsf{F}_b \\ (\mathsf{C}_a + \mathsf{D}_a\mathsf{F}_a) & (\mathsf{C}_b + \mathsf{D}_b\mathsf{F}_b) \end{bmatrix} \\ \begin{matrix} 0 & \qquad\qquad 0 \end{matrix} \end{bmatrix}, \qquad \tilde{\mathsf{D}} := \begin{bmatrix} \begin{bmatrix} \mathsf{V}_a \\ \mathsf{D}_a\mathsf{V}_a \end{bmatrix} & \begin{bmatrix} \mathsf{V}_b \\ \mathsf{D}_b\mathsf{V}_b \end{bmatrix} \\ \mathrm{I} & 0 \end{bmatrix},$$

$$\mathsf{Z} := \tilde{\mathsf{D}}^*\mathsf{J}\tilde{\mathsf{D}} + \mathsf{B}^*\mathsf{X}_\infty\mathsf{B} =: \begin{bmatrix} \mathsf{Z}_1 & \mathsf{Z}_2^* \\ \mathsf{Z}_2 & \mathsf{Z}_3 \end{bmatrix}, \qquad \mathsf{L} := \tilde{\mathsf{D}}^*\mathsf{J}\tilde{\mathsf{D}} + \mathsf{B}^*\mathsf{X}_\infty\mathsf{A}$$

and $\mathsf{A}_\bullet$, $\mathsf{B}_\bullet$, $\mathsf{C}_\bullet$, $\mathsf{D}_\bullet$, $\mathsf{F}_\bullet$ and $\mathsf{V}_\bullet$ ("$\bullet$" denotes a or b) are defined in equations (5.16) and (5.21).

Remark 5.7 Given finite-dimensional expressions (matrix representations) for $\grave{\mathsf{A}}$, $\grave{\mathsf{R}}$ and $\grave{\mathsf{Q}}$ and a computationally-tractable procedure for testing condition (*iii*), a bisection-search based on the conditions given in Theorem 5.6 can be used to compute to any desired accuracy the directed gap between any two systems in $\mathscr{P}_{\mathcal{U}\to\mathcal{Y}}$ that admit stabilisable and detectable, state-space realisations. Then since the gap is simply the maximum of two directed gaps, this constitutes a procedure for computing the gap between two LPTV systems. ♣

In the sequel the necessary finite-dimensional expressions for $\grave{\mathsf{A}}$, $\grave{\mathsf{R}}$ and $\grave{\mathsf{Q}}$ and a computationally-tractable test for condition (*iii*) are obtained. To this end, the following assumption is required to hold throughout.

Assumption 5.8 *It is assumed that $D_a(t) = D_b(t)$ for all t. In other words, the two systems P_a and P_b have the same instantaneous response.*

5.4.2 Computing $\grave{\mathsf{A}}$, $\grave{\mathsf{R}}$ and $\grave{\mathsf{Q}}$ in Theorem 5.6

In this sub-section, finite-dimensional expressions are derived for the operators $\grave{\mathsf{A}}$, $\grave{\mathsf{R}}$ and $\grave{\mathsf{Q}}$ appearing in Theorem 5.6. The formulae involve computing the value of a state-transition operator at only one point. The relevant state-transition operator is associated with a time-varying TPBVP that is expressible explicitly in terms of the periodically-time-varying, state-space realisations of the two LPTV systems P_a and P_b. The TPBVP characterisation is derived in a manner similar to the TPBVP analysis presented in [CG97] (cf. Chapter 6), where finite-dimensional expressions are derived for certain operators involved in the so-called SD-discretisation approach to H^∞ synthesis of SD control-systems. The derivation here is a long and tedious series of state-space manipulations and as such, the proof is deferred to Appendix C.1.

Lemma 5.9 *Given $\gamma > 0$, construct the following periodic, matrix function of time :*

$$[E(t)]_{i,j} := E_{ij}(t),$$

where the E_{ij}'s ($i, j = 1, \dots, 7$) are defined in Appendix C.2. Let

$$\mathsf{E} := \mathbf{\Phi}_E(h, 0)$$

and partition it conformably with E, where $\mathbf{\Phi}_E(t,\tau)$ denotes the state-transition operator associated with $E(t)$. Then under Assumption 5.8,

$$\grave{\mathsf{Q}} = -\begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix} \tilde{\mathsf{E}}^{-1} \begin{bmatrix} \left(\mathsf{E}_{43} - \begin{bmatrix} \mathsf{X}_a & 0 \end{bmatrix} \mathsf{E}_{23}\right) \\ \left(\mathsf{E}_{53} - \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix} \mathsf{E}_{13}\right) \\ \mathsf{E}_{63} \\ \left(\mathsf{E}_{73} - \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix} \mathsf{E}_{33}\right) \end{bmatrix}, \tag{5.29}$$

$$\grave{\mathsf{R}} = \begin{bmatrix} \mathsf{E}_{24} & \mathsf{E}_{26} \end{bmatrix} \begin{bmatrix} -\mathsf{E}_{64} & -\mathsf{E}_{66} \\ \left(\mathsf{E}_{44} - \begin{bmatrix} \mathsf{X}_a & 0 \end{bmatrix} \mathsf{E}_{24}\right) & \left(\mathsf{E}_{46} - \begin{bmatrix} \mathsf{X}_a & 0 \end{bmatrix} \mathsf{E}_{26}\right) \end{bmatrix}^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{5.30}$$

and

$$\begin{aligned}\grave{\mathsf{A}} &= (\mathsf{E}_{33}-\mathsf{E}_{23}) - \begin{bmatrix} (\mathsf{E}_{34}-\mathsf{E}_{24}) & (\mathsf{E}_{35}-\mathsf{E}_{25}) & (\mathsf{E}_{36}-\mathsf{E}_{26}) & (\mathsf{E}_{37}-\mathsf{E}_{27}) \end{bmatrix} \times \\ & \quad \tilde{\mathsf{E}}^{-1} \begin{bmatrix} \left(\mathsf{E}_{43} - \begin{bmatrix} \mathsf{X}_a & 0 \end{bmatrix} \mathsf{E}_{23}\right) \\ \left(\mathsf{E}_{53} - \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix} \mathsf{E}_{13}\right) \\ \mathsf{E}_{63} \\ \left(\mathsf{E}_{73} - \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix} \mathsf{E}_{33}\right) \end{bmatrix},\end{aligned} \tag{5.31}$$

where

$$\tilde{\mathsf{E}} := \begin{bmatrix} \left(\mathsf{E}_{44} - \begin{bmatrix} \mathsf{X}_a & 0 \end{bmatrix}\mathsf{E}_{24}\right) & \left(\mathsf{E}_{45} - \begin{bmatrix} \mathsf{X}_a & 0 \end{bmatrix}\mathsf{E}_{25}\right) & \left(\mathsf{E}_{46} - \begin{bmatrix} \mathsf{X}_a & 0 \end{bmatrix}\mathsf{E}_{26}\right) & \left(\mathsf{E}_{47} - \begin{bmatrix} \mathsf{X}_a & 0 \end{bmatrix}\mathsf{E}_{27}\right) \\ \left(\mathsf{E}_{54} - \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix}\mathsf{E}_{14}\right) & \left(\mathsf{E}_{55} - \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix}\mathsf{E}_{15}\right) & \left(\mathsf{E}_{56} - \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix}\mathsf{E}_{16}\right) & \left(\mathsf{E}_{57} - \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix}\mathsf{E}_{17}\right) \\ \mathsf{E}_{64} & \mathsf{E}_{65} & \mathsf{E}_{66} & \mathsf{E}_{67} \\ \left(\mathsf{E}_{74} - \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix}\mathsf{E}_{34}\right) & \left(\mathsf{E}_{75} - \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix}\mathsf{E}_{35}\right) & \left(\mathsf{E}_{76} - \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix}\mathsf{E}_{36}\right) & \left(\mathsf{E}_{77} - \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix}\mathsf{E}_{37}\right) \end{bmatrix},$$

X_a *is a self-adjoint, positive-semidefinite, stabilising solution to the algebraic, Riccati equation*

$$\mathsf{X}_a = \grave{\mathsf{A}}_a^* \mathsf{X}_a (\mathrm{I} + \grave{\mathsf{R}}_a \mathsf{X}_a)^{-1} \grave{\mathsf{A}}_a + \grave{\mathsf{Q}}_a,$$

and X_b *is a self-adjoint, positive-semidefinite, stabilising solution to the algebraic, Riccati equation*

$$\mathsf{X}_b = \grave{\mathsf{A}}_b^* \mathsf{X}_b (\mathrm{I} + \grave{\mathsf{R}}_b \mathsf{X}_b)^{-1} \grave{\mathsf{A}}_b + \grave{\mathsf{Q}}_b,$$

with

$$\grave{\mathsf{A}}_a = \begin{bmatrix} I & 0 \end{bmatrix} (\mathsf{E}_{33} - \mathsf{E}_{37}\mathsf{E}_{77}^{-1}\mathsf{E}_{73}) \begin{bmatrix} I \\ 0 \end{bmatrix}, \qquad \grave{\mathsf{A}}_b = \begin{bmatrix} 0 & I \end{bmatrix} (\mathsf{E}_{33} - \mathsf{E}_{37}\mathsf{E}_{77}^{-1}\mathsf{E}_{73}) \begin{bmatrix} 0 \\ I \end{bmatrix},$$

$$\grave{\mathsf{R}}_a = -\begin{bmatrix} I & 0 \end{bmatrix} \mathsf{E}_{33}\mathsf{E}_{77}^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix}, \qquad \grave{\mathsf{R}}_b = -\begin{bmatrix} 0 & I \end{bmatrix} \mathsf{E}_{33}\mathsf{E}_{77}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix},$$

$$\grave{\mathsf{Q}}_a = -\begin{bmatrix} I & 0 \end{bmatrix} \mathsf{E}_{77}^{-1}\mathsf{E}_{73} \begin{bmatrix} I \\ 0 \end{bmatrix} \qquad \text{and} \qquad \grave{\mathsf{Q}}_b = -\begin{bmatrix} 0 & I \end{bmatrix} \mathsf{E}_{77}^{-1}\mathsf{E}_{73} \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Proof : The proof, rather long and tedious, is deferred to Appendix C.1. ∎

5.4.3 Testing Condition (*iii*) in Theorem 5.6

Recall that condition (*iii*) in Theorem 5.6 involves testing

$$\mathsf{Z}_1 - \mathsf{Z}_2^*\mathsf{Z}_3^{-1}\mathsf{Z}_2 < 0,$$

where

$$\mathsf{Z} := \tilde{\mathsf{D}}^*\mathsf{J}\tilde{\mathsf{D}} + \mathsf{B}^*\mathsf{X}_\infty\mathsf{B} =: \begin{bmatrix} \mathsf{Z}_1 & \mathsf{Z}_2^* \\ \mathsf{Z}_2 & \mathsf{Z}_3 \end{bmatrix},$$

$\tilde{\mathsf{D}}$, J and B are defined in Theorem 5.6 and X_∞ is a self-adjoint, positive-semidefinite, stabilising solution to the finite-dimensional, algebraic, Riccati equation (5.27). It is straightforward to show that

$$\mathsf{Z}_1 - \mathsf{Z}_2^*\mathsf{Z}_3^{-1}\mathsf{Z}_2 = -\gamma^2\mathrm{I} + \mathsf{V}_a^*\tilde{W}^*(\mathrm{I} - \tilde{\mathsf{Y}}(\tilde{\mathsf{Y}}^*\tilde{\mathsf{Y}})^{-1}\tilde{\mathsf{Y}}^*)\tilde{W}\mathsf{V}_a, \tag{5.32}$$

where

$$\tilde{W} := \begin{bmatrix} \mathrm{I} \\ W \end{bmatrix}, \qquad W := \begin{bmatrix} \mathsf{D}_a \\ \mathsf{X}_\infty^{\frac{1}{2}} \begin{bmatrix} \mathsf{B}_a \\ 0 \end{bmatrix} \end{bmatrix}, \tag{5.33a}$$

$$\tilde{\mathsf{Y}} := \begin{bmatrix} \mathrm{I} \\ \mathsf{Y} \end{bmatrix}, \qquad \mathsf{Y} := \begin{bmatrix} \mathsf{D}_b \\ \mathsf{X}_\infty^{\frac{1}{2}} \begin{bmatrix} 0 \\ \mathsf{B}_b \end{bmatrix} \end{bmatrix}, \tag{5.33b}$$

$\mathsf{B}_\bullet$ and $\mathsf{D}_\bullet$ are defined in equations (5.16) and V_a is defined in equation (5.21a). Since $(\mathrm{I} - \tilde{\mathsf{Y}}(\tilde{\mathsf{Y}}^*\tilde{\mathsf{Y}})^{-1}\tilde{\mathsf{Y}}^*)$ is idempotent, it follows from equation (5.32) that

$$\mathsf{Z}_1 - \mathsf{Z}_2^*\mathsf{Z}_3^{-1}\mathsf{Z}_2 < 0$$

$$\Updownarrow$$

$$-\gamma^2\mathrm{I} + \mathsf{V}_a^*\tilde{W}^*(\mathrm{I} - \tilde{\mathsf{Y}}(\tilde{\mathsf{Y}}^*\tilde{\mathsf{Y}})^{-1}\tilde{\mathsf{Y}}^*)(\mathrm{I} - \tilde{\mathsf{Y}}(\tilde{\mathsf{Y}}^*\tilde{\mathsf{Y}})^{-1}\tilde{\mathsf{Y}}^*)\tilde{W}\mathsf{V}_a < 0.$$

This last condition is tantamount to

$$\left\|(\mathrm{I} - \tilde{\mathsf{Y}}(\tilde{\mathsf{Y}}^*\tilde{\mathsf{Y}})^{-1}\tilde{\mathsf{Y}}^*)\tilde{W}\mathsf{V}_a\right\| = \left\|\left(\mathrm{I} - \begin{bmatrix} \mathrm{I} \\ \mathsf{Y} \end{bmatrix}(\mathrm{I} + \mathsf{Y}^*\mathsf{Y})^{-1}\begin{bmatrix} \mathrm{I} & \mathsf{Y}^* \end{bmatrix}\right)\begin{bmatrix} \mathrm{I} \\ W \end{bmatrix}\mathsf{V}_a\right\| < \gamma.$$

Now note that the operator

$$\begin{bmatrix} (\mathrm{I} + \mathsf{Y}^*\mathsf{Y})^{-\frac{1}{2}}\begin{bmatrix} \mathrm{I} & \mathsf{Y}^* \end{bmatrix} \\ (\mathrm{I} + \mathsf{Y}\mathsf{Y}^*)^{-\frac{1}{2}}\begin{bmatrix} -\mathsf{Y} & \mathrm{I} \end{bmatrix} \end{bmatrix}$$

is unitary and therefore, that

$$\left\| \left(\mathrm{I} - \begin{bmatrix} \mathrm{I} \\ \mathrm{Y} \end{bmatrix} (\mathrm{I}+\mathrm{Y}^*\mathrm{Y})^{-1} \begin{bmatrix} \mathrm{I} & \mathrm{Y}^* \end{bmatrix} \right) \begin{bmatrix} \mathrm{I} \\ \mathrm{W} \end{bmatrix} \mathrm{V}_a \right\| < \gamma$$

$$\Updownarrow$$

$$\left\| \begin{bmatrix} (\mathrm{I}+\mathrm{Y}^*\mathrm{Y})^{-\frac{1}{2}} \begin{bmatrix} \mathrm{I} & \mathrm{Y}^* \end{bmatrix} \\ (\mathrm{I}+\mathrm{Y}\mathrm{Y}^*)^{-\frac{1}{2}} \begin{bmatrix} -\mathrm{Y} & \mathrm{I} \end{bmatrix} \end{bmatrix} \left(\mathrm{I} - \begin{bmatrix} \mathrm{I} \\ \mathrm{Y} \end{bmatrix} (\mathrm{I}+\mathrm{Y}^*\mathrm{Y})^{-1} \begin{bmatrix} \mathrm{I} & \mathrm{Y}^* \end{bmatrix} \right) \begin{bmatrix} \mathrm{I} \\ \mathrm{W} \end{bmatrix} \mathrm{V}_a \right\| < \gamma$$

$$\Updownarrow$$

$$\left\| (\mathrm{I}+\mathrm{Y}\mathrm{Y}^*)^{-\frac{1}{2}} (\mathrm{W}-\mathrm{Y}) \mathrm{V}_a \right\| < \gamma.$$

Under Assumption 5.8, $D_a(t) = D_b(t)$ for all t and hence, the operator $(\mathrm{W}-\mathrm{Y})$ is compact.[3] It follows that $(\mathrm{I}+\mathrm{Y}\mathrm{Y}^*)^{-\frac{1}{2}}(\mathrm{W}-\mathrm{Y})\mathrm{V}_a$ is also compact and therefore, that its norm can be determined by calculating its largest singular value (cf. Sub-Section 2.2.2 or [Kre89b]).

Recall that a number $\sigma \in \mathbb{R}^+$ is a singular value of $(\mathrm{I}+\mathrm{Y}\mathrm{Y}^*)^{-\frac{1}{2}}(\mathrm{W}-\mathrm{Y})\mathrm{V}_a$ if and only if there exist a non-zero $\mathring{\vec{x}} \in \mathrm{L}^2_{\mathbb{H}}(\mathcal{U})$ such that

$$\mathrm{V}_a^*(\mathrm{W}-\mathrm{Y})^*(\mathrm{I}+\mathrm{Y}\mathrm{Y}^*)^{-1}(\mathrm{W}-\mathrm{Y})\mathrm{V}_a \mathring{\vec{x}} = \sigma^2 \mathring{\vec{x}}.$$

Since $\mathrm{V}_a = \mathrm{V}_a^*$ is invertible, this is equivalent to the existence of a non-zero $\mathring{\vec{z}}$ $(= \mathrm{V}_a^{-1}\mathring{\vec{x}})$ such that

$$(\mathrm{W}-\mathrm{Y})^*(\mathrm{I}+\mathrm{Y}\mathrm{Y}^*)^{-1}(\mathrm{W}-\mathrm{Y})\mathrm{V}_a^2 \mathring{\vec{z}} = \sigma^2 \mathring{\vec{z}}. \tag{5.34}$$

For $\sigma > 0$, such a $\mathring{\vec{z}}$ can be constructed (as follows) in terms of the solution to a TPBVP. To see this, define

$$\mathring{\vec{w}} := (\mathrm{W}-\mathrm{Y})^*(\mathrm{I}+\mathrm{Y}\mathrm{Y}^*)^{-1}(\mathrm{W}-\mathrm{Y})\sigma^{-1}\mathring{\vec{u}} \tag{5.35a}$$

and

$$\mathring{\vec{u}} := \mathrm{V}_a^2 \mathring{\vec{z}}. \tag{5.35b}$$

---

[3]Under Assumption 5.8, the action of $\begin{bmatrix} \mathring{\vec{y}} \\ \tilde{y} \end{bmatrix} = (\mathrm{W}-\mathrm{Y})\mathring{\vec{u}}$ is completely characterised by the integral operators

$$\begin{aligned} \mathring{\vec{y}}(\theta) &= \begin{bmatrix} C_a & -C_b \\ 0 & 0 \end{bmatrix} \int_0^\theta \mathbf{\Phi}_{\begin{bmatrix} A_a & 0 \\ 0 & A_b \end{bmatrix}}(\theta,\tau) \begin{bmatrix} B_a & 0 \\ 0 & B_b \end{bmatrix} \mathring{\vec{u}}(\tau)\, d\tau, \\ \tilde{y} &= \begin{bmatrix} 0 \\ \mathrm{X}_\infty^{\frac{1}{2}} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \end{bmatrix} \int_0^h \mathbf{\Phi}_{\begin{bmatrix} A_a & 0 \\ 0 & A_b \end{bmatrix}}(h,\tau) \begin{bmatrix} B_a & 0 \\ 0 & B_b \end{bmatrix} \mathring{\vec{u}}(\tau)\, d\tau, \end{aligned}$$

which are both compact [Kre89a], since there is no direct feedthrough.

It then follows from equations (5.16) and (5.33), that the operator mapping $\vec{u}^{\circ}$ to $\vec{w}^{\circ}$ in equation (5.35a) can be expressed in terms of the following TPBVP:

$$\vec{w}^{\circ} = \begin{bmatrix} 0 & \begin{bmatrix} B_a^* & -B_b^* \end{bmatrix} \end{bmatrix} \begin{bmatrix} \vec{a}^{\circ} \\ \vec{b}^{\circ} \end{bmatrix} \tag{5.36a}$$

$$\begin{aligned} \frac{d}{d\theta}\left(\begin{bmatrix} \vec{a}^{\circ} \\ \vec{b}^{\circ} \end{bmatrix}\right) &= \begin{bmatrix} \underline{A} & -\begin{bmatrix} 0 \\ B_b \end{bmatrix} R_b^{-1} \begin{bmatrix} 0 & B_b^* \end{bmatrix} \\ -\begin{bmatrix} C_a^* \\ C_a^* \end{bmatrix} \tilde{R}_b^{-1} \begin{bmatrix} C_a & C_b \end{bmatrix} & -\underline{A}^* \end{bmatrix} \begin{bmatrix} \vec{a}^{\circ} \\ \vec{b}^{\circ} \end{bmatrix}(\theta) \\ &\quad + \begin{bmatrix} \begin{bmatrix} B_a \\ -B_b \end{bmatrix} \\ 0 \end{bmatrix} \sigma^{-1} \vec{u}^{\circ}(\theta), \end{aligned} \tag{5.36b}$$

with boundary conditions

$$\vec{a}^{\circ}(0) = 0 \quad \text{and} \quad \vec{b}^{\circ}(h) = \mathsf{X}_{\infty} \vec{a}^{\circ}(h),$$

where

$$\underline{A} := \begin{bmatrix} A_a & 0 \\ 0 & A_b \end{bmatrix} - \begin{bmatrix} 0 \\ B_b \end{bmatrix} R_b^{-1} D_b^* \begin{bmatrix} C_a & C_b \end{bmatrix},$$

$R_\bullet :=(I + D_\bullet^* D_\bullet)$ and $\tilde{R}_\bullet :=(I + D_\bullet D_\bullet^*)$.[4] Similarly, the operator mapping $\vec{z}^{\circ}$ to $\vec{u}^{\circ}$ in equation (5.35b) is completely characterised by the TPBVP:

$$\vec{u}^{\circ} = R_a^{-1} \vec{z}^{\circ} + \begin{bmatrix} -R_a^{-1} D_a^* C_a & R_a^{-1} B_a^* \end{bmatrix} \begin{bmatrix} \vec{e}^{\circ} \\ \vec{g}^{\circ} \end{bmatrix}, \tag{5.37a}$$

$$\begin{aligned} \frac{d}{d\theta}\left(\begin{bmatrix} \vec{e}^{\circ} \\ \vec{g}^{\circ} \end{bmatrix}\right) &= \begin{bmatrix} (A_a - B_a R_a^{-1} D_a^* C_a) & B_a R_a^{-1} B_a^* \\ C_a^* \tilde{R}_a^{-1} C_a & -(A_a - B_a R_a^{-1} D_a^* C_a)^* \end{bmatrix} \begin{bmatrix} \vec{e}^{\circ} \\ \vec{g}^{\circ} \end{bmatrix}(\theta) \\ &\quad + \begin{bmatrix} B_a R_a^{-1} \\ C_a^* D_a R_a^{-1} \end{bmatrix} \vec{z}^{\circ}(\theta), \end{aligned} \tag{5.37b}$$

with boundary conditions

$$\vec{e}^{\circ}(0) = 0 \quad \text{and} \quad \vec{g}^{\circ}(h) = -\mathsf{X}_a \vec{e}^{\circ}(h),$$

[4]Recall that under Assumption 5.8, $D_a = D_b$. However, for clarity of the derivations it is convenient to maintain a notational distinction between them.

where X_a is a self-adjoint, positive-semidefinite, stabilising solution to the algebraic, Riccati equation (5.22). Moreover, from equations (5.34) and (5.35), it follows that $\vec{z}^\circ = \sigma^{-1}\vec{w}^\circ$. With substitution of this into equations (5.36) and (5.37), elementary algebraic manipulation yields the following TPBVP characterisation of the $\vec{z}^\circ$'s satisfying equation (5.34):

$$\vec{z}^\circ = \sigma^{-1}\begin{bmatrix} 0 & \begin{bmatrix} 0 & \begin{bmatrix} B_a^* & -B_b^* \end{bmatrix} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \vec{r}^\circ \\ \vec{v}^\circ \end{bmatrix}, \tag{5.38a}$$

$$\frac{d}{d\theta}\left(\begin{bmatrix} \vec{r}^\circ \\ \vec{v}^\circ \end{bmatrix}\right) = \begin{bmatrix} \check{A}^\sigma & \check{B}^\sigma(\check{R}^\sigma)^{-1}(\check{B}^\sigma)^* \\ (\check{C}^\sigma)^*(\check{Q}^\sigma)^{-1}\check{C}^\sigma & -(\check{A}^\sigma)^* \end{bmatrix} \begin{bmatrix} \vec{r}^\circ \\ \vec{v}^\circ \end{bmatrix}(\theta), \tag{5.38b}$$

with boundary conditions

$$\vec{r}^\circ(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v}^\circ(h) = \begin{bmatrix} -\mathsf{X}_a & 0 \\ 0 & \mathsf{X}_\infty \end{bmatrix} \vec{r}^\circ(h), \tag{5.38c}$$

where

$$\vec{r}^\circ := \begin{bmatrix} \vec{e}^\circ \\ \vec{a}^\circ \end{bmatrix}, \qquad \vec{v}^\circ := \begin{bmatrix} \vec{g}^\circ \\ \vec{b}^\circ \end{bmatrix},$$

$$\check{R}^\sigma := \begin{bmatrix} \sigma^2 R_a & 0 \\ 0 & -R_b \end{bmatrix}, \qquad \check{Q}^\sigma := \begin{bmatrix} \sigma^2 \tilde{R}_a & 0 \\ 0 & -\tilde{R}_b \end{bmatrix},$$

$$\check{B}^\sigma := \begin{bmatrix} \sigma B_a & 0 \\ B_a & 0 \\ -B_b & B_b \end{bmatrix}, \qquad \check{C}^\sigma := \begin{bmatrix} \sigma C_a & 0 & 0 \\ 0 & C_a & C_b \end{bmatrix}$$

and

$$\check{A}^\sigma := \begin{bmatrix} A_a & 0 & 0 \\ 0 & A_a & 0 \\ 0 & 0 & A_b \end{bmatrix} - \check{B}^\sigma(\check{R}^\sigma)^{-1}\begin{bmatrix} D_a & 0 \\ 0 & -D_b \end{bmatrix}\check{C}^\sigma.$$

For $\vec{z}^\circ$ in equation (5.38a) to be non-zero, the TPBVP must have non-trivial solution. With

$$\check{E}^\sigma(t) := \begin{bmatrix} \check{A}^\sigma & \check{B}^\sigma(\check{R}^\sigma)^{-1}(\check{B}^\sigma)^* \\ (\check{C}^\sigma)^*(\check{Q}^\sigma)^{-1}\check{C}^\sigma & -(\check{A}^\sigma)^* \end{bmatrix}(t)$$

and [5]

$$\check{\mathsf{E}}^\sigma := \mathbf{\Phi}_{\check{E}^\sigma}(h,0), \tag{5.40}$$

[5]Recall that $\mathbf{\Phi}_{\cdot}(t,\tau)$ denotes the state-transition operator associated with $\cdot$.

existence of a non-trivial solution implies

$$\mathcal{K}\left(\check{\mathsf{E}}^{\sigma}_{22} - \begin{bmatrix} -\mathsf{X}_a & 0 \\ 0 & \mathsf{X}_\infty \end{bmatrix} \check{\mathsf{E}}^{\sigma}_{12}\right) \neq \{0\},$$

where $\check{\mathsf{E}}^{\sigma} =: \begin{bmatrix} \check{\mathsf{E}}^{\sigma}_{11} & \check{\mathsf{E}}^{\sigma}_{12} \\ \check{\mathsf{E}}^{\sigma}_{21} & \check{\mathsf{E}}^{\sigma}_{22} \end{bmatrix}$ and the partition is conformal with that of $\check{E}^{\sigma}(t)$. To see this, note that

$$\begin{bmatrix} \mathring{\vec{r}}(h) \\ \mathring{\vec{v}}(h) \end{bmatrix} = \begin{bmatrix} \check{\mathsf{E}}^{\sigma}_{11} & \check{\mathsf{E}}^{\sigma}_{12} \\ \check{\mathsf{E}}^{\sigma}_{21} & \check{\mathsf{E}}^{\sigma}_{22} \end{bmatrix} \begin{bmatrix} \mathring{\vec{r}}(0) \\ \mathring{\vec{v}}(0) \end{bmatrix},$$

which combined with the boundary conditions (5.38c), gives

$$\left(\check{\mathsf{E}}^{\sigma}_{22} - \begin{bmatrix} -\mathsf{X}_a & 0 \\ 0 & \mathsf{X}_\infty \end{bmatrix} \check{\mathsf{E}}^{\sigma}_{12}\right) \mathring{\vec{v}}(0) = 0.$$

Now suppose that $\mathcal{K}\left(\check{\mathsf{E}}^{\sigma}_{22} - \begin{bmatrix} -\mathsf{X}_a & 0 \\ 0 & \mathsf{X}_\infty \end{bmatrix} \check{\mathsf{E}}^{\sigma}_{12}\right) = \{0\}$. It then follows that $\mathring{\vec{v}}(0) = 0$ and hence, since $\mathring{\vec{r}}(0) = 0$, that the TPBVP has only the trivial solution $\begin{bmatrix} \mathring{\vec{r}} \\ \mathring{\vec{v}} \end{bmatrix} \equiv 0$. This contradicts the initial hypothesis that

$$0 \neq \mathring{\vec{z}} = \sigma^{-2} \begin{bmatrix} 0 & \begin{bmatrix} 0 & \begin{bmatrix} B_a^* & -B_b^* \end{bmatrix} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \mathring{\vec{r}} \\ \mathring{\vec{v}} \end{bmatrix}.$$

Thus, if $\mathring{\vec{z}}$ is non-zero then necessarily, $\mathcal{K}\left(\check{\mathsf{E}}^{\sigma}_{22} - \begin{bmatrix} -\mathsf{X}_a & 0 \\ 0 & \mathsf{X}_\infty \end{bmatrix} \check{\mathsf{E}}^{\sigma}_{12}\right) \neq \{0\}$. The latter condition is also sufficient. That is, if

$$\det\left(\check{\mathsf{E}}^{\sigma}_{22} - \begin{bmatrix} -\mathsf{X}_a & 0 \\ 0 & \mathsf{X}_\infty \end{bmatrix} \check{\mathsf{E}}^{\sigma}_{12}\right) = 0, \tag{5.41}$$

then there exists a non-zero $\mathring{\vec{z}}$ satisfying equation (5.34) and hence, σ is a singular value of $(\mathrm{I} + \mathsf{Y}\mathsf{Y}^*)^{-\frac{1}{2}}(\mathsf{W} - \mathsf{Y})\mathsf{V}_a$. Again, this can be seen by contradiction. Let equation (5.41) hold. Then there exists a $\mathring{\vec{v}}(0) := \begin{bmatrix} \mathring{\vec{g}}(0) \\ \mathring{\vec{b}}(0) \end{bmatrix} \neq 0$, such that

$$\left(\check{\mathsf{E}}^{\sigma}_{22} - \begin{bmatrix} -\mathsf{X}_a & 0 \\ 0 & \mathsf{X}_\infty \end{bmatrix} \check{\mathsf{E}}^{\sigma}_{12}\right) \begin{bmatrix} \mathring{\vec{g}}(0) \\ \mathring{\vec{b}}(0) \end{bmatrix} = 0$$

and consequently, a corresponding non-trivial solution to equation (5.38b) with boundary conditions (5.38c). Now suppose that $\mathring{\vec{z}} \equiv 0$. Then $\mathring{\vec{u}}$ in equation (5.35b) is identically zero and equations

(5.36a) and (5.37a) (which define the TPBVP (5.38)), become

$$\frac{d}{d\theta}\left(\begin{bmatrix} \vec{\mathring{a}} \\ \vec{\mathring{b}} \end{bmatrix}\right) = \begin{bmatrix} \underline{A} & -\begin{bmatrix} 0 \\ B_b \end{bmatrix} R_b^{-1} \begin{bmatrix} 0 & B_b^* \end{bmatrix} \\ -\begin{bmatrix} C_a^* \\ C_a^* \end{bmatrix} \tilde{R}_b^{-1} \begin{bmatrix} C_a & C_b \end{bmatrix} & -\underline{A}^* \end{bmatrix} \begin{bmatrix} \vec{\mathring{a}} \\ \vec{\mathring{b}} \end{bmatrix}(\theta), \tag{5.42a}$$

$$\frac{d}{d\theta}\left(\begin{bmatrix} \vec{\mathring{e}} \\ \vec{\mathring{g}} \end{bmatrix}\right) = \begin{bmatrix} (A_a - B_a R_a^{-1} D_a^* C_a) & B_a R_a^{-1} B_a^* \\ C_a^* \tilde{R}_a^{-1} C_a & -(A_a - B_a R_a^{-1} D_a^* C_a)^* \end{bmatrix} \begin{bmatrix} \vec{\mathring{e}} \\ \vec{\mathring{g}} \end{bmatrix}(\theta), \tag{5.42b}$$

with boundary conditions

$$\vec{\mathring{a}}(0) = 0, \quad \vec{\mathring{b}}(h) = \mathsf{X}_\infty \vec{\mathring{a}}(h), \quad \vec{\mathring{e}}(0) = 0 \text{ and } \vec{\mathring{g}}(h) = -\mathsf{X}_a \vec{\mathring{e}}(h). \tag{5.42c}$$

Now consider the operator

$$\begin{bmatrix} \mathsf{S}^*\mathsf{S} & 0 \\ 0 & \mathsf{B}_a \mathsf{V}_a^2 \mathsf{B}_a^* \end{bmatrix} : \mathcal{X}_a \oplus \mathcal{X}_b \oplus \mathcal{X}_a \to \mathcal{X}_a \oplus \mathcal{X}_b \oplus \mathcal{X}_a,$$

where

$$\mathsf{S} := (\mathsf{I} + \mathsf{Y}\mathsf{Y}^*)^{-\frac{1}{2}} \begin{bmatrix} \mathsf{C}_a \quad \mathsf{C}_b \\ \mathsf{X}_\infty^{\frac{1}{2}} \begin{bmatrix} \mathsf{A}_a & 0 \\ 0 & \mathsf{A}_b \end{bmatrix} \end{bmatrix}$$

and $\mathsf{A}_\bullet$, $\mathsf{B}_\bullet$ and $\mathsf{C}_\bullet$ are defined in equations (5.16). It is readily verifiable that equations (5.42a) and (5.42b) with boundary conditions

$$\vec{\mathring{a}}(0) = x_1, \quad \vec{\mathring{b}}(h) = \mathsf{X}_\infty \vec{\mathring{a}}(h), \quad \vec{\mathring{e}}(0) = 0 \text{ and } \vec{\mathring{g}}(h) = -\mathsf{X}_a \vec{\mathring{e}}(h) + x_2, \tag{5.43}$$

characterise

$$\begin{bmatrix} \mathsf{S}^*\mathsf{S} & 0 \\ 0 & \mathsf{B}_a \mathsf{V}_a^2 \mathsf{B}_a^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \vec{\mathring{b}}(0) \\ \vec{\mathring{e}}(h) \end{bmatrix}. \tag{5.44}$$

Then since the boundary conditions (5.42c) are identical to the boundary conditions (5.43) when $x_1 = 0$ and $x_2 = 0$, it follows by linearity and equation (5.44) that $\vec{\mathring{b}}(0) = 0$ and $\vec{\mathring{e}}(h) = 0$. The boundary conditions (5.42c) then imply that $\vec{\mathring{e}}(0) = \vec{\mathring{g}}(h) = 0$ and hence, using equation (5.42b), that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \vec{\mathring{e}}(h) \\ \vec{\mathring{g}}(h) \end{bmatrix} = \mathbf{\Phi}_{\begin{bmatrix} (A_a - B_a R_a^{-1} D_a^* C_a) & B_a R_a^{-1} B_a^* \\ C_a^* \tilde{R}_a^{-1} C_a & -(A_a - B_a R_a^{-1} D_a^* C_a)^* \end{bmatrix}}(h,0) \begin{bmatrix} 0 \\ \vec{\mathring{g}}(0) \end{bmatrix}.$$

Since the state-transition operator

$$\mathbf{\Phi}\begin{bmatrix} (A_a - B_a R_a^{-1} D_a^* C_a) & B_a R_a^{-1} B_a^* \\ C_a^* \tilde{R}_a^{-1} C_a & -(A_a - B_a R_a^{-1} D_a^* C_a)^* \end{bmatrix}^{(t,\tau)}$$

is invertible for all t and τ, this gives $\vec{g}^{\circ}(0) = 0$. That is,

$$\vec{v}^{\circ}(0) = \begin{bmatrix} \vec{g}^{\circ}(0) \\ \vec{b}^{\circ}(0) \end{bmatrix} = 0,$$

which is a contradiction, since by the initial hypothesis $\vec{v}^{\circ}(0) \neq 0$. Hence, there must exist a non-zero $\vec{z}^{\circ}$ satisfying equation (5.34), which is the desired result. This is all summarised in the following lemma:

Lemma 5.10 *A real number $\sigma > 0$ is a singular value of $(\mathrm{I} + \mathsf{Y}\mathsf{Y}^*)^{-\frac{1}{2}}(\mathsf{W} - \mathsf{Y})\mathsf{V}_a$ if and only if*

$$\det\left(\check{\mathsf{E}}^{\sigma}_{22} - \begin{bmatrix} -\mathsf{X}_a & 0 \\ 0 & \mathsf{X}_{\infty} \end{bmatrix} \check{\mathsf{E}}^{\sigma}_{12}\right) = 0, \tag{5.45}$$

where $\check{\mathsf{E}}^{\sigma} =: \begin{bmatrix} \check{\mathsf{E}}^{\sigma}_{11} & \check{\mathsf{E}}^{\sigma}_{12} \\ \check{\mathsf{E}}^{\sigma}_{21} & \check{\mathsf{E}}^{\sigma}_{22} \end{bmatrix}$ is defined in equation (5.40).

By Lemma 5.10, it can be seen that testing the norm of $(\mathrm{I}+\mathsf{Y}\mathsf{Y}^*)^{-\frac{1}{2}}(\mathsf{W}-\mathsf{Y})\mathsf{V}_a$ and hence, condition (iii) of Theorem 5.6, is equivalent to testing the largest $\sigma > 0$ that satisfies condition (5.45). As a step towards testing the latter, recall that the set of singular values of a compact operator is discrete, with the only possible accumulation point being 0 [Kre89b]. Therefore, by Lemma 5.10, the zeros of

$$d(\mu) := \det\left(\check{\mathsf{E}}^{\frac{1}{\mu}}_{22} - \begin{bmatrix} -\mathsf{X}_a & 0 \\ 0 & \mathsf{X}_{\infty} \end{bmatrix} \check{\mathsf{E}}^{\frac{1}{\mu}}_{12}\right)$$

must constitute a discrete set with no finite accumulation points. Furthermore, note that $\check{\mathsf{E}}^{\frac{1}{\mu}}$ and hence $d(\mu)$, are analytic functions of μ. So with $\mathrm{wno}_r(d)$ defined to be the number of clockwise encirclements of the origin made by $d(\mu)$ as μ traverses the standard Nyquist contour of radius r,[6] the following corollary of Lemma 5.10 is a simple consequence of the Principle of the Argument [MH87].[7]

Corollary 5.11 *Condition (iii) of Theorem 5.6 is satisfied, or equivalently given $\gamma > 0$*

$$\|(\mathrm{I} + \mathsf{Y}\mathsf{Y}^*)^{-\frac{1}{2}}(\mathsf{W} - \mathsf{Y})\mathsf{V}_a\| < \gamma,$$

[6] A semi-circle (centred at the origin) into the right half plane and of radius r.

[7] Prof. Bassam Bamieh must be acknowledged for first suggesting the use of winding numbers in this context.

if and only if

$$\mathrm{wno}_{\frac{1}{\gamma}}(d) = 0.$$

Remark 5.12 Although it is not necessary to do so when computing the directed gap, it is possible to compute $\|(\mathsf{I} + \mathsf{YY}^*)^{-\frac{1}{2}}(\mathsf{W} - \mathsf{Y})\mathsf{V}_a\|$ to any desired accuracy using Corollary 5.11. That is, one could simply perform a bisection-search along $r \in \mathbb{R}^+/0$ to find the point at which $\mathrm{wno}_r(d)$ changes from 0 to a value greater than 0. ♣

Remark 5.13 As an aside, note that if X_a were equal to the 11-block of X_∞, then

$$\|(\mathsf{I} + \mathsf{YY}^*)^{-\frac{1}{2}}(\mathsf{W} - \mathsf{Y})\mathsf{V}_a\| < \gamma$$

would be equivalent to the gap between the two finite-horizon, linear operators W and Y being less than γ. As such, the techniques developed in this subsection could be used to compute the gap between the compression of (possibly time-varying) state-space systems. ♣

Remark 5.14 Although it is not explicitly shown here, the technique developed in this chapter could be used to compute the gap between an LTI, continuous-time controller and a low-pass filtered, sampled and held (sampled-data) approximation. The formulae would change slightly because it is not possible to express a sampled and held, LSI, discrete-time system directly in terms of a system of differential equations. However, given a state-space realisation of the LSI, discrete-time component of the sampled-data approximation (and of the low-pass filter usually used before the sampling device) it is possible to construct a state-space system of difference equations that characterises the evolution of the sampled-data approximation (in the time-lifted domain). Using such a state-space realisation, the results presented in Sub-Section 5.3.1 could be used to construct a normalised, right-coprime factorisation and hence, strong-right representation of the graph of the sampled-data approximation. The rest would then follow in exactly the same way. ♣

5.5 Summary

Exploiting the equivalence of testing a bound on the directed-gap between two LPTV systems and solving a particular LSI, full-information, $\ell^2_{\mathbb{Z}^+}(\cdot)$-synthesis problem, a procedure for computing the directed gap between LPTV systems that admit stabilisable and detectable, state-space realisations, is developed in this chapter. Much of the chapter is devoted to establishing computationally-tractable tests for a necessary and sufficient condition that characterises the existence of a solution to the equivalent synthesis problem, derived from the necessary and sufficient condition given in Proposition 2.22 for the generic LSI, full-information, $\ell^2_{\mathbb{Z}^+}(\cdot)$-synthesis problem. These tests can be used in a bisection-search procedure for computing the directed gap (and hence gap) to any desired accuracy.

The equivalent full-information, $\ell^2_{\mathbb{Z}+}(\cdot)$-synthesis problem associated with bounding the directed-gap involves a generalised plant constructed from isometric, strong-right representations of the time-lifted graphs of the two systems concerned. To this end, it is shown how to construct a state-space realisation of a strong-right representation of an LPTV system's time-lifted graph. This involves obtaining a normalised, right-coprime factorisation of the time-lifted-equivalent system concerned. A state-space realisation of the relevant coprime factors can be expressed explicitly in terms of the system's original, continuous-time, state-space realisation and the stabilising solution to an *algebraic*, Riccati equation.

6

H$^\infty$ *SD Synthesis and Related Numerical Issues*

6.1 Introduction

This chapter is basically an extension of Section 2.5. All of the material presented in Section 2.5 (including specific state-space realisations defined there) is used in this chapter without re-statement. As such, it is suggested that the reader be familiar with that section before continuing.

Consider the control-system configuration shown in Figure 6.1, where G is an LTI, continuous-time generalised-plant, K is an LSI, discrete-time controller, $\mathbf{A_D}$ is an ideal, periodic (period h) sampling-device and $\mathbf{D_A}$ is a zeroth-order hold-device synchronised with $\mathbf{A_D}$. The task of synthesis-

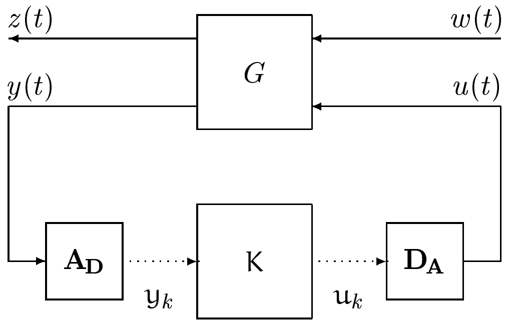

Figure 6.1 *SD Control System*

ing an internally-stabilising controller K that achieves some bound on the $\mathrm{L}^2_{\mathbb{R}+}$-induced norm of the closed-loop operator $\mathbf{F}_\ell(G, \mathbf{D_A}\mathsf{K}\mathbf{A_D})$, mapping w to z, is complicated by the mixture of continuous-time and discrete-time signals and the fact that the closed-loop is periodically time-varying. Recall however, as discussed in Section 2.3.2, that periodically-time-varying, continuous-time systems are equivalent, via the time-lifting isomorphism, to shift-invariant systems that evolve in discrete-time.

It is therefore not surprising, that the LPTV, SD synthesis problem can be restated as an equivalent LSI problem in discrete-time (cf. Section 2.5 for the details of this). Although this equivalent problem involves a mixture of finite-dimensional and infinite-dimensional input and output spaces, there exist various methods for determining a completely finite-dimensional equivalent (cf. Proposition 2.28 for example and [BP92, Toi92, CF95, KH93, Toi93]). The main purpose of this chapter is to present a new, compact derivation of state-space formulae for constructing a state-space realisation of the SD-equivalent generalised-plant defined in Proposition 2.28 (cf. also Remark 2.29). Furthermore, numerical issues are discussed and numerically robust formulae obtained.

Perhaps the most simple explicit formulae for the state-space matrices of the finite-dimensional, equivalent, discrete-time, synthesis problem are given in [Toi93] and involve computation of only a single matrix exponential. In [Toi93] a finite-dimensional, discrete-time, equivalent problem is derived without using time-lifting. Instead classical, linear-quadratic, optimal-control theory is used to give a closed-loop characterisation of the worst-case, inter-sample disturbance. The relationship between the time-lifting and the linear-quadratic, optimal control approaches is examined in [TS97]. There, it is noted that the time-lifting based approach can be interpreted as consisting of a lifting transformation followed by an open-loop characterisation of the worst-case, inter-sample disturbance. In this way the two methods were said to be equivalent, although expressions for the equivalent, finite-dimensional, discrete-time problem were different. In this chapter, using only the time-lifting technique, essentially identical (except for the accommodation of a direct feed-through term D_{11} here) state-space formulae to those obtained by [Toi93] are developed and hence, the equivalence of the two approaches follows directly. The formulation developed here involves solving a set of TPBVPs, which are used to characterise the behaviour of certain state-space operators that describe the infinite-dimensional, time-lifted, discrete-time, equivalent problem (cf. Proposition 2.24). These TPBVPs lead to state-space formulae that involve the computation of only a single, finite-dimensional matrix-exponential.

The required matrix exponentiation is of a Hamiltonian structured matrix expressible explicitly in terms of the state-space matrices of the original generalised-plant. Since the eigenvalues of a Hamiltonian matrix are symmetric about the imaginary axis, direct computation of its exponential can be dominated by large terms if any of the eigenvalues have large real part. These large terms can cancel or become small in the formulae used to determine the state-space matrices of the finite-dimensional, equivalent problem. Hence, without care the formulae initially derived can be numerically sensitive. With this in mind, it is shown here how to re-structure the state-space formulae to give a numerically robust procedure for computation.

Briefly, the structure of the chapter is as follows: in Section 6.2 a new, compact derivation of formulae for constructing a state-space realisation of the SD-equivalent generalised-plant defined in Proposition 2.28, is presented; in Section 6.3, it is shown how to re-structure the new formulae for numerical robustness; in Section 6.4, two assumptions required to make this possible are discussed; finally, a brief summary is given in Section 6.5.

6.2 State-Space Formulae for the SD-Equivalent Generalised-Plant

The purpose of the section is to present a new, compact derivation of formulae for computing the state-space matrices of the SD-equivalent generalised-plant $\breve{\mathrm{G}}$, defined in Proposition 2.28 (cf. Section 2.5 of Chapter 2). The formulae are expressible explicitly in terms of the state-space matrices of the original generalised-plant G (cf. equation (2.17)) and are similar to those presented in [Toi93, TS97]. Finite-dimensional expressions for the state-space matrices of $\breve{\mathrm{G}}$ are constructed by solving a set of TPBVPs, which characterise the combination of integral operator expressions used in the state-space description of $\breve{\mathrm{G}}$ in Proposition 2.28. The formulae derived accommodate direct feedthrough from the disturbance inputs w to the controlled outputs z (that is, $D_{11} \neq 0$).

It follows from equations (2.26–2.37), that in order to explicitly determine the state-space matrices of $\breve{\mathrm{G}}$, finite-dimensional expressions are required for

$$\gamma^{-2}\mathrm{B}_1(\mathrm{I}-\gamma^{-2}\mathrm{D}_{11}^*\mathrm{D}_{11})^{-1}\mathrm{B}_1^*, \tag{6.1}$$

$$\gamma^2\begin{bmatrix} \mathrm{C}_1^* \\ \mathrm{D}_{12}^* \end{bmatrix}(\mathrm{I}-\gamma^{-2}\mathrm{D}_{11}\mathrm{D}_{11}^*)^{-1}\begin{bmatrix} \mathrm{C}_1 & \mathrm{D}_{12} \end{bmatrix} \tag{6.2}$$

and

$$\gamma^{-2}\mathrm{B}_1\mathrm{D}_{11}^*(\mathrm{I}-\gamma^{-2}\mathrm{D}_{11}\mathrm{D}_{11}^*)^{-1}\begin{bmatrix} \mathrm{C}_1 & \mathrm{D}_{12} \end{bmatrix}, \tag{6.3}$$

which can be obtained as described below. Given $\gamma > \|\mathrm{D}_{11}\|$, define

$$\grave{Q} := e^{h\grave{E}} := \exp\left\{h\begin{bmatrix} -\grave{A}^* & -\grave{C}^*\grave{C} \\ \grave{B}\grave{B}^* & \grave{A} \end{bmatrix}\right\}, \tag{6.4}$$

where

$$\begin{aligned}
\grave{A} &:= \bar{A}+\gamma^{-2}\begin{bmatrix} B_1 \\ 0 \end{bmatrix}R^{-1}D_{11}^*\begin{bmatrix} C_1 & D_{12} \end{bmatrix}, \\
\grave{C}^*\grave{C} &:= \begin{bmatrix} C_1^* \\ D_{12}^* \end{bmatrix}S^{-1}\begin{bmatrix} C_1 & D_{12} \end{bmatrix}, \\
\grave{B}\grave{B}^* &:= \gamma^{-2}\begin{bmatrix} B_1 \\ 0 \end{bmatrix}R^{-1}\begin{bmatrix} B_1^* & 0 \end{bmatrix}
\end{aligned}$$

$$\bar{A} \;:=\; \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}. \tag{6.5}$$

$S := (I - \gamma^{-2} D_{11} D_{11}^*)$ and $R := (I - \gamma^{-2} D_{11}^* D_{11})$. Partition $\grave{Q} =: \begin{bmatrix} \grave{Q}_{11} & \grave{Q}_{12} \\ \grave{Q}_{21} & \grave{Q}_{22} \end{bmatrix}$ conformably with $\grave{E}$.

Theorem 6.1 *The state-space matrices corresponding to the state-space realisation of the SD-equivalent generalised-plant* $\breve{\mathsf{G}}$ *given in equation (2.35), can be characterised as follows:*

$$\tilde{\mathsf{A}} = \begin{bmatrix} I_n & 0 \end{bmatrix} (\grave{Q}_{11}^*)^{-1} \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \tag{6.6}$$

$$\tilde{\mathsf{B}}_2 = \begin{bmatrix} I_n & 0 \end{bmatrix} (\grave{Q}_{11}^*)^{-1} \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}, \tag{6.7}$$

$\breve{\mathsf{B}}_1$ *a matrix satisfying*

$$\breve{\mathsf{B}}_1 \breve{\mathsf{B}}_1^* = \begin{bmatrix} I_n & 0 \end{bmatrix} \grave{Q}_{21} \grave{Q}_{11}^{-1} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \tag{6.8}$$

and $\breve{\mathsf{C}}_1$ *and* $\breve{\mathsf{D}}_{12}$ *matrices satisfying*

$$\begin{bmatrix} \breve{\mathsf{C}}_1^* \\ \breve{\mathsf{D}}_{12}^* \end{bmatrix} \begin{bmatrix} \breve{\mathsf{C}}_1 & \breve{\mathsf{D}}_{12} \end{bmatrix} = -\gamma^2 \grave{Q}_{11}^{-1} \grave{Q}_{12}. \tag{6.9}$$

Proof : By Proposition 2.28, it is required to determine finite-dimensional expressions for the operators given in equations (6.1), (6.2) and (6.3). Consider the first operator

$$\gamma^{-2} \mathsf{B}_1 (\mathsf{I} - \gamma^{-2} \mathsf{D}_{11}^* \mathsf{D}_{11})^{-1} \mathsf{B}_1^* := z \mapsto x,$$

where $z \in \mathbb{R}^n$. Define

$$\begin{aligned} \mathring{\vec{y}} \;&:=\; \gamma^{-1} \mathsf{B}_1^* z, && (6.10\text{a}) \\ \mathring{\vec{r}} \;&:=\; \gamma^{-1} \mathsf{D}_{11} \mathring{\vec{u}} && (6.10\text{b}) \end{aligned}$$

and

$$\begin{aligned} \mathring{\vec{u}} \;&:=\; (\mathsf{I} - \gamma^{-2} \mathsf{D}_{11}^* \mathsf{D}_{11})^{-1} \mathring{\vec{y}}, && (6.10\text{c}) \\ &=\; \gamma^{-1} \mathsf{D}_{11}^* \mathring{\vec{r}} + \gamma^{-1} \mathsf{B}_1^* z, && (6.10\text{d}) \end{aligned}$$

where equation (6.10d) follows from equations (6.10b) and (6.10c). Then clearly, $x = \gamma^{-1} \mathsf{B}_1 \mathring{\vec{u}}$. Now from equations (2.24) and (6.10b) it follows that for $\theta \in \mathbb{H}$

$$\begin{aligned} \frac{d\mathring{\vec{a}}}{d\theta} \;&=\; A \mathring{\vec{a}}(\theta) + \gamma^{-1} B_1 \mathring{\vec{u}}(\theta); \qquad \mathring{\vec{a}}(0) = 0, && (6.11\text{a}) \\ \mathring{\vec{r}}(\theta) \;&=\; C_1 \mathring{\vec{a}}(\theta) + \gamma^{-1} D_{11} \mathring{\vec{u}}(\theta). && (6.11\text{b}) \end{aligned}$$

Furthermore, using equations (2.21) and (2.24), the final-condition response of

$$\begin{aligned}\frac{d\mathring{\vec{e}}}{d\theta} &= -A^*\mathring{\vec{e}}(\theta) - C_1^*\mathring{\vec{r}}(\theta); \qquad \mathring{\vec{e}}(h) = z, &(6.12a)\\ \mathring{\vec{u}}(\theta) &= \gamma^{-1}B_1^*\mathring{\vec{e}}(\theta) + \gamma^{-1}D_{11}^*\mathring{\vec{r}}(\theta), &(6.12b)\end{aligned}$$

gives $\gamma^{-1}\mathrm{B}_1^* z$ and the zero, final-condition response gives $\gamma^{-1}\mathrm{D}_{11}^*\mathring{\vec{r}}$, which is precisely equation (6.10d). Then from equations (2.21) and (6.11),

$$x = \gamma^{-1}\mathrm{B}_1\mathring{\vec{u}} = \mathring{\vec{a}}(h).$$

Combining equations (6.11) and (6.12) yields a TPBVP. It follows from the boundary conditions, that

$$\begin{bmatrix} z \\ \mathring{\vec{a}}(h) \end{bmatrix} = \begin{bmatrix} \mathring{\vec{e}}(h) \\ \mathring{\vec{a}}(h) \end{bmatrix} = Q(h)\begin{bmatrix} \mathring{\vec{e}}(0) \\ 0 \end{bmatrix}, \tag{6.13}$$

where $Q(t)$ is defined in equation (2.34). Hence,[1]

$$\mathring{\vec{a}}(h) = Q_{21}(h)Q_{11}(h)^{-1}z.$$

That is, $Q_{21}(h)Q_{11}(h)^{-1}$ constitutes a finite-dimensional expression for the first operator. Now by the structure of $\grave{E}$ it follows that

$$Q_{21}(h)Q_{11}(h)^{-1} = \begin{bmatrix} I_n & 0 \end{bmatrix}\grave{Q}_{21}\grave{Q}_{11}^{-1}\begin{bmatrix} I_n \\ 0 \end{bmatrix},$$

which is equation equation (6.8), as claimed.

Before continuing, a well-known result from matrix theory is required [VL78]. Let M_{11} and M_{22} be square matrices, then for $t \geq 0$

$$\exp\left\{t\begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22}\end{bmatrix}\right\} = \begin{bmatrix} e^{tM_{11}} & \grave{M}_{12} \\ 0 & e^{tM_{22}}\end{bmatrix}, \tag{6.14}$$

where $\grave{M}_{12} = \int_0^t e^{(t-\tau)M_{11}}M_{12}e^{\tau M_{22}}\,d\tau$. Now consider the second and third operators

$$\gamma^2\begin{bmatrix} \mathrm{C}_1^* \\ \mathrm{D}_{12}^* \end{bmatrix}(\mathrm{I} - \gamma^{-2}\mathrm{D}_{11}\mathrm{D}_{11}^*)^{-1}\begin{bmatrix} \mathrm{C}_1 & \mathrm{D}_{12}\end{bmatrix} := z \mapsto x_1$$

and

$$\gamma^{-2}\mathrm{B}_1\mathrm{D}_{11}^*(\mathrm{I} - \gamma^{-2}\mathrm{D}_{11}\mathrm{D}_{11}^*)^{-1}\begin{bmatrix} \mathrm{C}_1 & \mathrm{D}_{12}\end{bmatrix} := z \mapsto x_2$$

[1] Recall that by Remark 2.26, $Q_{11}(h)$ is invertible.

respectively. Define $\mathring{v} := \begin{bmatrix} \mathsf{C}_1 & \mathsf{D}_{12} \end{bmatrix} z$, for some $z \in \mathbb{R}^{n+m_2}$. It then follows from equations (2.22), (2.25) and (6.14), that for $\theta \in \mathbb{H}$

$$\mathring{v}(\theta) = \begin{bmatrix} C_1 & D_{12} \end{bmatrix} e^{\theta \bar{A}} z, \tag{6.15}$$

where $\bar{A}$ is defined in equation (6.5). Now define

$$\mathring{g} := (\mathrm{I} - \gamma^{-2} \mathsf{D}_{11} \mathsf{D}_{11}^*)^{-1} \mathring{v}, \tag{6.16}$$

so that

$$x_1 = \gamma^2 \begin{bmatrix} \mathsf{C}_1^* \\ \mathsf{D}_{12}^* \end{bmatrix} \mathring{g} \tag{6.17}$$

and

$$x_2 = \gamma^{-2} \mathsf{B}_1 \mathsf{D}_{11}^* \mathring{g}. \tag{6.18}$$

With

$$\mathring{w}(\theta) := \gamma^{-1} \mathsf{D}_{11}^* \mathring{g}(\theta),$$

it follows from equation (2.24) that for $\theta \in \mathbb{H}$

$$\begin{aligned} \frac{d\mathring{m}}{d\theta} &= -\bar{A}^* \mathring{m}(\theta) - \begin{bmatrix} C_1^* \\ D_{12}^* \end{bmatrix} \mathring{g}(\theta); \qquad \mathring{m}(h) = 0, && (6.19a) \\ \mathring{w}(\theta) &= \begin{bmatrix} \gamma^{-1} B_1^* & 0 \end{bmatrix} \mathring{m}(\theta) + \gamma^{-1} D_{11}^* \mathring{g}(\theta). && (6.19b) \end{aligned}$$

Furthermore, from equations (6.16) and (6.15),

$$\begin{aligned} \mathring{g}(\theta) &= \mathring{v}(\theta) + \gamma^{-1} (\mathsf{D}_{11} \mathring{w})(\theta) \\ &= \begin{bmatrix} C_1 & D_{12} \end{bmatrix} e^{\theta \bar{A}} z + \gamma^{-1} (\mathsf{D}_{11} \mathring{w})(\theta). \end{aligned}$$

Again, using equation (2.24) this can be expressed as

$$\begin{aligned} \frac{d\mathring{n}}{d\theta} &= \bar{A} \mathring{n}(\theta) + \begin{bmatrix} \gamma^{-1} B_1 \\ 0 \end{bmatrix} \mathring{w}(\theta); \qquad \mathring{n}(0) = z, && (6.20a) \\ \mathring{g}(\theta) &= \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \mathring{n}(\theta) + \gamma^{-1} D_{11} \mathring{w}(\theta), && (6.20b) \end{aligned}$$

where the zero, initial-condition response gives $\gamma^{-1} \mathsf{D}_{11} \mathring{w}$ and the initial-condition response gives $\begin{bmatrix} C_1 & D_{12} \end{bmatrix} e^{\theta \bar{A}} z$. The systems (6.20) and (6.19) form a TPBVP, from which it follows that

$$\begin{bmatrix} 0 \\ \mathring{n}(h) \end{bmatrix} = \begin{bmatrix} \mathring{m}(h) \\ \mathring{n}(h) \end{bmatrix} = \grave{Q} \begin{bmatrix} \mathring{m}(0) \\ z \end{bmatrix}, \tag{6.21}$$

where $\grave{Q}$ is defined in equation (6.4). Now from equations (6.17), (6.18), (6.19) and (6.20),

$$x_1 = \gamma^2 \begin{bmatrix} \mathsf{C}_1^* \\ \mathsf{D}_{12}^* \end{bmatrix} \vec{g} = \gamma^2 \vec{m}(0), \tag{6.22}$$

and

$$x_2 = \gamma^{-2} \mathsf{B}_1 \mathsf{D}_{11}^* \vec{g} = \gamma^{-1} \mathsf{B}_1 \vec{w} = \begin{bmatrix} I_n & 0 \end{bmatrix} \{ \vec{n}(h) - e^{h\bar{A}} z \}. \tag{6.23}$$

Recall the partitioning of $\grave{Q}$. Then from equation (6.21), $\vec{m}(0) = -\grave{Q}_{11}^{-1} \grave{Q}_{12} z$ and combining this with equation (6.22) characterises a finite-dimensional expression for the second operator. Similarly,

$$\vec{n}(h) = \{ \grave{Q}_{22} - \grave{Q}_{21} \grave{Q}_{11}^{-1} \grave{Q}_{12} \} z. \tag{6.24}$$

Define

$$J = \begin{bmatrix} 0 & -I_{n+m_2} \\ I_{n+m_2} & 0 \end{bmatrix}, \tag{6.25}$$

noting that $-\grave{E}^* = J \grave{E} J^{-1}$ ($\grave{E}$ is Hamiltonian), so that

$$(\grave{Q}^*)^{-1} = e^{-h\grave{E}^*} = J \grave{Q} J^{-1}.$$

That is, $\grave{Q}^* J \grave{Q} J^{-1} = J \grave{Q} J^{-1} \grave{Q}^* = I$, which yields the relationship

$$\grave{Q}_{22} - \grave{Q}_{21} \grave{Q}_{11}^{-1} \grave{Q}_{12} = (\grave{Q}_{11}^*)^{-1}.$$

Combining this with equations (6.23) and (6.24) then gives a finite-dimensional expression for the third operator. Furthermore, from equation (2.27),

$$\begin{aligned} \tilde{\mathsf{A}} &= e^{hA} + \begin{bmatrix} I_n & 0 \end{bmatrix} \{ (\grave{Q}_{11}^*)^{-1} - e^{h\bar{A}} \} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} I_n & 0 \end{bmatrix} (\grave{Q}_{11}^*)^{-1} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \end{aligned}$$

and from equation (2.29),

$$\begin{aligned} \tilde{\mathsf{B}}_2 &= \int_0^h e^{(h-t)A} dt B_2 + \begin{bmatrix} I_n & 0 \end{bmatrix} \{ (\grave{Q}_{11}^*)^{-1} - e^{h\bar{A}} \} \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix} \\ &= \begin{bmatrix} I_n & 0 \end{bmatrix} (\grave{Q}_{11}^*)^{-1} \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}. \end{aligned}$$

To see that $\grave{Q}_{11}$ is invertible, note that from equations (6.21) and (6.14),

$$\grave{Q}_{11} = \begin{bmatrix} Q_{11}(h) & 0 \\ ? & I_{m_2} \end{bmatrix},$$

where Q_{11} is defined as before and "?" denotes an irrelevant sub-matrix. Thus, $\grave{Q}_{11}$ is invertible if and only if Q_{11} is, which is the case by Remark 2.26. This completes the proof. ∎

6.3 Re-Structuring for Numerical Robustness

In this section, the state-space formulae presented in Theorem 6.1 are re-structured to avoid numerical problems that can arise when calculating the matrix exponential defined there.

As detailed in Theorem 6.1, computing a state-space realisation of $\breve{\mathrm{G}}$ involves taking the matrix exponential of the Hamiltonian-structured matrix $\grave{E}$. Since $\grave{E}$ has Hamiltonian structure, its eigenvalues are symmetric about the imaginary axis. Therefore, if $\grave{E}$ has eigenvalues with large (with respect to the sampling frequency), positive real-part, the computation of $\grave{Q} := e^{h\grave{E}}$ can be numerically sensitive, with the calculation being dominated by large terms.[2] These large terms cancel or become very small in equations (6.6-6.9), which characterise a state-space realisation of $\breve{\mathrm{G}}$, making any procedure based on these equations directly, numerically sensitive on the whole. In what follows, equation (6.6-6.9) are re-structured to avoid direct computation of the full matrix exponential $e^{h\grave{E}}$. The aim of the re-structuring is to restrict all matrix exponential calculations to involve only matrices with all eigenvalues having real part less than or equal to some number of order 1. By similar argument, the following re-structuring procedure should also be adopted to modify the formulae given in Lemma 2.25, which can be used to compute bounds on $\|\mathrm{D}_{11}\|$ (cf. Corollary 2.27).

First some fundamental results from matrix theory [GVL89].

Lemma 6.2 (Schur Decomposition) *If $A \in \mathbb{C}^{n\times n}$ then there exists a unitary matrix $Q \in \mathbb{C}^{n\times n}$ such that*

$$Q^*AQ = T = D + N,$$

where $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ and $N \in \mathbb{C}^{n\times n}$ is strictly upper triangular. Furthermore, Q can be chosen so that the eigenvalues λ_i appear in any order along the diagonal.

Lemma 6.3 (Block Diagonalisation) *Let $T \in \mathbb{C}^{n\times n}$ be partitioned as follows:*

$$T = \begin{array}{cc} \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} & \begin{array}{c} p \\ q \end{array} \\ \begin{array}{cc} \;\; p & \quad q \end{array} & \end{array} . \tag{6.26}$$

Define the linear transformation $\phi : \mathbb{C}^{p\times q} \to \mathbb{C}^{p\times q}$ by

$$\phi(X) = T_{11}X - XT_{22},$$

[2] For example, this situation could occur if the original generalised-plant G has stable or unstable modes that are fast relative to the sampling period.

where $X \in \mathbb{C}^{p\times q}$. *Then* ϕ *is non-singular if and only if* $\mathrm{spec}(T_{11}) \cap \mathrm{spec}(T_{22}) = \emptyset$. *If* ϕ *is non-singular and* Y *is defined by*

$$Y = \begin{bmatrix} I_p & Z \\ 0 & I_q \end{bmatrix}, \qquad \phi(Z) = -T_{12},$$

then $Y^{-1}TY = \mathrm{diag}(T_{11}, T_{22})$.

Remark 6.4 Lemma 6.3 has a natural extension via an inductive argument to block, upper triangular matrices with finer partitioning than shown in the lemma. ♣

Let $M \in \mathbb{C}^{2(n+m_2)\times 2(n+m_2)}$, be an invertible matrix such that

$$\grave{E} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} \end{bmatrix}, \tag{6.27}$$

where $\tilde{M} := M^{-1}$,

$$\begin{aligned} \mathrm{spec}(T_{11}) &\subset \mathbb{C}_{>\alpha} := \{s : \mathrm{Re}(s) > \alpha\}, \\ \mathrm{spec}(T_{22}) &\subset \bar{\mathbb{C}}_{\leq\alpha} := \{s : \mathrm{Re}(s) \leq \alpha\}, \end{aligned}$$

$\alpha \in [0, l/h]$ (l of order 1) and the matrices M and $\tilde{M}$ are partitioned conformably with $\grave{E}$. That such a matrix exists follows by Lemmas 6.2 and 6.3. Decomposing the matrix $\grave{E}$ into its spectral components involves first obtaining an ordered Schur decomposition, the accuracy of which, in certain circumstances, can be improved by balancing the original matrix [WR71]. The balancing process involves applying similarity transformations to the original matrix so that corresponding rows and columns have approximately the same vector norms. As implemented in MATLAB[3], the similarity transformations can often be ill-conditioned, especially when there are both relatively very large and very small, non-zero elements in the original matrix [MAT92]. Since the computation of $\grave{E}$ is likely to be embedded in an iterative synthesis or norm-analysis procedure, which could give rise to very small entries that are supposed to be zero, the bad effects of using ill-conditioned balancing transformations are readily observable. As such, caution should be exercised when using such transformations. Also, $\grave{E}$ involves "squared" terms like $\grave{B}\grave{B}^*$ and $\grave{C}^*\grave{C}$, which can accentuate differences between the size of elements in $\grave{B}$ and those in $\grave{C}$. It is therefore recommended that the original system G be scaled prior to forming $\grave{E}$, so that corresponding rows and columns of the state-space matrices are of comparable size in norm. Such an algorithm (`syscl`) is implemented in the μ-tools[4] tool-box of MATLAB. Having performed the Schur Decomposition the next step required

[3]Copyright Maths Works Inc.
[4]Copyright Musyn, 1993.

to obtain the decomposition in equation (6.27) is a Block Triangularisation. The choice of the real Schur form obtained in the previous step and its partitioning determines the sensitivity of the Sylvester equations that must be solved in the diagonalisation process. There can be substantial loss of accuracy whenever the spectral sets $\mathrm{spec}(T_{ii})$ are insufficiently separated [GNVL79]. However, algorithms exist which dynamically determine the partitioning to limit the loss of accuracy [BS79]. Before going on to show how this spectral decomposition can be used to restructure the state-space formulae derived in the previous section, the following technical result is required.

Lemma 6.5 *M_{11} has full column-rank if (A, C_1) has no unobservable modes in $\mathbb{C}_{>\alpha}$. Similarly, $\tilde{M}_{11}$ has full row-rank if (A, B_1) has no uncontrollable modes in $\mathbb{C}_{>\alpha}$.*

Proof : From equation (6.27) it follows that

$$\grave{E}\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} = \begin{bmatrix} -\grave{A}^* & -\grave{C}^*\grave{C} \\ \grave{B}\grave{B}^* & \grave{A} \end{bmatrix}\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} = \begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} T_{11}.$$

Since T_{11} has full rank and is square with all eigenvalues having real part strictly greater than α, the proof that M_{11} has full column rank if (A, C_1) has no unobservable modes in $\mathbb{C}_{>\alpha}$, is roughly the same as the proof that the Algebraic Riccati equation

$$A'X + XA + XRX + Q = 0,$$

has stabilising solution if (A, R) is stabilisable (cf. [ZDG95, Theorem 13.6]). See Appendix D.1 for the full proof. The proof that $\tilde{M}_{11}$ has full row rank if (A, B_1) has no uncontrollable modes in $\mathbb{C}_{>\alpha}$, follows similarly. ∎

In view of Lemma 6.5, the following assumptions are made:

(A1) (A, B_1) has no uncontrollable modes in $\mathbb{C}_{>\alpha}$;

(A2) (A, C_1) has no unobservable modes in $\mathbb{C}_{>\alpha}$.

For the sake of continuity, justification of these assumptions is postponed to the next section.

With the matrix $\grave{E}$ decomposed into its $\mathbb{C}_{>\alpha}$ and $\mathbb{C}_{\leq\alpha}$ spectral components, it is now possible to restructure the formulae given in Theorem 6.1, to avoid direct computation of the matrix exponential $e^{hT_{11}}$. Let p denote the dimension of T_{11} and $r := n + m_2 - p$. Since under assumptions (A1) and (A2) the matrix M_{11} has full column rank and $\tilde{M}_{11}$ has full row rank, the following invertible matrices are well-defined:

$$\begin{bmatrix} M_{11}^L \\ M_{11}^\perp \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{M}_{11}^R & \tilde{M}_{11}^\perp \end{bmatrix},$$

such that

$$\begin{bmatrix} M_{11}^L \\ M_{11}^\perp \end{bmatrix} M_{11} = \begin{bmatrix} I_p \\ 0_{r,p} \end{bmatrix} \quad \text{and} \quad \tilde{M}_{11} \begin{bmatrix} \tilde{M}_{11}^R & \tilde{M}_{11}^\perp \end{bmatrix} = \begin{bmatrix} I_p & 0_{p,r} \end{bmatrix},$$

where $0_{i,j}$ denotes an $i \times j$ matrix of zeros. Now define the following matrices:

$$\begin{aligned}
\Omega_L &:= \left(\begin{bmatrix} e^{-hT_{11}} M_{11}^L \\ M_{11}^\perp \end{bmatrix} M_{12} e^{hT_{22}} \tilde{M}_{21} + \begin{bmatrix} I_p \\ 0_{r,p} \end{bmatrix} \tilde{M}_{11} \right)^{-1}, \\
\Omega_R &:= \left(M_{11} \begin{bmatrix} I_p & 0_{p,r} \end{bmatrix} + M_{12} e^{hT_{22}} \tilde{M}_{21} \begin{bmatrix} \tilde{M}_{11}^R e^{-hT_{11}} & \tilde{M}_{11}^\perp \end{bmatrix} \right)^{-1}, \\
\Psi &:= \begin{bmatrix} \tilde{M}_{11}^R e^{-hT_{11}} & \tilde{M}_{11}^\perp \end{bmatrix} \Omega_R, && (6.28) \\
\Phi &:= \left(M_{22} e^{hT_{22}} \tilde{M}_{21} \begin{bmatrix} \tilde{M}_{11}^R e^{-hT_{11}} & \tilde{M}_{11}^\perp \end{bmatrix} + M_{21} \begin{bmatrix} I_p & 0_{p,r} \end{bmatrix} \right) \Omega_R && (6.29)
\end{aligned}$$

and

$$\Upsilon := \Omega_L \left(\begin{bmatrix} e^{-hT_{11}} M_{11}^L \\ M_{11}^\perp \end{bmatrix} M_{12} e^{hT_{22}} \tilde{M}_{22} + \begin{bmatrix} I_p \\ 0_{r,p} \end{bmatrix} \tilde{M}_{12} \right). \tag{6.30}$$

Note that none of the matrices above involve the exponential of a matrix with eigenvalues having real part greater than l, a number of order 1.

Theorem 6.6 *The state-space matrices of the SD-equivalent generalised-plant* $\breve{\mathsf{G}}$ *are given by:*

$$\tilde{\mathsf{A}} = \begin{bmatrix} I_n & 0 \end{bmatrix} \Psi^* \begin{bmatrix} I_n \\ 0 \end{bmatrix},$$

$$\tilde{\mathsf{B}}_2 = \begin{bmatrix} I_n & 0 \end{bmatrix} \Psi^* \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix},$$

$\breve{\mathsf{B}}_1$ *a matrix satisfying* $\breve{\mathsf{B}}_1 \breve{\mathsf{B}}_1^* = \begin{bmatrix} I_n & 0 \end{bmatrix} \Phi \begin{bmatrix} I_n \\ 0 \end{bmatrix}$ *and*

$\breve{\mathsf{C}}_1$ *and* $\breve{\mathsf{D}}_{12}$ *matrices satisfying* $\begin{bmatrix} \breve{\mathsf{C}}_1^* \\ \breve{\mathsf{D}}_{12}^* \end{bmatrix} \begin{bmatrix} \breve{\mathsf{C}}_1 & \breve{\mathsf{D}}_{12} \end{bmatrix} = -\gamma^2 \Upsilon.$

Proof : Note from equation (6.27) that

$$\grave{Q}_{11}^{-1} \left(M_{11} e^{hT_{11}} \tilde{M}_{11} + M_{12} e^{hT_{22}} \tilde{M}_{21} \right) = I.$$

Post-multiplying both sides by $\begin{bmatrix} \tilde{M}_{11}^R e^{-hT_{11}} & \tilde{M}_{11}^\perp \end{bmatrix}$ gives

$$\grave{Q}_{11}^{-1} \Omega_R^{-1} = \begin{bmatrix} \tilde{M}_{11}^R e^{-hT_{11}} & \tilde{M}_{11}^\perp \end{bmatrix},$$

and hence from equation (6.28), $\Psi = \grave{Q}_{11}^{-1}$.

Let $\Phi = \grave{Q}_{21}\grave{Q}_{11}^{-1}$, and note from equation (6.27) that

$$\Phi\left(M_{11}e^{hT_{11}}\tilde{M}_{11} + M_{12}e^{hT_{22}}\tilde{M}_{21}\right) = \left(M_{21}e^{hT_{11}}\tilde{M}_{11} + M_{22}e^{hT_{22}}\tilde{M}_{21}\right).$$

Post-multiplying both sides by $\begin{bmatrix} \tilde{M}_{11}^{R}e^{-hT_{11}} & \tilde{M}_{11}^{\perp} \end{bmatrix}$ gives

$$\Phi\Omega_R^{-1} = \left(M_{22}e^{hT_{22}}\tilde{M}_{21}\begin{bmatrix} \tilde{M}_{11}^{R}e^{-hT_{11}} & \tilde{M}_{11}^{\perp} \end{bmatrix} + M_{21}\begin{bmatrix} I_p & 0_{p,r} \end{bmatrix}\right),$$

which is equation (6.29).

Similarly, let $\Upsilon = \grave{Q}_{11}^{-1}\grave{Q}_{12}$, and note from equation (6.27) that

$$\left(M_{11}e^{hT_{11}}\tilde{M}_{11} + M_{12}e^{hT_{22}}\tilde{M}_{21}\right)\Upsilon = \left(M_{11}e^{hT_{11}}\tilde{M}_{12} + M_{12}e^{hT_{22}}\tilde{M}_{22}\right).$$

Pre-multiplying both sides by $\begin{bmatrix} e^{-hT_{11}}M_{11}^{L} \\ M_{11}^{\perp} \end{bmatrix}$ gives

$$\Omega_L^{-1}\Upsilon = \left(\begin{bmatrix} e^{-hT_{11}}M_{11}^{L} \\ M_{11}^{\perp} \end{bmatrix}M_{12}e^{hT_{22}}\tilde{M}_{22} + \begin{bmatrix} I_p \\ 0_{r,p} \end{bmatrix}\tilde{M}_{12}\right),$$

which is equation (6.30). ∎

6.4 Comments Concerning Assumptions (A1) and (A2)

Under assumptions (A1) and (A2), the matrices M_{11} and $\tilde{M}_{11}$ have full column and full row rank respectively. This is required to enable re-structuring of the formulae given in Theorem 6.1. In this section, it is argued that the assumptions (A1) and (A2) are reasonable.

There are two cases in which the assumptions can be violated. The first case is when any unstable mode, which is unobservable (and/or uncontrollable) from the controlled outputs (disturbance inputs), has a time-constant greater than or equal to the sampling period divided by some number l, of order 1. The second case is when any unstable mode, which is unobservable (and/or uncontrollable) from the controlled outputs (disturbance inputs), has time constant less than the sampling period divided by some number l, of order 1.

In the first case, it is sufficient to simply choose α to be larger than the real part of the unobservable (and/or uncontrollable) mode. In this case such an α will be less than or equal to l/h and therefore, lie in the required interval $[0, l/h]$.

Remark 6.7 With this in mind, a suitable choice for α is the smallest number in $[0, l/h]$, such that (A, C_1) and (A, B_1) have no unobservable and uncontrollable modes respectively, in the set $\{s : \alpha < \mathrm{Re}(s) \leq l/h\}$, where l is a number of order 1. If this cannot be determined exactly then setting $\alpha = l/h$ will suffice. ♣

The second case is more interesting and is now illustrated with a numerical example. Consider a problem for which the original generalised-plant has state-space realisation

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{array}\right],$$

with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 50 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \text{ and } C_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

and the sampling period is set to $h = 1$ sec. The generalised plant G has an unstable mode at 50 which is unobservable from the controlled output. Note that the unstable mode is fast (time constant 0.02 sec) relative to the sampling period (1 sec). In this case

$$\grave{E} = \begin{bmatrix} 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & -50 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 50 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which has eigenvalues $0, -1.7321, 1.7321, -50, 50,$ and 0, two of which are faster than the sampling frequency. For simplicity let $l = 1$. Since in this example the real part of the unobservable mode is greater than the sampling frequency, it is not possible to select α (which must lie in $[0, l/h]$) to lie to the right of it. Thus (A1) is violated for all possible choices of α in the sensible interval $[0, l/h]$. However, it is suggested that from a synthesis point of view, in this case it is the structure of the original problem that is not sensible. The unstable mode is excitable by the disturbance, yet this is not observed from the controlled output and hence, is not "observed" in the optimisation problem formulated to bound the norm of the controlled output for worst case disturbances. Since this unstable mode has a time constant of 0.02 seconds, in one sampling period it can increase by a large amount ($e^{50} = 5.18E + 21$). It is reasonable then to assume that this will require a large control action in the next sampling period, which if not penalised at the controlled output (that is $D_{12} = 0$) would be admissible. Furthermore, if penalised at the controlled output (that is D_{12} full column rank) this would result in very poor performance. Additionally, one would intuitively expect that it be very difficult to stabilise a system with such a fast unstable pole using such a slow sampling frequency. Similar arguments hold for when (A, B_1) has an uncontrollable mode that is faster than the sampling frequency.

In view of the cases considered, it is claimed that the assumptions (A1) and (A2) are reasonable in the context of H^∞ SD-synthesis.

6.5 Summary

In this chapter a new, compact derivation of formulae for H^∞, SD synthesis is presented. It is shown that these formulae can be re-structured to give a numerically-robust procedure for computing a state-space realisation of the SD-equivalent generalised-plant defined in Proposition 2.28.

The formulae derived are essentially identical (except for the accommodation of a direct feed-through term, D_{11}, here) to those derived in [Toi93] using classical, linear-quadratic, optimal-control theory. This indicates directly that the time-lifting approach and the classical, linear-quadratic, optimal-control-theoretic approach are equivalent, as was also pointed out in [TS97]. The formulae are simple, requiring computation of only a single matrix exponential. It is the simplicity of the formulae obtained that facilitates the re-structuring summarised in Theorem 6.6, for numerical robustness.

7

Concluding Remarks

The original motivation for this dissertation stems from the wide-spread use of digital hardware to implement control strategies for continuous-time processes. This invariably leads to closed-loops that exhibit time-varying behaviour, which is often periodic in nature. Although the mathematical analysis is more involved, the periodic structure of the time-variation exhibited by systems studied in this dissertation can be exploited in a manner similar to time-invariance, yielding powerful results analogous to many known to hold for LTI systems. The main contributions of this work and directions for future research are given below.

7.1 Contributions

- A new notion of frequency response is developed for LPTV systems. It is defined in terms of the average power of the asymptotic response to sinusoidal inputs of a single frequency and it can be characterised in terms of the singular values of a frequency-dependent, *finite-dimensional* matrix, called the Average-Power Gain Matrix. The performance-indicating properties of this new notion of frequency response are quite intuitive and it can be used to derive bounds on performance-indicating, closed-loop operators, which facilitate the design of parametric weights employed in a new H^{∞}-loopshaping based design procedure proposed for SD, control-system development.

- It is shown that the graph of a stabilisable, LPTV system can be expressed as the range (respectively kernel) of a stable, LPTV system that has a stable, LPTV left-inverse (respectively right-inverse). This representation is only unique up to right (respectively left) multiplication by stably-invertible factors and is called a strong-right (respectively strong-left) representation. It resembles the right (respectively left), coprime-factor representations known to exist for certain classes of LTI systems.

- Given a well-posed, LPTV plant/controller pair, a useful characterisation of closed-loop stability is obtained in terms of strong-right and strong-left representations the plant and controller graphs. In turn, this leads to a Youla-style parameterisation of stabilising controllers.

- Quantitative, robust-stability results are obtained for LPTV, closed-loop systems in terms of the gap metric. To this end, a formula, expressible as an optimisation problem over H^∞, is derived for the directed gap between two LPTV systems. Given a nominal, LPTV plant and a stabilising, LPTV controller, the largest gap-ball of systems centred at the nominal plant, which the controller can be guaranteed to stabilise, is identified. Further to this, the maximally-tolerable, simultaneous gap-perturbation to both the nominal plant and controller is characterised. Importantly, all of these results apply to the special case of LTI systems under periodic, SD control.

- Qualitatively, it is shown that the topology induced by the gap metric on a general class of LPTV systems, is the weakest with respect to which closed-loop stability is a robust property and closed-loop performance varies continuously. That is, the gap metric on LPTV systems induces the graph topology.

- A numerical procedure involving only solutions to finite-dimensional, *algebraic*, Riccati equations and evaluation of state-transition matrices at only single points in time, is developed for computing (to any desired accuracy) the directed gap (and hence gap) between LPTV, continuous-time systems that admit stabilisable and detectable, state-space realisations.

- A new, compact derivation of state-space formulae is presented for H^∞, SD synthesis. It is shown that these formulae can be easily restructured for numerical robustness.

7.2 Directions for Future Research

- Several case studies, in which the new procedure proposed for SD, control-system development is to be employed, are required to gain explicit insight into the design issues involved with the development of such control systems.

- Computation of the gap between LPTV, state-space systems involves calculating the state-transition operator associated with an "almost" Hamiltonian-structured, time-periodic matrix at one point. It would be useful to determine if a restructuring for numerical robustness of the relevant formulae is necessary or indeed possible.

- In Section 2.4.3, a state-space solution to the LSI, full-information, $\ell^2_{\mathbb{Z}+}(\cdot)$-synthesis problem is given for state-space systems with possibly infinite-dimensional input and output spaces, but importantly, a finite-dimensional state-space. As seen throughout this work, such state-space systems frequently appear in the study of LPTV, continuous-time systems. An interesting direction of research would be to seek a sensible handle on causality for these equivalent, time-lifted (discrete-time), state-space representations, by which it would be possible to use the solution to the LSI, full-information, $\ell^2_{\mathbb{Z}+}(\cdot)$-synthesis problem given here and the corresponding $\mathrm{H}^\infty_{\mathbb{D}}$-filtering result, to obtain a state-space solution to the LPTV, output-feedback, $\mathrm{L}^2_{\mathbb{R}+}(\cdot)$-synthesis problem, involving only finite-dimensional, *algebraic*, Riccati equations. Such a solution would be easier to work with than a solution involving Riccati, differential equalities.

- Another useful direction for future research would be to develop a ν-gap (cf. [Vin93a]) type metric for LPTV systems, with the same qualitative properties as the gap metric, but by which sharper quantitative results known to hold for LTI systems, could perhaps be obtained for LPTV systems.

A

Appendix to Chapter 2

A.1 Proof of Proposition 2.20

Let $F := (R + B^*XB)^{-1}(S + B^*XA)$, then a useful identity used in what follows is

$$A - BF = A - B(R + B^*XB)^{-1}(S + B^*XA) = A_1 - B(R + B^*XB)^{-1}B^*XA_1. \tag{A.1}$$

Proof : First note that stabilisability of (A, B) is necessary by definition. Now, since

$$0 \leq \begin{bmatrix} I & -S^*R^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} Q & S^* \\ S & R \end{bmatrix} \begin{bmatrix} I & 0 \\ -R^{-1}S & I \end{bmatrix} = \begin{bmatrix} Q - S^*R^{-1}S & 0 \\ 0 & R \end{bmatrix},$$

it follows that $Q - S^*R^{-1}S \geq 0$ and hence, that $\tilde{X} = 0$ is a solution to the Riccati inequality

$$-\tilde{X} + A^*\tilde{X}A + Q - (S + B^*\tilde{X}A)^*(R + B^*\tilde{X}B)^{-1}(S + B^*\tilde{X}A) \geq 0,$$

such that $(R + B\tilde{X}B) > 0$. It then follows by [LR95, Theorem 13.11], that under the assumption (A, B) is stabilisable, there exists a self-adjoint solution $X \geq 0$ to the algebraic, Riccati equation (2.7) such that $\operatorname{spec}(A - BF) \subset \overline{\mathbb{D}}$ (the closed unit disc). So stabilisability of (A, B) is almost sufficient in that it guarantees existence of a self-adjoint, positive-semidefinite solution, but leaves open the possibility $\operatorname{spec}(A - BF) \cap \mathbb{T} \neq \emptyset$. In what follows, it is shown that given a self-adjoint, positive-semidefinite solution to equation (2.7), $\operatorname{spec}(A - BF) \subset \mathbb{D}$ if and only if (Q_1, A_1) has no unobservable modes on $\mathbb{T}$, by which the result claimed holds.

First note that equation (2.7) can be re-written as

$$X - (A - BF)^*X(A - BF) = Q - S^*R^{-1}S + (F - R^{-1}S)^*R(F - R^{-1}S). \tag{A.2}$$

Now let there be no unobservable modes of (Q_1, A_1) on $\mathbb{T}$ and suppose there exists a $\lambda \in \mathbb{T}$ and corresponding $x \in \mathcal{X}$, such that $(A - BF)x = \lambda x$. Then from equation (A.2), it follows that

$$(1 - |\lambda|^2)\langle x, Xx\rangle_{\mathcal{X}} = \left\langle x, (Q - S^*R^{-1}S + (F - R^{-1}S)^*R(F - R^{-1}S))x\right\rangle_{\mathcal{X}} = 0.$$

Equivalently,

$$\left\langle x, (Q - S^*R^{-1}S)x\right\rangle_{\mathcal{X}} = -\left\langle (F - R^{-1}S)x, R(F - R^{-1}S)x\right\rangle_{\mathcal{X}},$$

which since $Q_1 = (Q - S^*R^{-1}S) \geq 0$ and $R > 0$, implies that $\left\langle (F - R^{-1}S)x, R(F - R^{-1}S)x\right\rangle_{\mathcal{X}} = 0$ and that

$$\langle x, Q_1 x\rangle_{\mathcal{X}} = 0. \tag{A.3}$$

Therefore, $(F - R^{-1}S)x = 0$ and consequently, $Fx = R^{-1}Sx$ and

$$(A - BF)x = (A - BR^{-1}S)x =: A_1 x = \lambda x. \tag{A.4}$$

Furthermore, it follows from equation (A.3), that $x \in (\mathcal{R}_{Q_1})^{\perp} = \mathcal{K}_{(Q_1)^* = Q_1}$. Combined with equation (A.4), this implies that $\mathcal{K}_{(\lambda I - A_1)} \cap \mathcal{K}_{(Q_1)} \neq \{0\}$, which is a contradiction to the initial hypothesis. So if (Q_1, A_1) has no unobservable modes on $\mathbb{T}$, then $\mathrm{spec}(A - BF) \subset \mathbb{D}$. It now remains to show that the converse also holds.

Let $\mathrm{spec}(A - BF) \subset \mathbb{D}$ and suppose there exists an $x \in \mathcal{X}$ and $\lambda \in \mathbb{T}$ such that $A_1 x = \lambda x$ and $Q_1 x = 0$ (that is, λ corresponds to an unobservable mode of (Q_1, A_1)). Now, recall from Proposition 2.18, that X is also a solution to

$$X \;=\; A_1^* X A_1^* + Q_1 - A_1^* X B (R + B^* X B)^{-1} B^* X A_1$$

and hence that,

$$0 = \frac{(1 - |\lambda|^2)}{|\lambda|^2}\langle x, Xx\rangle_{\mathcal{X}} = \left\langle B^* X x, (R + B^* X B)^{-1} B^* X x\right\rangle_{\mathcal{X}}.$$

Since $R + B^*XB > 0$ (which follows from $R > 0$ and $X \geq 0$), it follows that $B^*Xx = 0$. Using this and the identity in equation (A.1), yields $(A - BF)x = A_1 x - B(R + B^*XB)^{-1}B^*XA_1 x = \lambda x$, which contradicts $\mathrm{spec}(A - BF) \subset \mathbb{D}$. Hence, if $\mathrm{spec}(A - BF) \subset \mathbb{D}$, then the pair (Q_1, A_1) necessarily has no unobservable modes on $\mathbb{T}$. ∎

A.2 Proof of Proposition 2.22

Sufficiency: In the ensuing analysis it is convenient to define

$$\phi_k := \begin{bmatrix} w_k \\ u_k \end{bmatrix} \quad \text{and} \quad \psi_k := \begin{bmatrix} z_k \\ w_k \end{bmatrix}, \tag{A.5}$$

so that

$$\begin{bmatrix} x_{k+1} \\ \psi_k \end{bmatrix} = \begin{bmatrix} A & B \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} x_k \\ \phi_k \end{bmatrix}.$$

Now consider the following algebraic identity:

$$\begin{aligned}
&\langle z_k, z_k\rangle_{\mathcal{Z}} - \gamma^2 \langle w_k, w_k\rangle_{\mathcal{W}} + \langle x_{k+1}, X_\infty x_{k+1}\rangle_{\mathcal{X}} \\
&= \left\langle \begin{bmatrix} x_{k+1} \\ \psi_k \end{bmatrix}, \begin{bmatrix} X_\infty & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} x_{k+1} \\ \psi_k \end{bmatrix} \right\rangle_{\mathcal{X}\oplus\mathcal{Z}\oplus\mathcal{W}} \\
&= \left\langle \begin{bmatrix} x_k \\ \phi_k \end{bmatrix}, \begin{bmatrix} A & B \\ \tilde{C} & \tilde{D} \end{bmatrix}^* \begin{bmatrix} X_\infty & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} A & B \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} x_k \\ \phi_k \end{bmatrix} \right\rangle_{\mathcal{X}\oplus\mathcal{W}\oplus\mathcal{U}} \\
&= \left\langle \begin{bmatrix} x_k \\ \phi_k \end{bmatrix}, \begin{bmatrix} A^*X_\infty A + \tilde{C}^*J\tilde{C} & L^* \\ L & Z \end{bmatrix} \begin{bmatrix} x_k \\ \phi_k \end{bmatrix} \right\rangle_{\mathcal{X}\oplus\mathcal{W}\oplus\mathcal{U}},
\end{aligned} \tag{A.6}$$

where for the moment X_∞ is an arbitrary, self-adjoint, positive-semidefinite operator, J is defined at the top of Proposition 2.22 and the operators $Z =: \begin{bmatrix} Z_1 & Z_2^* \\ Z_2 & Z_3 \end{bmatrix}$ and $L =: \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ (partitioned conformably with ϕ) are defined in equations (2.14) and (2.15) respectively.

Suppose that $X_\infty \geq 0$ is a solution to the algebraic, Riccati equation (2.13) such that $\nabla := Z_1 - Z_2^* Z_3^{-1} Z_2 < 0$ and $\operatorname{spec}(A - BZ^{-1}L) \subset \mathbb{D}$. Now note that $Z_3 = D_2^* D_2 + B_2^* X_\infty B_2$ and since $D_2^* D_2 > 0$ and $X_\infty \geq 0$, that $Z_3 > 0$. Therefore, (cf. Remark 2.19) Z_3 is boundedly invertible and it is possible to write

$$\begin{bmatrix} Z_1 & Z_2^* \\ Z_2 & Z_3 \end{bmatrix} = \begin{bmatrix} I & Z_2^* Z_3^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \nabla & 0 \\ 0 & Z_3 \end{bmatrix} \begin{bmatrix} I & 0 \\ Z_3^{-1} Z_2 & I \end{bmatrix}. \tag{A.7}$$

With $\nabla < 0$ (which follows by the original hypothesis), ∇ is boundedly invertible (cf. Remark 2.19) and it follows from equation (A.7), that Z is too. A Schur decomposition of the operator on the right-hand side of (A.6) is therefore possible, giving

$$\begin{aligned}
&\langle z_k, z_k\rangle_{\mathcal{Z}} - \gamma^2 \langle w_k, w_k\rangle_{\mathcal{W}} + \langle x_{k+1}, X_\infty x_{k+1}\rangle_{\mathcal{X}} \\
&= \left\langle \begin{bmatrix} I & 0 \\ Z^{-1}L & I \end{bmatrix} \begin{bmatrix} x_k \\ \phi_k \end{bmatrix}, \begin{bmatrix} A^*X_\infty A + \tilde{C}^*J\tilde{C} - L^*Z^{-1}L & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} I & 0 \\ Z^{-1}L & I \end{bmatrix} \begin{bmatrix} x_k \\ \phi_k \end{bmatrix} \right\rangle_{\mathcal{X}\oplus\mathcal{W}\oplus\mathcal{U}}.
\end{aligned}$$

Defining

$$\begin{bmatrix} w_k^\star \\ u_k^\star \end{bmatrix} := -\begin{bmatrix} \nabla^{-1}L_\nabla & 0 \\ Z_3^{-1}L_2 & Z_3^{-1}Z_2 \end{bmatrix}\begin{bmatrix} x_k \\ w_k \end{bmatrix},$$

where $L_\nabla = L_1 - Z_2^* Z_3^{-1} Z_2$, it follows that since $X_\infty \geq 0$ is a solution to the algebraic, Riccati equation (2.13),

$$\begin{aligned}
&\langle z_k, z_k\rangle_{\mathcal{Z}} - \gamma^2\langle w_k, w_k\rangle_{\mathcal{W}} + \langle x_{k+1}, X_\infty x_{k+1}\rangle_{\mathcal{X}} \\
&= \ \langle x_k, X_\infty x_k\rangle_{\mathcal{X}} + \langle (w_k - w_k^\star), \nabla(w_k - w_k^\star)\rangle_{\mathcal{W}} + \langle (u_k - u_k^\star), Z_3(u_k - u_k^\star)\rangle_{\mathcal{U}}. \qquad (A.8)
\end{aligned}$$

Note, that $u^\star$ has the form of a causal, LSI, full-information control-law. Moreover, if $u = u^\star$ is a stabilising control-law, then by definition $\lim_{k\to\infty} x_k = 0$ and consequently,

$$\sum_{k=0}^{\infty}\{\langle x_k, X_\infty x_k\rangle_{\mathcal{X}} - \langle x_{k+1}, X_\infty x_{k+1}\rangle_{\mathcal{X}}\} = \langle \tilde{x}, X_\infty \tilde{x}\rangle_{\mathcal{X}}.$$

Combining this with equation (A.8), leads to

$$\begin{aligned}
\|z\|^2_{\ell^2_{\mathbb{Z}^+}(\mathcal{Z})} - \gamma^2\|w\|^2_{\ell^2_{\mathbb{Z}^+}(\mathcal{W})} &\leq \langle \tilde{x}, X_\infty\tilde{x}\rangle_{\mathcal{X}} - \alpha\|w - w^\star\|^2_{\ell^2_{\mathbb{Z}^+}(\mathcal{W})} \\
(&= -\alpha\|w - w^\star\|^2_{\ell^2_{\mathbb{Z}^+}(\mathcal{W})} \text{ when } \tilde{x} = 0), \qquad (A.9)
\end{aligned}$$

where $\alpha > 0$ is any real number such that $\nabla \leq \alpha I$. So it remains to show that $u = u^\star$ is indeed stabilising and that there exist an $\epsilon > 0$ such that $\alpha\|w - w^\star\|^2_{\ell^2_{\mathbb{Z}^+}(\mathcal{W})} \geq \epsilon\|w\|^2_{\ell^2_{\mathbb{Z}^+}(\mathcal{W})}$. That this is true, follows by the initial hypothesis that $\mathrm{spec}(A - BZ^{-1}L) \subset \mathbb{D}$. To see this, first note that with $u = u^\star$,

$$\begin{bmatrix} x_{k+1} \\ z_k \end{bmatrix} = \begin{bmatrix} A - B_2Z_3^{-1}L_2 & B_1 - B_2Z_3^{-1}Z_2 \\ C - D_2Z_3^{-1}L_2 & D_1 - D_2Z_3^{-1}Z_2 \end{bmatrix}\begin{bmatrix} x_k \\ w_k \end{bmatrix}$$

and hence, that $u = u^\star$ is stabilising if $\mathrm{spec}(A - B_2Z_3^{-1}L_2) \subset \mathbb{D}$. Now note that

$$\begin{bmatrix} A & B_2 \\ C & D_2 \end{bmatrix}^*\begin{bmatrix} X_\infty & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A & B_2 \\ C_1 & D_2 \end{bmatrix} = \begin{bmatrix} X_\infty + L_\nabla^*\nabla^{-1}L_\nabla & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} L_2^* \\ Z_3 \end{bmatrix} Z_3^{-1}\begin{bmatrix} L_2 & Z_3 \end{bmatrix}.$$

Multiplying this on the right by $\begin{bmatrix} I \\ -Z_3^{-1}L_2 \end{bmatrix}$ and on the left by $\begin{bmatrix} I & -L_2^*Z_3^{-1} \end{bmatrix}$, gives

$$X_\infty = (A - B_2Z_3^{-1}L_2)^*X_\infty(A - B_2Z_3^{-1}L_2) + (C - D_2Z_3^{-1}L_2)^*(C - D_2Z_3^{-1}L_2) - L_\nabla^*\nabla^{-1}L_\nabla.$$

Recalling that $\nabla < 0$ (by the initial hypothesis), it follows that $((C - D_2Z_3^{-1}L_2)^*(C - D_2Z_3^{-1}L_2) - L_\nabla^*\nabla^{-1}L_\nabla) > 0$ and hence, that any eigenvalue λ of $(A - B_2Z_3^{-1}L_2)$ and corresponding eigenvector $x \in \mathcal{X}$, satisfy

$$(1 - |\lambda|^2)\langle x, X_\infty x\rangle_{\mathcal{X}} = \left\langle x, ((C - D_2Z_3^{-1}L_2)^*(C - D_2Z_3^{-1}L_2) - L_\nabla^*\nabla^{-1}L_\nabla)x\right\rangle_{\mathcal{X}} \geq 0. \qquad (A.10)$$

Since $\mathsf{X}_\infty \geq 0$ this implies that $\text{spec}(\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2) \subset \overline{\mathbb{D}}$. In fact, this can be strengthened to $\text{spec}(\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2) \subset \mathbb{D}$ if $(\mathsf{L}_\nabla, \mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)$ has no unobservable modes on $\mathbb{T}$. To see this, suppose that under the additional hypothesis that $(\mathsf{L}_\nabla, \mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)$ has no unobservable modes on $\mathbb{T}$, there exist an eigenvalue of $(\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)$ on the unit circle and denote the corresponding eigenvector by $x \in \mathcal{X}$. Then from equation (A.10), $\langle \mathsf{L}_\nabla \mathsf{x}, \nabla^{-1}\mathsf{L}_\nabla \mathsf{x}\rangle_\mathcal{X} = \langle (\mathsf{C} - \mathsf{D}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)x, (\mathsf{C} - \mathsf{D}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)x\rangle_\mathcal{Z} \geq 0$ and hence, since $\nabla^{-1} < 0$, it follows that $\mathsf{L}_\nabla x = 0$, contradicting the additional hypothesis. So in terms of showing $\mathsf{u} = \mathsf{u}^\star$ is stabilising, it is sufficient to show that $\text{spec}(\mathsf{A} - \mathsf{B}\mathsf{Z}^{-1}\mathsf{L}) \subset \mathbb{D}$ implies $(\mathsf{L}_\nabla, \mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)$ has no unobservable modes on $\mathbb{T}$. This is indeed the case, as can be seen by noting that

$$\mathsf{A} - \mathsf{B}\mathsf{Z}^{-1}\mathsf{L} = (\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2) - (\mathsf{B}_1 - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{Z}_2)\nabla^{-1}\mathsf{L}_\nabla \tag{A.11}$$

and supposing there exists a $|\lambda| = 1$ and $x \in \mathcal{X}$ such that $(\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)\mathsf{x} = \lambda x$ and $\mathsf{L}_\nabla x = 0$. Then using equation (A.11), it follows that $(\mathsf{A} - \mathsf{B}\mathsf{Z}^{-1}\mathsf{L})x = \lambda x - 0 = \lambda x$, contradicting the original hypothesis $\text{spec}(\mathsf{A} - \mathsf{B}\mathsf{Z}^{-1}\mathsf{L}) \subset \mathbb{D}$.

To complete the sufficiency proof it remains to show that there exist an $\epsilon > 0$ such that $\alpha\|w - w^\star\|^2_{\ell^2_{\mathbb{Z}+}(\mathcal{W})} \geq \epsilon\|w\|^2_{\ell^2_{\mathbb{Z}+}(\mathcal{W})}$. Let $W : \mathcal{W} \to \mathcal{W}$ denote the operator mapping w to $(w - w^\star)$ and note that it has state-space realisation

$$\begin{bmatrix} \mathsf{x}_{k+1} \\ w_k - w_k^\star \end{bmatrix} = \begin{bmatrix} \mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2 & \mathsf{B}_1 - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{Z}_2 \\ \nabla^{-1}\mathsf{L}_\nabla & \mathsf{I} \end{bmatrix} \begin{bmatrix} \mathsf{x}_k \\ w_k \end{bmatrix}.$$

Since $\text{spec}(\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2) \subset \mathbb{D}$, W is a bounded operator. In fact it has bounded inverse, since $\text{spec}(\mathsf{A} - \mathsf{B}\mathsf{Z}^{-1}\mathsf{L}) \subset \mathbb{D}$ and using equation (A.11), a state-space realisation for W^{-1} is

$$\begin{bmatrix} \mathsf{s}_{k+1} \\ w_k \end{bmatrix} = \begin{bmatrix} \mathsf{A} - \mathsf{B}\mathsf{Z}^{-1}\mathsf{L} & \mathsf{B}_1 - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{Z}_2 \\ -\nabla^{-1}\mathsf{L}_\nabla & \mathsf{I} \end{bmatrix} \begin{bmatrix} \mathsf{s}_k \\ w_k - w_k^\star \end{bmatrix}.$$

It follows that $\frac{\alpha}{\|W^{-1}\|^2}\|w\|_{\ell^2_{\mathbb{Z}+}(\mathcal{W})} \leq \alpha\|w - w^\star\|_{\ell^2_{\mathbb{Z}+}(\mathcal{W})}$ and therefore, that a suitable ϵ is $\epsilon = \frac{\alpha}{\|W^{-1}\|^2} > 0$ (cf. equation (A.9)). This completes the proof of sufficiency.

Necessity: The approach taken by Green and Limebeer [GL95, Appendix B] to prove necessity is based on the solution of the finite-horizon problem with appropriate terminal-state penalty. That is, to define a causal, LSI control-law $\mathsf{u} = \mathsf{K}\begin{bmatrix} \mathsf{x} \\ w \end{bmatrix}$ over the finite-horizon $[0, N]$, such that given $\gamma > 0$ and $\mathsf{X}_{tsp} \geq 0$,

$$\|z\|^2_{\ell^2_{[0,N]}(\mathcal{Z})} - \gamma^2\|w\|^2_{\ell^2_{[0,N]}(\mathcal{W})} + \langle \mathsf{x}_{N+1}, \mathsf{X}_{tsp}\mathsf{x}_{N+1}\rangle_\mathcal{X} \leq -\epsilon\|w\|^2_{\ell^2_{[0,N]}(\mathcal{W})} \tag{A.12}$$

for all $w \in \ell^2_{[0,N]}(\mathcal{W})$ and some $\epsilon > 0$, when $\tilde{x} = 0$. It can be shown [GL95, Theorem B.2.1], that such a control-law exists if and only if

$$\begin{aligned} \mathsf{Z}_{k,3}^{\{N+1,\mathsf{X}_{tsp}\}} &> 0, \quad k = 0,\dots,N \\ \nabla_k^{\{N+1,\mathsf{X}_{tsp}\}} &\leq -\alpha\mathrm{I}, \quad k = 0,\dots,N \end{aligned}$$

for some $\alpha > 0$, where

$$\begin{aligned} \mathsf{Z}_k^{\{N+1,\mathsf{X}_{tsp}\}} &:= \tilde{\mathsf{D}}^*\mathsf{J}\tilde{\mathsf{D}} + \mathsf{B}^*\mathsf{X}_{k+1}^{\{\mathrm{N}+1,\mathsf{X}_{tsp}\}}\mathsf{B} =: \begin{bmatrix} \mathsf{Z}_{k,1}^{\{N+1,\mathsf{X}_{tsp}\}} & (\mathsf{Z}_{k,2}^{\{N+1,\mathsf{X}_{tsp}\}})^* \\ \mathsf{Z}_{k,2}^{\{N+1,\mathsf{X}_{tsp}\}} & \mathsf{Z}_{k,3}^{\{N+1,\mathsf{X}_{tsp}\}} \end{bmatrix}, \\ \mathsf{L}_k^{\{N+1,\mathsf{X}_{tsp}\}} &:= \tilde{\mathsf{D}}^*\mathsf{J}\tilde{\mathsf{C}} + \mathsf{B}^*\mathsf{X}_{k+1}^{\{N+1,\mathsf{X}_{tsp}\}}\mathsf{A} =: \begin{bmatrix} \mathsf{L}_{k,1}^{\{N+1,\mathsf{X}_{tsp}\}} \\ \mathsf{L}_{2,\mathrm{k}}^{\{N+1,\mathsf{X}_{tsp}\}} \end{bmatrix}, \\ \nabla_k^{\{N+1,\mathsf{X}_{tsp}\}} &:= \mathsf{Z}_{k,1}^{\{N+1,\mathsf{X}_{tsp}\}} - (\mathsf{Z}_{k,2}^{\{N+1,\mathsf{X}_{tsp}\}})^*(\mathsf{Z}_{k,3}^{\{N+1,\mathsf{X}_{tsp}\}})^{-1}\mathsf{Z}_{k,2}^{\{N+1,\mathsf{X}_{tsp}\}} \end{aligned}$$

and $\mathsf{X}_k^{\{N+1,\mathsf{X}_{tsp}\}}$ satisfies the Riccati difference equation

$$\begin{aligned} \mathsf{X}_k^{\{N+1,\mathsf{X}_{tsp}\}} &= \mathsf{A}^*\mathsf{X}_{k+1}^{\{N+1,\mathsf{X}_{tsp}\}}\mathsf{A} + \tilde{\mathsf{C}}^*\mathsf{J}\tilde{\mathsf{C}} - (\mathsf{L}_k^{\{N+1,\mathsf{X}_{tsp}\}})^*(\mathsf{Z}_k^{\{N+1,\mathsf{X}_{tsp}\}})^{-1}\mathsf{L}_k^{\{N+1,\mathsf{X}_{tsp}\}}, \\ \mathsf{X}_{N+1}^{\{N+1,\mathsf{X}_{tsp}\}} &= \mathsf{X}_{tsp}. \end{aligned} \tag{A.13}$$

In this case, $\mathsf{X}_k^{\{N+1,\mathsf{X}_{tsp}\}} \geq 0$ for $k = 0,\dots,N+1$.

Now assume that $(\mathsf{A},\mathsf{B}_2)$ is stabilisable and

$$\mathcal{K}\begin{bmatrix} \mathsf{A} - e^{j\theta}\mathrm{I} & \mathsf{B}_2 \\ \mathsf{C} & \mathsf{D}_2 \end{bmatrix} = \{0\}$$

for all $\theta \in [0,2\pi)$ and suppose that a causal, LSI, stabilising, full-information control-law exists such that the infinite-horizon objective (2.12) is achieved. Then for all $w \in \ell^2_{[0,N]}(\mathcal{W})$ (that is, $w_k = 0$ for $k \geq N+1$),

$$\begin{aligned} -\epsilon\|w\|^2_{\ell^2_{[0,N]}(\mathcal{W})} &\geq \|z\|^2_{\ell^2_{\mathbb{Z}^+}(\mathcal{Z})} - \gamma^2\|w\|^2_{\ell^2_{\mathbb{Z}^+}(\mathcal{W})} \\ &\geq \|z\|^2_{\ell^2_{[0,N]}(\mathcal{Z})} - \gamma^2\|w\|^2_{\ell^2_{[0,N]}(\mathcal{W})} + \min_{\text{stab. ctrl-laws}} \sum_{k=\mathrm{N}+1}^{\infty} \langle z_k, z_k\rangle_{\mathcal{Z}} \\ &= \|z\|^2_{\ell^2_{[0,N]}(\mathcal{Z})} - \gamma^2\|w\|^2_{\ell^2_{[0,N]}(\mathcal{W})} + \langle \mathsf{x}_{N+1}, \mathsf{X}_2\mathsf{x}_{N+1}\rangle_{\mathcal{X}}, \end{aligned}$$

where $\mathsf{X}_2 \geq 0$ is the stabilising solution to the algebraic, Riccati equation (2.10) and the last equality follows by Proposition 2.21. By the finite-horizon results, $\mathsf{X}_k^{\{N+1,\mathsf{X}_2\}} \geq 0$ and

$$\nabla_k^{\{N+1,\mathsf{X}_2\}} \leq -\alpha\mathrm{I} \tag{A.14}$$

for all $k \leq N+1$ and some $\alpha \geq \epsilon$ (without loss of generality it is possible to take $\alpha = \epsilon$ since this is independent of the problem horizon.) Furthermore, it can be shown that $\mathsf{X}_k^{\{N+1,\mathsf{X}_2\}}$ is uniformly[1] bounded above and monotonically increasing in N. Therefore, it follows that

$$\mathsf{X}_\infty = \lim_{N\to\infty} \mathsf{X}_k^{\{N+1,\mathsf{X}_2\}} \geq 0$$

exists, is independent of k and a is solution to the algebraic, Riccati equation (2.13). Moreover, since $\alpha \geq \epsilon$ is independent of the horizon length, $\nabla = \lim_{N\to\infty} \nabla_k^{\{N+1,\mathsf{X}_2\}} < 0$, and by monoticity $(\mathsf{X}_\infty - \mathsf{X}_2) \geq 0$. So it remains to show that $\mathrm{spec}(\mathsf{A} - \mathsf{B}\mathsf{Z}^{-1}\mathsf{L}) \subset \mathbb{D}$. This is done in two steps: first it is shown that it is necessary for $\mathrm{spec}(\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)$ to lie inside $\mathbb{D}$, which is then used this to show that $\mathrm{spec}(\mathsf{A} - \mathsf{B}\mathsf{Z}^{-1}\mathsf{L}) \subset \mathbb{D}$.

First note that $\Omega := \mathsf{X}_\infty - \mathsf{X}_2 \geq 0$ satisfies

$$\Omega = (\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)^*(\Omega + \Omega\mathsf{B}_2(\mathsf{Z}_3 - \mathsf{B}_2^*\Omega\mathsf{B}_2)^{-1}\mathsf{B}_2^*\Omega)(\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2) - \mathsf{L}_\nabla^*\nabla^{-1}\mathsf{L}_\nabla.$$

Now since $\mathsf{Z}_3 - \mathsf{B}_2^*\Omega\mathsf{B}_2 = \mathsf{D}_2^*\mathsf{D}_2 + \mathsf{B}_2^*\mathsf{X}_2\mathsf{B}_2$, it is positive-definite, invertible (cf. Remark 2.19) and therefore, has positive-definite square-root [Kre89b, Theorem 9.4-2]. Furthermore, since $\nabla < 0$, $((\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)^*\Omega\mathsf{B}_2(\mathsf{Z}_3 - \mathsf{B}_2^*\Omega\mathsf{B}_2)^{-1}\mathsf{B}_2^*\Omega(\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2) - \mathsf{L}_\nabla^*\nabla^{-1}\mathsf{L}_\nabla) \geq 0$ and hence, any eigenvalue λ of $(\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)$ and corresponding eigenvector $x \in \mathcal{X}$, satisfy

$$\begin{aligned}
&(1 - |\lambda|^2)\langle x, \Omega x\rangle_{\mathcal{X}} \\
&\quad = \left\langle x, ((\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)^*\Omega\mathsf{B}_2(\mathsf{Z}_3 - \mathsf{B}_2^*\Omega\mathsf{B}_2)^{-1}\mathsf{B}_2^*\Omega(\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2) - \mathsf{L}_\nabla^*\nabla^{-1}\mathsf{L}_\nabla)x\right\rangle_{\mathcal{X}} \\
&\quad \geq 0. \qquad \text{(A.15)}
\end{aligned}$$

Since $\Omega \geq 0$, this implies $\mathrm{spec}(\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2) \subset \overline{\mathbb{D}}$. In fact, this can be strengthened to $\mathrm{spec}(\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2) \subset \mathbb{D}$ if $(\mathsf{B}_2^*\Omega(\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2), \mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)$ has no unobservable modes on the unit circle of the complex plane. To see this, suppose that under the hypothesis that $(\mathsf{B}_2^*\Omega(\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2), \mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)$ has no unobservable modes on $\mathbb{T}$, there exists an eigenvalue of $(\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)$ on $\mathbb{T}$ and denote the corresponding eigen-vector by $x \in \mathcal{X}$. Since $\nabla^{-1} < 0$, it follows from equation (A.15), that

$$\|(\mathsf{Z}_3 - \mathsf{B}_2^*\Omega\mathsf{B}_2)^{-\frac{1}{2}}\mathsf{B}_2^*\Omega(\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)x\|_{\mathcal{X}}^2 \quad - \quad \left\langle \mathsf{L}_\nabla x, \nabla^{-1}\mathsf{L}_\nabla x\right\rangle_{\mathcal{X}} \leq 0,$$

This implies that $(\mathsf{Z}_3 - \mathsf{B}_2^*\Omega\mathsf{B}_2)^{-\frac{1}{2}}\mathsf{B}_2^*\Omega(\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)x = 0$, which since $\mathcal{K}_{(\mathsf{Z}_3 - \mathsf{B}_2^*\Omega\mathsf{B}_2)^{-\frac{1}{2}}} = \{0\}$ implies that $\mathsf{B}_2^*\Omega(\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)x = 0$, contradicting the hypothesis. So in terms of showing that spec(A −

[1] Independently of N (and k by time-invariance).

$\mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2) \subset \mathbb{D}$, it remains to show that $(\mathsf{B}_2^*\Omega(\mathsf{A}-\mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2), \mathsf{A}-\mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)$ has no unobservable modes on $\mathbb{T}$. This is indeed the case, as can be seen by noting that the spectrum of

$$\begin{aligned} &\mathsf{A}-\mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2+\mathsf{B}_2(\mathsf{Z}_3-\mathsf{B}_2^*\Omega\mathsf{B}_2)^{-1}\mathsf{B}_2^*\Omega(\mathsf{A}-\mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2) \\ &\quad = \mathsf{A}-\mathsf{B}_2(\mathsf{D}_2^*\mathsf{D}_2+\mathsf{B}_2^*\mathsf{X}_2\mathsf{B}_2)^{-1}(\mathsf{D}_2\mathsf{C}+\mathsf{B}_2^*\mathsf{X}_2\mathsf{A}) \end{aligned} \tag{A.16}$$

lies inside $\mathbb{D}$, since it corresponds to the optimally-controlled dynamics of the quadratic-regulator problem. Now, suppose that there exists a $|\lambda|=1$ and $x \in \mathcal{X}$ such that $(\mathsf{A}-\mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)x = \lambda x$ and $\mathsf{B}_2^*\Omega(\mathsf{A}-\mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)x = 0$. Then using equation (A.16), it follows that

$$(\mathsf{A}-\mathsf{B}_2(\mathsf{D}_2^*\mathsf{D}_2+\mathsf{B}_2^*\mathsf{X}_2\mathsf{B}_2)^{-1}(\mathsf{D}_2\mathsf{C}+\mathsf{B}_2^*\mathsf{X}_2\mathsf{A}))x = \lambda x - 0,$$

which is a contradiction. Hence, $(\mathsf{B}_2^*\Omega(\mathsf{A}-\mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2), \mathsf{A}-\mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)$ has no unobservable modes on $\mathbb{T}$ and $\operatorname{spec}(\mathsf{A}-\mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2) \subset \mathbb{D}$.

Now, in so far as the proof of necessity goes, it remains to show that $\operatorname{spec}(\mathsf{A}-\mathsf{B}\mathsf{Z}^{-1}\mathsf{L}) \subset \mathbb{D}$. To this end, recall that

$$\mathsf{A}-\mathsf{B}\mathsf{Z}^{-1}\mathsf{L} = (\mathsf{A}-\mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)-(\mathsf{B}_1-\mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{Z}_2)\nabla^{-1}\mathsf{L}_\nabla \tag{A.17}$$

and consider any $x \in \mathcal{K}_\Omega$. Then from equation (A.15), it follows that

$$0 \le \left\langle x, (\mathsf{A}-\mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)^*\Omega\mathsf{B}_2(\mathsf{Z}_3-\mathsf{B}_2^*\Omega\mathsf{B}_2)^{-1}\mathsf{B}_2^*\Omega(\mathsf{A}-\mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)x\right\rangle_{\mathcal{X}} = \left\langle \mathsf{L}_\nabla x, \nabla^{-1}\mathsf{L}_\nabla x\right\rangle_{\mathcal{X}},$$

which since $\nabla^{-1} < 0$ implies that $\mathsf{L}_\nabla x = 0$. Hence, from equation (A.17),

$$x \in \mathcal{K}_\Omega \Rightarrow (\mathsf{A}-\mathsf{B}\mathsf{Z}^{-1}\mathsf{L})\mathsf{x} = (\mathsf{A}-\mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2)\mathsf{x}. \tag{A.18}$$

Now, define $\Omega_k :=(\mathsf{X}_\infty - \mathsf{X}_k^{\{N+1,\mathsf{X}_2\}})$ and note that Ω_k satisfies the difference equation

$$\Omega_k = (\mathsf{A}-\mathsf{B}\mathsf{Z}^{-1}\mathsf{L})^*\Omega_{k+1}(\mathsf{I}-\mathsf{B}\mathsf{Z}^{-1}\mathsf{B}^*\Omega_{k+1})^{-1}(\mathsf{A}-\mathsf{B}\mathsf{Z}^{-1}\mathsf{L}),$$

with boundary condition $\Omega_{N+1} = \Omega :=(\mathsf{X}_\infty - \mathsf{X}_2)$. Furthermore, $\lim_{N\to\infty} \Omega_k = 0$ $(k \le N)$. It follows that for each $\lambda \in \operatorname{spec}(\mathsf{A}-\mathsf{B}\mathsf{Z}^{-1}\mathsf{L})$ and corresponding eigen-vector $x \in \mathcal{X}$,

$$\langle x, \Omega_N x\rangle_{\mathcal{X}} = |\lambda|^2\left\langle x, \Omega(\mathsf{I}-\mathsf{B}\mathsf{Z}^{-1}\mathsf{B}^*\Omega)^{-1}x\right\rangle_{\mathcal{X}}. \tag{A.19}$$

If $\lambda = 0$ then it is immediate that $x \in \mathcal{K}_{\Omega_N}$. In fact, taking the limit as $N \to \infty$ in equation (A.19), gives $\left\langle x, \Omega(\mathsf{I}-\mathsf{B}\mathsf{Z}^{-1}\mathsf{B}^*\Omega)^{-1}x\right\rangle_{\mathcal{X}} = 0$ and hence, $x \in \mathcal{K}_{\Omega_N}$ for $\lambda \ne 0$ as well. If it can now be shown that $\mathcal{K}_{\Omega_N} \subset \mathcal{K}_{\Omega}$, then it would follow from equation (A.18), that $\lambda \subset \mathbb{D}$, since $\mathrm{spec}(\mathsf{A} - \mathsf{B}_2\mathsf{Z}_3^{-1}\mathsf{L}_2) \subset \mathbb{D}$. To see that this is true, note that

$$\|z\|^2_{\ell^2_{[k,N]}(\mathcal{Z})} - \gamma^2\|w\|^2_{\ell^2_{[k,N]}(\mathcal{W})} + \langle \mathsf{x}_{N+1}, \mathsf{X}_2\mathsf{x}_{N+1}\rangle_{\mathcal{X}} = \tag{A.20}$$

$$\left\langle \mathsf{x}_k, \mathsf{X}_k^{\{N+1,\mathsf{X}_2\}}\mathsf{x}_k\right\rangle_{\mathcal{X}} + \sum_{i=k}^{N} \langle (w_i - w_i^\star), \nabla(w_i - w_i^\star)\rangle_{\mathcal{W}} \tag{A.21}$$

and

$$\|z\|^2_{\ell^2_{[k,\infty)}(\mathcal{Z})} - \gamma^2\|w\|^2_{\ell^2_{[k,\infty)}(\mathcal{W})} + \langle \mathsf{x}_k, \mathsf{X}_\infty\mathsf{x}_k\rangle_{\mathcal{X}} = \sum_{i=k}^{\infty} \langle (w_i - w_i^\star), \nabla(w_i - w_i^\star)\rangle_{\mathcal{W}}. \tag{A.22}$$

Consider the signal

$$w_i = \begin{cases} 0 & i < k \\ \tilde{w}_i & k \le i \le N \\ 0 & k > N \end{cases}$$

for arbitrary $\tilde{w}_i \in \mathcal{W}$, $k \le i \le N$. Then

$$\|z\|^2_{\ell^2_{[N+1,\infty)}(\mathcal{Z})} - \langle \mathsf{x}_{N+1}, \mathsf{X}_\infty\mathsf{x}_{N+1}\rangle_{\mathcal{X}} = \sum_{i=N+1}^{\infty} \langle (w_i - w_i^\star), \nabla(w_i - w_i^\star)\rangle_{\mathcal{W}}.$$

Subtracting equations (A.20) and (A.22), it follows that

$$\langle \mathsf{x}_{N+1}, (\mathsf{X}_\infty - \mathsf{X}_2)\mathsf{x}_{N+1}\rangle_{\mathcal{X}} = \left\langle \mathsf{x}_k, (\mathsf{X}_\infty - \mathsf{X}_k^{\{N+1,\mathsf{X}_2\}})\mathsf{x}_k\right\rangle_{\mathcal{X}}.$$

Since $(\mathsf{X}_\infty - \mathsf{X}_2) \ge 0$, it is possible to conclude from this that $\mathcal{K}_{\left(\mathsf{X}_\infty - \mathsf{X}_k^{\{N+1,\mathsf{X}_2\}}\right)} \subset \mathcal{K}_{(\mathsf{X}_\infty - \mathsf{X}_2)}$ and hence, that $\mathrm{spec}(\mathsf{A} - \mathsf{B}\mathsf{Z}^{-1}\mathsf{L}) \subset \mathbb{D}$. This completes the proof of necessity. ∎

B

Appendix to Chapter 3

B.1 Proof of Lemma 3.9

Before going on to prove Lemma 3.9, a few remarks concerning LTI systems and the signals spaces on which they are defined is required (for further details cf. [Fra86]). First note that the Hilbert space of continuous-time signals $\mathrm{L}^2_{\mathbb{R}+}(\mathcal{U})$ is isomorphic to the Hardy space $\mathrm{H}^2_{\mathbb{C}+}(\mathcal{U})$ of functions $\hat{f}(s) : \mathbb{C}^+ \to \mathcal{U}$ that are analytic in $\mathbb{C}^+$ and for which

$$\frac{1}{2\pi}\int_{\mu-j\infty}^{\mu+j\infty} \left\langle \hat{f}(j\omega), \hat{f}(j\omega)\right\rangle_{\mathcal{U}} \, d\omega$$

is uniformly bounded for all $\mu > 0$. The isomorphism between $\mathrm{L}^2_{\mathbb{R}+}(\mathcal{U})$ and $\mathrm{H}^2_{\mathbb{C}+}(\mathcal{U})$ is the Laplace Transform, which for $f(t) \in \mathrm{L}^2_{\mathbb{R}+}(\mathcal{U})$ is defined by

$$\hat{f}(s) :=(\mathbf{L}f)(s) := \int_0^\infty f(t)e^{-st} \, dt.$$

By identifying functions in $\mathrm{H}^2_{\mathbb{C}+}(\mathcal{U})$ with their boundary values on the imaginary axis, $\mathrm{H}^2_{\mathbb{C}+}(\mathcal{U})$ can be embedded as a closed subspace in the Hilbert space $\mathrm{L}^2_{j\mathbb{R}}(\mathcal{U})$ of functions $\hat{f}(j\omega) : \mathbb{R} \to \mathcal{U}$ that satisfy $\left\langle \hat{f}, \hat{f}\right\rangle_{\mathrm{L}^2_{j\mathbb{R}}(\mathcal{U})} < \infty$, where for $\hat{f}, \hat{g} \in \mathrm{L}^2_{j\mathbb{R}}(\mathcal{U})$

$$\left\langle \hat{f}, \hat{g}\right\rangle_{\mathrm{L}^2_{j\mathbb{R}}(\mathcal{U})} :=\frac{1}{2\pi}\int_{-\infty}^{\infty} \left\langle \hat{f}(j\omega), \hat{g}(j\omega)\right\rangle_{\mathcal{U}} \, d\omega.$$

Furthermore, $\mathrm{L}^2_{j\mathbb{R}}(\mathcal{U})$ is isomorphic to the doubly-infinite, continuous-resolution, Hilbert space $\mathrm{L}^2_{\mathbb{R}}(\mathcal{U})$ of functions $f : \mathbb{R} \to \mathcal{U}$ that satisfy

$$\int_{-\infty}^{\infty} \langle f(t), f(t)\rangle_{\mathcal{U}} \, dt < \infty.$$

The isomorphism between $\mathrm{L}^2_{j\mathbb{R}}(\mathcal{U})$ and $\mathrm{L}^2_{\mathbb{R}}(\mathcal{U})$ is the Fourier transform defined by

$$\hat{f}(j\omega) :=(\mathbf{F}\hat{f})(j\omega) := \int_{-\infty}^{\infty} f(t)e^{-j\omega t} \, dt$$

for all $f(t) \in \mathrm{L}^2_{\mathbb{R}}(\mathcal{U})$. By analytic continuation, the Fourier-Transform isomorphism takes functions in $\mathrm{L}^2_{\mathbb{R}+}(\mathcal{X}) \subset \mathrm{L}^2_{\mathbb{R}}(\mathcal{U})$ to $\mathrm{H}^2_{\mathbb{C}+}(\mathcal{U})$ and functions in $\mathrm{L}^2_{\mathbb{R}\text{-}}(\mathcal{X}) := \mathrm{L}^2_{\mathbb{R}}(\mathcal{U}) \ominus \mathrm{L}^2_{\mathbb{R}+}(\mathcal{X})$ to $\mathrm{H}^{2\perp}_{\mathbb{C}+}(\mathcal{U}) := \mathrm{L}^2_{j\mathbb{R}}(\mathcal{U}) \ominus \mathrm{H}^2_{\mathbb{C}+}(\mathcal{U})$.

Now as for discrete-time LSI systems, continuous-time LTI systems are equivalent via the Laplace-Transform isomorphism to multiplication operators defined on subspaces of $\mathrm{H}^2_{\mathbb{C}+}(\mathcal{U})$. Furthermore, the multiplication operator equivalent of an LTI system is bounded (that is, its domain is the whole of $\mathrm{H}^2_{\mathbb{C}+}(\mathcal{U})$ and it has finite induced-norm) if and only if its frequency domain symbol is an element of $\mathrm{H}^\infty_{\mathbb{C}+}(\mathcal{B}_{\mathcal{U}\to\mathcal{Y}})$, which is the Hardy space of $\mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$-valued functions $\hat{F}(s) : \mathbb{C}^+ \to \mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$ that are analytic and bounded in the right-half plane $\mathbb{C}^+$. That is, an LTI system $F : \mathcal{D}_F \subset \mathrm{L}^2_{\mathbb{R}+}(\mathcal{U}) \to \mathrm{L}^2_{\mathbb{R}+}(\mathcal{Y})$ is stable if and only if it is equivalent via the Laplace-Transform isomorphism to a multiplication operator $\mathbf{M}_{\hat{F}}$ with $\hat{F}(s) \in \mathrm{H}^\infty_{\mathbb{C}+}(\mathcal{B}_{\mathcal{U}\to\mathcal{Y}})$.[1] Note also, that for any function $\hat{F}(j\omega) : j\mathbb{R} \to \mathcal{B}_{\mathcal{U}\to\mathcal{Y}}$ the multiplication operator defined by $(\mathbf{M}_{\hat{F}}\hat{f})(j\omega) := \hat{F}(j\omega)\hat{f}(j\omega)$ for all $\hat{f} \in \mathrm{L}^2_{j\mathbb{R}}(\mathcal{U})$ is bounded and equivalent via the Fourier-Transform isomorphism to an LTI system on the doubly-infinite, continuous-time signal-space $\mathrm{L}^2_{\mathbb{R}}(\mathcal{U})$.

In the following proof, the notion of inner and co-inner functions is required. Let $\mathcal{U}$ and $\mathcal{Y}$ be finite-dimensional Hilbert-spaces. Then a function $\hat{F}(s) \in \mathrm{H}^\infty_{\mathbb{C}+}(\mathcal{B}_{\mathcal{U}\to\mathcal{Y}})$ is said to be inner if $\hat{F}^\sim(s)\hat{F}(s) = I$ and co-inner if $\hat{F}(s)\hat{F}^\sim(s) = I$, where $\hat{F}^\sim(s) := F(-s)^T$ and the superscript T denotes matrix transpose (without conjugation). It can be shown that $(\mathbf{M}_{\hat{F}})^* = \mathbf{\Pi}_{\mathrm{H}^2_{\mathbb{C}+}(\mathcal{U})}\mathbf{M}_{\hat{F}^\sim}$, where recall that the superscript $*$ denotes the Hilbert space adjoint, which in this case is the adjoint on $\mathrm{H}^2_{\mathbb{C}+}(\mathcal{U})$. To see this, note that $\mathbf{M}_{\hat{F}^\sim}$ is the $\mathrm{L}^2_{j\mathbb{R}}(\mathcal{U})$ adjoint of $\mathbf{M}_{\hat{F}}$ and hence, that for any $\hat{f} \in \mathrm{H}^2_{\mathbb{C}+}(\mathcal{U})$ and $\hat{g} \in \mathrm{H}^2_{\mathbb{C}+}(\mathcal{Y})$,

$$\begin{aligned}
\left\langle \hat{g}, \mathbf{M}_{\hat{F}}\hat{f} \right\rangle_{\mathrm{H}^2_{\mathbb{C}+}(\mathcal{Y})} = \left\langle \hat{g}, \mathbf{M}_{\hat{F}}\hat{f} \right\rangle_{\mathrm{L}^2_{j\mathbb{R}}(\mathcal{Y})} = \left\langle \mathbf{M}_{\hat{F}^\sim}\hat{g}, \hat{f} \right\rangle_{\mathrm{L}^2_{j\mathbb{R}}(\mathcal{U})} &= \left\langle \mathbf{\Pi}_{\mathrm{H}^2_{\mathbb{C}+}(\mathcal{U})}\mathbf{M}_{\hat{F}^\sim}\hat{g}, \hat{f} \right\rangle_{\mathrm{L}^2_{j\mathbb{R}}(\mathcal{U})} \\
&= \left\langle \mathbf{\Pi}_{\mathrm{H}^2_{\mathbb{C}+}(\mathcal{U})}\mathbf{M}_{\hat{F}^\sim}\hat{g}, \hat{f} \right\rangle_{\mathrm{H}^2_{\mathbb{C}+}(\mathcal{U})},
\end{aligned}$$

where all of the equalities follow readily from the fact that $\mathrm{H}^2_{\mathbb{C}+}(\mathcal{Y}) \subset \mathrm{L}^2_{j\mathbb{R}}(\mathcal{U})$. That is, $(\mathbf{M}_{\hat{F}})^* = \mathbf{\Pi}_{\mathrm{H}^2_{\mathbb{C}+}(\mathcal{U})}\mathbf{M}_{\hat{F}^\sim}$ as required. So, it follows immediately that if $\hat{F}$ is inner, then $F := \mathbf{L}^{-1}\mathbf{M}_{\hat{F}}\mathbf{L}$ is an isometry on $\mathrm{L}^2_{\mathbb{R}+}(\mathcal{U})$. With this established it is now possible to prove the lemma:

Proof : (of Lemma 3.9) That $\|GX\| = \|X\|$ if $\hat{G}$ is inner is obvious, since in this case G is an isometry on $\mathrm{L}^2_{\mathbb{R}+}(\mathcal{U})$. The rest of the proof to show that $\|XG\| = \|X\|$ when $\hat{G}$ is co-inner, follows an idea suggested in [Vin96]. Since all operators concerned are linear

$$\|XG\| = \|G^*X^*\| = \|\mathbf{\Pi}_{\mathrm{L}^2_{\mathbb{R}+}(\mathcal{U})}\mathbf{F}^{-1}\mathbf{M}_{\hat{G}^\sim}\mathbf{F}X^*\|,$$

[1] Recall the similar result for discrete-time LSI systems (cf. Proposition 2.2)

where all adjoint above are taken on $\mathrm{L}^2_{\mathbb{R}+}(\mathcal{U})$. Furthermore,

$$\begin{aligned}\|\mathbf{\Pi}_{\mathrm{L}^2_{\mathbb{R}+}(\mathcal{U})}\mathbf{F}^{-1}\mathbf{M}_{\hat{G}^\sim}\mathbf{F}X^*\| &\leq \sup_{0\neq y\in\mathrm{L}^2_{\mathbb{R}+}(\mathcal{Y})} \frac{\langle \mathbf{M}_{\hat{G}^\sim}\mathbf{F}X^*y, \mathbf{M}_{\hat{G}^\sim}\mathbf{F}X^*y\rangle_{\mathrm{L}^2_{j\mathbb{R}}(\mathcal{U})}}{\langle y,y\rangle_{\mathrm{L}^2_{\mathbb{R}+}(\mathcal{Y})}} \\ &= \|X^*\| = \|X\|,\end{aligned}$$

where the second equality follows from the fact that $\hat{G}\hat{G}^\sim = I$ and the fact that $\mathbf{M}_{\hat{G}}$ is the $\mathrm{L}^2_{j\mathbb{R}}(\mathcal{U})$ adjoint of $\mathbf{M}_{\hat{G}^\sim}$. Consequently, it is now sufficient to show that

$$\|X\| \leq \|XG\|.$$

Without loss of generality suppose $\|X\| = 1$, which implies that for any $\delta > 0$ there exists a $w \in \mathrm{L}^2_{\mathbb{R}+}(\mathcal{W})$ with $\|w\|_{\mathrm{L}^2_{\mathbb{R}+}(\mathcal{W})} < 1$ such that

$$\|Xw\|_{\mathrm{L}^2_{\mathbb{R}+}(\mathcal{Y})} > 1 - \delta.$$

Furthermore, since X is a bounded, linear operator it is Lipschitz-continuous and hence, there exists an $\epsilon > 0$ such that

$$\|X\tilde{w}\|_{\mathrm{L}^2_{\mathbb{R}+}(\mathcal{Y})} > 1 - \delta \tag{B.1}$$

for all $\tilde{w} \in \mathrm{L}^2_{\mathbb{R}+}(\mathcal{W})$ that satisfy $\|w-\tilde{w}\|_{\mathrm{L}^2_{\mathbb{R}+}(\mathcal{W})} < \epsilon$. Now for notational convenience let $w_{[k]} := \mathbf{U}_{kh}w$ ($k \in \mathbb{Z}^+$), where $\mathbf{U}_{kh}$ is the unilateral shift. Then since X is LTPV (with period h)

$$\|Xw_{[k]}\|_{\mathrm{L}^2_{\mathbb{R}+}(\mathcal{Y})} = \|Xw\|_{\mathrm{L}^2_{\mathbb{R}+}(\mathcal{Y})} > 1 - \delta, \tag{B.2}$$

for all $k \in \mathbb{Z}^+$. Also, since $\mathbf{M}_{\hat{G}}$ is a co-isometry on $\mathrm{L}^2_{j\mathbb{R}}(\mathcal{W})$, it follows that G is a co-isometry on $\mathrm{L}^2_{\mathbb{R}}(\mathcal{W})$ and thus, that there exists a $u \in \mathrm{L}^2_{\mathbb{R}}(\mathcal{U})$ with $\|u\|_{\mathrm{L}^2_{\mathbb{R}}(\mathcal{U})} = \|w\|_{\mathrm{L}^2_{\mathbb{R}+}(\mathcal{W})}$ such that

$$w = Gu$$

and $w_{[k]} = Gu_{[k]}$ where $u_{[k]} := \mathbf{U}_{kh}u$. Then since $u \in \mathrm{L}^2_{\mathbb{R}}(\mathcal{U})$ there exists a $n_\epsilon \in \mathbb{Z}^+$ such that $\|\mathbf{\Pi}_{\mathrm{L}^2_{\mathbb{R}-}(\mathcal{U})}u_{[n_\epsilon]}\|_{\mathrm{L}^2_{\mathbb{R}}(\mathcal{U})} < \epsilon$.[2] Hence, with $u^+_{[n_\epsilon]} := \mathbf{\Pi}_{\mathrm{L}^2_{\mathbb{R}+}(\mathcal{U})}u_{[n_\epsilon]} \in \mathrm{L}^2_{\mathbb{R}+}(\mathcal{U})$ and $\tilde{w}_{[n_\epsilon]} := Gu^+_{[n_\epsilon]}$ it follows that

$$\|w_{[n_\epsilon]} - \tilde{w}_{[n_\epsilon]}\|_{\mathrm{L}^2_{\mathbb{R}+}(\mathcal{W})} = \|G(u_{[n_\epsilon]} - u^+_{[n_\epsilon]})\|_{\mathrm{L}^2_{\mathbb{R}}(\mathcal{W})} = \|G\mathbf{\Pi}_{\mathrm{L}^2_{\mathbb{R}-}(\mathcal{U})}u_{[n_\epsilon]}\|_{\mathrm{L}^2_{\mathbb{R}}(\mathcal{W})} < \epsilon.$$

[2]To see this note that for any $v \in \mathrm{L}^2_{\mathbb{R}}(\mathcal{U})$, $\lim_{\tau\to-\infty}\int_{-\infty}^{\tau} \langle v(t), v(t)\rangle_{\mathcal{U}}\, dt = 0$. Define $v^-_{[k]} := \mathbf{\Pi}_{\mathrm{L}^2_{\mathbb{R}-}(\mathcal{U})}\mathbf{U}_{kh}v$. Then

$$\|v^-_{[k]}\|_{\mathrm{L}^2_{\mathbb{R}}(\mathcal{U})} = \int_{-\infty}^{-kh} \langle v(t), v(t)\rangle_{\mathcal{U}}\, dt$$

and it follows that $\lim_{k\to\infty}\|v^-_{[k]}\|_{\mathrm{L}^2_{\mathbb{R}}(\mathcal{U})} = 0$.

So from equations (B.1) and (B.2),

$$\|X\tilde{w}_{[n_\epsilon]}\|_{\mathrm{L}^2_{\mathbb{R}+}(\mathcal{Y})} = \|XGu^+_{[n_\epsilon]}\|_{\mathrm{L}^2_{\mathbb{R}+}(\mathcal{Y})} > 1 - \delta.$$

Then since $\|u^+_{[n_\epsilon]}\|_{\mathrm{L}^2_{\mathbb{R}+}(\mathcal{U})} \leq \|u\|_{\mathrm{L}^2_{\mathbb{R}}(\mathcal{U})} = \|w\|_{\mathrm{L}^2_{\mathbb{R}+}(\mathcal{W})} < 1$ and it is possible to construct such a $u^+_{[n_\epsilon]} \in \mathrm{L}^2_{\mathbb{R}+}(\mathcal{U})$ for any $\delta > 0$, it follows that

$$\begin{aligned} \|XG\| &:= \sup_{\|u\|_{\mathrm{L}^2_{\mathbb{R}+}(\mathcal{U})} \leq 1} \|XGu\|_{\mathrm{L}^2_{\mathbb{R}+}(\mathcal{U})} \\ &\geq 1 = \|X\|, \end{aligned}$$

as required. ∎

B.2 Computing the Average-Power Gain Matrix for SD Systems

In this section state-space formulae are derived for computing the average-power gain matrix of an SD control-system. The formulae derived are expressible explicitly in terms of the state-space matrices of the continuous-time plant and discrete-time controller. Specifically, the stable, closed-

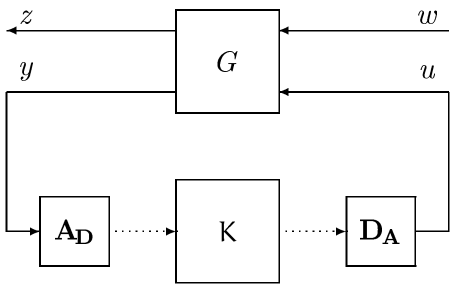

Figure B.1 *SD Closed-Loop*

loop system shown in Figure B.1 is considered, where $\mathbf{A_D}$ is an ideal, h-periodic sampling device synchronised with the zero-order hold $\mathbf{D_A}$ and for fixed $t \in \mathbb{R}$ the signals $w(t) \in \mathbb{R}^{m_1}$, $u(t) \in \mathbb{R}^{m_2}$, $z(t) \in \mathbb{R}^{p_1}$ and $y(t) \in \mathbb{R}^{p_2}$. Let the LTI, generalised plant G and the LSI controller K have state-space realisations

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & 0 & 0 \end{array}\right] \quad \text{and} \quad \mathsf{K} = \left[\begin{array}{c|c} \mathsf{A}_d & \mathsf{B}_d \\ \hline \mathsf{C}_d & \mathsf{D}_d \end{array}\right],$$

where the realisation for G has n states and that for K has l states. In the sequel, state-space formulae are derived for computing the average-power gain matrix of the stable, closed-loop system $P := \mathbf{F}_\ell(G, \mathbf{D_A}\mathsf{K}\mathbf{A_D}) \in \mathscr{P}_{\mathbb{R}^{m_1} \to \mathbb{R}^{p_1}}$. The formulae derived are expressible explicitly in terms of the state-space realisations of G and K.

Recall that the average-power gain matrix is defined in terms of the transfer operator. To characterise this for the system shown in Figure B.1, it is first necessary to obtain a state-space realisation of

$$\mathsf{P} := \mathbf{W} P \mathbf{W}^{-1} := \mathbf{W} \mathbf{F}_\ell(G, \mathbf{D_A} \mathsf{K} \mathbf{A_D}) \mathbf{W}^{-1} \in \mathscr{D}_{\mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{m_1}) \to \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{p_1})}.$$

This can be achieved using the technique used in section 2.5. Absorbing the ideal sample and hold devices into G and the expressing the evolution of the closed-loop in terms of the state at the sampling instances kh, yields the following discrete-time, LSI state-space realisation of P:

$$\begin{bmatrix} \mathsf{x}_{k+1} \\ \mathsf{s}_{k+1} \end{bmatrix} = \mathsf{A}_{cl} \begin{bmatrix} \mathsf{x}_k \\ \mathsf{s}_k \end{bmatrix} + \mathsf{B}_{cl}\, \vec{w}_k(\theta); \quad \begin{bmatrix} \mathsf{x}_0 \\ \mathsf{s}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{B.3a}$$

$$\vec{z}_k(\theta) = \mathsf{C}_{cl} \begin{bmatrix} \mathsf{x}_k \\ \mathsf{s}_k \end{bmatrix} + \mathsf{D}_{cl}\, \vec{w}_k(\theta), \tag{B.3b}$$

where

$$\begin{aligned}
\mathsf{A}_{cl} &:= \begin{bmatrix} \mathsf{A} + \mathsf{B}_2 \mathsf{D}_d C_2 & \mathsf{B}_2 \mathsf{C}_d \\ \mathsf{B}_d C_2 & \mathsf{A}_d \end{bmatrix} : \mathbb{R}^{n+l} \to \mathbb{R}^{n+l}, \\
\mathsf{B}_{cl} &:= \begin{bmatrix} \mathsf{B}_1 \\ 0 \end{bmatrix} : \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{m_1}) \to \mathbb{R}^{n+l}, \\
\mathsf{C}_{cl} &:= \begin{bmatrix} \mathsf{C}_1 + \mathsf{D}_{12} \mathsf{D}_d C_2 & \mathsf{D}_{12} \mathsf{C}_d \end{bmatrix} : \mathbb{R}^{n+l} \to \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{p_1}), \\
\mathsf{D}_{cl} &:= \mathsf{D}_{11} : \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{m_1}) \to \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{p_1}),
\end{aligned}$$

and for any $x \in \mathbb{R}^{n+l}$, $u \in \mathbb{R}^{m_2}$ and $\vec{w} \in \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{m_1})$

$$\mathsf{A} : \mathbb{R}^n \to \mathbb{R}^n, \qquad \mathsf{A}x := e^{hA}x, \tag{B.4a}$$

$$\mathsf{B}_2 : \mathbb{R}^{m_2} \to \mathbb{R}^n, \qquad \mathsf{B}_2 u := \int_0^h e^{(h-\tau)A}\, d\tau\, B_2 u, \tag{B.4b}$$

$$\mathsf{B}_1 : \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{m_1}) \to \mathbb{R}^n, \qquad \mathsf{B}_1 \vec{w} := \int_0^h e^{(h-\tau)A} B_1 \vec{w}(\tau)\, d\tau, \tag{B.4c}$$

$$\mathsf{C}_1 : \mathbb{R}^n \to \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{p_1}), \qquad (\mathsf{C}_1 x)(\theta) := C_1 e^{\theta A} x, \tag{B.4d}$$

$$\mathsf{D}_{11} : \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{m_1}) \to \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{p_1}), \qquad (\mathsf{D}_{11} \vec{w})(\theta) := D_{11} \vec{w}(\theta) + \int_0^\theta C_1 e^{(\theta-\tau)A} B_1 \vec{w}(\tau)\, d\tau \tag{B.4e}$$

and

$$\mathsf{D}_{12} : \mathbb{R}^{m_2} \to \mathrm{L}^2_{\mathbb{H}}(\mathbb{R}^{p_1}), \qquad (\mathsf{D}_{12} u)(\theta) := D_{12} u + \int_0^\theta C_1 e^{(\theta-\tau)A}\, d\tau B_2 u. \tag{B.4f}$$

The transfer operator $\hat{\mathsf{P}}(\lambda)$ is defined to be the symbol of the multiplication operator equivalent of P, where the equivalence is via the $\mathbf{Z}$-Transform isomorphism. Using the state-space realisation given in equation (B.3), it is possible to characterise $\hat{\mathsf{P}}(\lambda)$ in terms of A_{cl}, B_{cl}, C_{cl} and D_{cl} as follows: Let $\vec{z} = \mathsf{P}\vec{w}$, then using equation (B.3) it is straightforward to show that

$$\hat{\vec{z}}(\lambda) := (\mathbf{Z}\,\vec{z})(\lambda) := \sum_{k=0}^{\infty} \vec{z}_k \lambda^k = \sum_{k=0}^{\infty} \lambda^k \left(\sum_{n=0}^{k-1} \mathsf{C}_{cl}\mathsf{A}_{cl}^{k-n-1}\mathsf{B}_{cl}\vec{w}_n + \mathsf{D}_{cl}\vec{w}_k \right) + \mathsf{D}_{cl}\vec{w}_0,$$

which since $\mathrm{spec}(\mathsf{A}_{cl}) \subset \mathbb{D}$ (because P is stable), converges (with respect to $\|\cdot\|_{\mathrm{L}^2_{\mathbb{H}}(\bullet)}$) for all $\lambda \in \mathbb{D}$. Simple rearrangement of terms yields

$$\begin{aligned} \hat{\vec{z}}(\lambda) &= \left(\sum_{l=0}^{\infty} (\lambda^{l+1}\mathsf{C}_{cl}\mathsf{A}_{cl}^{l}\mathsf{B}_{cl}) + \mathsf{D}_{cl} \right) \sum_{q=0}^{\infty} \vec{w}_q \lambda^q \\ &= (\mathsf{C}_{cl}\lambda(I - \lambda\mathsf{A}_{cl})^{-1}\mathsf{B}_{cl} + D_{cl})\hat{\vec{w}}(\lambda) =: \hat{\mathsf{P}}(\lambda)\hat{\vec{w}}(\lambda). \end{aligned}$$

It then follows that (cf. equations (3.2–3.5))

$$\begin{aligned} \mathbf{\Theta}_P(j\omega)\mathbf{\Theta}_P(j\omega) &:= \frac{1}{h}(\mathbf{\Gamma}_P(j\omega))^*\mathbf{\Gamma}_P(j\omega) := (\Omega(j\omega))^*(\hat{\mathsf{P}}(e^{j\omega_0 h}))^*\hat{\mathsf{P}}(e^{j\omega_0 h})\Omega(j\omega), \\ &= \frac{1}{h}(\Omega(j\omega))^*(\mathsf{B}_{cl}^*\hat{\mathsf{A}}(e^{j\omega_0 h})^*\mathsf{C}_{cl}^* + \mathsf{D}_{cl}^*)(\mathsf{C}_{cl}\hat{\mathsf{A}}_{cl}(e^{j\omega_0})\mathsf{B}_{cl} + \mathsf{D}_{cl})\Omega(j\omega), \quad \text{(B.5)} \end{aligned}$$

where $\omega_0 := \omega - \left]\frac{\omega h}{2\pi}\right[\frac{2\pi}{h}$ and $\hat{\mathsf{A}}(e^{j\omega_0})^* := (e^{j\omega_0 h}I - \mathsf{A}_{cl})^{-1}$. By expanding the expression in equation (B.5), it follows that it is necessary to determine finite-dimensional expressions for the following operators:

$$\begin{gathered} \mathsf{A}, \quad \mathsf{B}_2, \quad \mathsf{B}_1\Omega(j\omega), \quad \begin{bmatrix} \mathsf{C}_1^* \\ \mathsf{D}_{12}^* \end{bmatrix} \begin{bmatrix} \mathsf{C}_1 & \mathsf{D}_{12} \end{bmatrix}, \\ \Omega(j\omega)^*\mathsf{D}_{11}^*\mathsf{D}_{11}\Omega(j\omega), \quad \Omega(j\omega)^*\mathsf{D}_{11}^*\begin{bmatrix} \mathsf{C}_1 & \mathsf{D}_{12} \end{bmatrix}. \end{gathered}$$

This can be achieved using equations (B.4) and a technique used in [CG97] (cf. also Chapters 5 and 6), to characterise finite-dimensional combinations of these types of integral operator expressions in terms of TPBVPs. Subsequent computation of $\mathbf{\Theta}_P(j\omega)$ is then a simple matter of standard finite-dimensional matrix arithmetic. State-Space formulae for the required finite-dimensional expressions are given in Theorem B.1 below. Note that the formulae are expressed explicitly in terms of the state-space matrices of the realisations of G and K and involve a single matrix exponential of a Hamiltonian-structured matrix. Since the resultant formula are similar to those obtained for the SD, H^∞-synthesis problem (cf. [CG97] and Theorem 6.1), it is possible to re-structure the formulae presented here in a similar manner to that described in Section 6.3, to render the formulae numerically robust.

The following matrices are required in Theorem B.1. Let $\breve{A} := \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}$,

$$\begin{aligned} \tilde{A} &:= \begin{bmatrix} \breve{A} & \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \\ 0 & (j\omega I_m) \end{bmatrix}, \qquad \tilde{C} := \begin{bmatrix} C_1 & D_{12} & D_{11} \end{bmatrix}, \\ E &:= \begin{bmatrix} \tilde{A} & 0 \\ -\tilde{C}'\tilde{C} & -\tilde{A}' \end{bmatrix} \end{aligned} \tag{B.6}$$

and

$$Q := \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} = e^{hE}, \tag{B.7}$$

where the partitioning of Q conforms with that of E.

Theorem B.1 *Recall that m_1 (respectively n) is the number of inputs to (respectively the number of states of the realisation of) 11-block of G and that m_2 is the number of outputs from K_d. For notational convenience, define $i := n + m_2$, $k := n + m_1$ and $m := m_1 + m_2$. Then,*

$$\begin{aligned} \Omega(j\omega)^*\mathsf{D}_{11}^*\mathsf{D}_{11}\Omega(j\omega) &= -\begin{bmatrix} 0_{m_1,i} & I_m \end{bmatrix} Q_{22}^{-1}Q_{21} \begin{bmatrix} 0_{i,m_1} \\ I_{m_1} \end{bmatrix}, \\ \Omega(j\omega)^*\mathsf{D}_{11}^* \begin{bmatrix} \mathsf{C}_1 & \mathsf{D}_{12} \end{bmatrix} &= -\begin{bmatrix} 0_{m_1,i} & I_m \end{bmatrix} Q_{22}^{-1}Q_{21} \begin{bmatrix} I_k \\ 0_{m_2,k} \end{bmatrix}, \\ \begin{bmatrix} \mathsf{C}_1^* \\ \mathsf{D}_{12}^* \end{bmatrix} \begin{bmatrix} \mathsf{C}_1 & \mathsf{D}_{12} \end{bmatrix} &= -\begin{bmatrix} I_k & 0_{k,m_2} \end{bmatrix} Q_{22}^{-1}Q_{21} \begin{bmatrix} I_k \\ 0_{m_2,k} \end{bmatrix}, \\ \mathsf{B}_1\Omega(j\omega) &= \begin{bmatrix} I_n & 0_{n,m} \end{bmatrix} Q_{11} \begin{bmatrix} 0_{i,m_1} \\ I_{m_1} \end{bmatrix}, \\ \mathsf{A} &= \begin{bmatrix} I_n & 0_{n,m} \end{bmatrix} Q_{11} \begin{bmatrix} I_n \\ 0_{m,n} \end{bmatrix}, \\ \mathsf{B}_2 &= \begin{bmatrix} I_n & 0_{n,m} \end{bmatrix} Q_{11} \begin{bmatrix} 0_{n,m_2} \\ I_{m_2} \\ 0_{m_1,m_2} \end{bmatrix}, \end{aligned}$$

where $0_{p,r}$ denotes an $p \times p$ matrix of zeros and I_p the $p \times p$ identity.

Proof : Consider $\tilde{z} = \Omega(j\omega)^*\mathsf{D}_{11}^*\mathsf{D}_{11}\Omega(j\omega)z$ for some $z \in \mathbb{R}^{m_1}$ and let $\vec{u} := \Omega(j\omega)z$, $\vec{y} := \mathsf{D}_{11}u$ and $\vec{v} := \mathsf{D}_{11}^*\vec{y}$. By the definition of $\Omega(j\omega)$ (cf. equation (3.3)), $\vec{u} := \Omega(j\omega)z$ can be characterised as

$$\frac{d\vec{a}}{d\theta} = (j\omega I_{m_1})\,\vec{a}(\theta); \quad \vec{a}(0) = z, \tag{B.8a}$$

$$u(\theta) = \vec{a}(\theta). \tag{B.8b}$$

It follows then, that $\tilde{z} = \Omega(j\omega)^* \mathring{\vec{v}}$ can be expressed as

$$
\begin{aligned}
\frac{d\mathring{\vec{b}}}{d\theta} &= -(j\omega I_{m_1})^* \mathring{\vec{b}}(\theta) - \mathring{\vec{v}}(\theta); \quad \mathring{\vec{b}}(h) = 0, && \text{(B.9a)}\\
\tilde{z} &= \mathring{\vec{b}}(0). && \text{(B.9b)}
\end{aligned}
$$

From equation (B.4e), it follows that $\mathring{\vec{y}} = \mathrm{D}_{11}\, \mathring{\vec{u}}$ and respectively $\mathring{\vec{v}} = \mathrm{D}_{11}^*\, \mathring{\vec{y}}$, can be characterised by the system of equations

$$
\begin{aligned}
\frac{d\mathring{\vec{g}}}{d\theta} &= \breve{A}\,\mathring{\vec{g}}(\theta) + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \mathring{\vec{u}}(\theta); \quad \mathring{\vec{g}}(0) = 0, && \text{(B.10a)}\\
\mathring{\vec{y}}(\theta) &= \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \mathring{\vec{g}}(\theta) + D_{11}\, \mathring{\vec{u}}(\theta), && \text{(B.10b)}
\end{aligned}
$$

respectively

$$
\begin{aligned}
\frac{d\mathring{\vec{e}}}{d\theta} &= -\breve{A}^*\, \mathring{\vec{e}}(\theta) - \begin{bmatrix} C_1^* \\ D_{12}^* \end{bmatrix} \mathring{\vec{y}}(\theta); \quad \mathring{\vec{e}}(h) = 0, && \text{(B.11a)}\\
\mathring{\vec{v}}(\theta) &= \begin{bmatrix} B_1^* & 0 \end{bmatrix} \mathring{\vec{e}}(\theta) + D_{11}^*\, \mathring{\vec{y}}(\theta). && \text{(B.11b)}
\end{aligned}
$$

Combining equations (B.8–B.11) gives the TPBVP

$$
\frac{d}{d\theta}\left(\begin{bmatrix} \mathring{\vec{g}} \\ \mathring{\vec{a}} \\ \mathring{\vec{e}} \\ \mathring{\vec{b}} \end{bmatrix} \right) = E \begin{bmatrix} \mathring{\vec{g}} \\ \mathring{\vec{a}} \\ \mathring{\vec{e}} \\ \mathring{\vec{b}} \end{bmatrix} (\theta), \quad \begin{bmatrix} \mathring{\vec{g}}(0) \\ \mathring{\vec{a}}(0) \\ \mathring{\vec{e}}(h) \\ \mathring{\vec{b}}(h) \end{bmatrix} = \begin{bmatrix} 0 \\ z \\ 0 \\ 0 \end{bmatrix}
$$

where E is defined in equation (B.6). The solution to this TPBVP satisfies

$$
\left[\begin{array}{c} \mathring{\vec{g}}(h) \\ \mathring{\vec{a}}(h) \\ \hline \mathring{\vec{e}}(h) \\ \mathring{\vec{b}}(h) \end{array}\right] = e^{tE} \left[\begin{array}{c} \mathring{\vec{g}}(0) \\ \mathring{\vec{a}}(0) \\ \hline \mathring{\vec{e}}(0) \\ \mathring{\vec{b}}(0) \end{array}\right].
$$

Using the terminal condition $\begin{bmatrix} \mathring{\vec{e}}(h) \\ \mathring{\vec{b}}(h) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ it follows that

$$
\begin{bmatrix} \mathring{\vec{e}}(0) \\ \mathring{\vec{b}}(0) \end{bmatrix} = -Q_{22}^{-1} Q_{21} \begin{bmatrix} \mathring{\vec{g}}(0) \\ \mathring{\vec{a}}0 \end{bmatrix} = -Q_{22}^{-1} Q_{21} \begin{bmatrix} 0 \\ z \end{bmatrix}.
$$

where Q is defined in equation (B.7). Hence, noting that $\tilde{z} = \mathring{\vec{b}}(0) = \Omega(j\omega)^* \mathrm{D}_{11}^* \mathrm{D}_{11} \Omega(j\omega) z$ gives

$$
\Omega(j\omega)^* \mathrm{D}_{11}^* \mathrm{D}_{11} \Omega(j\omega) = -\begin{bmatrix} 0_{m,l} & I_m \end{bmatrix} Q_{22}^{-1} Q_{21} \begin{bmatrix} 0_{l,m} \\ I_m \end{bmatrix}.
$$

The proof follows in a similar manner for the other operators and is therefore omitted. ∎

C

Appendix to Chapter 5

C.1 Proof of Lemma 5.9

Although Assumption 5.8 holds throughout, in what follows it is convenient to maintain the notational distinction between D_a and D_b. First consider the operator

$$\begin{aligned}
\grave{\mathsf{Q}} :=& \tilde{\mathsf{C}}^* \left(\mathrm{I} - \tilde{\mathsf{D}}\left(\tilde{\mathsf{D}}^*\mathsf{J}\tilde{\mathsf{D}}\right)\tilde{\mathsf{D}}^*\right)\tilde{\mathsf{C}} = \\
& \begin{bmatrix} \mathsf{F}_a^* & (\mathsf{C}_a^* + \mathsf{F}_a^*\mathsf{D}_a^*) \\ \mathsf{F}_b^* & (\mathsf{C}_b^* + \mathsf{F}_b^*\mathsf{D}_b^*) \end{bmatrix} \left(\begin{bmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{bmatrix} \right. \\
& - \begin{bmatrix} \mathrm{I} & \mathrm{I} \\ \mathsf{D}_a & \mathsf{D}_b \end{bmatrix} \left(\begin{bmatrix} \mathrm{I} & \mathsf{D}_a^* \\ \mathrm{I} & \mathsf{D}_b^* \end{bmatrix} \begin{bmatrix} \mathrm{I} & \mathrm{I} \\ \mathsf{D}_a & \mathsf{D}_b \end{bmatrix} - \begin{bmatrix} \gamma^2\mathsf{V}_a^{-2} & 0 \\ 0 & 0 \end{bmatrix} \right)^{-1} \left. \begin{bmatrix} \mathrm{I} & \mathsf{D}_a^* \\ \mathrm{I} & \mathsf{D}_b^* \end{bmatrix} \right) \times \\
& \begin{bmatrix} \mathsf{F}_a & \mathsf{F}_b \\ (\mathsf{C}_a + \mathsf{D}_a\mathsf{F}_a) & (\mathsf{C}_b + \mathsf{D}_b\mathsf{F}_b) \end{bmatrix} := z \mapsto x, \qquad \text{(C.1)}
\end{aligned}$$

where C_a, C_b, D_a, D_b, F_a, F_b and V_a are defined in equations (5.16) and (5.21) and $z \in \mathcal{X}_a \oplus \mathcal{X}_b$. Let

$$\mathring{\vec{y}}_1 := \begin{bmatrix} \mathrm{I} & \mathsf{D}_a^* \\ \mathrm{I} & \mathsf{D}_b^* \end{bmatrix} \mathring{\vec{y}}, \qquad \text{(C.2)}$$

$$\mathring{\vec{u}}_1 := \left(\begin{bmatrix} \mathrm{I} & \mathsf{D}_a^* \\ \mathrm{I} & \mathsf{D}_b^* \end{bmatrix} \begin{bmatrix} \mathrm{I} & \mathrm{I} \\ \mathsf{D}_a & \mathsf{D}_b \end{bmatrix} - \begin{bmatrix} \gamma^2\mathsf{V}_a^{-2} & 0 \\ 0 & 0 \end{bmatrix} \right)^{-1} \mathring{\vec{y}}_1, \qquad \text{(C.3)}$$

$$\mathring{\vec{u}} := \begin{bmatrix} \mathrm{I} & \mathrm{I} \\ \mathsf{D}_a & \mathsf{D}_b \end{bmatrix} \mathring{\vec{u}}_1 \qquad \text{(C.4)}$$

and

$$
\begin{aligned}
\vec{\mathring{v}}_1 &:= \begin{bmatrix} \gamma^2 \mathsf{V}_a^{-2} & 0 \\ 0 & 0 \end{bmatrix} \vec{\mathring{u}}_1 = \begin{bmatrix} \gamma^2(\mathsf{I}+\mathsf{D}_a^*\mathsf{D}_a+\mathsf{B}_a^*\mathsf{X}_a\mathsf{B}_a) & 0 \\ 0 & 0 \end{bmatrix} \vec{\mathring{u}}_1 && \text{(C.5a)} \\
&= \begin{bmatrix} \mathsf{I} & \mathsf{D}_a^* \\ \mathsf{I} & \mathsf{D}_b^* \end{bmatrix} \vec{\mathring{u}} - \vec{\mathring{y}}_1 = \begin{bmatrix} \mathsf{I} & \mathsf{D}_a^* \\ \mathsf{I} & \mathsf{D}_b^* \end{bmatrix} (\vec{\mathring{u}} - \vec{\mathring{y}}), && \text{(C.5b)}
\end{aligned}
$$

where $\vec{\mathring{y}}$ will be defined later and equation (C.5b) follows from equation (C.3). It follows from equations (5.16) and (C.4) that for $0 \le \theta \le h$, [1]

$$
\begin{aligned}
\frac{d\vec{\mathring{a}}}{d\theta} &= \begin{bmatrix} A_a & 0 \\ 0 & A_b \end{bmatrix} \vec{\mathring{a}}(\theta) + \begin{bmatrix} B_a & 0 \\ 0 & B_b \end{bmatrix} \vec{\mathring{u}}_1(\theta); \quad \vec{\mathring{a}}(0) = 0, && \text{(C.6a)} \\
\vec{\mathring{u}}(\theta) &= \begin{bmatrix} 0 & 0 \\ C_a & C_b \end{bmatrix} \vec{\mathring{a}}(\theta) + \begin{bmatrix} I & I \\ D_a & D_b \end{bmatrix} \vec{\mathring{u}}_1(\theta). && \text{(C.6b)}
\end{aligned}
$$

Furthermore (cf. equation (C.5b)),

$$
\begin{aligned}
\frac{d\vec{\mathring{b}}}{d\theta} &= -\begin{bmatrix} A_a^* & 0 \\ 0 & A_b^* \end{bmatrix} \vec{\mathring{b}}(\theta) - \begin{bmatrix} 0 & C_a^* \\ 0 & C_b^* \end{bmatrix} (\vec{\mathring{u}} - \vec{\mathring{y}})(\theta); \quad \vec{\mathring{b}}(h) = 0, && \text{(C.7a)} \\
\vec{\mathring{v}}_1(\theta) &= \begin{bmatrix} B_a^* & 0 \\ 0 & B_b^* \end{bmatrix} \vec{\mathring{b}}(\theta) + \begin{bmatrix} I & D_a^* \\ I & D_b^* \end{bmatrix} (\vec{\mathring{u}} - \vec{\mathring{y}})(\theta) && \text{(C.7b)}
\end{aligned}
$$

and similarly (cf. equation (C.5a)),

$$
\begin{aligned}
\frac{d\vec{\mathring{c}}}{d\theta} &= -A_a^* \vec{\mathring{c}}(\theta) - \begin{bmatrix} C_a^* C_a & 0 \end{bmatrix} \vec{\mathring{a}}(\theta) - \begin{bmatrix} C_a^* D_a & 0 \end{bmatrix} \vec{\mathring{u}}_1(\theta); \quad \vec{\mathring{c}}(h) = \begin{bmatrix} \mathsf{X}_a & 0 \end{bmatrix} \vec{\mathring{a}}(h), && \text{(C.8a)} \\
\vec{\mathring{v}}_1(\theta) &= \begin{bmatrix} \gamma^2 B_a^* \\ 0 \end{bmatrix} \vec{\mathring{c}}(\theta) + \begin{bmatrix} \gamma^2 D_a^* C_a & 0 \\ 0 & 0 \end{bmatrix} \vec{\mathring{a}}(\theta) + \begin{bmatrix} \gamma^2(I + D_a^* D_a) & 0 \\ 0 & 0 \end{bmatrix} \vec{\mathring{u}}_1(\theta), && \text{(C.8b)}
\end{aligned}
$$

where X_a is a self-adjoint, positive-semidefinite, stabilising solution to the algebraic Riccati equation (5.22). Using the two expressions for $\vec{\mathring{v}}_1$ in equations (C.7b) and (C.8b), $\vec{\mathring{u}}_1$ can be expressed in terms of $\vec{\mathring{a}}$, $\vec{\mathring{b}}$, $\vec{\mathring{c}}$ and $\vec{\mathring{y}}$ only. The following TPBVP characterisation of the operator mapping $\vec{\mathring{y}}$ to $\vec{\mathring{u}}$ can then be obtained via elementary state-space manipulation of equations (C.6-C.8):

$$
\begin{aligned}
\frac{d\vec{\mathring{a}}}{d\theta} = {} & \left(\begin{bmatrix} A_a & 0 \\ 0 & A_b \end{bmatrix} - \begin{bmatrix} B_a & 0 \\ 0 & B_b \end{bmatrix} R^{-1} \begin{bmatrix} (1-\gamma^2) D_a^* C_a & D_a^* C_b \\ D_b^* C_a & D_b^* C_b \end{bmatrix} \right) \vec{\mathring{a}}(\theta) \\
& - \begin{bmatrix} B_a & 0 \\ 0 & B_b \end{bmatrix} R^{-1} \begin{bmatrix} B_a^* & 0 \\ 0 & B_b^* \end{bmatrix} \vec{\mathring{b}}(\theta) \\
& + \begin{bmatrix} B_a & 0 \\ 0 & B_b \end{bmatrix} R^{-1} \begin{bmatrix} \gamma^2 B_a^* \\ 0 \end{bmatrix} \vec{\mathring{c}}(\theta) + \begin{bmatrix} B_a & 0 \\ 0 & B_b \end{bmatrix} R^{-1} \begin{bmatrix} I & D_a^* \\ I & D_b^* \end{bmatrix} \vec{\mathring{y}}(\theta), && \text{(C.9a)}
\end{aligned}
$$

[1]For notational convenience time (θ) dependence of the state-space matrices is suppressed.

$$\begin{aligned}
\frac{d\vec{\mathring{b}}}{d\theta} &= -\begin{bmatrix} 0 & C_a^* \\ 0 & C_b^* \end{bmatrix} \left(\begin{bmatrix} 0 & 0 \\ C_a & C_b \end{bmatrix} - \begin{bmatrix} I & I \\ D_a & D_b \end{bmatrix} R^{-1} \begin{bmatrix} (1-\gamma^2)D_a^* C_a & D_a^* C_b \\ D_b^* C_a & D_b^* C_b \end{bmatrix} \right) \vec{\mathring{a}}(\theta) \\
&+ \left(\begin{bmatrix} -A_a^* & 0 \\ 0 & -A_b^* \end{bmatrix} + \begin{bmatrix} 0 & C_a^* \\ 0 & C_b^* \end{bmatrix} \begin{bmatrix} I & I \\ D_a & D_b \end{bmatrix} R^{-1} \begin{bmatrix} B_a^* & 0 \\ 0 & B_b^* \end{bmatrix} \right) \vec{\mathring{b}}(\theta) \\
&- \begin{bmatrix} 0 & C_a^* \\ 0 & C_b^* \end{bmatrix} \begin{bmatrix} I & I \\ D_a & D_b \end{bmatrix} R^{-1} \begin{bmatrix} \gamma^2 B_a^* \\ 0 \end{bmatrix} \vec{\mathring{c}}(\theta) \\
&+ \begin{bmatrix} 0 & C_a^* \\ 0 & C_b^* \end{bmatrix} \left(I - \begin{bmatrix} I & I \\ D_a & D_b \end{bmatrix} R^{-1} \begin{bmatrix} I & D_a^* \\ I & D_b^* \end{bmatrix} \right) \vec{\mathring{y}}(\theta), \qquad \text{(C.9b)}
\end{aligned}$$

$$\begin{aligned}
\frac{d\vec{\mathring{c}}}{d\theta} &= \left(\begin{bmatrix} C_a^* D_a & 0 \end{bmatrix} R^{-1} \begin{bmatrix} (1-\gamma^2)D_a^* C_a & D_a^* C_b \\ D_b^* C_a & D_b^* C_b \end{bmatrix} - \begin{bmatrix} C_a^* C_a & 0 \end{bmatrix} \right) \vec{\mathring{a}}(\theta) \\
&\begin{bmatrix} C_a^* D_a & 0 \end{bmatrix} R^{-1} \begin{bmatrix} B_a^* & 0 \\ 0 & B_b^* \end{bmatrix} \vec{\mathring{b}}(\theta) - \left(A_a^* + \begin{bmatrix} C_a^* D_a & 0 \end{bmatrix} R^{-1} \begin{bmatrix} \gamma^2 B_a^* \\ 0 \end{bmatrix} \right) \vec{\mathring{c}}(\theta) \\
&- \begin{bmatrix} C_A^* D_a & 0 \end{bmatrix} R^{-}1 \begin{bmatrix} I & D_a^* \\ I & D_b^* \end{bmatrix} \vec{\mathring{y}}(\theta) \qquad \text{(C.9c)}
\end{aligned}$$

with boundary conditions

$$\vec{\mathring{a}}(0) = 0, \quad \vec{\mathring{b}}(h) = 0, \quad \vec{\mathring{c}}(h) = \begin{bmatrix} \mathsf{X}_a & 0 \end{bmatrix} \vec{\mathring{a}}(h), \qquad \text{(C.9d)}$$

and

$$\begin{aligned}
\vec{\mathring{u}}(\theta) &= \left(\begin{bmatrix} 0 & 0 \\ C_a & C_b \end{bmatrix} - \begin{bmatrix} I & I \\ D_a & D_b \end{bmatrix} R^{-1} \begin{bmatrix} (1-\gamma^2)D_a^* C_a & D_a^* C_b \\ D_b^* C_a & D_b^* C_b \end{bmatrix} \right) \vec{\mathring{a}}(\theta) \\
&- \begin{bmatrix} I & I \\ D_a & D_b \end{bmatrix} R^{-1} \begin{bmatrix} B_a^* & 0 \\ 0 & B_b^* \end{bmatrix} \vec{\mathring{b}}(\theta) + \begin{bmatrix} I & I \\ D_a & D_b \end{bmatrix} R^{-1} \begin{bmatrix} \gamma^2 B_a^* \\ 0 \end{bmatrix} \vec{\mathring{c}}(\theta) \\
&+ \begin{bmatrix} I & I \\ D_a & D_b \end{bmatrix} R^{-1} \begin{bmatrix} I & D_a^* \\ I & D_b^* \end{bmatrix} \vec{\mathring{y}}(\theta), \qquad \text{(C.9e)}
\end{aligned}$$

where

$$R := \begin{bmatrix} (1-\gamma^2)(I + D_a^* D_a) & (I + D_a^* D_b) \\ (I + D_b^* D_a) & (I + D_b^* D_b) \end{bmatrix}.$$

To see that R is invertible, recall that by Assumption 5.8 $D_a(t) = D_b(t)$ for all t and hence, note that

$$R = \begin{bmatrix} (1-\gamma^2)I & I \\ I & I \end{bmatrix} \begin{bmatrix} (I + D_a^* D_a) & 0 \\ 0 & (I + D_a^* D_a) \end{bmatrix}.$$

Given $\gamma > 0$, both terms on the right-hand side are clearly invertible and the invertibility of R follows.

Now, define

$$
\begin{aligned}
\vec{y}^{\circ} &:= \begin{bmatrix} \mathsf{F}_a & \mathsf{F}_b \\ (\mathsf{C}_a + \mathsf{D}_a\mathsf{F}_a) & (\mathsf{C}_b + \mathsf{D}_b\mathsf{F}_b) \end{bmatrix} z \qquad \text{(C.10)} \\
&= \left(\begin{bmatrix} 0 & 0 \\ \mathsf{C}_a & \mathsf{C}_b \end{bmatrix} - \begin{bmatrix} \mathrm{I} & \mathrm{I} \\ \mathsf{D}_a & \mathsf{D}_b \end{bmatrix} \begin{bmatrix} \mathsf{V}_a^2 & 0 \\ 0 & \mathsf{V}_b^2 \end{bmatrix} \begin{bmatrix} (\mathsf{B}_a^*\mathsf{X}_a\mathsf{A}_a + \mathsf{D}_a^*\mathsf{C}_a) & 0 \\ 0 & (\mathsf{B}_b^*\mathsf{X}_b\mathsf{A}_b + \mathsf{D}_b^*\mathsf{C}_b) \end{bmatrix} \right) z,
\end{aligned}
$$

for some some arbitrary $z \in \mathfrak{X}_a \oplus \mathfrak{X}_b$, where A_a and A_b are defined in equation (5.16) and X_b is a self-adjoint, positive-semidefinite, stabilising solution to the algebraic, Riccati equation (5.22) with the subscript a replaced by b. Moreover, let

$$
\vec{v}^{\circ} := \begin{bmatrix} \mathsf{V}_a^2 & 0 \\ 0 & \mathsf{V}_b^2 \end{bmatrix} \begin{bmatrix} (B_a^*\mathsf{X}_a\mathsf{A}_a + \mathsf{D}_a^*\mathsf{C}_a) & 0 \\ 0 & (B_b^*\mathsf{X}_b\mathsf{A}_b + \mathsf{D}_b^*\mathsf{C}_b) \end{bmatrix} z. \qquad \text{(C.11)}
$$

Then, using equation (5.21a), it follows that

$$
\begin{aligned}
0 &= \begin{bmatrix} (B_a^*\mathsf{X}_a\mathsf{A}_a + \mathsf{D}_a^*\mathsf{C}_a) & 0 \\ 0 & (B_b^*\mathsf{X}_b\mathsf{A}_b + \mathsf{D}_b^*\mathsf{C}_b) \end{bmatrix} z \\
&\quad + \begin{bmatrix} (\mathrm{I} + \mathsf{D}_a^*\mathsf{D}_a + B_a^*\mathsf{X}_a\mathsf{B}_a) & 0 \\ 0 & (\mathrm{I} + \mathsf{D}_b^*\mathsf{D}_b + B_b^*\mathsf{X}_b\mathsf{B}_b) \end{bmatrix} (-\vec{v}^{\circ}). \qquad \text{(C.12)}
\end{aligned}
$$

Again, this can all be expressed in terms of a TPBVP. From equation (5.16), it follows that the initial-condition response of

$$
\begin{aligned}
\frac{d\vec{e}^{\circ}}{d\theta} &= \begin{bmatrix} A_a & 0 \\ 0 & A_a \end{bmatrix} \vec{e}^{\circ}(\theta) + \begin{bmatrix} B_a & 0 \\ 0 & B_b \end{bmatrix} (-\vec{v}^{\circ})(\theta); \quad \vec{e}^{\circ}(0) = z, \qquad \text{(C.13a)} \\
\vec{w}^{\circ}(\theta) &= \begin{bmatrix} C_a & 0 \\ 0 & C_b \end{bmatrix} \vec{e}^{\circ}(\theta) + \begin{bmatrix} D_a & 0 \\ 0 & D_b \end{bmatrix} (-\vec{v}^{\circ})(\theta), \qquad \text{(C.13b)}
\end{aligned}
$$

gives $\begin{bmatrix} \mathsf{C}_a & 0 \\ 0 & \mathsf{C}_b \end{bmatrix} z + \begin{bmatrix} \mathsf{A}_a & 0 \\ 0 & \mathsf{A}_b \end{bmatrix} z$, and that the zero, initial-condition response gives $\begin{bmatrix} \mathsf{B}_a & 0 \\ 0 & \mathsf{B}_b \end{bmatrix} (-\vec{v}^{\circ}) + \begin{bmatrix} \mathsf{D}_a & 0 \\ 0 & \mathsf{D}_b \end{bmatrix} (-\vec{v}^{\circ})$. Similarly, from equation (C.12),

$$
\begin{aligned}
\frac{d\vec{g}^{\circ}}{d\theta} &= -\begin{bmatrix} A_a & 0 \\ 0 & A_b \end{bmatrix} \vec{g}^{\circ}(\theta) - \begin{bmatrix} C_a & 0 \\ 0 & C_b \end{bmatrix} \vec{w}^{\circ}(\theta); \quad \vec{g}^{\circ}(h) = \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix} \vec{e}^{\circ}(h), \qquad \text{(C.14a)} \\
0 &= \begin{bmatrix} B_a^* & 0 \\ 0 & B_b^* \end{bmatrix} \vec{g}^{\circ}(\theta) + \begin{bmatrix} D_a^* & 0 \\ 0 & D_b^* \end{bmatrix} \vec{w}^{\circ}(\theta) + (-\vec{v}^{\circ})(\theta). \qquad \text{(C.14b)}
\end{aligned}
$$

Then, elementary manipulation of equations (C.13) and (C.14) yields the following TPBVP characterisation of the operator mapping z to $\vec{y}^{\circ}$:

$$\begin{aligned}\frac{d\vec{e}^{\circ}}{d\theta} &= \begin{bmatrix}(A_a - B_aR_a^{-1}D_a^*C_a) & 0\\ 0 & (A_b - B_bR_b^{-1}D_b^*C_b)\end{bmatrix}\vec{e}^{\circ}(\theta)\\ &\quad -\begin{bmatrix}B_aR_a^{-1}B_a^* & 0\\ 0 & B_bR_b^{-1}B_b^*\end{bmatrix}\vec{g}^{\circ}(\theta),\end{aligned} \tag{C.15a}$$

$$\begin{aligned}\frac{d\vec{g}^{\circ}}{d\theta} &= -\begin{bmatrix}C_a^*\tilde{R}_a^{-1}C_a & 0\\ 0 & C_b^*\tilde{R}_b^{-1}C_b\end{bmatrix}\vec{e}^{\circ}(\theta)\\ &\quad -\begin{bmatrix}(A_a^* - C_a^*D_aR_a^{-1}B_a^*) & 0\\ 0 & (A_b^* - C_b^*D_bR_b^{-1}B_b^*)\end{bmatrix}\vec{g}^{\circ}(\theta),\end{aligned} \tag{C.15b}$$

with boundary conditions

$$\vec{e}^{\circ}(0) = z, \quad \vec{g}^{\circ}(h) = \begin{bmatrix}\mathsf{X}_a & 0\\ 0 & \mathsf{X}_b\end{bmatrix}\vec{e}^{\circ}(h) \tag{C.15c}$$

and

$$\begin{aligned}\vec{y}^{\circ}(\theta) &= \left(\begin{bmatrix}0 & 0\\ C_A & C_b\end{bmatrix} - \begin{bmatrix}I & I\\ D_a & D_b\end{bmatrix}\begin{bmatrix}R_a^{-1} & 0\\ 0 & R_b^{-1}\end{bmatrix}\begin{bmatrix}D_a & 0\\ 0 & D_b\end{bmatrix}\begin{bmatrix}C_a & 0\\ 0 & C_b\end{bmatrix}\right)\vec{e}^{\circ}(\theta)\\ &\quad -\begin{bmatrix}I & I\\ D_a & D_b\end{bmatrix}\begin{bmatrix}R_a^{-1} & 0\\ 0 & R_b^{-1}\end{bmatrix}\begin{bmatrix}B_a^* & 0\\ 0 & B_b^*\end{bmatrix}\vec{g}^{\circ}(\theta),\end{aligned} \tag{C.15d}$$

where $R_a := I + D_a^*D_a$, $R_b := I + D_b^*D_b$, $\tilde{R}_a := I + D_aD_a^*$ and $\tilde{R}_b := I + D_bD_b^*$.

Note that from equation (C.1), (C.4) and (C.10),

$$\begin{aligned}\mathsf{x} &:= \tilde{\mathsf{C}}^*\left(\mathrm{I} - \tilde{\mathsf{D}}\left(\tilde{\mathsf{D}}^*\mathsf{J}\tilde{\mathsf{D}}\right)\tilde{\mathsf{D}}^*\right)\tilde{\mathsf{C}}z \qquad\qquad (\text{C.16})\\ &= \begin{bmatrix}\mathsf{F}_a^* & (\mathsf{C}_a^* + \mathsf{F}_a^*\mathsf{D}_a^*)\\ \mathsf{F}_b^* & (\mathsf{C}_b^* + \mathsf{F}_b^*\mathsf{D}_b^*)\end{bmatrix}(\vec{y}^{\circ} - \vec{u}^{\circ})\\ &= \left(\begin{bmatrix}0 & \mathsf{C}_a^*\\ 0 & \mathsf{C}_b^*\end{bmatrix}\right.\\ &\quad \left. -\begin{bmatrix}(\mathsf{A}_a^*\mathsf{X}_a\mathsf{B}_a + \mathsf{C}_a^*\mathsf{D}_a) & 0\\ 0 & (\mathsf{A}_b^*\mathsf{X}_b\mathsf{B}_b + \mathsf{C}_b^*\mathsf{D}_b)\end{bmatrix}\begin{bmatrix}\mathsf{V}_a^2 & 0\\ 0 & \mathsf{V}_b^2\end{bmatrix}\begin{bmatrix}\mathrm{I} & \mathsf{D}_a^*\\ \mathrm{I} & \mathsf{D}_b^*\end{bmatrix}\right)(\vec{y}^{\circ} - \vec{u}^{\circ}).\end{aligned}$$

Now define

$$\vec{r}^{\circ} := \begin{bmatrix}\mathsf{V}_a^2 & 0\\ 0 & \mathsf{V}_b^2\end{bmatrix}\begin{bmatrix}\mathrm{I} & \mathsf{D}_a^*\\ \mathrm{I} & \mathsf{D}_b\end{bmatrix}(\vec{y}^{\circ} - \vec{u}^{\circ}), \tag{C.17}$$

so that

$$x = \begin{bmatrix} 0 & \mathsf{C}_a^* \\ 0 & \mathsf{C}_b^* \end{bmatrix} (\vec{y}^\circ - \vec{u}^\circ) + \begin{bmatrix} (\mathsf{A}_a^* \mathsf{X}_a \mathsf{B}_a + \mathsf{C}_a^* \mathsf{D}_a) & 0 \\ 0 & (\mathsf{A}_b^* \mathsf{X}_b \mathsf{B}_b + \mathsf{C}_b^* \mathsf{D}_b) \end{bmatrix} (-\vec{r}^\circ).$$

As in the preceding paragraphs it is possible to express all of this in terms of a TPBVP.

$$\frac{d\vec{m}^\circ}{d\theta} = \begin{bmatrix} A_a & 0 \\ 0 & A_b \end{bmatrix} \vec{m}^\circ(\theta) + \begin{bmatrix} B_a & 0 \\ 0 & B_b \end{bmatrix} (-\vec{r}^\circ)(\theta); \quad \vec{m}^\circ(0) = 0, \tag{C.18a}$$

$$\vec{p}^\circ = \begin{bmatrix} C_a & 0 \\ 0 & C_b \end{bmatrix} \vec{m}^\circ(\theta) + \begin{bmatrix} D_a & 0 \\ 0 & D_b \end{bmatrix} (-\vec{r}^\circ)(\theta), \tag{C.18b}$$

$$\begin{aligned} \frac{d\vec{n}^\circ}{d\theta} &= -\begin{bmatrix} A_a^* & 0 \\ 0 & A_b^* \end{bmatrix} \vec{n}^\circ(\theta) - \begin{bmatrix} C_a^* & 0 \\ 0 & C_b^* \end{bmatrix} \vec{p}^\circ(\theta) \\ &\quad - \begin{bmatrix} 0 & C_a^* \\ 0 & C_b^* \end{bmatrix} (\vec{y}^\circ - \vec{u}^\circ)(\theta); \quad \vec{n}^\circ(h) = \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix} \vec{m}^\circ(h), \end{aligned} \tag{C.19a}$$

$$x = \vec{n}^\circ(0). \tag{C.19b}$$

Note that in equation (C.19), the final-condition response plus the response to the forcing term $\vec{p}^\circ$ gives $\begin{bmatrix} (\mathsf{A}_a^* \mathsf{X}_a \mathsf{B}_a + \mathsf{C}_a^* \mathsf{D}_a) & 0 \\ 0 & (\mathsf{A}_b^* \mathsf{X}_b \mathsf{B}_b + \mathsf{C}_b^* \mathsf{D}_b) \end{bmatrix} (-\vec{r}^\circ)$ and that the response to the forcing term $(\vec{y}^\circ - \vec{u}^\circ)$ gives $\begin{bmatrix} 0 & \mathsf{C}_a^* \\ 0 & \mathsf{C}_b^* \end{bmatrix} (\vec{y}^\circ - \vec{u}^\circ)$. Using equation (C.17), $\vec{r}^\circ$ can be expressed in terms of $\vec{m}^\circ$, $\vec{n}^\circ$ and $(\vec{y}^\circ - \vec{u}^\circ)$ only:

$$\vec{r}^\circ(\theta) = \begin{bmatrix} R_a^{-1} & 0 \\ 0 & R_b^{-1} \end{bmatrix} \left(\begin{bmatrix} B_a^* & 0 \\ 0 & B_b^* \end{bmatrix} \vec{n}^\circ(\theta) + \begin{bmatrix} D_a^* C_a & 0 \\ 0 & D_b^* C_b \end{bmatrix} \vec{m}^\circ(\theta) + \begin{bmatrix} I & D_a^* \\ I & D_b^* \end{bmatrix} (\vec{y}^\circ - \vec{u}^\circ) \right).$$

Substituting this into, and further manipulating equations (C.18) and (C.19), yields the following TPBVP characterisation of the operator mapping $(\vec{y}^\circ - \vec{u}^\circ)$ to x:

$$\begin{aligned} \frac{d\vec{m}^\circ}{d\theta} &= \begin{bmatrix} (A_a - B_a R_a^{-1} D_a^* C_a) & 0 \\ 0 & (A_b - B_b R_b^{-1} D_b^* C_b) \end{bmatrix} \vec{m}^\circ(\theta) - \begin{bmatrix} B_a R_a^{-1} B_a^* & 0 \\ 0 & B_b R_b^{-1} B_b^* \end{bmatrix} \vec{n}^\circ(\theta) \\ &\quad - \begin{bmatrix} B_a & 0 \\ 0 & B_b \end{bmatrix} \begin{bmatrix} R_a^{-1} & 0 \\ 0 & R_b^{-1} \end{bmatrix} \begin{bmatrix} I & D_a^* \\ I & D_b^* \end{bmatrix} (\vec{y}^\circ - \vec{u}^\circ)(\theta), \end{aligned} \tag{C.20a}$$

$$\begin{aligned} \frac{d\vec{n}^\circ}{d\theta} &= -\begin{bmatrix} C_a^* \tilde{R}_a^{-1} C_a & 0 \\ 0 & C_b^* \tilde{R}_b^{-1} C_b \end{bmatrix} \vec{m}^\circ(\theta) - \begin{bmatrix} (A_a^* - C_a^* D_a R_a^{-1} B_a^*) & 0 \\ 0 & (A_b^* - C_b^* D_b R_b^{-1} B_b^*) \end{bmatrix} \vec{n}^\circ(\theta) \\ &\quad - \left(\begin{bmatrix} 0 & C_a^* \\ 0 & C_b^* \end{bmatrix} - \begin{bmatrix} C_a^* & 0 \\ 0 & C_b^* \end{bmatrix} \begin{bmatrix} D_a & 0 \\ 0 & D_b \end{bmatrix} \begin{bmatrix} R_a^{-1} & 0 \\ 0 & R_b^{-1} \end{bmatrix} \begin{bmatrix} I & D_a^* \\ I & D_b^* \end{bmatrix} \right) (\vec{y}^\circ - \vec{u}^\circ)(\theta), \end{aligned} \tag{C.20b}$$

with boundary conditions

$$\vec{m}^{\circ}(0) = 0, \quad \vec{n}^{\circ}(h) = \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix} \vec{m}^{\circ}(h) \tag{C.20c}$$

and

$$x = \vec{n}^{\circ}(0). \tag{C.20d}$$

Finally, combining equations (C.9-C.20) yields the TPBVP

$$\frac{d}{d\theta}\left(\begin{bmatrix} \vec{m}^{\circ} \\ \vec{a}^{\circ} \\ \vec{e}^{\circ} \\ \vec{c}^{\circ} \\ \vec{n}^{\circ} \\ \vec{b}^{\circ} \\ \vec{g}^{\circ} \end{bmatrix}\right) = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} & E_{17} \\ E_{21} & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} & E_{27} \\ E_{31} & E_{32} & E_{33} & E_{34} & E_{35} & E_{36} & E_{37} \\ E_{41} & E_{42} & E_{43} & E_{44} & E_{45} & E_{46} & E_{47} \\ E_{51} & E_{52} & E_{53} & E_{54} & E_{55} & E_{56} & E_{57} \\ E_{61} & E_{62} & E_{63} & E_{64} & E_{65} & E_{66} & E_{67} \\ E_{71} & E_{72} & E_{73} & E_{74} & E_{75} & E_{76} & E_{77} \end{bmatrix} \begin{bmatrix} \vec{m}^{\circ} \\ \vec{a}^{\circ} \\ \vec{e}^{\circ} \\ \vec{c}^{\circ} \\ \vec{n}^{\circ} \\ \vec{b}^{\circ} \\ \vec{g}^{\circ} \end{bmatrix}(\theta),$$

with boundary conditions

$$\begin{bmatrix} \vec{m}^{\circ}(0) \\ \vec{a}^{\circ}(0) \\ \vec{e}^{\circ}(0) \\ \vec{c}^{\circ}(h) \\ \vec{n}^{\circ}(h) \\ \vec{b}^{\circ}(h) \\ \vec{g}^{\circ}(h) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z \\ \begin{bmatrix} \mathsf{X}_a & 0 \end{bmatrix} \vec{a}^{\circ}(h) \\ \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix} \vec{m}^{\circ}(h) \\ 0 \\ \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix} \vec{e}^{\circ}(h) \end{bmatrix},$$

and

$$x = \vec{n}^{\circ}(0),$$

where the E_{ij}'s are defined in Appendix C.2. This completely characterises the operator

$$\tilde{\mathsf{C}}^* \left(\mathsf{I} - \tilde{\mathsf{D}} \left(\tilde{\mathsf{D}}^* \mathsf{J} \tilde{\mathsf{D}}\right) \tilde{\mathsf{D}}^*\right) \tilde{\mathsf{C}} := z \mapsto x.$$

Let $\mathsf{E} := \mathbf{\Phi}_{E(t)}(h,0)$, where $\mathbf{\Phi}_{E(t)}(t,\tau)$ is the state-transition operator associated with $E(t)$. It follows readily that

$$-\tilde{\mathsf{E}}\begin{bmatrix} \vec{\mathring{c}}(0) \\ \vec{\mathring{n}}(0) \\ \vec{\mathring{b}}(0) \\ \vec{\mathring{g}}(0) \end{bmatrix} = \begin{bmatrix} \left(\mathsf{E}_{43} - \begin{bmatrix} \mathsf{X}_a & 0 \end{bmatrix}\mathsf{E}_{23}\right) \\ \left(\mathsf{E}_{53} - \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix}\mathsf{E}_{13}\right) \\ \mathsf{E}_{63} \\ \left(\mathsf{E}_{73} - \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix}\mathsf{E}_{33}\right) \end{bmatrix} z, \tag{C.21}$$

where

$$\tilde{\mathsf{E}} = \begin{bmatrix} \left(\mathsf{E}_{44} - \begin{bmatrix} \mathsf{X}_a & 0 \end{bmatrix}\mathsf{E}_{24}\right) & \left(\mathsf{E}_{45} - \begin{bmatrix} \mathsf{X}_a & 0 \end{bmatrix}\mathsf{E}_{25}\right) & \left(\mathsf{E}_{46} - \begin{bmatrix} \mathsf{X}_a & 0 \end{bmatrix}\mathsf{E}_{26}\right) & \left(\mathsf{E}_{47} - \begin{bmatrix} \mathsf{X}_a & 0 \end{bmatrix}\mathsf{E}_{27}\right) \\ \left(\mathsf{E}_{54} - \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix}\mathsf{E}_{14}\right) & \left(\mathsf{E}_{55} - \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix}\mathsf{E}_{15}\right) & \left(\mathsf{E}_{56} - \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix}\mathsf{E}_{16}\right) & \left(\mathsf{E}_{57} - \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix}\mathsf{E}_{17}\right) \\ \mathsf{E}_{64} & \mathsf{E}_{65} & \mathsf{E}_{66} & \mathsf{E}_{67} \\ \left(\mathsf{E}_{74} - \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix}\mathsf{E}_{34}\right) & \left(\mathsf{E}_{75} - \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix}\mathsf{E}_{35}\right) & \left(\mathsf{E}_{76} - \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix}\mathsf{E}_{36}\right) & \left(\mathsf{E}_{77} - \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix}\mathsf{E}_{37}\right) \end{bmatrix}.$$

To see that $\tilde{\mathsf{E}}$ is invertible, take $z = 0$ and suppose that there exists a non-zero

$$\begin{bmatrix} \vec{\mathring{c}}_0 \\ \vec{\mathring{n}}_0 \\ \vec{\mathring{b}}_0 \\ \vec{\mathring{g}}_0 \end{bmatrix} \in \mathcal{K}_{\tilde{\mathsf{E}}}. \tag{C.22}$$

This clearly satisfies equation (C.21) and since $z = 0$, the unique solution to equations (C.13) and (C.14) is $\vec{\mathring{y}} \equiv \vec{\mathring{e}} \equiv \vec{\mathring{g}} \equiv 0$. Consequently, the unique solution to equations (C.6-C.8) is $\vec{\mathring{a}} \equiv \vec{\mathring{b}} \equiv \vec{\mathring{c}} \equiv 0$ and thus, the only possible non-zero element of (C.22) is $\vec{\mathring{n}}_0$. This contradicts linearity of the operator mapping z to $x = \vec{\mathring{n}}_0$ and hence, $\mathcal{K}_{\tilde{\mathsf{E}}} = \{0\}$. Equation (5.29) (which constitutes a finite-dimensional characterisation of $\grave{\mathsf{Q}}$) now follows directly from equation (C.21).

Consider the operator $\grave{\mathsf{A}}$. Let z be an arbitrary element in $\mathcal{X}_a \oplus \mathcal{X}_b$ and recall that

$$\begin{aligned} \grave{\mathsf{A}} z &= \left(\mathsf{A} - \mathsf{B}\left(\tilde{\mathsf{D}}^*\mathsf{J}\tilde{\mathsf{D}}\right)^{-1}\tilde{\mathsf{D}}^*\mathsf{J}\tilde{\mathsf{C}}\right) z \\ &= \begin{bmatrix} (\mathsf{A}_a + \mathsf{B}_a\mathsf{F}_a) & 0 \\ 0 & (\mathsf{A}_b + \mathsf{B}_b\mathsf{F}_b) \end{bmatrix} z + \begin{bmatrix} \mathsf{B}_a & 0 \\ 0 & \mathsf{B}_b \end{bmatrix} \vec{\mathring{u}}_1 \\ &= \begin{bmatrix} \mathsf{A}_a & 0 \\ 0 & \mathsf{A}_b \end{bmatrix} z + \begin{bmatrix} \mathsf{B}_a & 0 \\ 0 & \mathsf{B}_b \end{bmatrix}(-\vec{\mathring{v}}) - \begin{bmatrix} \mathsf{B}_a & 0 \\ 0 & \mathsf{B}_b \end{bmatrix} \vec{\mathring{u}}_1 \\ &= \vec{\mathring{e}}(h) - \vec{\mathring{a}}(h), \end{aligned}$$

where $\vec{u}^{\circ}_1$, $\vec{a}^{\circ}$, $\vec{v}^{\circ}$ and $\vec{e}^{\circ}$ are defined in equations (C.3), (C.6a), (C.11) and (C.13a) respectively. Since z is arbitrary, the finite-dimensional characterisation of $\grave{\mathrm{A}}$ given in equation (5.31) follows directly from equation (C.21).

Similarly, consider the bounded, linear operator

$$\begin{aligned}\grave{\mathrm{R}} &:= \mathrm{B}\left(\tilde{\mathrm{D}}^*\mathrm{J}\tilde{\mathrm{D}}\right)\mathrm{B}^* \\ &= \begin{bmatrix} \mathrm{B}_a & 0 \\ 0 & \mathrm{B}_b \end{bmatrix}\left(\begin{bmatrix} \mathrm{I} & \mathrm{D}_a^* \\ \mathrm{I} & \mathrm{D}_b^* \end{bmatrix}\begin{bmatrix} \mathrm{I} & \mathrm{I} \\ \mathrm{D}_a & \mathrm{D}_b \end{bmatrix} - \begin{bmatrix} \gamma^2\mathrm{V}_a^{-2} & 0 \\ 0 & 0 \end{bmatrix}\right)^{-1}\begin{bmatrix} \mathrm{B}_a^* & 0 \\ 0 & \mathrm{B}_b^* \end{bmatrix} :=z \mapsto x,\end{aligned}$$

for $z \in \mathcal{X}_a \oplus \mathcal{X}_b$. If the initial condition in equation (C.13a) is set to 0 then $\vec{e}^{\circ}(\theta) \equiv \vec{g}^{\circ}(\theta) \equiv \vec{y}^{\circ}(\theta) \equiv 0$. Now replacing equation (C.2) by

$$\vec{y}^{\circ}_1 = \begin{bmatrix} \mathrm{B}_a^* & 0 \\ 0 & \mathrm{B}_b^* \end{bmatrix} z,$$

equation (C.5) becomes

$$\begin{aligned}\vec{v}^{\circ}_1 &:= \begin{bmatrix} \gamma^2\mathrm{V}_a^{-2} & 0 \\ 0 & 0 \end{bmatrix}\vec{u}^{\circ}_1 = \begin{bmatrix} \gamma^2(\mathrm{I}+\mathrm{D}_a^*\mathrm{D}_a+\mathrm{B}_a^*\mathrm{X}_a\mathrm{B}_a) & 0 \\ 0 & 0 \end{bmatrix}\vec{u}^{\circ}_1 && \text{(C.23a)} \\ &= \begin{bmatrix} \mathrm{I} & \mathrm{D}_a^* \\ \mathrm{I} & \mathrm{D}_b^* \end{bmatrix}\vec{u}^{\circ} - \vec{y}^{\circ}_1 = \begin{bmatrix} \mathrm{I} & \mathrm{D}_a^* \\ \mathrm{I} & \mathrm{D}_b^* \end{bmatrix}\vec{u}^{\circ} + \begin{bmatrix} \mathrm{B}_a^* & 0 \\ 0 & \mathrm{B}_b^* \end{bmatrix}(-z). && \text{(C.23b)}\end{aligned}$$

It then follows from equation (5.16) and (C.23b) that

$$\begin{aligned}\frac{d\vec{b}^{\circ}}{d\theta} &= -\begin{bmatrix} A_a^* & 0 \\ 0 & A_b^* \end{bmatrix}\vec{b}^{\circ}(\theta) - \begin{bmatrix} 0 & C_a^* \\ 0 & C_b^* \end{bmatrix}\vec{u}^{\circ}(\theta); \quad \vec{b}^{\circ}(h) = -z, && \text{(C.24a)} \\ \vec{v}^{\circ}_1(\theta) &= \begin{bmatrix} B_a^* & 0 \\ 0 & B_b^* \end{bmatrix}\vec{b}^{\circ}(\theta) + \begin{bmatrix} I & D_a^* \\ I & D_b^* \end{bmatrix}\vec{u}^{\circ}(\theta). && \text{(C.24b)}\end{aligned}$$

The only differences between equation (C.24) and equation (C.7) are the final condition and since it is identically 0, $\vec{y}^{\circ}$ has been omitted in (C.24). Similarly, using equation (C.23a), $\vec{v}^{\circ}_1$ can be expressed in terms of $\vec{a}^{\circ}$ and $\vec{c}^{\circ}$ defined in equations (C.6) and (C.8). Furthermore, it can be shown that

$$x = \vec{a}^{\circ}(h).$$

Since $\vec{m}^{\circ}$ and $\vec{n}^{\circ}$ do not come into the equations defining $\vec{a}^{\circ}$, $\vec{b}^{\circ}$ and $\vec{c}^{\circ}$, they can be ignored. Hence, it follows that the action of the operator $\grave{\mathrm{R}}$ can be expressed in terms of a TPBVP with the same system of differential equations a the TPBVP characterisation of $\grave{\mathsf{Q}}$ but different boundary conditions, given by

$$\begin{bmatrix} \vec{m}(0) \\ \vec{a}(0) \\ \vec{e}(0) \\ \vec{c}(h) \\ \theta(h) \\ \vec{b}(h) \\ \vec{g}(h) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \begin{bmatrix} \mathsf{X}_a & 0 \end{bmatrix} \vec{a}(h) \\ \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix} \vec{m}(h) \\ -z \\ \begin{bmatrix} \mathsf{X}_a & 0 \\ 0 & \mathsf{X}_b \end{bmatrix} \vec{e}(h) \end{bmatrix}.$$

Consequently, it is possible to write

$$\grave{\mathsf{R}} = \begin{bmatrix} \mathsf{E}_{24} & \mathsf{E}_{26} \end{bmatrix} \begin{bmatrix} -\mathsf{E}_{64} & -\mathsf{E}_{66} \\ \left(\mathsf{E}_{44} - \begin{bmatrix} \mathsf{X}_a & 0 \end{bmatrix} \mathsf{E}_{24}\right) & \left(\mathsf{E}_{46} - \begin{bmatrix} \mathsf{X}_a & 0 \end{bmatrix} \mathsf{E}_{26}\right) \end{bmatrix}^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

It remains to establish finite-dimensional expressions for the operators that appear in the two algebraic, Riccati equations associated with the normalised, coprime-factorisation of the two, original, LPTV systems. Consider

$$x := \begin{bmatrix} \grave{\mathsf{R}}_a & 0 \\ 0 & \grave{\mathsf{R}}_b \end{bmatrix} z := \begin{bmatrix} \mathsf{B}_a(\mathsf{I} + \mathsf{D}_a^*\mathsf{D}_a)^{-1}\mathsf{B}_a^* & 0 \\ 0 & \mathsf{B}_b(\mathsf{I} + \mathsf{D}_b^*\mathsf{D}_b)^{-1}\mathsf{B}_b^* \end{bmatrix} z,$$

for some arbitrary $z \in \mathcal{X}_a \oplus \mathcal{X}_b$. Define

$$\vec{w} := \begin{bmatrix} (\mathsf{I} + \mathsf{D}_a^*\mathsf{D}_a)^{-1} & 0 \\ 0 & (\mathsf{I} + \mathsf{D}_b^*\mathsf{D}_b)^{-1} \end{bmatrix} \begin{bmatrix} \mathsf{B}_a^* & 0 \\ 0 & \mathsf{B}_b^* \end{bmatrix} z, \tag{C.25a}$$

and

$$\vec{v} := \begin{bmatrix} \mathsf{D}_a & 0 \\ 0 & \mathsf{D}_b \end{bmatrix} \vec{w}. \tag{C.25b}$$

Then, from equation (C.25a),

$$\vec{w} = \begin{bmatrix} \mathsf{B}_a^* & 0 \\ 0 & \mathsf{B}_b^* \end{bmatrix} z - \begin{bmatrix} \mathsf{D}_a^* & 0 \\ 0 & \mathsf{D}_b^* \end{bmatrix} \vec{v}. \tag{C.25c}$$

Using equations (5.16), it follows from equation (C.25b), that

$$\frac{d\vec{e}}{d\theta} = \begin{bmatrix} A_a & 0 \\ 0 & A_a \end{bmatrix} \vec{e}(\theta) + \begin{bmatrix} B_a & 0 \\ 0 & B_b \end{bmatrix} \vec{w}(\theta); \quad \vec{e}(0) = 0, \tag{C.26a}$$

$$\vec{w}(\theta) = \begin{bmatrix} C_a & 0 \\ 0 & C_b \end{bmatrix} \vec{e}(\theta) + \begin{bmatrix} D_a & 0 \\ 0 & D_b \end{bmatrix} \vec{w}(\theta), \tag{C.26b}$$

and from equation (C.25c), that

$$\frac{d\mathring{\vec{g}}}{d\theta} = -\begin{bmatrix} A_a^* & 0 \\ 0 & A_b^* \end{bmatrix}\mathring{\vec{g}}(\theta) - \begin{bmatrix} C_a^* & 0 \\ 0 & C_b^* \end{bmatrix}\mathring{\vec{v}}(\theta); \quad \mathring{\vec{g}}(h) = -z, \tag{C.27a}$$

$$\mathring{\vec{w}} = -\begin{bmatrix} B_a^* & 0 \\ 0 & B_b^* \end{bmatrix}\mathring{\vec{g}}(\theta) - \begin{bmatrix} D_a^* & 0 \\ 0 & D_b^* \end{bmatrix}\mathring{\vec{v}}(\theta), \tag{C.27b}$$

where the final-condition response gives $\begin{bmatrix} \mathsf{B}_a^* & 0 \\ 0 & \mathsf{B}_b^* \end{bmatrix} z$ and the zero, final-condition response gives $-\begin{bmatrix} \mathsf{D}_a^* & 0 \\ 0 & \mathsf{D}_b^* \end{bmatrix}\mathring{\vec{v}}$. Furthermore,

$$x := \begin{bmatrix} \mathsf{B}_a & 0 \\ 0 & \mathsf{B}_b \end{bmatrix}\mathring{\vec{w}} = \mathring{\vec{e}}(h).$$

Elementary manipulation of equations (C.27) and (C.26) leads to a TPBVP with dynamics described by equations (C.15a) and (C.15b) subject to the boundary conditions

$$\mathring{\vec{e}}(0) = 0 \quad \text{and} \quad \mathring{\vec{g}}(h) = -z.$$

By the structure of E and consequently, that of $\mathsf{E} := \mathbf{\Phi}_{E(t)}(h, 0)$, it follows that

$$\begin{bmatrix} \mathring{\vec{e}}(h) \\ \mathring{\vec{g}}(h) \end{bmatrix} = \begin{bmatrix} \mathsf{E}_{33} & \mathsf{E}_{37} \\ \mathsf{E}_{73} & \mathsf{E}_{77} \end{bmatrix}\begin{bmatrix} \mathring{\vec{e}}(0) \\ \mathring{\vec{g}}(0) \end{bmatrix} \tag{C.28}$$

and hence, since z is arbitrary, that

$$\begin{bmatrix} \grave{\mathsf{R}}_a & 0 \\ 0 & \grave{\mathsf{R}}_b \end{bmatrix} = -\mathsf{E}_{37}\mathsf{E}_{77}^{-1}.$$

Note that $\begin{bmatrix} \mathsf{E}_{33} & \mathsf{E}_{37} \\ \mathsf{E}_{73} & \mathsf{E}_{77} \end{bmatrix}$ is invertible since it is simply the state-transmission operator associated with the differential system described by equations (C.15a) and (C.15b). To see that E_{77} is also invertible, suppose there exist a non-zero $\tilde{\mathring{\vec{g}}} \in \mathcal{K}_{\mathsf{E}_{77}}$ and take $z = \mathring{\vec{g}}(h) = 0$. Then by linearity, $x = \mathring{\vec{e}}(h) = 0$ and hence from equation (C.28) that

$$\begin{bmatrix} \mathsf{E}_{37} \\ \mathsf{E}_{77} \end{bmatrix}\tilde{\mathring{\vec{g}}} = 0,$$

which contradicts the invertibility of $\begin{bmatrix} \mathsf{E}_{33} & \mathsf{E}_{37} \\ \mathsf{E}_{73} & \mathsf{E}_{77} \end{bmatrix}$. Hence, $\mathcal{K}_{\mathsf{E}_{77}} = \{0\}$ and E_{77} is invertible.

The expression for $\grave{\mathsf{Q}}_a$, $\grave{\mathsf{Q}}_b$, $\grave{\mathsf{A}}_a$ and $\grave{\mathsf{A}}_b$ presented in the lemma statement follow similarly. This completes the proof of Lemma 5.9. ∎

C.2 Definition of Matrices Used in Lemma 5.9

For notational convenience time-dependence is not expressed explicitly.

$$
\begin{aligned}
R_a &:= I + D_a^* D_a, \quad \tilde{R}_a := I + D_a D_a^*, \\
R_b &:= I + D_b^* D_b, \quad \tilde{R}_b := I + D_b D_b^*, \\
R &:= \begin{bmatrix} (1-\gamma^2) R_a & I + D_a^* D_b \\ I + D_b^* D_a & R_b \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
S_1 &:= \begin{bmatrix} B_a & 0 \\ 0 & B_b \end{bmatrix} \begin{bmatrix} R_a^{-1} & 0 \\ 0 & R_b^{-1} \end{bmatrix} \begin{bmatrix} I & D_a^* \\ I & D_b^* \end{bmatrix}, \\
S_2 &:= \left(I - \begin{bmatrix} I & I \\ D_a & D_b \end{bmatrix} R^{-1} \begin{bmatrix} I & D_a^* \\ I & D_b^* \end{bmatrix} \right), \\
S_3 &:= \left(\begin{bmatrix} 0 & 0 \\ C_a & C_b \end{bmatrix} - \begin{bmatrix} I & I \\ D_a & D_b \end{bmatrix} \begin{bmatrix} R_a^{-1} & 0 \\ 0 & R_b^{-1} \end{bmatrix} \begin{bmatrix} D_a^* & 0 \\ 0 & D_b^* \end{bmatrix} \begin{bmatrix} C_a & 0 \\ 0 & C_b \end{bmatrix} \right), \\
S_4 &:= \left(\begin{bmatrix} 0 & 0 \\ C_a & C_b \end{bmatrix} - \begin{bmatrix} I & I \\ D_a & D_b \end{bmatrix} R^{-1} \begin{bmatrix} (1-\gamma^2) D_a^* C_a & D_a^* C_b \\ D_b^* C_a & D_b^* C_b \end{bmatrix} \right),
\end{aligned}
$$

$$
\begin{aligned}
E_{11} &:= \begin{bmatrix} A_a - B_a R_a^{-1} D_a^* C_a & 0 \\ 0 & A_b - B_b R_b^{-1} D_b^* C_b \end{bmatrix}, \\
E_{12} &:= S_1 S_2 S_4, \\
E_{13} &:= -S_1 S_2 S_3, \\
E_{14} &:= S_1 \begin{bmatrix} I & I \\ D_a & D_b \end{bmatrix} R^{-1} \begin{bmatrix} \gamma^2 B_a^* \\ 0 \end{bmatrix}, \\
E_{15} &:= - \begin{bmatrix} B_a R_a^{-1} B_a^* & 0 \\ 0 & B_b R_b^{-1} B_b^* \end{bmatrix}, \\
E_{16} &:= -S_1 \begin{bmatrix} I & I \\ D_a & D_b \end{bmatrix} R^{-1} \begin{bmatrix} B_a^* & 0 \\ 0 & B_b^* \end{bmatrix}, \\
E_{17} &:= S_1 S_2 S_1^*,
\end{aligned}
$$

$$\begin{aligned}
E_{21} &:= 0,\\
E_{22} &:= \left(\begin{bmatrix} A_a & 0 \\ 0 & A_b \end{bmatrix} - \begin{bmatrix} B_a & 0 \\ 0 & B_b \end{bmatrix} R^{-1} \begin{bmatrix} (1-\gamma^2)D_a^*C_a & D_a^*C_b \\ D_b^*C_a & D_b^*C_b \end{bmatrix}\right),\\
E_{23} &:= \begin{bmatrix} B_a & 0 \\ 0 & B_b \end{bmatrix} R^{-1} \begin{bmatrix} I & D_a^* \\ I & D_b^* \end{bmatrix} S_3,\\
E_{24} &:= \begin{bmatrix} B_a & 0 \\ 0 & B_b \end{bmatrix} R^{-1} \begin{bmatrix} \gamma^{-2}B_a^* \\ 0 \end{bmatrix},\\
E_{25} &:= 0,\\
E_{26} &:= -\begin{bmatrix} B_a & 0 \\ 0 & B_b \end{bmatrix} R^{-1} \begin{bmatrix} B_a^* & 0 \\ 0 & B_b^* \end{bmatrix},\\
E_{27} &:= -\begin{bmatrix} B_a & 0 \\ 0 & B_b \end{bmatrix} R^{-1} \begin{bmatrix} I & D_a^* \\ I & D_b^* \end{bmatrix} S_1^*,
\end{aligned}$$

$$\begin{aligned}
E_{31} &:= 0,\\
E_{32} &:= 0,\\
E_{33} &:= E_{11},\\
E_{34} &:= 0,\\
E_{35} &:= 0,\\
E_{36} &:= 0,\\
E_{37} &:= E_{15},
\end{aligned}$$

$$\begin{aligned}
E_{41} &:= 0,\\
E_{42} &:= \left(\begin{bmatrix} C_a^*D_a & 0 \end{bmatrix} R^{-1} \begin{bmatrix} (1-\gamma^2)D_a^*C_a & D_a^*C_b \\ D_b^*C_a & D_b^*C_b \end{bmatrix} - \begin{bmatrix} C_a^*C_a & 0 \end{bmatrix}\right),\\
E_{43} &:= -\begin{bmatrix} C_a^*D_a & 0 \end{bmatrix} R^{-1} \begin{bmatrix} I & D_a^* \\ I & D_b^* \end{bmatrix} S_3,\\
E_{44} &:= -\left(A_a^* + \begin{bmatrix} C_a^*D_a & 0 \end{bmatrix} R^{-1} \begin{bmatrix} \gamma^2B_a^* \\ 0 \end{bmatrix}\right),\\
E_{45} &:= 0,\\
E_{46} &:= \begin{bmatrix} C_a^*D_a & 0 \end{bmatrix} R^{-1} \begin{bmatrix} B_a^* & 0 \\ 0 & B_b^* \end{bmatrix},\\
E_{47} &:= \begin{bmatrix} C_a^*D_a & 0 \end{bmatrix} R^{-1} \begin{bmatrix} I & D_a^* \\ I & D_b^* \end{bmatrix} S_1^*,
\end{aligned}$$

$$
\begin{aligned}
E_{51} &:= -\begin{bmatrix} C_a^* \tilde{R}_a^{-1} C_a & 0 \\ 0 & C_b^* \tilde{R}_b^{-1} C_b \end{bmatrix}, \\
E_{52} &:= S_3^* S_4, \\
E_{53} &:= -S_3^* S_2 S_3, \\
E_{54} &:= S_3^* \begin{bmatrix} I & I \\ D_a & D_b \end{bmatrix} R^{-1} \begin{bmatrix} \gamma^2 B_a \\ 0 \end{bmatrix}, \\
E_{55} &:= -E_{11}^*, \\
E_{56} &:= -S_3^* \begin{bmatrix} I & I \\ D_a & D_b \end{bmatrix} R^{-1} \begin{bmatrix} B_a^* & 0 \\ 0 & B_b^* \end{bmatrix}, \\
E_{57} &:= S_3^* S_2 S_1^*,
\end{aligned}
$$

$$
\begin{aligned}
E_{61} &:= 0, \\
E_{62} &:= -\begin{bmatrix} 0 & C_a^* \\ 0 & C_b^* \end{bmatrix} S_4, \\
E_{63} &:= \begin{bmatrix} 0 & C_a^* \\ 0 & C_b^* \end{bmatrix} S_2 S_3, \\
E_{64} &:= -\begin{bmatrix} 0 & C_a^* \\ 0 & C_b^* \end{bmatrix} \begin{bmatrix} I & I \\ D_a & D_b \end{bmatrix} R^{-1} \begin{bmatrix} \gamma^2 B_a^* \\ 0 \end{bmatrix}, \\
E_{65} &:= 0, \\
E_{66} &:= -\left(\begin{bmatrix} A_a^* & 0 \\ 0 & A_b^* \end{bmatrix} - \begin{bmatrix} 0 & C_a^* \\ 0 & C_b^* \end{bmatrix} \begin{bmatrix} I & I \\ D_a & D_b \end{bmatrix} R^{-1} \begin{bmatrix} B_a^* & 0 \\ 0 & B_b^* \end{bmatrix} \right), \\
E_{67} &:= -\begin{bmatrix} 0 & C_a^* \\ 0 & C_b^* \end{bmatrix} S_2 S_1^*
\end{aligned}
$$

$$
\begin{aligned}
E_{71} &:= 0, \\
E_{72} &:= 0, \\
E_{73} &:= E_{51}, \\
E_{74} &:= 0, \\
E_{75} &:= 0, \\
E_{76} &:= 0, \\
E_{77} &:= -E_{11}^*.
\end{aligned}
$$

D

Appendix to Chapter 6

D.1 Proof of Lemma 6.5

Before proving Lemma 6.5, the following technical result is required.

Lemma D.1 *The matrix* $M_{11}^* M_{21} = M_{21}^* M_{11}$ *and* $\tilde{M}_{11}\tilde{M}_{12}^* = \tilde{M}_{12}\tilde{M}_{11}^*$.

Proof : This proof is similar to that of Theorem 13.3 in [ZDG95]. From equation (6.27), it follows that

$$\grave{E}\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} = \begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} T_{11}$$

and therefore, that

$$\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}^* J\grave{E}\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} = \begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}^* J\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} T_{11}, \tag{D.1}$$

where J is defined in equation (6.25). Since $\grave{E}$ is Hamiltonian, $J\grave{E}$ is symmetric and hence, so is the right-hand side of the equation (D.1). That is,

$$\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}^* J\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} T_{11} = -T_{11}^*\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}^* J\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix},$$

which leads to the Lyapunov equation

$$(-M_{11}^* M_{21} + M_{21}^* M_{11})T_{11} - (-T_{11}^*)(-M_{11}^* M_{21} + M_{21}^* M_{11}) = 0.$$

Since $\text{spec}(T_{11}) \cap \text{spec}(-T_{11}^*) = \emptyset$, this has unique solution

$$-M_{11}^* M_{21} + M_{21}^* M_{11} = 0,$$

and therefore, $M_{21}^* M_{11}$ is symmetric. Similarly, by noting from equation (6.27) that

$$\begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \end{bmatrix} \grave{E} = T_{11} \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \end{bmatrix},$$

it follows that $\tilde{M}_{11}\tilde{M}_{12}^*$ is symmetric. ∎

Proof of Lemma 6.5: This proof is similar to that of Theorem 13.6 in [ZDG95]. From equation (6.27), it follows that

$$\grave{E}\begin{bmatrix} M_{11} \\ M_{12} \end{bmatrix} = \begin{bmatrix} -\grave{A}^* & -\grave{C}^*\grave{C} \\ \grave{B}\grave{B}^* & \grave{A} \end{bmatrix}\begin{bmatrix} M_{11} \\ M_{12} \end{bmatrix} = \begin{bmatrix} M_{11} \\ M_{12} \end{bmatrix} T_{11}, \tag{D.2}$$

and therefore, that

$$-\grave{A}^* M_{11} - \grave{C}^*\grave{C} M_{21} = M_{11} T_{11}. \tag{D.3}$$

Let (A, C_1) have no unobservable modes in $\mathbb{C}_{>\alpha}$ and suppose that M_{11} is column-rank deficient. That is, $\mathcal{K}_{M_{11}} \neq \{0\}$. For any $x \in \mathcal{K}_{M_{11}}$, post-multiply (D.3) by x and pre-multiply by $x^* M_{21}^*$. Then

$$-x^* M_{21}^* \grave{A}^* M_{11} x - x^* M_{21}^* \grave{C}^* \grave{C} M_{21} x = x^* M_{21}^* M_{11} T_{11} x,$$

which by the symmetry of $M_{21}^* M_{11}$ (cf. Lemma D.1) implies that

$$-x^* M_{21}^* \grave{C}^* \grave{C} M_{21} x = 0.$$

Since $M_{21}^* \grave{C}^* \grave{C} M_{21}$ is positive semi-definite, this implies $\grave{C} M_{21} x = 0$. So simply post-multiplying (D.3) by x gives $M_{11} T_{11} x = 0$. Since T_{11} has full rank, it follows that $\mathcal{K}_{M_{11}}$ is T_{11}-invariant. That is there exists a $\lambda \in \mathbb{C}$ and a corresponding eigenvector $v \in \mathcal{K}_{M_{11}}$, such that

$$T_{11} v = \lambda v.$$

Note that since $\mathrm{spec}(T_{11}) \subset \mathbb{C}_{>\alpha}$, $\lambda > \alpha \geq 0$. Now, from equation (D.2) it follows that

$$\grave{B}\grave{B}^* M_{11} + \grave{A} M_{21} = M_{21} T_{11},$$

which when post-multiplied by v gives,

$$(\grave{A} - \lambda I) M_{21} v = 0.$$

Recall that for $v \in \mathcal{K}_{M_{11}}$, $\grave{C} M_{21} v = 0$ and hence, note that

$$\begin{bmatrix} \grave{A} - \lambda I \\ \grave{C} \end{bmatrix} M_{21} v = 0. \tag{D.4}$$

Furthermore,

$$\begin{bmatrix} \grave{A} - \lambda I \\ \grave{C} \end{bmatrix} = \begin{bmatrix} I & \gamma^{-2} B_1 R^{-1} D_{11}^* & 0 \\ 0 & 0 & -I \\ 0 & S^{-\frac{1}{2}} & 0 \end{bmatrix} \begin{bmatrix} A - \lambda I & B_2 \\ C_1 & D_{12} \\ 0 & \lambda I \end{bmatrix}, \tag{D.5}$$

where importantly the first matrix on the right-hand side has full rank. Since $\lambda > \alpha \geq 0$ and (A, C_1) has no unobservable modes in $\mathbb{C}_{>\alpha}$, λI has full rank and $\begin{bmatrix} A - \lambda I \\ C_1 \end{bmatrix}$ has full column rank. Thus, it follows from equation (D.5), that $\begin{bmatrix} \grave{A} - \lambda I \\ \grave{C} \end{bmatrix}$ also has full column rank. From equation (D.4), this then implies that $M_{21} v = 0$, which since $M_{11} v = 0$ contradicts the fact that $\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}$ has full column rank. Therefore, if (A, C_1) has no unobservable modes in $\mathbb{C}_{>\alpha}$, M_{11} has full column rank.

Similarly, noting that

$$\begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \end{bmatrix} \grave{E} = T_{11} \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \end{bmatrix},$$

it can be shown that if (A, B_1) has no uncontrollable modes in $\mathbb{C}_{>\alpha}$, then $\tilde{M}_{11}$ has full row rank.
∎

Bibliography

[AHI93] M. Araki, T. Hagiwara, and Y. Ito. Frequency-response of sampled-data systems II: Closed-loop consideration. In *Proceedings of the 12th IFAC World Congress*, pages 293–296, 1993.

[Ahl79] L. V. Ahlfors. *Complex Analysis.* Mathematics Series. McGraw-Hill International Editions, Singapore, 1979.

[AI93] M. Araki and Y. Ito. Frequency-response of sampled-data systems I: Open-loop consideration. In *Proceedings of the 12th IFAC World Congress*, pages 289–292, 1993.

[Bel62] R. E. Bellman. *Applied Dynamic Programming.* Princeton University Press, New Jersey, 1962.

[BGPK85] R. R. Bitmead, M. R. Gevers, I. R. Petersen, and R. J. Kaye. Monoticity and stabilisability properties of solutions of the Riccati difference equation: Propositions, lemmas, theorems, fallacious conjectures and counterexamples. *Systems and Control Letters*, 5:309–315, 1985.

[BLW91] S. Bittanti, A. J. Laub, and J. C. Willems, editors. *The Riccati Equation*, Berlin, 1991. Springer Verlag.

[Bol90] B. Bollobás. *Linear Analysis : An Introductory Course.* Cambridge Mathematical Textbooks. Cambridge University Press, 1990.

[BP92] B. A. Bamieh and J. B. Pearson. A general framework for linear periodic systems with applications to H_∞ sampled-data control. *IEEE Transactions of Automatic Control*, 37:418–435, 1992.

[BPFT91] B. A. Bamieh, J. B. Pearson, B. A. Francis, and A. R. Tannenbaum. A lifting technique for linear periodic systems with applications to sampled-data control. *Systems and Control Letters*, 17:79–88, 1991.

[Bro70] R. W. Brockett. *Finite-Dimensional Linear Systems.* Wiley, New York, 1970.

[BS79] C. Bavely and G. W. Stewart. An algorithm for computing reducing subspaces by block diagonalisation. *SIAM Journal of Numerical Analysis*, 16:359–367, 1979.

[CD91] F. M. Callier and C. A. Desoer. *Linear Systems Theory.* Springer Verlag, New York, 1991.

[CF90] T. Chen and B. A. Francis. On the L_2-induced norm of a sampled-data system. *Systems and Control Letters*, 15:211–219, 1990.

[CF91] T. Chen and B. A. Francis. Input-output stability of sampled-data systems. *IEEE Transactions on Automatic Control*, 36:50–58, 1991.

[CF95] T. Chen and B. A. Francis. *Optimal Sampled-Data Control Systems.* Communications and Control Engineering Series. Springer-Verlag, London, 1995.

[CG96] M. W. Cantoni and K. Glover. A design-framework for continuous-time systems under sampled-data control. In *Proceedings of the 35th IEEE CDC*, pages 458–463, 1996.

[CG97] M. W. Cantoni and K. Glover. H_∞ sampled-data synthesis and related numerical issues. *Automatica*, 33:2233–2241, 1997.

[Che84] C. T. Chen. *Linear Systems Theory and Design.* Holt, Rinehart and Winston, 1984.

[DFT92] J. C. Doyle, B. A. Francis, and A. R. Tannenbaum. *Feedback Control Theory.* Maxwell Macmillan International, Oxford, 1992.

[DG84] C. A. Desoer and C. L. Gustafson. Algebraic theory of linear multivariable feedback systems. *IEEE Transactions on Automatic Control*, 29:909–917, 1984.

[DGG86] C. E. DeSouza, M. R. Gevers, and G. C. Goodwin. Riccati equations in optimal filtering of nonstabilisable systems having singular state transition matrices. *IEEE Transaction on Automatic Control*, 31:831–838, 1986.

[DGKF89] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis. State space solutions to standard H_2 and H_∞ control problems. *IEEE Transaction on Automatic Control*, 34:831–847, 1989.

[DGS93] J. C. Doyle, T. T. Georgiou, and M. C. Smith. The parallel projection operators of a nonlinear feedback system. *Systems and Control Letters*, 20:79–85, 1993.

[DL70] P. Dorato and A. H. Levis. Optimal linear regulators: The discrete-time case. *IEEE Transactions on Automatic Control*, 16:613–620, 1970.

[DLJS80] C. A. Desoer, Murray Liu, R-W., J., and R. Saeks. Feedback system design: The fractional representation approach to analysis and synthesis. *IEEE Transactions on Automatic Control*, 25:399–412, 1980.

[DS88] N. Dunford and J. T. Schwartz. *Linear Operators, Part I : General Theory and Part II : Spectral Theory–Self Adjoint Operators in Hilbert Space*. Wiley Classics Library. Wiley, New York, 1988.

[DS93] W. N Dale and M. C. Smith. Stabilizability and existence of system representations for discrete-time time-varying systems. *SIAM Journal of Control and Optimization*, 31:1538–1557, 1993.

[Dul94] G. E. Dullerud. *Control of Uncertain Sampled-Data Systems*. PhD thesis, University of Cambridge, England, 1994.

[DV75] C. A. Desoer and M. Vidyasagar. *Feedback Systems: Input–Output Properties*. Academic Press, New York, 1975.

[ES85] A. K. El-Sakkary. The gap metric: Robustness of stabilisation of feedback systems. *IEEE Transactions on Automatic Control*, 30:240–247, 1985.

[FF90] C. Foiaş and A. Frazho. *The Commutant Lifting Approach to Interpolation Problems*, volume 44 of *Operator Theory : Advances and Applications*. Birkhäuser Verlag, Berlin, 1990.

[FG88] B. A. Francis and T. T. Georgiou. Stability theory for linear time-invariant plants with periodic digital control. *IEEE Transactions on Automatic Control*, 33:820–83, 1988.

[FGS93] C. Foiaş, T. T. Georgiou, and M. C. Smith. Robust stability of feedback systems: A geometric approach using the gap metric. *SIAM Journal of Control and Optimization*, 31:1518–1537, 1993.

[Fra86] B. A. Francis. *A Course in* H^∞ *Control Theory*. Springer-Verlag, Berlin, 1986.

[FS82] A. Feintuch and R. Saeks. *System Theory : a Hilbert Space Approach*. Pure and Applied Mathematics,. Academic Press, 1982.

[GA94] P. Gahinet and P. Apkarian. A linear matrix inequality approach to H_∞ control. *International Journal of Robust and Nonlinear Control*, 4:421–448, 1994.

[Geo88] T. T. Georgiou. On the computation of the gap metric. *Systems and Control Letters*, 11:253–257, 1988.

[GL95] M. Green and D. J. N. Limebeer. *Linear Robust Control.* Information and Systems Science Series. Prentice Hall, New Jersey, 1995.

[GNVL79] G. H. Golub, S. Nash, and C. F. Van Loan. A hessenberg-schur method for the matrix problem $AX + XB = C$. *IEEE Transactions on Automatic Control*, 24:909–913, 1979.

[GS90] T. T. Georgiou and M. C. Smith. Optimal robustness in the gap metric. *IEEE Transactions on Automatic Control*, 35:673–686, 1990.

[GS93] T. T. Georgiou and M. C. Smith. Graphs, causality and stabilizability: Linear, shift-invariant systems on $\mathrm{L}^2([0,\infty))$. *Mathematics of Control, Signals and Systems*, 6:195–223, 1993.

[GS94] G. C. Goodwin and M. Salgado. Frequency domain sensitivity functions for continuous-time systems under sampled-data control. *Automatica*, 30:1263–1270, 1994.

[GVL89] G. H. Golub and C. F. Van Loan. *Matrix Computations.* Johns Hopkins Series in the Mathematical Sciences. The Johns Hopkins University Press, Baltimore, second edition, 1989.

[Hal82] P. R. Halmos. *A Hilbert Space Problem Book.* Springer Verlag, New York, 1982.

[Hof62] K. Hoffman. *Banach Spaces of Analytic Functions.* Prentice-Hall, New Jersey, 1962.

[Hyd95] R. A. Hyde. H^∞ *Aerospace Control Design - A VSTOL Flight Application.* Advances in Industrial Control Series. Springer-Verlag, 1995.

[IG91] P. A. Iglesias and K. Glover. State-space approach to discrete-time H_∞ control. *International Journal of Control*, 54:1031–1073, 1991.

[IOW97] W. Ionescu, C. Oară, and M. Weiss. General matrix pencil techniques for the solution of algebraic Riccati equations. *IEEE Transactions on Automatic Control*, 42:1085–1098, 1997.

[Kai80] T. Kailath. *Linear Systems.* Information and System Science Series. Prentice Hall, New Jersey, 1980.

[Kat66] T. Kato. *Perturbation Theory for Linear Operators.* Springer Verlag, New York, 1966.

[KF75] A. N. Kolmogorov and S. V. Fomin. *Introductory Functional Analysis.* Dover, New York, 1975.

[KH93] P. T. Kabamba and S. Hara. Worst-case analysis and design of sampled-data control systems. *IEEE Transactions on Automatic Control*, 38:1337–1357, 1993.

[KHI90] R. Kondo, S. Hara, and T. Itou. Characterisation of discrete-time H_∞ controllers via bilinear transformation. In *Proceedings of the 29th IEEE CDC*, pages 1763–1768, 1990.

[Kre89a] R. Kress. *Linear Integral Equations.* Number 82 in Applied Mathematical Sciences. Springer-Verlag, 1989.

[Kre89b] E. Kreyszig. *Introductory Functional Analysis With Applications.* Wiley Classics Library. Wiley, New York, 1989.

[Kuč72] V. Kučera. A contribution to matrix quadratic equations. *IEEE Transaction on Automatic Control*, 17:344–347, 1972.

[KVZ+72] M. A. Krasnosel'skii, G. M. Vainikko, P. P. Zabreiko, Rutitskii, Y. B., and V. Y. Stetsenko. *Approximate Solution of Operator Equations.* Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1972.

[Lax59] P. D. Lax. Translation invariant subspaces. *Acta Mathematica*, 101:163–178, 1959.

[LR95] P. Lancaster and L. Rodman. *Algebraic Riccati Equations.* Oxford Science Publications, Oxford, 1995.

[MAT92] The Maths Works Inc., Natick, Massachusetts. *High-Performance Numeric Computation and Visualisation Software: Reference Guide*, 1992.

[MG90] D. C. McFarlane and K. Glover. *Robust Controller Design Using Normalised Coprime Factorisation Plant Descriptions.* Lecture Notes in Control and Information Sciences. Springer-Verlag, Berlin, 1990.

[MH87] J. E. Marsden and M. J. Hoffman. *Basic Complex Analysis.* Freeman, New York, second edition, 1987.

[Nik86] N. K. Nikol'skii. *Treatise on the Shift Operator.* Springer-Verlag, 1986.

[NJB84] C. N. Nett, C. A. Jacobson, and N. J. Balas. A connection between state-spaces and doubly coprime fractional representations. *IEEE Transactions on Automatic Control*, 29:831–832, 1984.

[OS91] R. J. Ober and J. A. Sefton. Stability of control systems and graphs of linear systems. *Systems and Control Letters*, 17:265–280, 1991.

[Pap98] G. Papageorgiou. *Robust Control System Design:* H^∞ *Loop shaping and Aerospace Applications.* PhD thesis, University of Cambridge, Cambridge, England, 1998.

[PLS80] T. Pappas, A. J. Laub, and N. R. Sandell. On the numerical solution of the discrete-time algebraic Riccati equation. *IEEE Transaction on Automatic Control*, 25:631–641, 1980.

[RPK92] R. Ravi, A. M. Pascoal, and P. P Khargonekar. Normalized coprime factorizations for linear time-varying systems. *Systems and Control Letter*, 18:455–465, 1992.

[RR85] M. Rosenblum and J Rovnyak. *Hardy Classes and Operator Theory.* Oxford University Press, 1985.

[Rud86] W. Rudin. *Real and Complex Analysis.* Mathematics Series. McGraw-Hill International Editions, Singapore, 1986.

[Rud91] W. Rudin. *Functional Analysis.* Mathematics Series. McGraw-Hill International Editions, Singapore, 1991.

[SLC90] M. G. Safonov, D. N. J. Limebeer, and R. Y. Chiang. Simplifying the H^∞ theory via loop-shifting, matrix pencil and descriptor concepts. *International Journal of Control*, 50:2467–2488, 1990.

[Smi89] M. C. Smith. On stabilisation and the existence of coprime factorisations. *IEEE Transactions on Automatic Control*, 34:1005–1007, 1989.

[SNF70] B. Sz-Nagy and C. Foiaş. *Harmonic Analysis of Operators on Hilbert Space.* North-Holland Publishing Company, Amsterdam, 1970.

[SO93] J. A. Sefton and R. J. Ober. On the gap metric and coprime factor perturbations. *Automatica*, 29:723–734, 1993.

[Sut93] W. A. Sutherland. *Introduction to Metric and Topological Spaces.* Oxford Science Publications, Oxford, 1993.

[Toi92] H. T. Toivonen. Sampled-data control of continuous-time systems with an H_∞ optimality criterion. *Automatica*, 28:45–54, 1992.

[Toi93] H. T. Toivonen. Worst-case sampling for sampled-data H_∞ design. In *Proceedings of the 32nd IEEE CDC, San Antonio, Texas*, pages 337–342, 1993.

[TS97] H. T. Toivonen and M. F. Sagfors. The sampled-data H_∞ problem: A unified framework for discretization based methods and Riccati equation solution. *International Journal of Control*, 66:289–309, 1997.

[Vid78] M. Vidyasagar. *Nonlinear Systems Analysis*. Networks Series. Prentice Hall, New Jersey, 1978.

[Vid84] M. Vidyasagar. The graph metric for unstable plants and robustness estimates for feedback stability. *IEEE Transactions on Automatic Control*, 29:403–417, 1984.

[Vid85] M. Vidyasagar. *Control System Synthesis : A Factorization Approach*. The MIT Press, Cambridge, Massachusetts, 1985.

[Vin93a] G. Vinnicombe. Frequency domain uncertainty and the graph topology. *IEEE Transactions on Automatic Control*, 38:1371–1383, 1993.

[Vin93b] G. Vinnicombe. *Measuring the Robustness of Feedback Systems*. PhD thesis, University of Cambridge, England, 1993.

[Vin96] G. Vinnicombe. Personal communication, 1996.

[Vin98] G. Vinnicombe. *Uncertainty and Feedback*. Imperial College Press, London, In Press 1998.

[VK86] M. Vidyasagar and H. Kimura. Robust controllers for uncertain linear multivariable systems. *Automatica*, 22:85–94, 1986.

[VL78] C. F. Van Loan. Computing integrals involving the matrix exponential. *IEEE Transactions on Automatic Control*, 23:395–404, 1978.

[VSF83] M. Vidyasagar, H. Schneider, and B. A. Francis. Algebraic and topological aspects of feedback stabilisation. *IEEE Transactions on Automatic Control*, 27:880–894, 1983.

[Wil71] J. C. Willems. *The Analysis of Feedback Systems*, volume 62 of *Research Monographs*. The MIT Press, Cambridge, Massachusetts, 1971.

[Won85] W. M. Wonham. *Linear Multivariable Control*. Springer Verlag, Berlin, 1985.

[WR71] J. H. Wilkinson and C. Reinsch. *Handbook for Automatic Computation*, volume II Linear Algebra. Springer-Verlag, Berlin, 1971.

[YA94] Y. Yamamoto and M. Araki. Frequency responses for sampled-data systems - their equivalence and relationships. *Linear Algebra and its Applications*, 205-206:1319–1339, 1994.

[Yam94] Y. Yamamoto. A function space approach to sampled-data control systems and tracking problems. *IEEE Transactions on Automatic Control*, 39:703–713, 1994.

[YJB76] D. C. Youla, H. A. Jabr, and J. J. Bongiorno. Modern Wiener-Hopf design of optimal controllers - part II: The multivariable case. *IEEE Transactions on Automatic Control*, 21:319–338, 1976.

[YK96] Y. Yamamoto and P. P. Khargonekar. Frequency response of sampled-data systems. *IEEE Transactions on Automatic Control*, 41:166–176, 1996.

[You86] N. Young. The nevanlinna-pick problem for matrix-valued functions. *Journal of Operator Theory*, 15:239–265, 1986.

[Zam66] G. Zames. On the input-output stability for time-varying nonlinear feedback systems – Part I conditions using concepts of loop gain, conicity and positivity. *IEEE Transactions on Automatic Control*, 2:228–238, 1966.

[Zam81] G. Zames. Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms and approximate inverses. *IEEE Transactions on Automatic Control*, 26:301–320, 1981.

[ZDG95] K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control.* Prentice Hall, New Jersey, 1995.

[ZES80] G. Zames and A. K. El-Sakkary. Unstable systems and feedback: The gap metric. In *Proceedings of the Allerton Conference*, pages 380–385, 1980.