Module 4F2: Nonlinear Systems and Control

Lectures 4–5: Describing Functions

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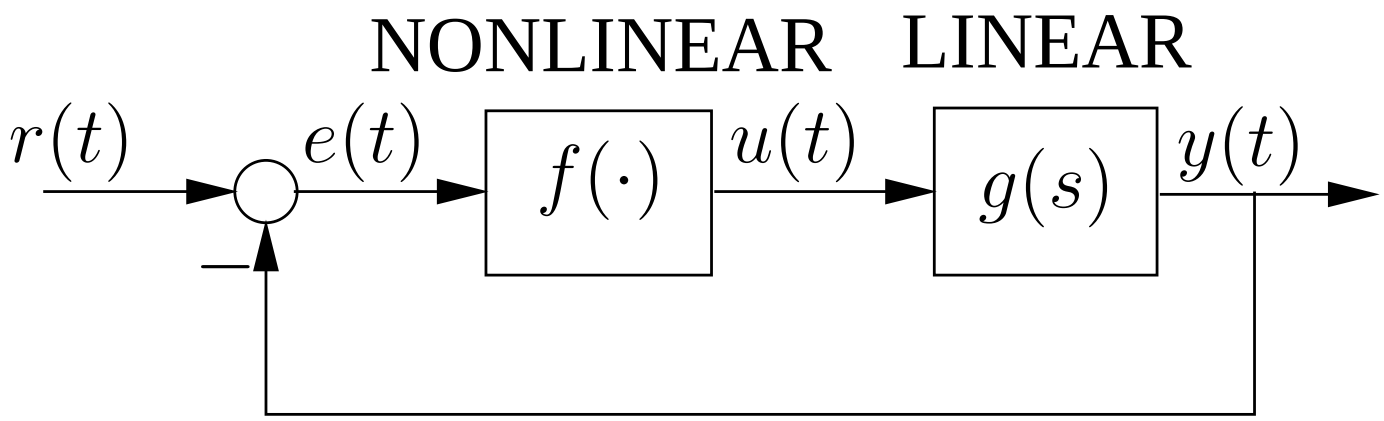
- Relies on a *low-pass* assumption.
- Generalises Nyquist stability theorem.

Nonlinear feedback system



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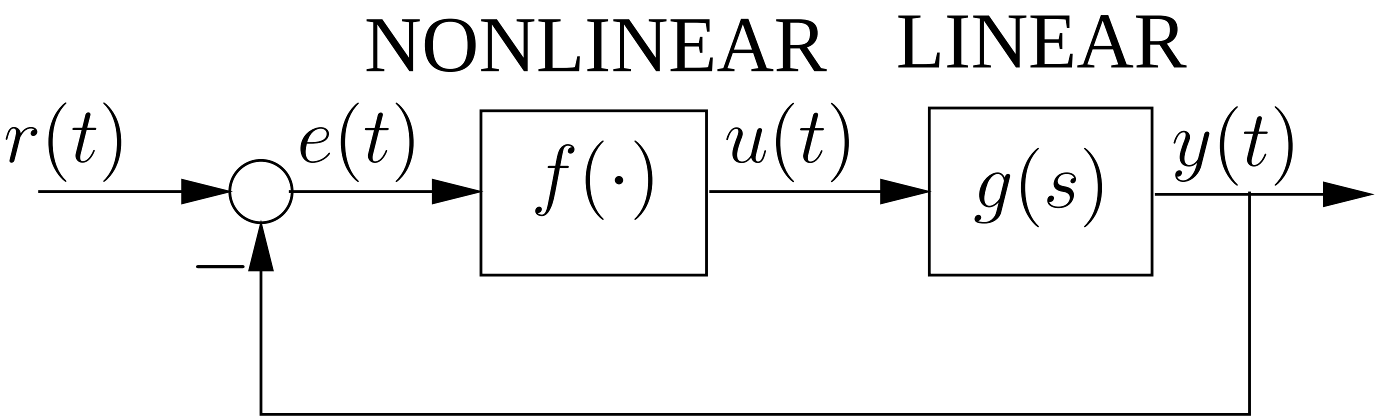

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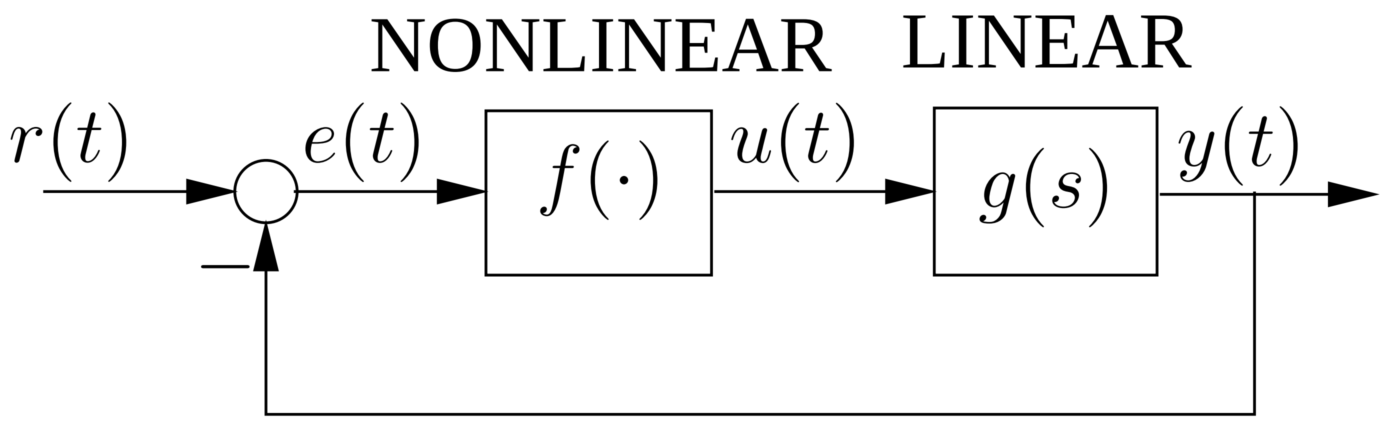

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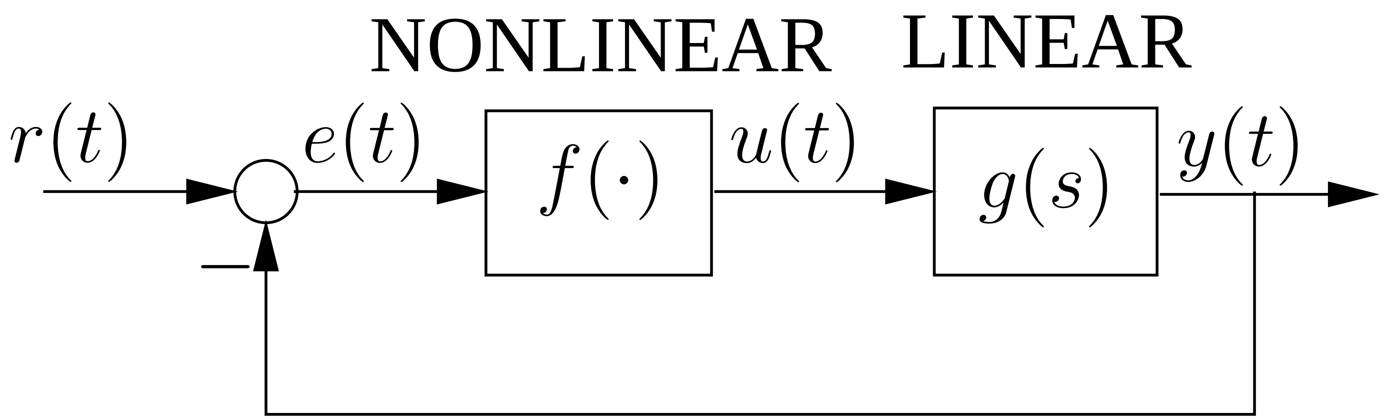

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- $u(t)$ is periodic — has Fourier series.
- Assume $g(s)$ is *low-pass* — $y(t)$ has first harmonic only.
- But $y(t) = -e(t)$ — harmonic balance.

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$$V_1 = \frac{1}{\pi} \int_0^{2\pi} f(E \sin \theta) \cos \theta \, d\theta.$$

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But

$$\begin{aligned} 0 \equiv & \; y(t) + e(t) \\ \Rightarrow \; 0 \equiv & \; \mathrm{Im}[(g(j\omega)(U_1 + jV_1) + E)e^{j\omega t}] = 0 \\ \Rightarrow \; 0 = & \; g(j\omega)(U_1 + jV_1) + E. \end{aligned}$$

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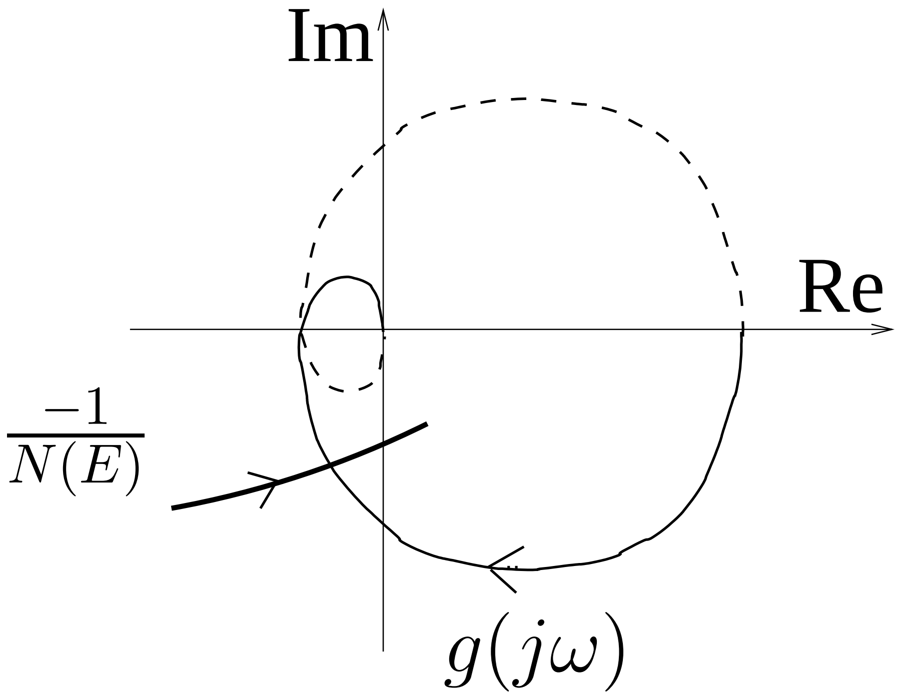
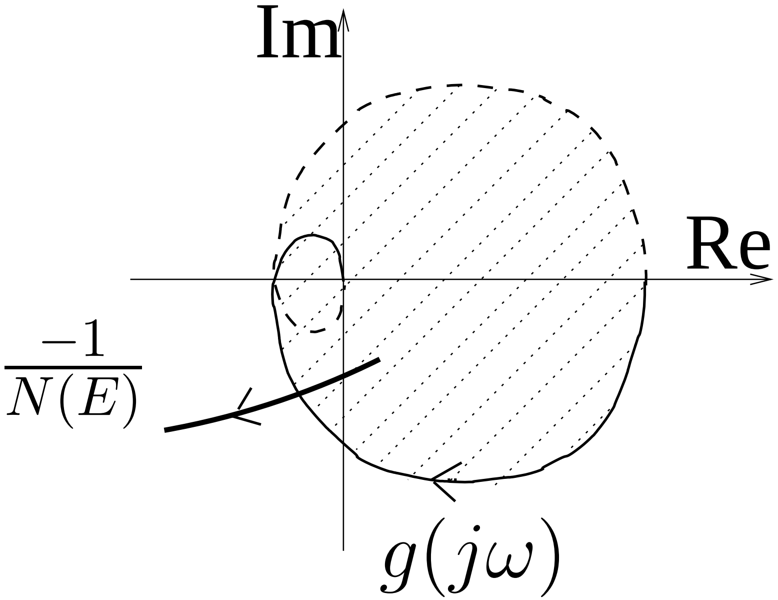
Think of $N(E)$ as *equivalent linear gain*.
Compare with Nyquist: $g(j\omega) = \frac{-1}{k}$.

Graphical interpretation

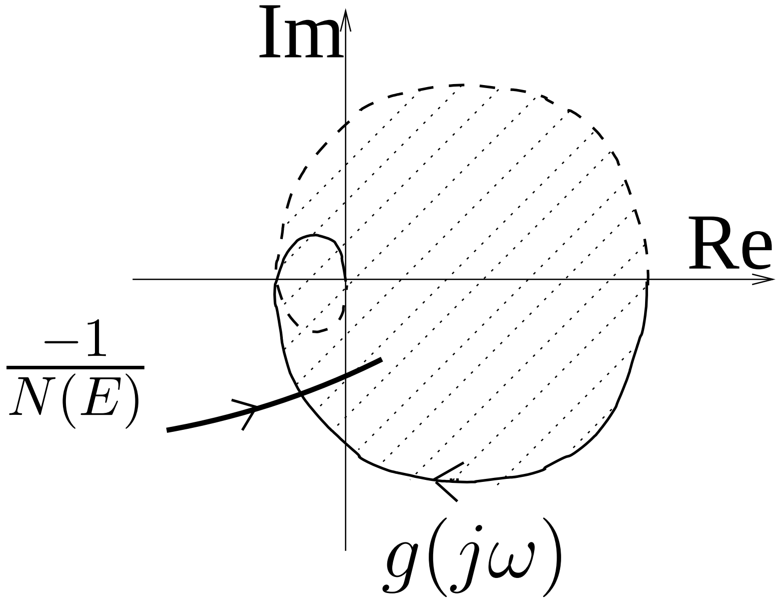


Arrows denote direction of increasing ω and E

Stability of limit cycle



Prediction: STABLE



Prediction: UNSTABLE

DF of relay nonlinearity

$$f(e) = \text{sign}(e)$$

$\text{sign}(\sin(\omega t))$ is odd. Hence $V_1 = 0$.

$$\begin{aligned}
U_1 &= \frac{1}{\pi}\int_0^{2\pi} \text{sign}(E\sin\theta)\sin\theta\, d\theta, \\
&= \frac{1}{\pi}\int_0^{\pi} \sin\theta\, d\theta - \frac{1}{\pi}\int_{\pi}^{2\pi} \sin\theta\, d\theta \\
&= \frac{4}{\pi}.
\end{aligned}$$

Hence

$$N(E) = \frac{4}{\pi E}$$

DF of polynomial nonlinearity

$$f(e) = e^n \qquad (n \text{ odd})$$

$$\sin^n \theta = \frac{2}{2^n} \sum_{k=0}^{\frac{n-1}{2}} (-1)^{\left(\frac{n-1}{2} - k\right)} \binom{n}{k} \sin([n - 2k]\theta)$$

Last term, $k = \frac{n-1}{2}$, gives fundamental component.

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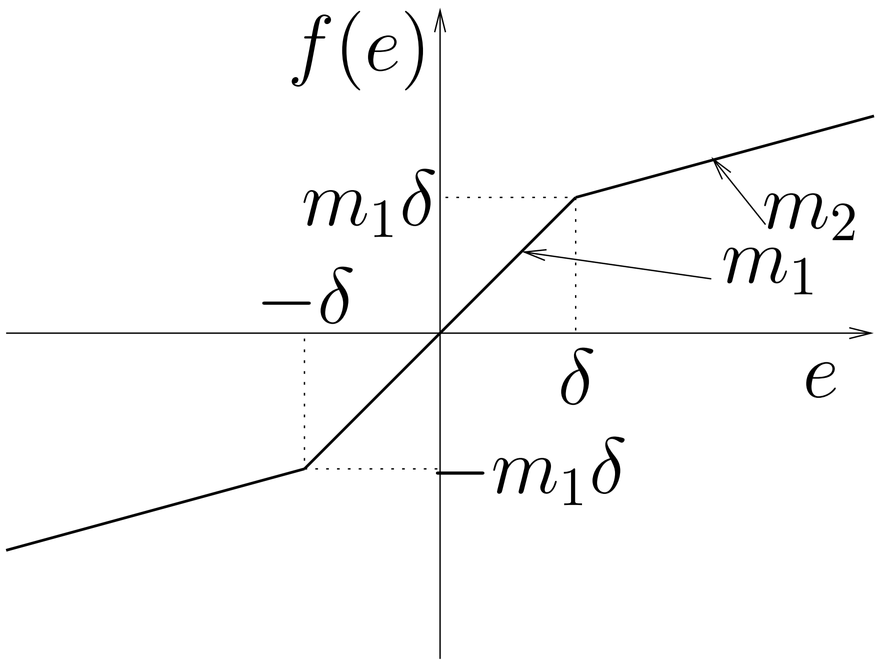
Example, $n = 3$: *(Maths Databook)*

$$(E \sin \theta)^3 = E^3 \left(\frac{3}{4} \sin \theta - \frac{1}{4} \sin(3\theta) \right)$$

so $U_1 = \frac{3E^3}{4}$, $V_1 = 0$, hence $\boxed{N(E) = \frac{3E^2}{4}}$.

Piecewise-linear non-linearity



$$f(e) = \begin{cases} m_1 e \text{ if } |e| < \delta \\ (m_1 - m_2)\delta + m_2 e \text{ if } e > \delta \\ (m_2 - m_1)\delta + m_2 e \text{ if } e < -\delta \end{cases}$$

Again $V_1 = 0$.
$E \leq \delta$: $U_1 = \frac{1}{2\pi} \int_0^{\pi} m_1 E \sin^2\theta \, d\theta = m_1 E$
$E > \delta$:

$$U_1 = \frac{4}{\pi} \int_0^{\pi/2} f(E \sin\theta) \sin\theta \, d\theta,$$

$$= \frac{4}{\pi} \int_0^{\sin^{-1}(\delta/E)} m_1 E \sin^2\theta \, d\theta +$$

$$\frac{4}{\pi} \int_{\sin^{-1}(\delta/E)}^{\pi/2} ((m_1 - m_2)\delta + m_2 E \sin\theta) \sin\theta \, d\theta$$

$$U_1 = \frac{2m_1 E}{\pi}\left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\sin^{-1}(\delta/E)} +$$

$$\frac{4(m_2 - m_1)\delta}{\pi}\left[\cos\theta\right]_{\sin^{-1}(\delta/E)}^{\pi/2} +$$

$$\frac{2m_2 E}{\pi}\left[\theta - \frac{1}{2}\sin(2\theta)\right]_{\sin^{-1}(\delta/E)}^{\pi/2}$$

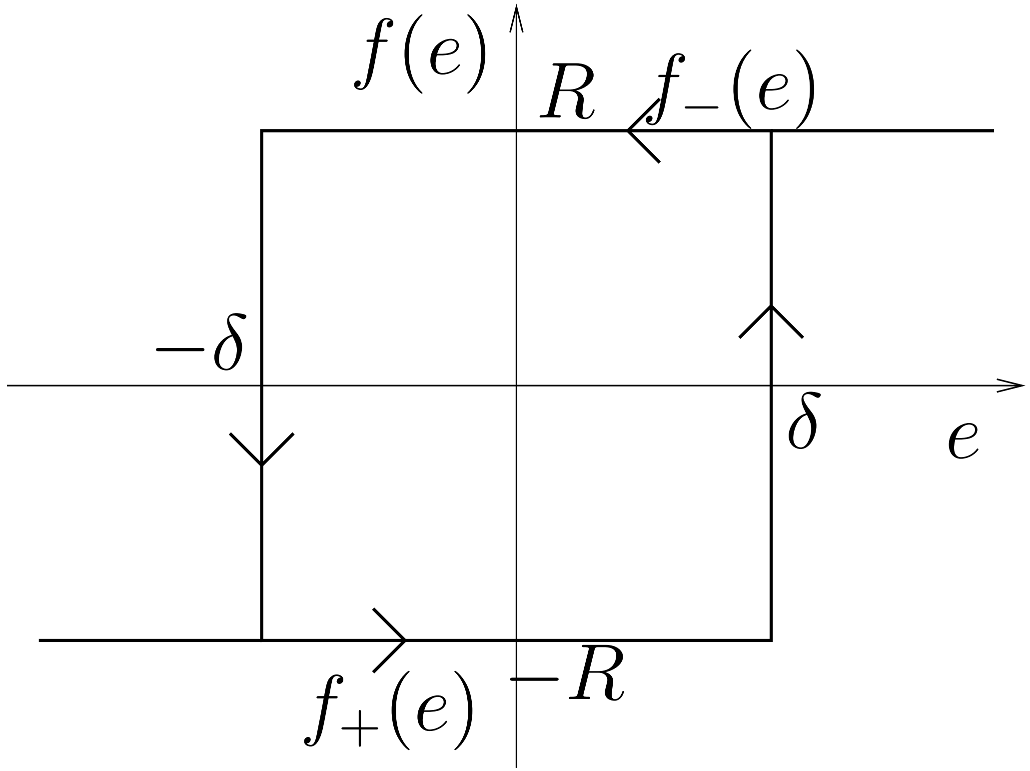
Using $\cos(\sin^{-1} x) = \sqrt{1-x^2}$ and $\sin(2\sin^{-1} x) = 2x\sqrt{1-x^2}$ gives:

$$U_1 = \frac{2(m_1 - m_2)E}{\pi}\left[\sin^{-1}(\frac{\delta}{E}) + \frac{\delta}{E}(1-(\frac{\delta}{E})^2)^{1/2}\right] + m_2 E.$$

Hence

$$N(E) = \begin{cases} m_1, \text{ if } E < \delta \\ \frac{2(m_1-m_2)}{\pi}\left[\sin^{-1}(\frac{\delta}{E}) + \frac{\delta}{E}(1-(\frac{\delta}{E})^2)^{1/2}\right] + m_2 \\ \qquad \text{if } E > \delta \end{cases}$$

Relay with hysteresis



Nonlinearity *with memory*.
Assume $E > \delta$.

$$N(E) = \frac{1}{\pi E} \int_{-\pi/2}^{3\pi/2} f(E \sin\theta)(\sin\theta + j\cos\theta)\, d\theta$$

Note limits of integration.

$$N(E) = \frac{1}{\pi E} \int_{-\pi/2}^{\pi/2} f_+(E \sin\theta)(\sin\theta + j\cos\theta)\, d\theta +$$
$$\frac{1}{\pi E} \int_{\pi/2}^{3\pi/2} f_-(E \sin\theta)(\sin\theta + j\cos\theta)\, d\theta$$

Real part of $N(E)$

Let $\nu = E\sin\theta$, $d\nu = E\cos\theta\, d\theta$.
Note $\cos\theta > 0$ if $-\pi/2 < \theta < \pi/2$,
and $\cos\theta < 0$ if $\pi/2 < \theta < 3\pi/2$.

$$\begin{aligned}
\mathrm{Re}N(E) &= \frac{1}{\pi E}\int_{-E}^{E} f_+(\nu)\left(\frac{\nu}{E}\right)\frac{d\nu}{E\sqrt{1-(\nu/E)^2}} \\
&\quad -\frac{1}{\pi E}\int_{E}^{-E} f_-(\nu)\left(\frac{\nu}{E}\right)\frac{d\nu}{E\sqrt{1-(\nu/E)^2}} \\
&= \frac{1}{\pi E}\int_{-E}^{E}[f_+(\nu)+f_-(\nu)]\left(\frac{\nu}{E}\right)\frac{d\nu}{E\sqrt{1-(\nu/E}
\end{aligned}$$

Imaginary part of $N(E)$

$$\begin{aligned}
\mathrm{Im}N(E) &= \frac{1}{\pi E}\int_{-E}^{E} f_+(\nu)\frac{d\nu}{E} + \frac{1}{\pi E}\int_{E}^{-E} f_-(\nu)\frac{d\nu}{E} \\
&= \frac{1}{\pi E^2}\int_{-E}^{E}[f_+(\nu) - f_-(\nu)]d\nu \\
&= -\frac{\Delta}{\pi E^2}
\end{aligned}$$

Δ is area enclosed by the ‘loop’.

$N(E)$ for the relay

$$\mathrm{Re}N(E) = \frac{4R}{\pi E}\int_{\delta}^{E}\frac{\nu\, d\nu}{E^2\sqrt{1-(\nu/E)^2}} = \frac{4R}{\pi E}\sqrt{1-(\frac{\delta}{E})^2}$$

hence

$$N(E) = \frac{4R}{\pi E}\left(\sqrt{1-(\frac{\delta}{E})^2} - \frac{j\delta}{E}\right).$$

Estimates of the DF

- *Example: Piecewise-linear nonlinearity:*

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- *Example: Relay with dead-zone:*

$$f(e) = \begin{cases} -1, & \text{if } e \leq -\delta \\ 0, & \text{if } |e| < \delta \\ +1, & \text{if } e \geq \delta \end{cases}$$

$N(E)$ increases rapidly for $|e|$ just above δ, then falls towards 0.