Achievable Dynamic Response for Automotive Active Suspensions[1]

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A complete set of constraints is derived for the road disturbance transfer functions in a quarter car model of an automotive active suspension, for typical choices of measured outputs. It is shown that any road disturbance responses which are achievable using "full state feedback" can be achieved, to within an arbitrary small tolerance, using a dynamic compensator measuring suspension deflection only. Also considered are the disturbance responses to loads acting on the sprung mass, and a complete set of constraints is derived for these. It is shown that road disturbance and load disturbance responses can be determined independently if suspension deflection and sprung mass velocity are measured. Indeed, any responses achievable separately with "full measurements" can be approximated together to an arbitrary small tolerance. Certain integral relationships are shown to follow from the derived transfer function constraints. These relationships imply fundamental limitations for certain responses (e.g. tyre deflection) no matter what measurements are available for feedback.

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Notation

m_s	sprung mass
m_u	unsprung mass
k_t	tyre spring stiffness
k_s	passive suspension spring stiffness
c_s	passive suspension damper coefficient
u	active or passive control force
f	load disturbance input
ω_1	tyre hop invariant frequency
ω_2	rattle space invariant frequency
z_r	vertical displacement of road disturbance input
z_s	vertical displacement of sprung mass
z_u	vertical displacement of unsprung mass

1 Introduction

Performance advantages of automotive active suspensions have now been conclusively demonstrated in practice [12], [8]. A number of different hardware configurations, control design methods and performance measures have been investigated and compared in the literature [6], [9], [1], [10]. Constraints and trade-offs have also been considered [7], [13], [5]. However, it appears that the merits and limitations of the various control schemes have not yet been fully analysed from a control systems point of view. In particular, the following issues have not received a complete treatment: (i) what is, and what is not, achievable with general dynamic compensation, and (ii) how much freedom is afforded by various choices of measurements? This paper considers these questions for a quarter car model of an automotive suspension. The approach taken is to determine a *complete* set of constraints on various transfer functions from the road and load disturbance inputs (where complete means that any transfer function which meets the constraints must correspond to an admissible control law). Such an approach has the potential to identify the degrees of freedom which are available to the designer, and to determine a minimum set of measurements needed to fully exploit the freedom.

The derivations of this paper make use of the factorization approach to feedback stability and the Youla parametrization of stabilizing controllers. The transfer function constraints naturally arise in the frequency domain in the form of invariant frequency points and restrictions on the leading terms of the Taylor series at zero and infinite frequency. Several of these types of constraint have been noted previously in [13], [5] where they are derived from an "invariant equation" satisfied by the system variables. In [7], lack of adjustability of some transfer function numerator coefficients, even for full sets of measurements, has been observed and cited as a design limitation. To obtain a complete set of constraints, as is done here, it is necessary to characterize the set of stabilizing controllers and to specify which measurements are available for feedback. Various choices of measurements are investigated ranging from suspension deflection only to a full set of state variables.

Integral relationships are shown to follow from the derived zero and infinite frequency constraints. These relationships imply fundamental limitations for certain frequency responses no matter what measurements are available for feedback. For example, the transfer function from road disturbances to tyre deflection (commonly used to evaluate road-holding) must be greater than one in modulus to at least the same extent that it is less than one, when measured in terms of the area on a Bode magnitude plot. Thus the designer has at most the freedom to "place" or "spread" the frequency ranges where disturbance amplification occurs. The differences in the constraints for differing measurements occur mainly in the higher order terms of the Taylor expansions at zero and infinite frequency. This leads to the fact that, any road disturbance responses which are achievable using "full state feedback" can be achieved, to within an arbitrary small tolerance, using a dynamic compensator measuring suspension deflection only. An example is given highlighting the fact that "optimizing" the road disturbance responses alone may be insufficient to give good all-round performance of a suspension system. That is, unsatisfactory performance may be experienced in other regards, e.g. poor behaviour under loading of the sprung mass. This motivates the separate study of load disturbance response.

As shown in [5], if one of the road disturbance responses is determined, the others are fixed automatically. The same is true of the load disturbance responses. Evidently there is one degree of freedom available for the response to each of the two disturbances. The question arises: to what extent are these degrees of freedom independent? It is shown that the pair of responses can be determined independently if two suitable measurements are available for feedback (e.g. suspension deflection and sprung mass velocity). Indeed, any responses achievable separately with "full measurements" can be approximated together to an arbitrary small tolerance. An example is given showing how such a controller can be calculated to give a "soft ride" while being stiff to applied loads.

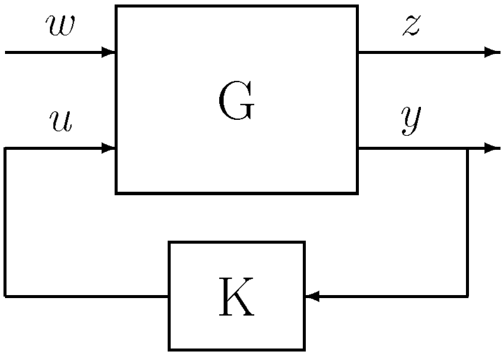

Figure 1: Standard block diagram.

2 Feedback Stability

In this section the factorization approach to feedback system stability is reviewed. This approach has been used extensively in modern frequency domain theories and in H_∞ control. The reader is referred to [2, Chapter 5] for a nice introduction and to [3], [11] for comprehensive treatments.

Consider the feedback configuration of Fig. 1. In this figure, G is the generalized plant, K is the controller, w is the exogenous input, u is the control input, z is the output to be controlled and y is the measured output. The transfer functions G and K are rational, possibly matrix-valued. The signals u, w, y, z are time-domain signals or their Laplace transforms, according to context. The transfer function of the generalized plant is partitioned conformably with u, w, y, z:

$$G(s) = \left(\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array} \right).$$

so that Fig. 1 represents the equations:

$$\begin{aligned} z &= G_{11}w + G_{12}u, \\ y &= G_{21}w + G_{22}u, \\ u &= Ky. \end{aligned}$$

The response from w to z is denoted by T_{zw} and can be calculated from the above equations as:

$$T_{zw} = [G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}] = [G_{11} + G_{12}(I - KG_{22})^{-1}KG_{21}]. \tag{1}$$

A (scalar) rational transfer function is said to be *stable* if all its poles lie in $\Re e(s) < 0$ and the numerator degree is less than or equal to the denominator degree. The set of such rational functions is denoted by $\mathcal{S}$ (which is in fact the set H_∞ restricted to the rational functions). A matrix transfer function is defined to be *stable* if all its elements belong to $\mathcal{S}$. The feedback configuration of Fig. 1 is defined to be *stable* if, when disturbance inputs are introduced at the two interconnection points, all the resulting closed-loop transfer functions are stable (see [3, Chapter 3]). It will be assumed that a K can be found to stabilize G. (This requires that G and G_{22} share the same unstable poles, which is invariably the case in practice). In this case, the set of *all* compensators K which stabilize G can be parametrized in terms of a coprime factorization of G_{22}—this is called the Youla parametrization. This parametrization is basic in the H_∞ control approach which seeks to minimize weighted norms on various closed-loop transfer functions over all possible stabilizing controllers. In this paper the Youla parametrization is used simply to characterize the set of achievable closed-loop transfer functions (without taking the further step of trying to optimize them).

The Youla parametrization is obtained as follows. Given G_{22} it is possible to find matrices N, M, $\tilde{N}$, $\tilde{M}$, X, Y, $\tilde{X}$ and $\tilde{Y}$ whose elements belong to $\mathcal{S}$ such that

$$G_{22} = NM^{-1} = \tilde{M}^{-1}\tilde{N}, \tag{2}$$

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I. \tag{3}$$

(For details on how to do this using state-space techniques, see [3, Chapter 4.4]). The factorizations (2) of G_{22} satisfying (3) are called right/left coprime factorizations over $\mathcal{S}$. It can be shown that matrices N, M, $\tilde{N}$ and $\tilde{M}$ satisfying (2) are coprime factorizations of G_{22} if and only if $\begin{pmatrix} M \\ N \end{pmatrix}$ and $\begin{pmatrix} \tilde{M}, \tilde{N} \end{pmatrix}$ have full rank for all s with $\Re e(s) \geq 0$ and at $s = \infty$. The Youla parametrization of all stabilizing controllers then takes the form

$$K = (Y - MQ)(X - NQ)^{-1} = (\tilde{X} - Q\tilde{N})^{-1}(\tilde{Y} - Q\tilde{M})$$

where Q ranges over all matrices of the appropriate dimension whose elements

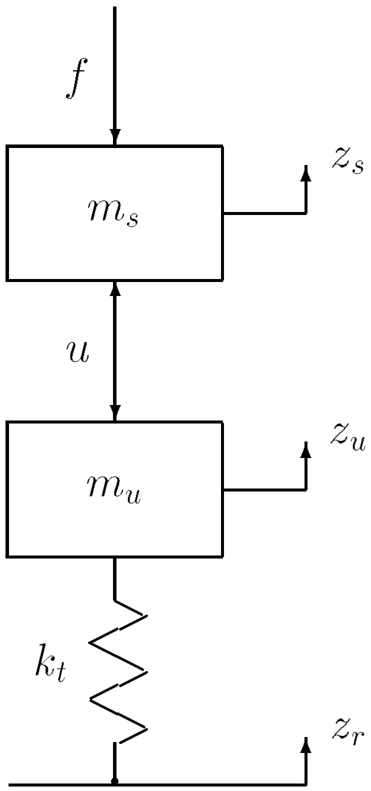

Figure 2: Quarter car model.

belong to $\mathcal{S}$ (see [3, Chapter 4]. With this parametrization, the transfer function from w to z takes a particularly convenient form which is linear in Q:

$$T_{zw} = G_{11} + G_{12}M(\tilde{Y} - Q\tilde{M})G_{21} = G_{11} + G_{12}(Y - MQ)\tilde{M}G_{21}.$$

In summary, as Q varies over the set of stable functions, this equation parametrizes *all achievable* transfer functions T_{zw}.

3 Quarter Car Model

The parametrization of the previous section will be illustrated for the quarter car model of Fig. 2. The control input u is a force (which in general will be a sum of active and passive forces, though the issues of hardware realization will not be considered here). Assuming that the tyre does not leave the ground, the equations of motion take the form:

$$m_s \ddot{z}_s = u - f, \tag{4}$$
$$m_u \ddot{z}_u = -u + k_t(z_r - z_u). \tag{5}$$

In this section it will be assumed that the load disturbance is absent ($f = 0$). The general form of the dynamic response will be computed from road displacement to (1) sprung mass position, (2) suspension deflection, (3) tyre deflection, assuming the measured output is $z_s - z_u$. This amounts to finding the general form of the following transfer functions:

$$H_1 = \frac{z_s}{z_r}, \tag{6}$$
$$H_2 = \frac{z_s - z_u}{z_r}, \tag{7}$$
$$H_3 = \frac{z_u - z_r}{z_r}. \tag{8}$$

Firstly, H_1 will be considered. Setting $w = z_r$, $z = z_s$, $y = z_s - z_u$ results in the following generalized plant:

$$G = \left(\begin{array}{c|c} 0 & \frac{1}{m_s s^2} \\ \hline \frac{-k_t}{m_u s^2 + k_t} & \frac{(m_s+m_u)s^2+k_t}{m_s s^2(m_u s^2+k_t)} \end{array} \right). \tag{9}$$

It can be verified that a pair of coprime factors for G_{22} is

$$N = \tilde{N} = \frac{(m_s + m_u)s^2 + k_t}{p(s)}, \tag{10}$$
$$M = \tilde{M} = \frac{m_s s^2(m_u s^2 + k_t)}{p(s)} \tag{11}$$

where $p(s)$ is any Hurwitz polynomial (i.e. all roots in $\Re e(s) < 0$) of degree 4. (It will not be necessary to calculate the corresponding $X = \tilde{X}$ and $Y = \tilde{Y}$ satisfying (3).) Thus

$$H_1 = T_{zw} = G_{12}(Y - MQ)\tilde{M}G_{21} = \frac{-k_t}{p}(Y - MQ). \tag{12}$$

Two sources of constraints on H_1 can be observed from (12). Firstly, since Y, M and Q all belong to $\mathcal{S}$, $Y - MQ$ is bounded near $s = \infty$. This means that

$H_1(s) = O(s^{-4})$ (i.e. $s^4 H_1(s)$ tends to a finite, possibly zero, limit as $s \to \infty$). Secondly, $Y - MQ$ is constrained by the fact that Q, being stable, can't invert the imaginary axis zeros of M. This means that

$$\begin{aligned} H_1(0) &= \left.\left(\frac{-k_t}{p(s)} Y(s)\right)\right|_{s=0}, \\ H_1(j\omega_1) &= \left.\left(\frac{-k_t}{p(s)} Y(s)\right)\right|_{s=j\omega_1}. \end{aligned}$$

where $\omega_1 = \sqrt{\frac{k_t}{m_u}}$ is the "tyre-hop" invariant frequency. (There is also a constraint on the first derivative of H_1 at 0 due to the double zero in $M(s)$ at $s = 0$ — this will be included below.) Since $XM - YN = 1$ then $\left(\frac{-k_t}{p} Y\right)(0) = 1$ and

$$\left(-(m_s + m_u)\frac{k_t}{m_u} + k_t\right) \left.\left(\frac{-1}{p(s)} Y(s)\right)\right|_{s=j\omega_1} = 1 \tag{13}$$

which gives $\left(\frac{-k_t}{p} Y\right)(j\omega_1) = -\frac{m_u}{m_s}$. In fact, it will be shown that this set of constraints is the *complete set* which an admissible H_1 must satisfy. That is, any complex function in $\mathcal{S}$ which satisfies these constraints corresponds to an admissible H_1 for some stabilizing control law. This result will now be stated formally, the full proof of which is given in the Appendix.

Theorem 1. Consider the quarter car model (4), (5) with $w = z_r$, $z = z_s$ and $y = z_s - z_u$, and let H_1 be *any* function in $\mathcal{S}$. Then $H_1 = T_{zw}$, for some stabilizing control law, if and only if:

1. $H_1(s) = O(s^{-4})$,
2. $H_1(0) = 1$ and $H_1'(0) = 0$,
3. $H_1(j\omega_1) = -\frac{m_u}{m_s}$.

A similar derivation to the above can be carried out for each of the transfer functions H_2 and H_3. Alternatively, Theorem 1 can be used to determine the required constraints using the invariant equation of [5]:

$$s^2(m_s z_s + m_u z_u) = k_t(z_r - z_u) \tag{14}$$

which is obtained from (4), (5) by eliminating u. It follows from (14) that H_1 determines H_2 completely, and vice versa, from the following identities

$$H_2 = \frac{1}{m_u s^2 + k_t}\Big(((m_s + m_u)s^2 + k_t)H_1 - k_t\Big), \tag{15}$$

$$H_1 = \frac{1}{(m_s+m_u)s^2+k_t}\left((m_u s^2+k_t)H_2+k_t\right). \tag{16}$$

From Theorem 1(3) and (15) it follows, as expected, that H_2 belongs to $\mathcal{S}$. Expanding H_2 in powers of s^{-1} and using Theorem 1(1) shows that

$$H_2 = -\frac{k_t}{m_u}s^{-2} + O(s^{-4}).$$

Using Theorem 1(2), which states that $H_1 = 1 + O(s^2)$, and expanding H_2 in powers of s shows that $H_2 = O(s^2)$, i.e. $H_2(0) = H_2'(0) = 0$. It can also be observed that H_2 is independent of H_1 at the "rattle-space" invariant frequency $s = j\omega_2 = j\sqrt{\frac{k_t}{m_s+m_u}}$. A brief calculation shows that $H_2(j\omega_2) = -\left(1+\frac{m_u}{m_s}\right)$. A set of constraints has been derived which any H_2 must necessarily satisfy. It remains to verify that these conditions are *sufficient* to determine a stabilizing feedback control law. This is achieved as follows. Using (16), each of the conditions (1)-(3) of Theorem 1 can be verified using similar derivations to the ones used to derive the constraints on H_2. Since the desired H_2 corresponds to an admissible H_1, then H_2 must also be admissible. The result is now formally stated.

Theorem 2. Consider the quarter car model (4), (5) with $w = z_r$, $z = z_s - z_u$ and $y = z_s - z_u$, and let H_2 be *any* function in $\mathcal{S}$. Then $H_2 = T_{zw}$, for some stabilizing control law, if and only if:

1. $H_2(s) = -\frac{k_t}{m_u}s^{-2} + O(s^{-4})$,
2. $H_2(0) = H_2'(0) = 0$,
3. $H_2(j\omega_2) = -\left(1+\frac{m_u}{m_s}\right)$.

A similar derivation can be carried out for H_3 using the equations

$$H_3 = \frac{-s^2}{m_u s^2+k_t}\left(m_s H_1 + m_u\right), \tag{17}$$

$$H_1 = \frac{-1}{m_s s^2}\left((m_u s^2+k_t)H_3 + m_u s^2\right). \tag{18}$$

The complete set of constraints obtained for H_3 are stated below.

Theorem 3. Consider the quarter car model (4), (5) with $w = z_r$, $z = z_u - z_r$ and $y = z_s - z_u$, and let H_3 be *any* function in $\mathcal{S}$. Then $H_3 = T_{zw}$, for some stabilizing control law, if and only if:

1. $H_3(s) = -1 + \frac{k_t}{m_u}s^{-2} + O(s^{-4})$,

2. $H_3(s) = -\frac{(m_s+m_u)}{k_t}s^2 + O(s^4)$.

In Theorems 1–3 above, the constraints on H_1, H_2 and H_3 were derived assuming that the measured output is $y = z_s - z_u$. Similar calculations can be carried out for different choices of measured outputs. The results for various other cases are given in Table 1. Outlines of the proofs are given in the Appendix. It is interesting to remark that the last row of Table 1 (where measurements of z_s, z_u and z_r are assumed) represents the minimum possible constraints. To see the latter fact, note first that $H_1(j\omega_1) = -m_u/m_s$ is implied by (16) as pointed out in [5]. Also note that, since K stabilizes G_{22} [3, Chapter 4], then $K(I - G_{22}K)^{-1}$ is bounded at $s = \infty$. Thus $H_1 = O(s^{-2})$ follows from (1) since $G_{11} = 0$, $G_{12} = O(s^{-2})$ and G_{21} is bounded at $s = \infty$ for any reasonable choice of measurement.

4 Performance Limitations for Quarter Car Model

In control law design for active suspensions it is clearly desirable to prevent the amplitudes of the frequency responses H_1, H_2 and H_3 (equations (6)–(8)) from being too large in any frequency range. A brief examination of Table 1 shows that the suspension deflection transfer function (H_2) must have modulus *strictly greater* than one at some frequencies (e.g. at ω_2)—indicating amplification of road disturbances. At first sight there seems to be no such requirement for H_1 and H_3. However, it turns out that the form of the Taylor series expansions at $s = 0$ and $s = \infty$ may force disturbance amplification to happen at some frequencies.

Consider first the tyre deflection transfer function H_3. In the cases where $H_3(s) = -1 + \frac{k_t}{m_u}s^{-2} + O(s^{-4})$ it is evident that $|H_3(j\omega)|$ must tend to 1 *from above* as ω tends to ∞. In case $H_3(s) = -1 + O(s^{-2})$, although the manner in which $|H_3(j\omega)|$ tends to 1 can be controlled, it still turns out the $|H_3(j\omega)|$ can't be made less than or equal to 1 at *all frequencies*. This is shown by the following result, which was proved by Freudenberg and Looze [4, Theorem 3] (see also [2, Chapter 6]) in the context of sensitivity reduction, and carries over to the present context essentially without modification (see Appendix).

Theorem 4. Let $H(s)$ belong to $\mathcal{S}$ and satisfy $H(s) = -1 + O(s^{-2})$ for large s. Let $z_1, \ldots z_n$ be the zeros of $H(s)$ with $\Re e(z_i) > 0$. Then

$$\int_0^\infty \ln |H(j\omega)| d\omega = \pi \sum_{i=1}^{n} \Re e(z_i). \qquad (19)$$

Clearly H_3 satisfies the conditions of Theorem 4 for any of the choices of measurement shown in Table 1. Since the right hand side of (19) is non-negative then it is not possible for $|H_3(j\omega)|$ to be less than or equal to one at all frequencies (since that would make the left hand side of the equation negative). *Thus, no matter what signals are used for feedback, the tyre deflection transfer function must amplify road disturbances at some frequencies.* Equation (19) is sometimes called the "area formula", since it says that the difference in area of the curve $\ln|H(j\omega)|$ above 0 and below 0 is fixed (and non-negative). As an example, Fig. 3 shows a plot of the tyre deflection transfer function for the quarter-car model with a passive spring-damper ($u = -k_s(z_s - z_u) - c_s(\dot{z}_s - \dot{z}_u)$). In this case, $H_3(s)$ has no right half plane zeros so the right hand side of (19) is zero. The parameter settings for the plot are: $m_s = 250$ Kg, $m_u = 35$ Kg, $k_t = 150 \times 10^3$ N/m, $k_s = 15 \times 10^3$ N/m and $c_s = 1.0 \times 10^3$ Ns/m.

A similar type of formula is valid when a transfer function has the expansion $H(s) = 1 + O(s^2)$ near $s = 0$. This result is stated below and a proof is given in the Appendix.

Theorem 5. Let $H(s)$ belong to $\mathcal{S}$ and satisfy $H(s) = 1 + O(s^2)$ for small s. Let $z_1, \ldots, z_n$ be the zeros of $H(s)$ with $\Re e(z_i) > 0$. Then

$$\int_0^\infty \ln|H(j\omega)| \frac{d\omega}{\omega^2} = \pi \sum_{i=1}^{n} \frac{\Re e(z_i)}{|z_i|^2}. \tag{20}$$

This result shows that, for the cases where $H_1(0) = 1$ and $H_1'(0) = 0$, $|H_1(j\omega)|$ cannot be less than or equal to 1 at all frequencies (since that would make the left hand side of (20) negative). The presence of the weighting factor $1/\omega^2$ in the integral shows that the "positive area" frequency ranges (i.e. where $|H_1(j\omega)|$ is greater than 1) have lower magnitude if they can be placed at lower frequencies. Thus, there does not seem to be an advantage in terms of ride quality, in these cases, to force $|H_1(j\omega)|$ to be less than one at low frequencies (by making $H_1''(j\omega) > 0$), since the positive area part will then need to occur at higher frequencies and will also have a correspondingly larger area.

An examination of Table 1 shows that, apart from the final three rows, which assume the states z_u and z_s are available for feedback, the only differences in the constraints occur in the higher order terms of the Taylor series expansions about $s = 0$ and $s = \infty$. In terms of the actual class of achievable frequency responses, these differences are minor in the following sense: any frequency responses which are achievable with the measurements $y = (z_s - z_u, \dot{z}_s, z_u - z_r, \dot{z}_u)^T$ (sometimes called "full state feedback" in this context) are achievable to within an arbitrary small tolerance using the measurement $y = z_s - z_u$ only. This fact is made precise as follows (see Appendix for the proof).

Theorem 6. Let H_1, H_2 and H_3 be the frequency responses (6), (7) and (8) obtained for the quarter car model (4), (5) with measurements $y = (z_s - z_u, \dot{z}_s, z_u - z_r, \dot{z}_u)^T$ and some stabilizing feedback law $u = K(s)y$. Take *any* $\epsilon > 0$. Then a feedback law $u = \overline{K}(s)(z_s - z_u)$ can be found with the corresponding frequency responses $\overline{H}_1$, $\overline{H}_2$ and $\overline{H}_3$ and satisfying $|H_i(j\omega) - \overline{H}_i(j\omega)| < \epsilon$ for each $i = 1, 2, 3$ and all ω.

The question arises whether the feedback laws implied by Theorem 6 give reasonable controllers from a practical point of view. The answer to this question is: possibly not. Problems may occur with other aspects of system performance which are independent of the road disturbance response. One of these aspects is the tolerance of the system to loads on the sprung mass. The example below demonstrates a case of such poor behaviour and motivates the study undertaken in the next section.

Example 1. The following parameters are used here for the quarter car model: $m_s = 250$ Kg, $m_u = 35$ Kg and $k_t = 150 \times 10^3$ N/m. The desired frequency responses $H_1(s)$, $H_2(s)$ and $H_3(s)$ are obtained by choosing certain suitable parameters in a "double skyhook" configuration:

$$u = -k_s(z_s - z_u) - c_1\dot{z}_s - c_2\dot{z}_u. \tag{21}$$

The following parameter values are chosen here: $k_s = 10 \times 10^3$ N/m, $c_1 = 4 \times 10^3$ Ns/m and $c_2 = -2 \times 10^3$ Ns/m, which correspond to quite a "soft ride" as measured by $H_1(s)$ and well-behaved suspension deflection and grip, as measured by $H_2(s)$ and $H_3(s)$. (The Bode plots are not displayed, but are almost identical with those in Fig. 6 obtained for the final design in the next section). To make H_1 admissible for the measurement $y = z_s - z_u$ it needs to be modified to satisfy the appropriate constraints. This is achieved by multiplying H_1 by the weighting function:

$$W_M(s) = 1 - \frac{\alpha s}{1 + Ms} \cdot \frac{1 + s^2/\omega_1^2}{(1 + s/\omega_1)^2} \cdot \frac{1 + \alpha^{-1}M^{-1}s}{1 + M^{-2}s} \tag{22}$$

where $\alpha = -(c_2 + c_1)/k_s$. (Briefly, the reasoning behind the weighting function is as follows. Direct calculation shows that

$$H_1(s) = \frac{-c_2k_ts + k_sk_t}{m_sm_us^4 + (c_1m_u - c_2m_s)s^3 + (k_sm_u + k_sm_s + k_tm_s)s^2 + c_1k_ts + k_sk_t},$$

so $H_1'(0) = \alpha$, and $W_M'(0) = -\alpha$ which ensures that $H_1 W_M = 1 + O(s^2)$. Additionally, $W_M(j\omega_1) = 1$ and $W_M(\infty) = 0$, which means that $H_1 W_M$ satisfies the required constraints at ω_1 and ∞. Finally, as M becomes large, W_M tends to 1 at finite frequencies.) The Bode plots of H_1 and $H_1 W_M$ are shown in Fig. 4 for $M = 20$. The compensator $K(s)$ is computed from $H_1 W_M$ directly using the equation (1)—see Appendix for details. The Bode plot of $K(s)$ is shown in Fig. 5. This compensator does indeed give responses for H_1, H_2 and H_3 which are very close to the desired ones. However, the gain of $K(s)$ is very low at low frequencies. Indeed, the d.c. gain equals 61.86, which means that the steady-state error in z_s to a 1 Newton load on the sprung mass is about 1.6cm! This is clearly a useless controller.

5 Load Response

Example 1 illustrates that the road disturbance responses can be quite misleading if they are used alone for assessing the performance of active suspension systems. One possible remedy for this situation is to consider the response to loads acting on the sprung mass. Accordingly, the quarter car model (4), (5) is considered now with the applied load f included (and with $z_r = 0$ temporarily). Two transfer functions of interest can be defined:

$$H_4 = \frac{z_s}{f}, \tag{23}$$

$$H_5 = \frac{z_u}{f}. \tag{24}$$

Since the transfer function $(z_s - z_u)/f$ equals $H_4 - H_5$, it will not be considered separately. The complete set of constraints which H_4 and H_5 satisfy, for various choices of measurements, is given in Table 2. The derivations are similar to those for the road disturbance responses, and are outlined in the Appendix. It is interesting to note that, once again, there is an invariant equation:

$$m_s s^2 z_s + (m_u s^2 + k_t) z_u = -f \tag{25}$$

(which is obtained from (4), (5) by eliminating u). It follows from (25) that H_4 determines H_5 completely, and vice versa, from the following identities

$$H_5 = \frac{-1}{m_u s^2 + k_t}\left(m_s s^2 H_4 + 1\right), \tag{26}$$

$$H_4 = \frac{-1}{m_s s^2}\left((m_u s^2 + k_t) H_5 + 1\right). \tag{27}$$

An inspection of Table 2 shows that the class of achievable transfer functions is distinctly wider when $\dot{z}_s$ is measured in addition to $z_s - z_u$, since H_5 is no longer constrained at ω_1. However, adding $\dot{z}_u$ also does not widen the class further. An examination of the form of H_5 around $s = 0$ shows that Theorem 5 is applicable. That is, *it is not possible to make* $|H_5(j\omega)|$ *less than* $1/k_t$ *at all frequencies*. See Fig. 7 for an example of the sort of peaking which must always occur in H_5.

The invariant equations (14) and (25) have shown that there is only one degree of freedom available in the design of the transfer functions H_1, H_2 and H_3, and one degree of freedom for H_4 and H_5. The question naturally arises: to what extent are these two degrees of freedom independent? The answer is that, providing *two* signals (suitably chosen) are available for feedback, they are completely independent. This is formally stated in the result below, a proof of which is given in the Appendix.

Theorem 7. Let H_1, H_2 and H_3 be the frequency responses (6), (7) and (8) obtained for the quarter car model (4), (5) with measurements $y = y_1 = (z_s - z_u, \dot{z}_s, z_u - z_r, \dot{z}_u)^T$ and some stabilizing feedback law $u = K_1(s)y_1$. Let H_4 and H_5 be the frequency responses (23) and (24) obtained with measurements $y = y_2 = (z_s - z_u, \dot{z}_s, z_u, \dot{z}_u)^T$ and some stabilizing feedback law $u = K_2(s)y_2$. Take *any* $\epsilon > 0$. Then a feedback law $u = \overline{K}(s)[z_s - z_u, \dot{z}_s]^T$ can be found with the corresponding frequency responses $\overline{H}_i$ $(i = 1, \ldots 5)$ and satisfying $|H_i(j\omega) - \overline{H}_i(j\omega)| < \epsilon$ for each $i = 1, \ldots 5$ and all ω.

Example 2. As an illustration of Theorem 7, a controller will be designed for the quarter car model in which the road disturbance transfer functions and the load transfer functions are independently "optimized". The resulting controller which implements these "target" transfer functions uses the measurements $z_s - z_u$ and $\dot{z}_s$ only. The quarter car model is used with the parameters of Example 1 ($m_s = 250$ Kg, $m_u = 35$ Kg and $k_t = 150 \times 10^3$ N/m). Also, the same set of transfer functions is taken for the road disturbance transmission paths (using the parameter values: $k_s = 10 \times 10^3$ N/m, $c_1 = 4 \times 10^3$ Ns/m and $c_2 = -2 \times 10^3$ Ns/m in the "double skyhook" configuration). Except for a small error at high frequencies, these transfer functions are the same as the corresponding ones in the final design—see Fig. 6 for the Bode plots. (The error arises due to the differing constraint at $s = \infty$ when $\dot{z}_u$ is not available for measurement.) Next, a pair of "target" transfer functions for the load disturbance response path is selected to be the ones obtained from the "double skyhook" feedback law (21) with: $k_s = 150 \times 10^3$ N/m, $c_1 = 12 \times 10^3$ Ns/m and $c_2 = -6 \times 10^3$ Ns/m. This makes the sprung mass quite stiff to applied loads (though the response to road disturbances would be unsatisfactory with this control law). See Fig. 7 for the Bode plots of the transfer functions $H_4(s)$ and $H_5(s)$ in the final design, which differ only slightly at high frequency from the "target" transfer functions.

A controller was designed to realise the above 5 target transfer functions together. This was done by solving backwards for $K(s)$ (see Appendix for details). The controller obtained was:

$$K(s) = -\frac{1}{d(s)}[n_1(s), n_2(s)] \tag{28}$$

where

$$\begin{aligned}
n_1(s) &= 10^5 \times [6.667s^5 + 1496s^4 + 68460s^3 + 1677000s^2 + 2.4 \times 10^7 s + 8.571 \times 10^7] \\
n_2(s) &= 10^4 \times [s^5 + 391.2s^4 + 23870s^3 + 1627000s^2 + 3.423 \times 10^7 s + 1.714 \times 10^8] \\
d(s) &= s^5 + 520.8s^4 + 71090s^3 + 2966000s^2 + 3.217 \times 10^7 s + 5.714 \times 10^7
\end{aligned}$$

The Bode plot of $K(s)$ is shown in Fig. 8.

6 Concluding Remarks

This paper has investigated the freedom available through dynamic compensation in determining the closed-loop road and load frequency responses, both separately and in combination, in a quarter car model of an automotive active suspension. The main results are summarized in Tables 1, 2 and Theorems 4–7. Theorems 1–3 state precisely how the constraints listed in the Tables are necessary and sufficient conditions. Several comments can now be made.

1. The paper has not sought to determine what are the "optimal" frequency responses for a given transmission path. Neither have scalar measures (such as the integral of squared modulus) been used to compare different control strategies. It is obvious that the notion of optimality and the suitability of various scalar measures of performance varies from one application to another, and indeed, good designs can be achieved without introducing such notions. It can be said that the frequency responses themselves are taken here as the "measures of performance".

2. The issues of sensitivity/robustness of the control system to variations in the model parameters, or to unmodelled dynamics, have not been taken into

account explicitly here. However, some simple tests for the controller (28) indicate satisfactory robustness properties, i.e. the frequency responses remain stable and within acceptable bounds for typical parameter changes. However, if robust regulation is required against unknown disturbances (e.g. z_s against unknown loads) then requirements on the measurements may change.

3. The paper has shown that the road disturbance responses alone are not sufficient to satisfactorily evaluate the performance of a suspension system. For this reason, Example 2 took account of the load disturbance responses as well. Although it seems reasonable that the road and load responses, taken together, are a "complete enough set" of indicators, this point requires further investigation.

4. The practical realization details for the control actuation and the associated energy requirements have not been considered. It has been assumed that sufficient "control authority" (to provide the desired control input force) is available over a reasonable bandwidth. The system is therefore assumed to be "fully active" in this sense. For the controller of the type (28) the actuator would probably need to have an adequate response up to about 200-300 rad/sec (say 30-50 Hz.) to properly implement the gain/phase characteristic of $K(s)$ in the mid-frequency range.

5. The paper has assessed the design freedom for certain disturbance response paths by computing the frequency domain constraints using Youla parametrization. For a quarter car model, this type of hand computation is feasible. Similar computations for more complex, full vehicle models, would be less tractable, though in principle possible. Alternative approaches to determine similar or equivalent information on design limitations, in these situations, need to be investigated.

6. The design approach used in Example 2 was to determine the desired frequency responses for the road and load disturbances separately and then to compute backwards to a controller. When such computations are carried out using polynomial manipulations, exact cancellation of various factors is required (and is guaranteed theoretically since the "target" transfer functions satisfy the appropriate constraints). It must be admitted that this procedure is not as convenient as would be desired. Moreover, the procedure is potentially quite susceptible to numerical problems for complex vehicle models (though difficulties were not experienced for the examples of this paper). Alternative approaches to achieve a similar end need to be investigated.

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Appendix

Proof of Theorem 1. Note that

$$\begin{aligned}\frac{-k_t}{p(s)}Y(s) &= -NY + \frac{s^2(m_s+m_u)}{p(s)}Y \\ &= 1 - XM + \frac{s^2(m_s+m_u)}{p(s)}Y. \end{aligned} \tag{29}$$

Thus $H_1 = \frac{-k_t}{p}(Y - MQ) = 1 + O(s^2)$. Hence $H_1'(0) = 0$. This completes the necessity part of the proof.

For the sufficiency part of the proof it must be shown that, given an H_1 satisfying the conditions of the theorem, then there is a Q in $\mathcal{S}$ such that $H_1 = \frac{-k_t}{p}(Y - MQ)$. Now solving for Q gives

$$Q = \left(Y + \frac{p}{k_t}H_1\right)M^{-1} = \left(\frac{k_t}{p}Y + H_1\right)\cdot\frac{p}{k_t}M^{-1}. \tag{30}$$

Since H_1 belongs to $\mathcal{S}$ it is sufficient to check that Q is analytic (i.e. has no poles) at the zeros of M and at $s = \infty$. Since $H_1(s) = 1 + O(s^2)$ it follows from (29) and the second expression in (30) that Q is analytic at $s = 0$. Similarly from (13), Q is analytic at $s = j\omega_1$. Finally, since $H_1(s) = O(s^{-4})$ and $M(\infty) \neq 0$, Q is analytic at $s = \infty$ using the first expression for Q in (30).

Verification of Table 1. For each different choice of measurements, the constraints on H_1 are derived using the same method as for Theorem 1. In each case, the constraints arise through the coefficient of Q in the parametrization of T_{zw}, and in particular through its zeros on the imaginary axis and at $s = \infty$. The sufficiency part of the proofs is also quite similar, being just a check that solving for Q in terms of H_1 gives a Q in $\mathcal{S}$. The corresponding constraints on H_2 and H_3 are derived as in Theorems 2 and 3 using (15), (16), (17) and (18). It should be noted that the values of the constraints at the invariant frequencies do not change when more measurements are added, so these values do not need to derived independently each time. Thus, only the generalized plant, the coprime factors of G_{22} and the parametrization of H_1 will be presented in each case.

(*Second row.*) For $w = z_r$, $z = z_s$ and $y = [z_s - z_u, \dot{z}_s - \dot{z}_u]^T$ the generalized plant is given by:

$$G = \left(\begin{array}{c|c} 0 & \frac{1}{m_s s^2} \\ \hline \frac{-k_t}{m_u s^2+k_t} & \frac{(m_s+m_u)s^2+k_t}{m_s s^2(m_u s^2+k_t)} \\ \frac{-sk_t}{m_u s^2+k_t} & \frac{(m_s+m_u)s^2+k_t}{m_s s(m_u s^2+k_t)} \end{array}\right).$$

It can be shown that the following are right and left coprime factorizations:

$$\begin{aligned} G_{22} &= \begin{pmatrix} \frac{(m_s+m_u)s^2+k_t}{p(s)} \\ \frac{s((m_s+m_u)s^2+k_t)}{p(s)} \end{pmatrix} \left(\frac{m_s s^2(m_u s^2+k_t)}{p(s)}\right)^{-1} \\ &= \begin{pmatrix} \frac{m_s k_t s^2}{p_3(s)} & \frac{m_s m_u s^3}{p_3(s)} \\ \frac{-s}{p_1(s)} & \frac{1}{p_1(s)} \end{pmatrix}^{-1} \begin{pmatrix} \frac{(m_s+m_u)s^2+k_t}{p_3(s)} \\ 0 \end{pmatrix} \end{aligned}$$

where $p_1(s)$, $p_3(s)$ and $p(s)$ are any Hurwitz polynomials of degree 1, 3 and 4 respectively. Thus

$$H_1 = T_{zw} = G_{12}(Y-MQ)\tilde{M}G_{21} = \frac{-k_t}{p_3}(Y-MQ)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{-k_t}{p_3}(Y_1 - MQ_1).$$

The only change from the first row is the rate of decay for large s which is now $O(s^{-3})$ due to p_3 having degree 3 (cf. (12)).

(*Third row.*) For $w = z_r$, $z = z_s$ and $y = [z_s - z_u, \dot{z}_s]^T$ the generalized plant is given by:

$$G = \left(\begin{array}{c|c} 0 & \frac{1}{m_s s^2} \\ \hline \frac{-k_t}{m_u s^2+k_t} & \frac{(m_s+m_u)s^2+k_t}{m_s s^2(m_u s^2+k_t)} \\ 0 & \frac{1}{m_s s} \end{array}\right).$$

It can be shown that the following are right and left coprime factorizations:

$$\begin{aligned} G_{22} &= \begin{pmatrix} \frac{(m_s+m_u)s^2+k_t}{p(s)} \\ \frac{s(m_u s^2+k_t)}{p(s)} \end{pmatrix} \left(\frac{m_s s^2(m_u s^2+k_t)}{p(s)}\right)^{-1} && (31) \\ &= \begin{pmatrix} \frac{m_s s(m_u s^2+k_t)}{p_3(s)} & \frac{-k_t m_s}{p_3(s)} \\ 0 & \frac{m_s s}{p_1(s)} \end{pmatrix}^{-1} \begin{pmatrix} \frac{(m_s+m_u)s}{p_3(s)} \\ \frac{1}{p_1(s)} \end{pmatrix} && (32) \end{aligned}$$

where $p_1(s)$, $p_3(s)$ and $p(s)$ are any Hurwitz polynomials of degree 1, 3 and 4 respectively. Thus

$$H_1 = T_{zw} = G_{12}M(\tilde{Y} - Q\tilde{M})G_{21}$$

$$= \frac{-k_t}{p}(\tilde{Y} - Q\tilde{M})\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{-k_t}{p}\left(\tilde{Y}_1 - Q_1\frac{m_s s(m_u s^2 + k_t)}{p_3}\right)$$

Here there is only a single power of s multiplying Q so there is no constraint on $H'(0)$.

(*Fourth row.*) For $w = z_r$, $z = z_s$ and $y = [z_s - z_u, \dot{z}_u]^T$ the generalized plant is given by:

$$G = \left(\begin{array}{c|c} 0 & \frac{1}{m_s s^2} \\ \hline \frac{-k_t}{m_u s^2 + k_t} & \frac{(m_s + m_u)s^2 + k_t}{m_s s^2 (m_u s^2 + k_t)} \\ \frac{s k_t}{m_u s^2 + k_t} & \frac{-s}{m_u s^2 + k_t} \end{array}\right).$$

It can be shown that the following are right and left coprime factorizations:

$$\begin{aligned} G_{22} &= \begin{pmatrix} \frac{(m_s + m_u)s^2 + k_t}{p(s)} \\ \frac{-m_s s^3}{p(s)} \end{pmatrix}\left(\frac{m_s s^2 (m_u s^2 + k_t)}{p(s)}\right)^{-1} \\ &= \begin{pmatrix} \frac{m_s s^2}{p_2(s)} & \frac{m_s s}{p_2(s)} \\ 0 & \frac{m_u s^2 + k_t}{p_2(s)} \end{pmatrix}^{-1}\begin{pmatrix} \frac{1}{p_2(s)} \\ \frac{-s}{p_2(s)} \end{pmatrix} \end{aligned}$$

where $p_2(s)$ and $p(s)$ are any Hurwitz polynomials of degree 2 and 4 respectively. Thus

$$\begin{aligned} H_1 = T_{zw} &= G_{12} M(\tilde{Y} - Q\tilde{M}) G_{21} \\ &= \frac{k_t}{p}(\tilde{Y} - Q\tilde{M})\begin{pmatrix} -1 \\ s \end{pmatrix} = \frac{k_t}{p}\tilde{Y}\begin{pmatrix} -1 \\ s \end{pmatrix} - Q_2\frac{k_t s(m_u s^2 + k_t)}{p_2 p}. \end{aligned}$$

(*Fifth row.*) For $w = z_r$, $z = z_s$ and $y = [z_s - z_u, \dot{z}_s, \dot{z}_u]^T$ the generalized plant is given by:

$$G = \left(\begin{array}{c|c} 0 & \frac{1}{m_s s^2} \\ \hline \frac{-k_t}{m_u s^2 + k_t} & \frac{(m_s + m_u)s^2 + k_t}{m_s s^2 (m_u s^2 + k_t)} \\ 0 & \frac{1}{m_s s} \\ \frac{s k_t}{m_u s^2 + k_t} & \frac{-s}{m_u s^2 + k_t} \end{array}\right).$$

For a right coprime factorization of G_{22}, $M = \frac{m_s s^2 (m_u s^2 + k_t)}{p(s)}$ and for the left factorization:

$$G_{22} = \begin{pmatrix} \frac{s}{p_1} & \frac{-1}{p_1} & \frac{1}{p_1} \\ 0 & \frac{m_s s}{p_1} & 0 \\ 0 & 0 & \frac{m_u s^2 + k_t}{p_2} \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ \frac{1}{p_1} \\ \frac{-s}{p_2} \end{pmatrix}$$

where $p_1(s)$, $p_2(s)$ and $p(s)$ are any Hurwitz polynomials of degree 1, 2 and 4 respectively. Thus

$$\begin{aligned} H_1 = T_{zw} &= G_{12}M(\tilde{Y} - Q\tilde{M})G_{21} \\ &= \frac{k_t}{p}(\tilde{Y} - Q\tilde{M}) \begin{pmatrix} -1 \\ 0 \\ s \end{pmatrix} = \tilde{Y} \begin{pmatrix} -1 \\ 0 \\ s \end{pmatrix} \frac{k_t}{p} - Q_3 \frac{k_t s(m_u s^2 + k_t)}{p_2 p}. \end{aligned}$$

(*Sixth row.*) For $w = z_r$, $z = z_s$ and $y = [z_s - z_u, z_u - z_r]^T$ the generalized plant is given by:

$$G = \left(\begin{array}{c|c} 0 & \frac{1}{m_s s^2} \\ \hline \frac{-k_t}{m_u s^2 + k_t} & \frac{(m_s + m_u)s^2 + k_t}{m_s s^2 (m_u s^2 + k_t)} \\ \frac{-m_u s^2}{m_u s^2 + k_t} & \frac{-1}{m_u s^2 + k_t} \end{array} \right).$$

For a right coprime factorization of G_{22}, $M = \frac{m_s s^2 (m_u s^2 + k_t)}{p(s)}$ and for the left factorization:

$$G_{22} = \begin{pmatrix} \frac{m_u m_s s^2}{p_2(s)} & \frac{-m_s k_t}{p_2(s)} \\ 0 & \frac{m_u s^2 + k_t}{p_2(s)} \end{pmatrix}^{-1} \begin{pmatrix} \frac{m_s + m_u}{p_2(s)} \\ \frac{-1}{p_2(s)} \end{pmatrix}$$

where $p_2(s)$ and $p(s)$ are any Hurwitz polynomials of degree 2 and 4 respectively. Thus

$$\begin{aligned} H_1 = T_{zw} &= G_{12}(Y - MQ)\tilde{M}G_{21} \\ &= \frac{-m_u}{m_s p_2}(Y - MQ) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{-m_u}{m_s p_2}(Y_2 - MQ_2). \end{aligned}$$

(*Seventh row.*) For $w = z_r$, $z = z_s$ and $y = [z_s - z_u, \dot{z}_s, z_u - z_r, \dot{z}_u]^T$ the generalized plant is given by:

$$G = \left(\begin{array}{c|c} 0 & \frac{1}{m_s s^2} \\ \hline \frac{-k_t}{m_u s^2 + k_t} & \frac{(m_s + m_u)s^2 + k_t}{m_s s^2 (m_u s^2 + k_t)} \\ 0 & \frac{1}{m_s s} \\ \frac{-m_u s^2}{m_u s^2 + k_t} & \frac{-1}{m_u s^2 + k_t} \\ \frac{s k_t}{m_u s^2 + k_t} & \frac{-s}{m_u s^2 + k_t} \end{array} \right).$$

For a right coprime factorization of G_{22}, $M = \frac{m_s s^2 (m_u s^2 + k_t)}{p(s)}$ and for the left factorization:

$$G_{22} = \begin{pmatrix} \frac{s}{p_1} & \frac{-1}{p_1} & 0 & \frac{1}{p_1} \\ 0 & \frac{m_s s}{p_1} & 0 & 0 \\ 0 & 0 & \frac{k_t}{p_1} & \frac{m_u s}{p_1} \\ 0 & 0 & \frac{-s}{p_1} & \frac{1}{p_1} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{1}{p_1} \\ \frac{-1}{p_1} \\ 0 \end{pmatrix} \tag{33}$$

where $p_1(s)$ and $p(s)$ are any Hurwitz polynomials of degree 1 and 4 respectively. Thus

$$\begin{aligned} H_1 = T_{zw} &= G_{12} M(\tilde{Y} - Q\tilde{M}) G_{21} \\ &= \tilde{Y} \begin{pmatrix} -k_t \\ 0 \\ -m_u s^2 \\ s k_t \end{pmatrix} \frac{1}{p} - Q_4 \frac{s(m_u s^2 + k_t)}{p_1 p}. \end{aligned}$$

(*Eighth row.*) For $w = z_r$, $z = z_s$ and $y = [z_s, z_u]^T$ the generalized plant is given by:

$$G = \left(\begin{array}{c|c} 0 & \frac{1}{m_s s^2} \\ \hline 0 & \frac{1}{m_s s^2} \\ \frac{k_t}{m_u s^2 + k_t} & \frac{-1}{m_u s^2 + k_t} \end{array} \right).$$

For a right coprime factorization of G_{22}, $M = \frac{m_s s^2 (m_u s^2 + k_t)}{p(s)}$ and for the left factorization:

$$G_{22} = \begin{pmatrix} \frac{m_s s^2}{p_2(s)} & 0 \\ 0 & \frac{m_u s^2 + k_t}{p_2(s)} \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{p_2(s)} \\ \frac{-1}{p_2(s)} \end{pmatrix}$$

where $p_2(s)$ and $p(s)$ are any Hurwitz polynomials of degree 2 and 4 respectively. Thus

$$\begin{aligned} H_1 = T_{zw} &= G_{12} M(\tilde{Y} - Q\tilde{M}) G_{21} \\ &= \frac{k_t}{p}(\tilde{Y} - Q\tilde{M}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{k_t}{p} \left(\tilde{Y}_2 - Q_2 \frac{m_u s^2 + k_t}{p_2} \right) \end{aligned}$$

Observe that there is no constraint at $s = 0$.

(*Tenth row.*) For $w = z_r$, $z = z_s$ and $y = [z_s, z_u, z_r]^T$ the generalized plant is given by:

$$G = \left(\begin{array}{c|c} 0 & \frac{1}{m_s s^2} \\ \hline 0 & \frac{1}{m_s s^2} \\ \frac{k_t}{m_u s^2 + k_t} & \frac{-1}{m_u s^2 + k_t} \\ 1 & 0 \end{array} \right).$$

For a right coprime factorization of G_{22}, $M = \frac{m_s s^2 (m_u s^2 + k_t)}{p(s)}$ and for the left factorization:

$$G_{22} = \begin{pmatrix} \frac{m_s s^2}{p_2(s)} & 0 & 0 \\ 0 & \frac{m_u s^2 + k_t}{p_2(s)} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{p_2(s)} \\ \frac{-1}{p_2(s)} \\ 0 \end{pmatrix}$$

where $p_2(s)$ and $p(s)$ are any Hurwitz polynomials of degree 2 and 4 respectively. Thus

$$\begin{aligned} H_1 = T_{zw} &= G_{12} M (\tilde{Y} - Q\tilde{M}) G_{21} \\ &= \tilde{Y} \begin{pmatrix} 0 \\ \frac{k_t}{p} \\ \frac{m_u s^2 + k_t}{p} \end{pmatrix} - \frac{m_u s^2 + k_t}{p} Q' \end{aligned}$$

where $Q' = Q_2 \frac{k_t}{p} + Q_3$. Note that Q' is an arbitrary element of $\mathcal{S}$.

Proof of Theorem 4. The result can be proved from first principles using exactly the same steps as [4]. Alternatively, [4, Theorem 3] can be translated directly into the current context as follows. Writing $-H(s) = S(s) = (1 + L(s))^{-1}$ means $L(s)$ satisfies (3.2) in [4]. Moreover, the poles of $L(s)$ are the zeros of $H(s)$. Closed-loop stability in [4] is equivalent to $S(s)$ belongs to $\mathcal{S}$.

Proof of Theorem 5. Note that $H(s)$ belongs to $\mathcal{S}$ if and only if $H(1/s)$ does. Writing $H(1/s) = S(s) = (1 + L(s))^{-1}$ gives $L(s) = O(s^{-2})$ for large s, which means that (3.2) is satisfied in [4]. Moreover, the poles of $L(s)$ are at the inverse locations to the zeros of $H(s)$. Thus

$$\int_0^\infty \ln |H(j/\phi)| d\phi = \pi \sum_{i=1}^n \Re e(z_i^{-1}) = \pi \sum_{i=1}^n \frac{\Re e(z_i)}{|z_i|^2}.$$

A change of variable $\omega = 1/\phi$ now gives the result.

Proof of Theorem 6. The notation $\|H\|_\infty = \sup_\omega |H(j\omega)|$ will be used in this proof. The idea of the proof is as follows. Let $H_1(s)$ satisfy $H_1(j\omega_1) = -\frac{m_u}{m_s}$,

$H_1(s) = O(s^{-2})$ and $H_1(0) = 1$ and suppose that $H_1'(0) = \alpha \neq 0$ (the case of $\alpha = 0$ will be dealt with separately). A family of functions $W_M(s)$ in $\mathcal{S}$ will be constructed which satisfy: (i) $W_M(s) = 1 - \alpha s + O(s^2)$, (ii) $W_M(j\omega_1) = 1$, (iii) $W_M(s) = O(s^{-2})$, (iv) $\|(W_M - 1)V\|_\infty \to 0$ as $M \to \infty$ for any V in $\mathcal{S}$ such that $V(s) \to 0$ as $s \to \infty$, (v) $|W_M'(j\omega_1)| \to 0$ as $M \to \infty$. Let $\overline{H}_1 = H_1 W_M$ and define $\overline{H}_2$, $\overline{H}_3$ as a function of $\overline{H}_1$ by the relations (15), (17). From (i), $\overline{H}_1 = 1 + O(s^2)$; from (ii), $\overline{H}_1(j\omega_1) = -\frac{m_u}{m_s}$; from (iii), $\overline{H}_1 = O(s^{-4})$. Therefore, $\overline{H}_1$ is admissible with $y = z_s - z_u$. From (iv) $\|\overline{H}_1 - H_1\|_\infty \to 0$ as $M \to \infty$. From (iv), (v) and (15), $\|\overline{H}_2 - H_2\|_\infty \to 0$ as $M \to \infty$, and similarly for $\overline{H}_3$. This proves the Theorem. It remains only to establish the existence of the sequence W_M. This is done by construction. Let

$$W_M = 1 - W_{1M}W_{2M}W_{3M} = 1 - \frac{\alpha s}{1 + Ms} \cdot \frac{1 + s^2/\omega_1^2}{(1 + s/\omega_1)^2} \cdot \frac{1 + \beta s + \alpha^{-1}M^{-3}s^2}{(1 + M^{-2}s)^2} \tag{34}$$

where $\beta = \alpha^{-1}M^{-1}(2 + 2\omega_1 M^{-2} + M^{-3})$. Properties (i) and (ii) follow immediately by direct computation and (iii) is established by writing W_M over a common denominator and showing that the coefficients of s^4 and s^5 both vanish in the numerator. Turning to (iv), note that $\|W_{1M}\|_\infty = |\alpha|/M \to 0$ as $M \to \infty$, $\|W_{2M}\|_\infty$ is fixed and bounded, and $|W_{3M}(j\omega)| \to 1$ uniformly on any compact interval $[0, \omega_0]$. Thus $|W_{1M}W_{2M}W_{3M}(j\omega)| \to 0$ on $[0, \omega_0]$ which means that $|((W_M - 1)V)(j\omega)| \to 0$ uniformly as $M \to \infty$ on $[0, \omega_0]$. Next note that

$$\begin{aligned}
|W_{1M}(j\omega)W_{3M}(j\omega)| &\leq |\alpha|M^{-1}\frac{1 + |\beta|\omega + |\alpha^{-1}M^{-3}|\omega^2}{1 + M^{-4}\omega^2} \\
&\leq |\alpha|M^{-1}\left(1 + \frac{|\beta|\omega}{1 + M^{-4}\omega^2} + |\alpha^{-1}|M\right) \\
&\leq |\alpha|M^{-1}(1 + |\beta|M^2/2 + |\alpha^{-1}|M) \\
&= |\alpha|M^{-1} + |1 + \omega_1 M^{-2} + M^{-3}/2| + 1
\end{aligned}$$

which is bounded above by a fixed constant, for all ω, as $M \to \infty$. Thus, on an interval $[\omega_0, \infty)$, $|((W_M - 1)V)(j\omega)|$ can be made uniformly small by choosing ω_0 large enough, since V vanishes at ∞. This now completes the justification of (iv). To see (v), note that $W_M'(j\omega_1) = (W_{1M}W_{2M}'W_{3M})(j\omega_1)$, and that $|W_{1M}(j\omega_1)| \to 0$ while the other two terms in the product are bounded. Finally, the case $\alpha = 0$ can be dealt with by multiplying H_1 by the weight (22) with $\alpha = 1$ and a second time with $\alpha = -1$.

Verification of Table 2. The proofs are similar to those for Table 1. Some brief details are presented below.

(*First row.*) For $w = f$, $z = z_s$ and $y = z_s - z_u$ the generalized plant is:

$$G = \left(\begin{array}{c|c} \frac{-1}{m_s s^2} & \frac{1}{m_s s^2} \\ \hline \frac{-1}{m_s s^2} & \frac{(m_s+m_u)s^2+k_t}{m_s s^2(m_u s^2+k_t)} \end{array} \right).$$

A pair of coprime factors for G_{22} is given as in (10), (11). Thus

$$\begin{aligned} H_4 &= G_{11} + G_{12}(Y - MQ)\tilde{M}G_{21} \\ &= \frac{-1}{m_s s^2}\left(1 + Y\frac{m_u s^2 + k_t}{p(s)}\right) + \left(\frac{m_u s^2 + k_t}{p}\right)^2 Q \qquad (35) \\ &= \frac{1}{p}\left(Y - (m_u s^2 + k_t)X\right) + \left(\frac{m_u s^2 + k_t}{p}\right)^2 Q \end{aligned}$$

The constraints on H_4 occur at $s = j\omega_1$ and at $s = \infty$. From (35) it follows that $H_4(s) = -\frac{1}{m_s s^2} + O(s^{-4})$. At $s = j\omega_1$ there is a double constraint with $H_4(j\omega_1) = \left(\frac{Y}{p}\right)(j\omega_1) = \frac{m_u}{m_s k_t}$ (from (35)). The constraint on the first derivative follows from expanding $\frac{-1}{m_s s^2}\left(1 + Y\frac{m_u s^2+k_t}{p(s)}\right)$ about $s = j\omega_1$. The constraints on H_5 can be deduced using (26) or calculated directly from the generalized plant:

$$G = \left(\begin{array}{c|c} 0 & \frac{-1}{m_u s^2+k_t} \\ \hline \frac{-1}{m_s s^2} & \frac{(m_s+m_u)s^2+k_t}{m_s s^2(m_u s^2+k_t)} \end{array} \right)$$

which gives $H_5 = \frac{1}{p}(Y - MQ)$.

(*Second row*). For $w = f$, $z = z_s$ and $y = [z_s - z_u, \dot{z}_s]^T$ the generalized plant is:

$$G = \left(\begin{array}{c|c} \frac{-1}{m_s s^2} & \frac{1}{m_s s^2} \\ \hline \frac{-1}{m_s s^2} & \frac{(m_s+m_u)s^2+k_t}{m_s s^2(m_u s^2+k_t)} \\ \frac{-1}{m_s s} & \frac{1}{m_s s} \end{array} \right).$$

A pair of coprime factors for G_{22} are given as in (31), (32). Thus

$$\begin{aligned} H_4 &= G_{11} + G_{12}(Y - MQ)\tilde{M}G_{21} \\ &= \frac{-1}{m_s s^2}\left(1 + Y\begin{pmatrix} \frac{m_u s}{p_3} \\ \frac{1}{p_1} \end{pmatrix}\right) + \frac{m_u s^2 + k_t}{p}Q\begin{pmatrix} \frac{m_u s}{p_3} \\ \frac{1}{p_1} \end{pmatrix}. \end{aligned}$$

(*Third row*). For $w = f$, $z = z_s$ and $y = [z_s - z_u, \dot{z}_s, z_u, \dot{z}_u]^T$ the generalized plant is:

$$G = \left(\begin{array}{c|c} \frac{-1}{m_s s^2} & \frac{1}{m_s s^2} \\ \hline \frac{-1}{m_s s^2} & \frac{(m_s+m_u)s^2+k_t}{m_s s^2(m_u s^2+k_t)} \\ \frac{-1}{m_s s} & \frac{1}{m_s s} \\ 0 & \frac{-1}{m_u s^2+k_t} \\ 0 & \frac{-s}{m_u s^2+k_t} \end{array} \right).$$

The left coprime factorization for G_{22} is given in (33) from which it follows that

$$\tilde{M}G_{21} = \frac{-1}{p_1} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus

$$\begin{aligned} H_4 &= G_{11} + G_{12}(Y - MQ)\tilde{M}G_{21} \\ &= \frac{-1}{m_s s^2}\left(1 + Y_2 \frac{1}{p_1}\right) + \frac{m_u s^2 + k_t}{p_1 p} Q_2. \end{aligned}$$

Proof of Theorem 7. Let $w = [z_r, f]^T$, $z = z_s$ and $y = [z_s - z_u, \dot{z}_s]^T$. This gives the following generalized plant:

$$G = \left(\begin{array}{cc|c} 0 & \frac{-1}{m_s s^2} & \frac{1}{m_s s^2} \\ \hline \frac{-k_t}{m_u s^2+k_t} & \frac{-1}{m_s s^2} & \frac{(m_s+m_u)s^2+k_t}{m_s s^2(m_u s^2+k_t)} \\ 0 & \frac{-1}{m_s s} & \frac{1}{m_s s} \end{array} \right). \tag{36}$$

The factorizations of G_{22} are given in (31), (32) from which it follows that

$$\tilde{M}G_{21} = \begin{pmatrix} \frac{-k_t m_s s}{p_3} & \frac{-m_u s}{p_3} \\ 0 & \frac{-1}{p_1} \end{pmatrix}.$$

Hence

$$\begin{aligned} [H_1, H_4] &= T_{zw} = \left[G_{11} + G_{12}M(\tilde{Y} - Q\tilde{M})G_{21}\right] \\ &= [G_{11} + G_{12}M\tilde{Y}G_{21}] + \frac{m_u s^2 + k_t}{p}\left[\frac{k_t m_s s}{p_3}Q_1, \frac{m_u s}{p_3}Q_1 + \frac{1}{p_1}Q_2\right] \end{aligned} \tag{37}$$

where $Q = [Q_1, Q_2]$. Note that Q_1 and Q_2 are completely independent free parameters in $\mathcal{S}$. From (37) it is evident that Q_1 can be chosen to obtain any H_1 satisfying $H_1(0) = 1$, $H_1(j\omega_1) = -m_u/m_s$ and $H_1(s) = O(s^{-4})$. (The constraints at $s = 0$ and $s = j\omega_1$ are necessarily the same as in Table 1, so they don't need to be recomputed from (37)). Now from Theorem 6, the *desired* H_1 can be approximated arbitrarily closely from the class of H_1's achieved in (37) (and in such a way that H_2 and H_3 are also approximated arbitrarily closely). So let Q_1 be chosen to achieve the desired approximation. Now define $Q_3 = \frac{p_1 m_u s}{p_3} Q_1 + Q_2$ and note that the second term in the expression for H_4 takes the form $\frac{(m_u s^2 + k_t)}{pp_1} Q_3$ where Q_3 is *arbitrary* in $\mathcal{S}$. As in the verification of the second row of Table 2, Q_3 can be chosen to give any H_4 which satisfies the constraints specified in the second row of Table 2. Hence Q_2 can be chosen to do the same. This completes the proof.

Controller computation. In Example 1, a controller $K(s)$ which uses the measurement $z_s - z_u$ was computed for a given admissible $H_1(s)$. This can be achieved as follows. Let $H_1(s) = n_1(s)/d_1(s)$ where n_1 and d_1 are polynomials. From the generalized plant (9) and equation (1) it can be verified that

$$K(s) = \frac{n_1(s) m_s s^2 (m_u s^2 + k_t)}{((m_s + m_u)s^2 + k_t) n_1(s) - k_t d_1(s)}.$$

The constraints which an admissible H_1 must satisfy ensure that $((m_s + m_u)s^2 + k_t)n_1(s) - k_t d_1(s)$ is divisible by $m_s s^2 (m_u s^2 + k_t)$ without remainder. Let this quotient be denoted by $q(s)$. Then $K(s) = n_1(s)/q(s)$.

In Example 2, a controller $K(s)$ which uses the measurements $[z_s - z_u, \dot{z}_s]^T$ was computed for given admissible $H_1(s) = n_1(s)/d_1(s)$ and $H_4(s) = n_4(s)/d_4(s)$. From the generalized plant formula (36) and equation (1) it follows that

$$G_{12}^{-1}\left((H_1, H_4) - G_{11}\right) G_{21}^{-1}(I - G_{22}K) = K$$

which can be solved to give

$$K = \left(\frac{(m_u s^2 + k_t) n_1 d_4}{n_1 d_4 + d_1 n_4 k_t}, \frac{(m_s s^2 n_4 + d_4) d_1 k_t - (m_u s^2 + k_t) n_1 d_4}{s(n_1 d_4 + d_1 n_4 k_t)} \right).$$

The constraints satisfied by H_1 and H_4 ensure that $n_1 d_4 + d_1 n_4 k_t$ is divisible exactly by $m_u s^2 + k_t$ (with quotient $q_1(s)$, say). Similarly, $(m_s s^2 n_4 + d_4)$ is divisible exactly by $m_u s^2 + k_t$ (with quotient $q_2(s)$, say). Finally, $q_2 d_1 k_t - n_1 d_4$ is divisible exactly by s (with quotient $q_3(s)$, say). This gives

$$K = (K_1, K_2) = \left(\frac{n_1 d_4}{q_1}, \frac{q_3}{q_1} \right).$$

In Example 2, the "target" transfer function for H_1 was not strictly admissible for the measurements $[z_s - z_u, \dot{z}_s]^T$, specifically, H_1 has third (rather than fourth) order roll-off, being obtained from a "double skyhook" feedback (see Table 1). One approach would be to introduce a weighting function as in Example 1 or the proof of Theorem 6. An alternative approach, in the case when H_1 is not admissible due only to the roll-off being not sufficiently fast, is to compute the controller as above, which results in $K(s)$ being unbounded at $s = \infty$, and then to introduce extra lags into the controller as a correction. This was the procedure carried out in Example 2 where an extra factor $1/(.003s + 1)$ was multiplied into the first entry of the controller to give the final $K_1(s)$.

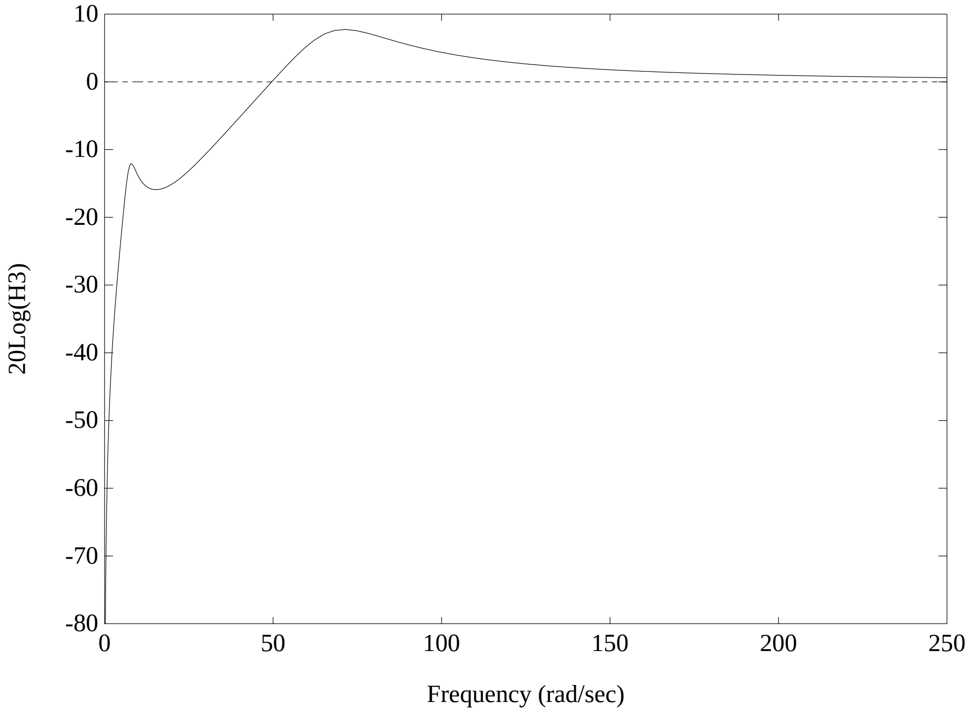

Figure 3: Graph of $20\mathrm{Log}_{10}|H_3(j\omega)|$ versus ω for passive spring-damper arrangement illustrating that area under curve integrates to zero.

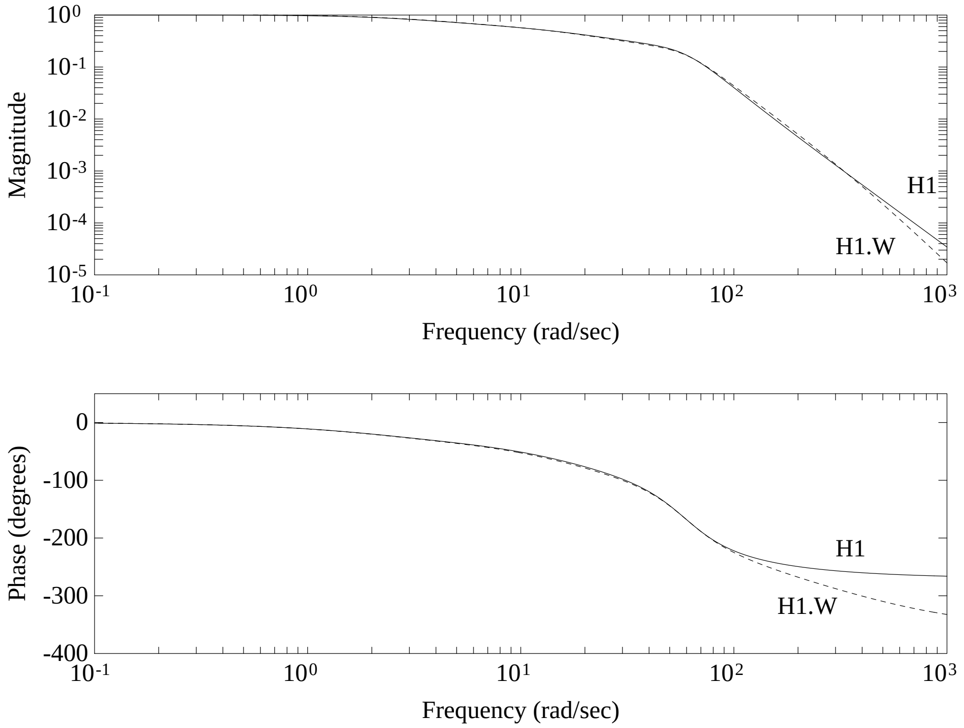

Figure 4: The transfer functions $H_1(s)$, $H_1(s)W_M(s)$ of Example 1.

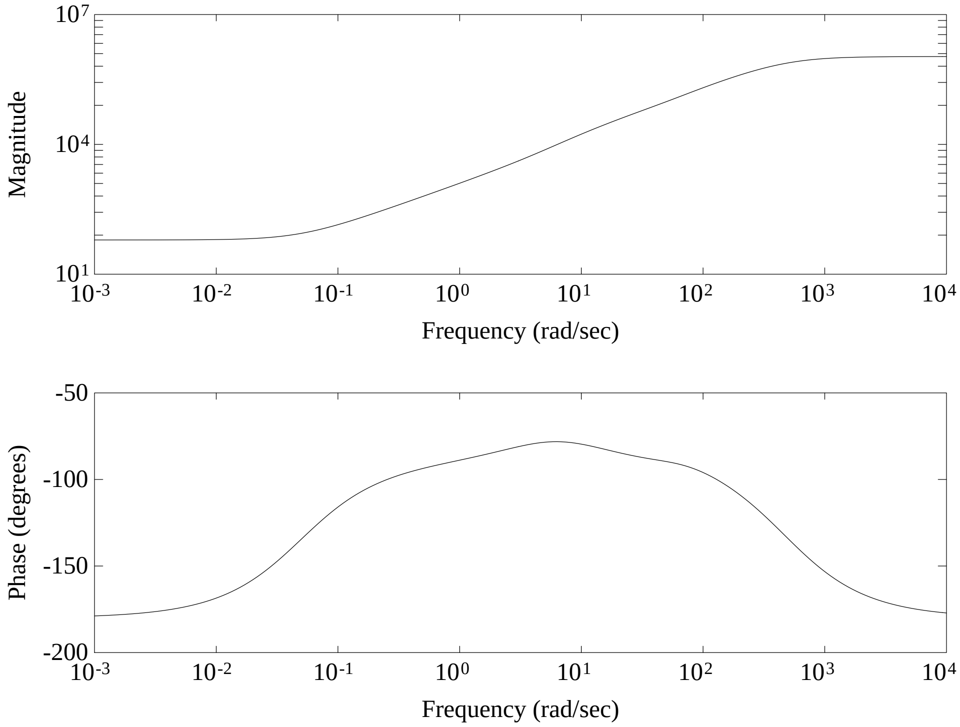

Figure 5: The controller $K(s)$ of Example 1.

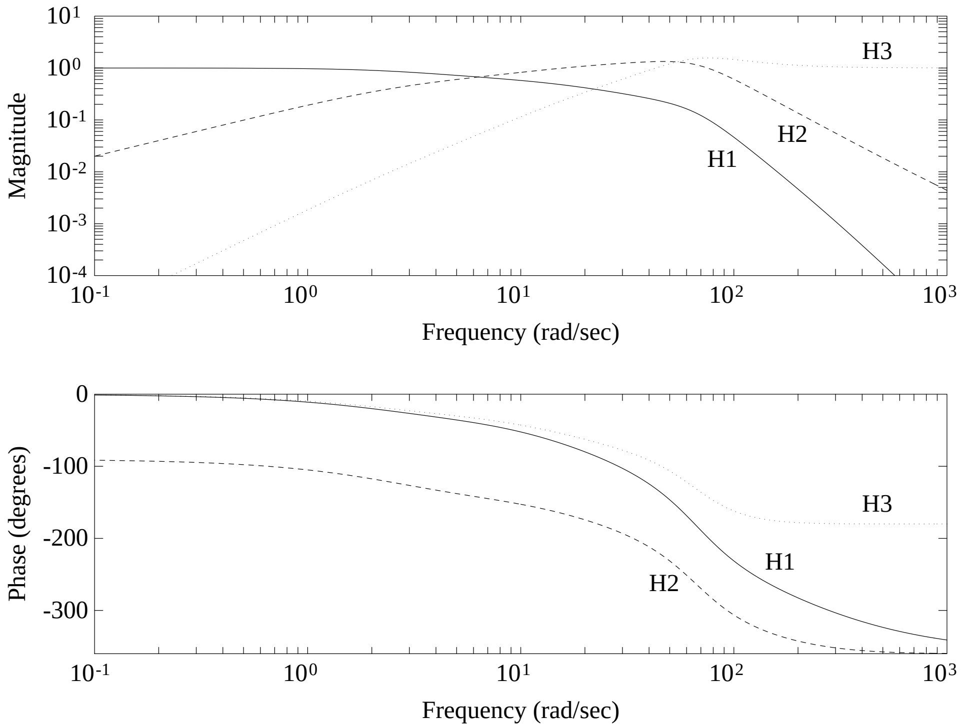

Figure 6: The transfer functions $H_1(s)$, $H_2(s)$ and $H_3(s)$ for the controller of Example 2.

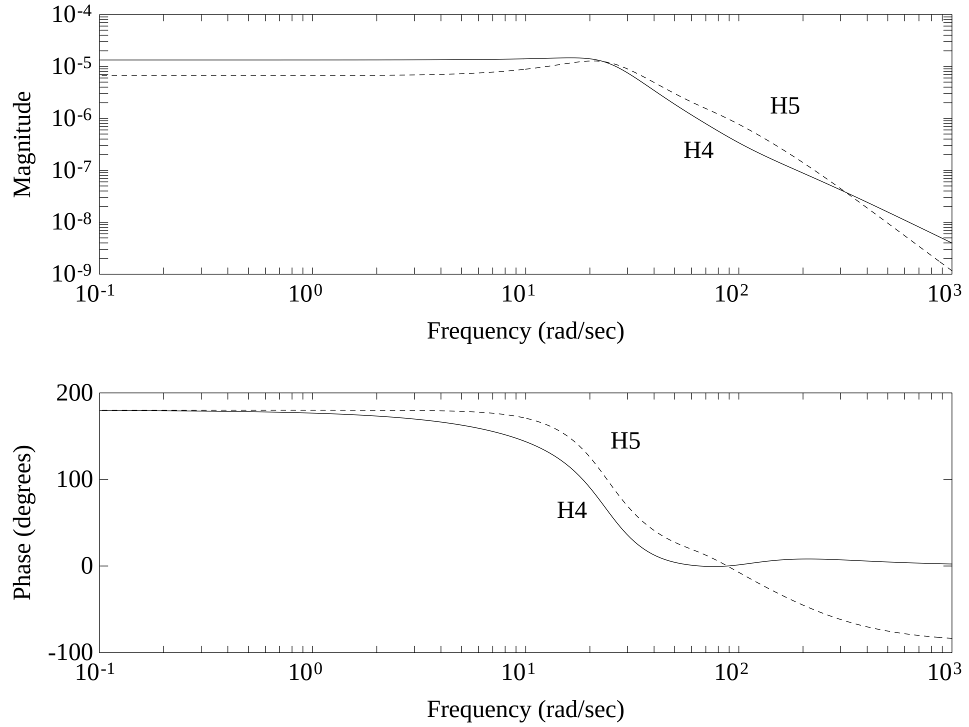

Figure 7: The transfer functions $H_4(s)$ and $H_5(s)$ for the controller of Example 2.

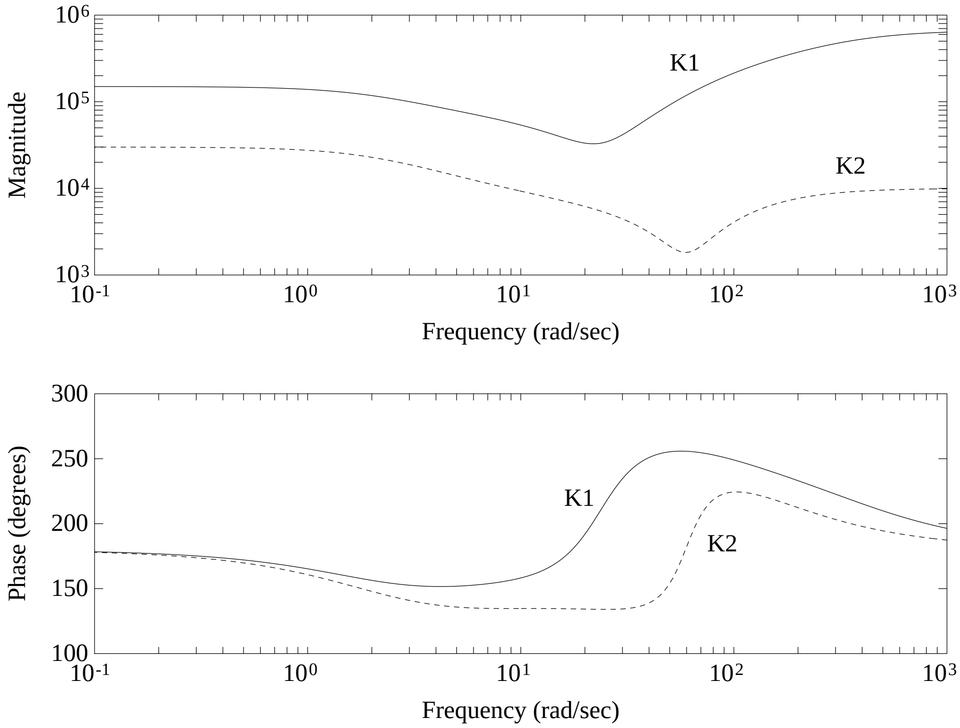

Figure 8: The transfer functions $K(s) = [K_1(s), K_2(s)]$ for the controller of Example 2.

measurements	H_1	H_2	H_3
(any)	$H_1(j\omega_1) = -\frac{m_u}{m_s}$	$H_2(j\omega_2) = -\left(1 + \frac{m_u}{m_s}\right)$	
$z_s - z_u$	$O(s^{-4})$, $H_1(0) = 1,\ H_1'(0) = 0$	$-\frac{k_t}{m_u}s^{-2} + O(s^{-4})$, $H_2(0) = H_2'(0) = 0$	$-1 + \frac{k_t}{m_u}s^{-2} + O(s^{-4})$, $-\frac{(m_s+m_u)}{k_t}s^2 + O(s^4)$
$z_s - z_u$, $\dot{z}_s - \dot{z}_u$	$O(s^{-3})$, $H_1(0) = 1,\ H_1'(0) = 0$	$-\frac{k_t}{m_u}s^{-2} + O(s^{-3})$, $H_2(0) = H_2'(0) = 0$	$-1 + \frac{k_t}{m_u}s^{-2} + O(s^{-3})$, $-\frac{(m_s+m_u)}{k_t}s^2 + O(s^4)$
$z_s - z_u$, $\dot{z}_s$	$O(s^{-4})$, $H_1(0) = 1$	$-\frac{k_t}{m_u}s^{-2} + O(s^{-4})$, $H_2(0) = 0$	$-1 + \frac{k_t}{m_u}s^{-2} + O(s^{-4})$, $-\frac{(m_s+m_u)}{k_t}s^2 + O(s^3)$
$z_s - z_u$, $\dot{z}_u$	$O(s^{-3})$, $H_1(0) = 1$	$-\frac{k_t}{m_u}s^{-2} + O(s^{-3})$, $H_2(0) = 0$	$-1 + \frac{k_t}{m_u}s^{-2} + O(s^{-3})$, $-\frac{(m_s+m_u)}{k_t}s^2 + O(s^3)$
$z_s - z_u$, $\dot{z}_s$, $\dot{z}_u$	same as $z_s - z_u$, $\dot{z}_u$		
$z_s - z_u$, $z_u - z_r$	$O(s^{-2})$, $H_1(0) = 1,\ H_1'(0) = 0$	$O(s^{-2})$, $H_2(0) = H_2'(0) = 0$	$-1 + O(s^{-2})$, $-\frac{(m_s+m_u)}{k_t}s^2 + O(s^4)$
$z_s - z_u$, $\dot{z}_s$, $z_u - z_r$, $\dot{z}_u$	$O(s^{-2})$, $H_1(0) = 1$	$O(s^{-2})$, $H_2(0) = 0$	$-1 + O(s^{-2})$, $-\frac{(m_s+m_u)}{k_t}s^2 + O(s^3)$
z_s, z_u	$O(s^{-4})$	$-\frac{k_t}{m_u}s^{-2} + O(s^{-4})$	$-1 + \frac{k_t}{m_u}s^{-2} + O(s^{-4})$ $H_3(0) = H_3'(0) = 0$
z_s, $\dot{z}_s$, z_u, $\dot{z}_u$,	$O(s^{-3})$	$-\frac{k_t}{m_u}s^{-2} + O(s^{-3})$	$-1 + \frac{k_t}{m_u}s^{-2} + O(s^{-3})$ $H_3(0) = H_3'(0) = 0$
z_s, z_u, z_r	$O(s^{-2})$	$O(s^{-2})$	$-1 + O(s^{-2})$ $H_3(0) = H_3'(0) = 0$

Table 1: Complete set of constraints on transfer functions H_1, H_2 and H_3. First row are always constraints.

measurements	H_4	H_5
(any)	$H_4(j\omega_1) = \frac{m_u}{m_s k_t}$	
$z_s - z_u$	$H_4'(j\omega_1) = \frac{2j}{m_s^2 \omega_1^3}(m_s + m_u)$, $-\frac{1}{m_s}s^{-2} + O(s^{-4})$	$H_5(j\omega_1) = \frac{m_u}{m_s k_t}$, $O(s^{-4})$, $-\frac{1}{k_t} + O(s^2)$
$z_s - z_u$, $\dot{z}_s$	$-\frac{1}{m_s}s^{-2} + O(s^{-3})$	$O(s^{-3})$, $-\frac{1}{k_t} + O(s^2)$
$z_s - z_u$, $\dot{z}_s$, z_u, $\dot{z}_u$	same as $z_s - z_u$, $\dot{z}_s$	

Table 2: Complete set of constraints on transfer functions H_4 and H_5. First row are always constraints.

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