Module 4F2: Nonlinear Systems and Control

Lectures 1 – 2: Dynamical Systems

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‘Nonlinear’ overview — 7L

- Nonlinear dynamical systems — 2 lectures
  - Continuous, discrete, hybrid
  - Examples
  - State-space description
  - Solutions, simulations
- Attractors, Stability, Lyapunov methods — 1.5 lectures
- Describing functions — 1.5 lectures
- Circle criterion for stability — 2 lectures
- 2 Examples papers, 1 Examples class

Dynamical system classification

Dynamical system: *Evolution of state over time.*

Types of state:

Continuous State x lives in Euclidean space $\mathbb{R}^n$ — familiar, eg from 3F2. Write $x \in \mathbb{R}^n$.

Discrete State q takes values in finite or countable set $\{q_1, q_2, \ldots\}$. *Example:* Light switch, $q \in \{ON, OFF\}$.

Hybrid Part of state lives in $\mathbb{R}^n$, other part has values in finite set. *Example:* Computer control of inverted pendulum.

Types of time:

Continuous $\dot{x} = Ax$ (linear) or $\dot{x} = f(x)$ (nonlinear).

Discrete $x_{k+1} = Ax_k$ (linear) or $x_{k+1} = f(x_k)$ (nonlinear).

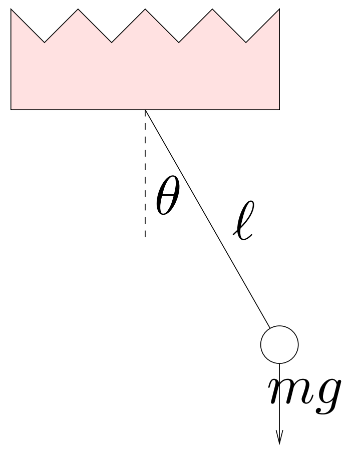
Hybrid System evolves over continuous time, but special things happen at particular instants.

We will deal mostly with:

Continuous-state, continuous-time, nonlinear

Example: Pendulum

Continuous-state, continuous-time, nonlinear



$$m\ell\ddot{\theta} + d\ell\dot{\theta} + mg\sin(\theta) = 0$$

Exercise: Derive this. Why is it *nonlinear*?

Solve the ODE:

Find

$$\theta(\cdot) : \mathbb{R} \to \mathbb{R}$$

such that

$$\theta(0) = \theta_0$$

$$\dot{\theta}(0) = \dot{\theta}_0$$

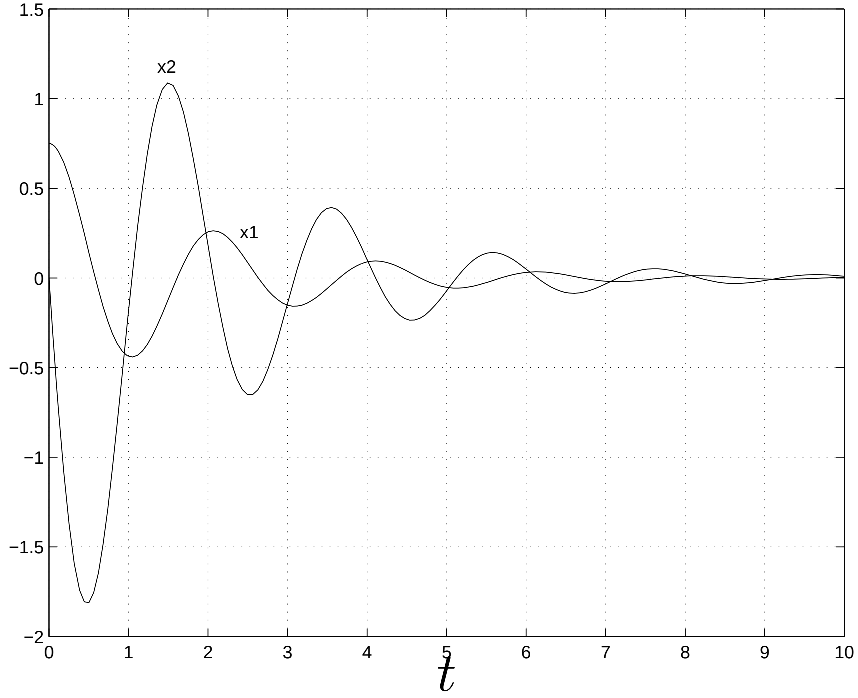
$$m\ell\ddot{\theta}(t) + d\ell\dot{\theta}(t) + mg\sin(\theta(t)) = 0, \ \forall t \in \mathbb{R}$$

Usually difficult to find solution analytically.
Find approximate solution by **simulation**.

Simulated solution

Parameters: $\ell = 1$, $m = 1$, $d = 1$, $g = 9.8$.

Initial conditions: $\theta(0) = 0.75$, $\dot{\theta}(0) = 0$.



State-space form:

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad n \geq 1$$

For the pendulum, $x \in \mathbb{R}^2$:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

which gives:

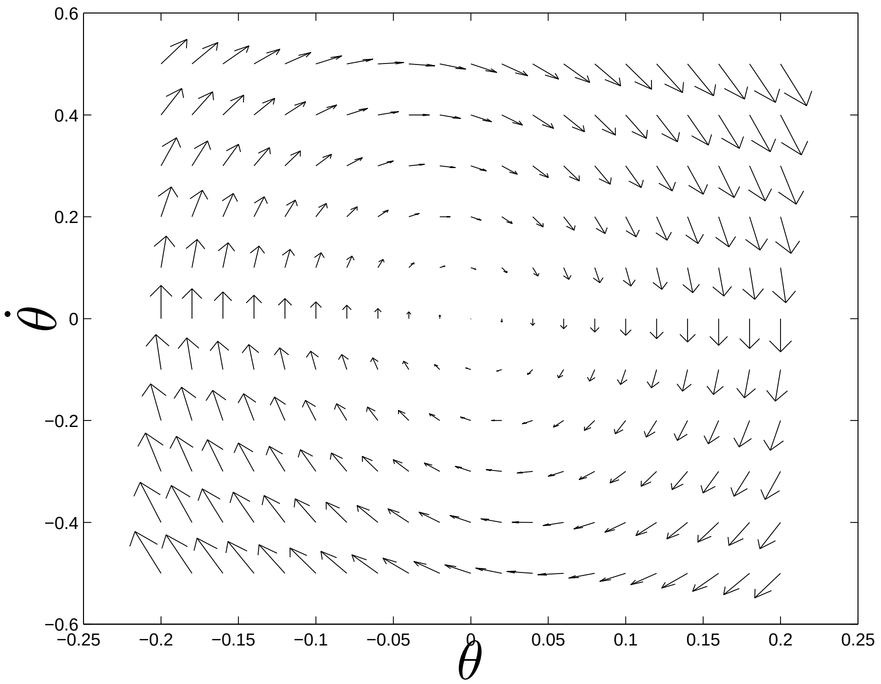
$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin(x_1) - \frac{d}{m}x_2 \end{bmatrix} = f(x)$$

x is **state**. This system has **dimension** 2.

Vector field

$$\dot{x} = f(x), \qquad f(\cdot) : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \text{is } \textbf{vector field}$$

$f(\cdot)$ assigns *velocity* vector to each state vector.

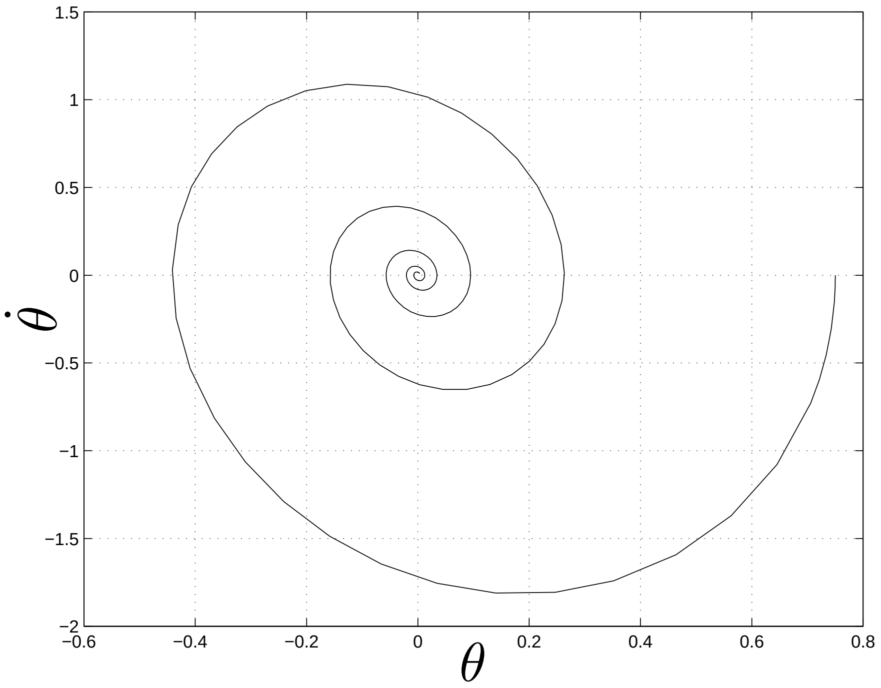


Solve the ODE (another view):

Find $x(\cdot) : \mathbb{R} \to \mathbb{R}^2$ such that

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} \theta_0 \\ \dot{\theta}_0 \end{bmatrix}$$

$$\dot{x}(t) = f(x(t)), \ \forall t \in \mathbb{R}.$$



Equilibrium states

For certain states $\hat{x} \in \mathbb{R}^n$,

$$f(\hat{x}) = 0$$

Hence system never leaves the state $\hat{x}$.
Such a state is an **equilibrium** state.

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Nonlinear system can have several equilibria.
Some stable, others unstable.

Linearisation

For θ close to 0: $\sin(\theta) \approx \theta$. Hence for θ close to 0:

$$m\ell\ddot{\theta} + d\ell\dot{\theta} + mg\theta = 0$$

or in state space form

$$\dot{x} = \begin{bmatrix} x_2 \\ -\frac{g}{\ell}x_1 - \frac{d}{m}x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{d}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = g(x)$$

Note that $g(x) = Ax$ ie linear state-space system.
A has eigenvalues in LHP — stable linear system.

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Exercise: Find linearisation near the other equilibrium. Examine its stability.

Example: Logistic map

Continuous-state, discrete-time, nonlinear.

$$x_{k+1} = ax_k(1 - x_k) = f(x_k)$$

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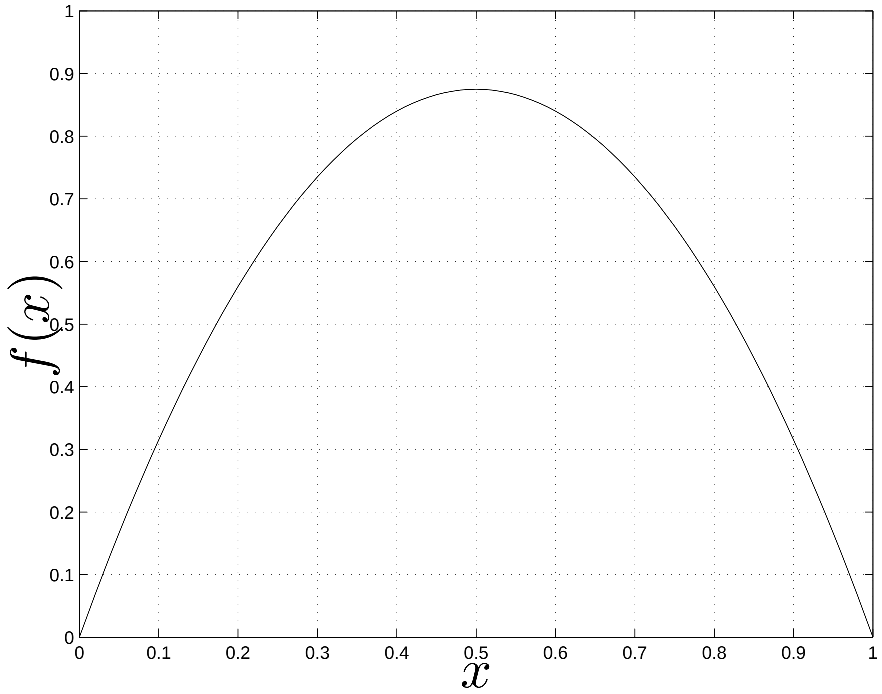
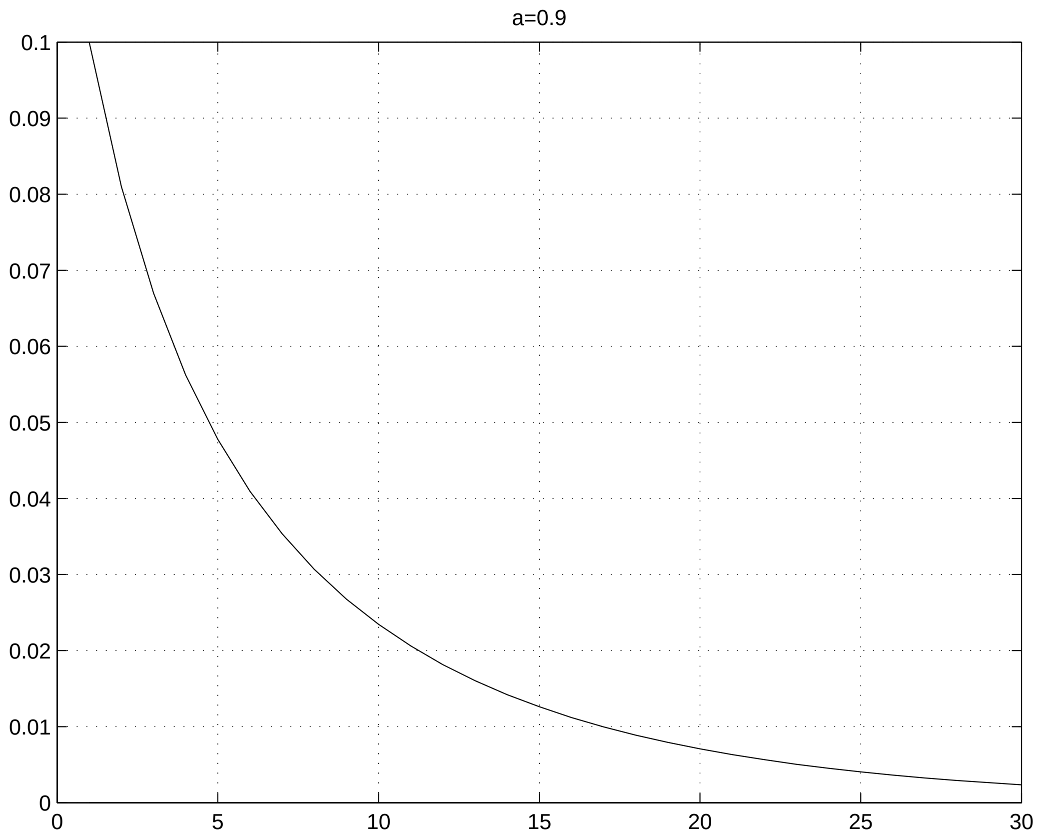
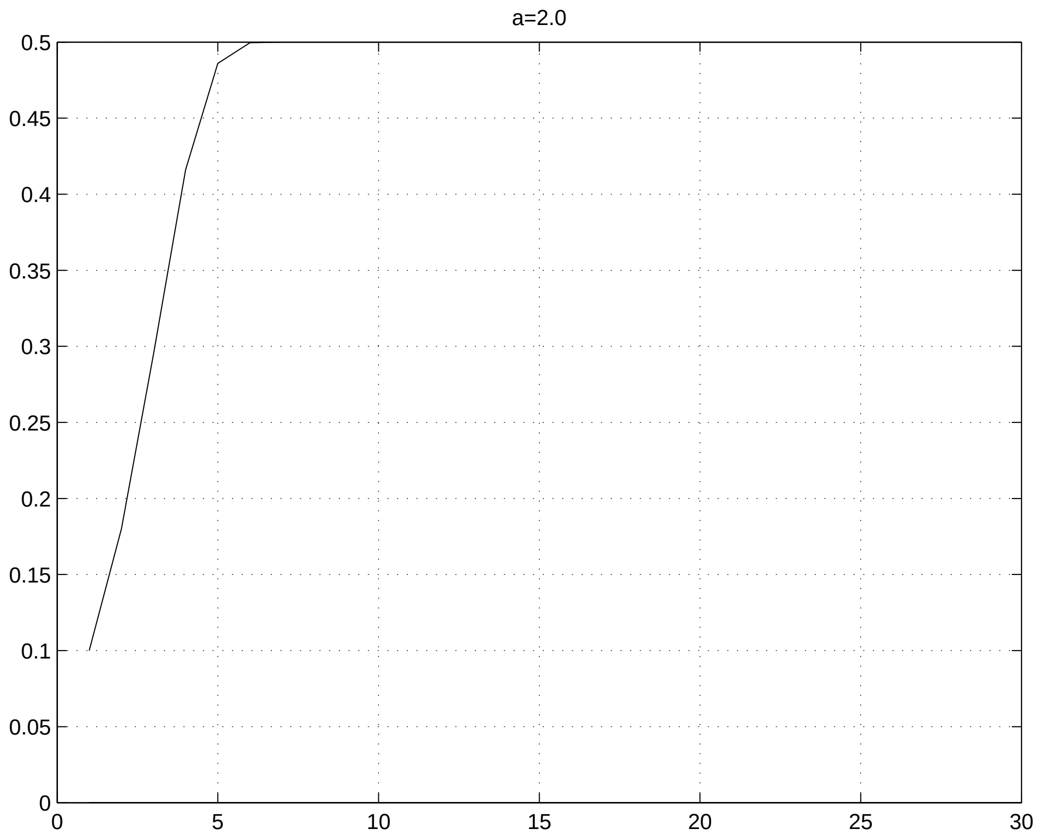


Equilibrium, Oscillation, Chaos



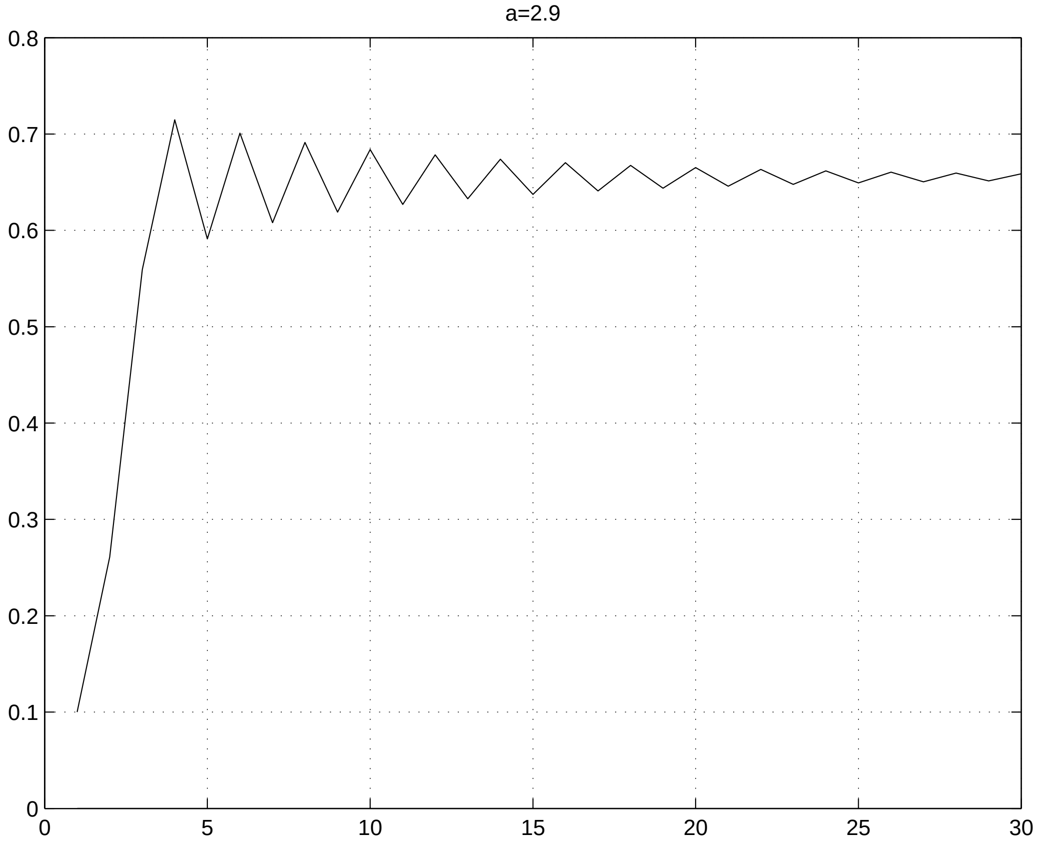
$0 \leq a < 1$: Decays to 0 for all x_0. $a = 0.9$

Equilibrium, Oscillation, Chaos



$1 \leq a \leq 3$: Tends to steady-state value. $a = 2.0$

Equilibrium, Oscillation, Chaos



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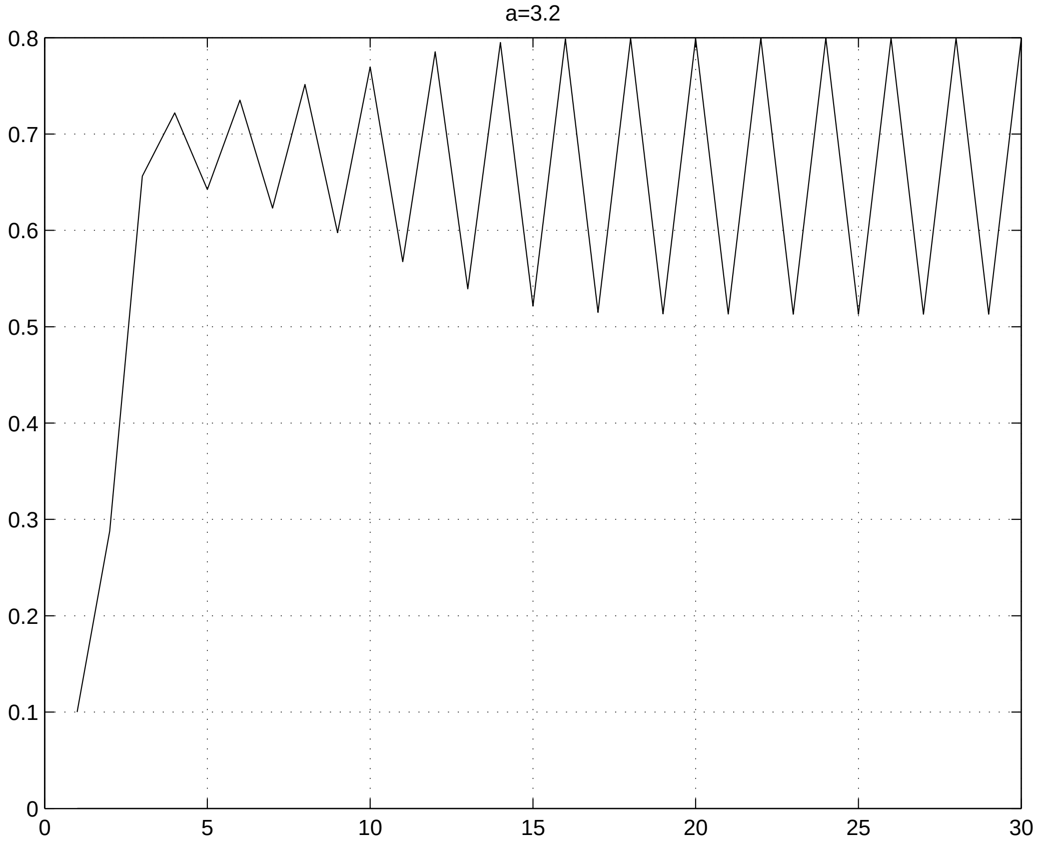
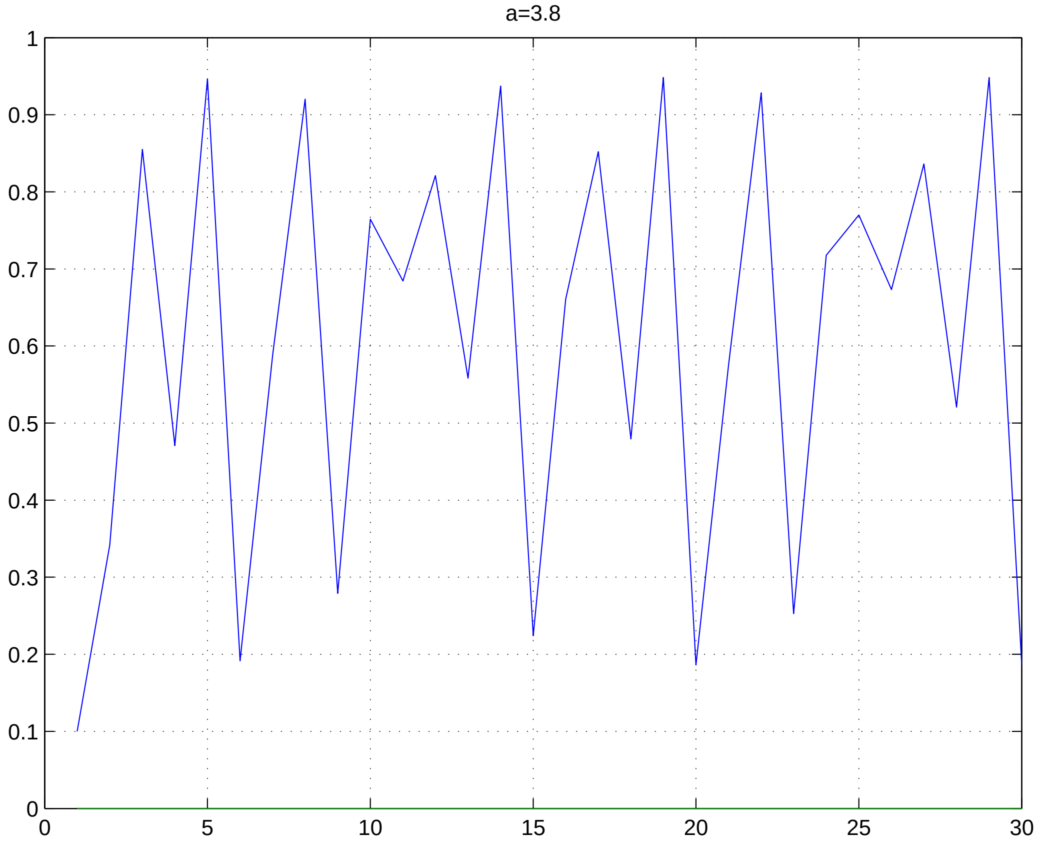


$3 < a \leq 1 + \sqrt{6} = 3.449$:
Tends to 2-period oscillation. $a = 3.2$

Equilibrium, Oscillation, Chaos



$1 + \sqrt{6} < a < 4$:
3-period, 4-period, ..., chaos.

$a = 3.8$

Example: Manufacturing cell

A discrete-state system.

Possible states: Idle (I), Working (W), Down (D).

Possible events:		
	p	part arrives
	c	complete processing
	f	failure
	r	repair

Abstract description of machine

$$q \in Q = \{I, W, D\}, \qquad \sigma \in \Sigma = \{p, c, f, r\}$$

State transition relation:

$$\delta : Q \times \Sigma \to Q$$

$$\delta(I, p) = W, \delta(W, c) = I, \delta(W, f) = D, \delta(D, r) = I.$$

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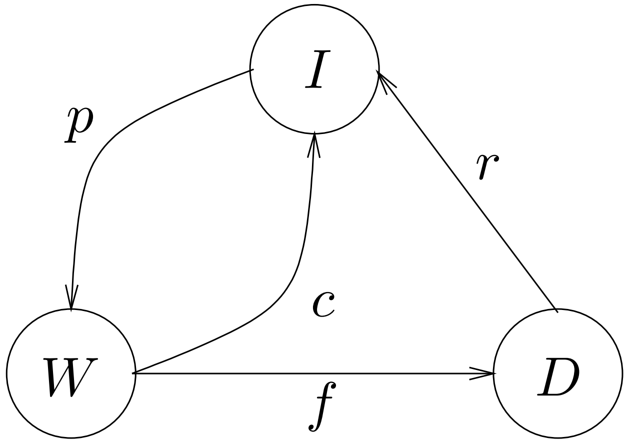


Example: Thermostat

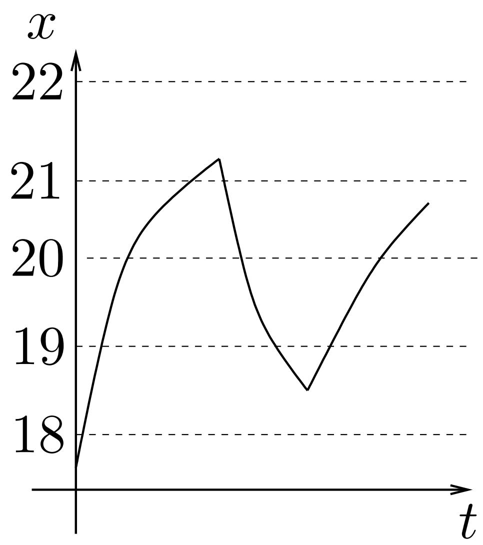
A hybrid system.

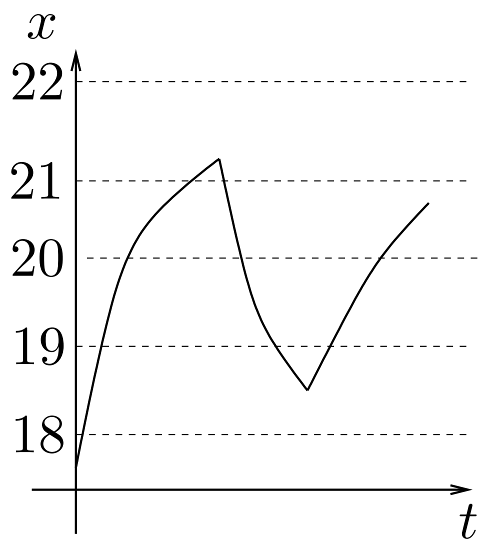
$x \in \mathbb{R}$: Room temperature,
$q \in \{ON, OFF\}$: Heater state.

Heater off: $q = OFF$, $\dot{x} = -ax$
Heater on: $q = ON$, $\dot{x} = -a(x - 30)$

Use hysteresis to prevent 'chattering':

```
if x<19, q := ON,
elseif x>21, q := OFF,
end
```


x
22
21
20
19
18
t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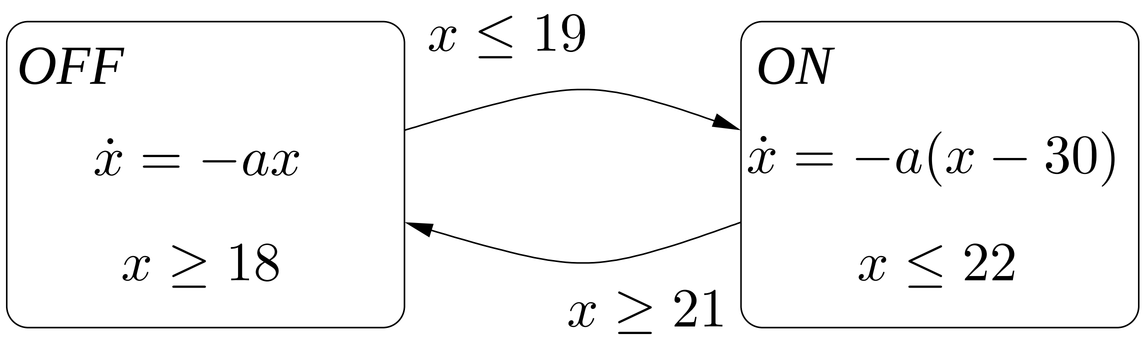
OFF
$\dot{x} = -ax$
$x \geq 18$
$x \leq 19$
ON
$\dot{x} = -a(x - 30)$
$x \leq 22$
$x \geq 21$

State-space form

States: $x_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$
Inputs: $u_j \in \mathbb{R}$,$j = 1, 2, \ldots, m$
Outputs: $y_k \in \mathbb{R}$, $k = 1, 2, \ldots, p$

$\dot{x} = f(x, u, t), \qquad y = h(u, x, t),$ vector functions

Special case: $\dot{x} = f(x)$ **autonomous**

What do $\dot{x} = f(x, u, t)$ and $y = h(u, x, t)$ mean?

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, \ldots, x_n, u_1, \ldots, u_m, t) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, \ldots, x_n, u_1, \ldots, u_m, t)
\end{aligned}
$$

$$
\begin{aligned}
y_1 &= h_1(x_1, \ldots, x_n, u_1, \ldots, u_m, t) \\
&\vdots \\
y_p &= h_p(x_1, \ldots, x_n, u_1, \ldots, u_m, t)
\end{aligned}
$$

Existence, Uniqueness

$\dot{x} = -sign(x), \; x(0) = 0$ — No solutions

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$\dot{x} = 3x^{2/3},\ x(0) = 0$ — Multiple solutions

For any $a \geq 0, \quad x(t) = \begin{cases} (t-a)^3 & t \geq a \\ 0 & t \leq a \end{cases}$

Existence, Uniqueness

$\dot{x} = -\textit{sign}(x),\ x(0) = 0$ — No solutions

$\dot{x} = 3x^{2/3},\ x(0) = 0$ — Multiple solutions

For any $a \geq 0$, $\quad x(t) = \begin{cases} (t-a)^3 & t \geq a \\ 0 & t \leq a \end{cases}$

$\dot{x} = 1 + x^2,\ x(0) = 0$ — Finite escape time
One solution: $x(t) = \tan(t)$

Lipschitz continuity

Definition 1 *A function* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is* **Lipschitz continuous** *if* $\exists \lambda > 0$ *such that* $\forall x, \hat{x} \in \mathbb{R}^n$

$$\|f(x) - f(\hat{x})\| < \lambda \|x - \hat{x}\|$$

Lipschitz continuity

Definition 2 *A function* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is* **Lipschitz continuous** *if* $\exists \lambda > 0$ *such that* $\forall x, \hat{x} \in \mathbb{R}^n$

$$\|f(x) - f(\hat{x})\| < \lambda \|x - \hat{x}\|$$

Theorem 2 (Existence & Uniqueness of Solutions) *If* f *is Lipschitz continuous, then*

$$\dot{x} = f(x), \quad x(0) = x_0$$

has a unique solution $x(\cdot) : [0, T] \to \mathbb{R}^n$ *for all* $T \geq 0$ *and all* $x_0 \in \mathbb{R}^n$.

Simulation

Theorem 3 (Continuity with Initial State) *Assume f is Lipschitz continuous with Lipschitz constant λ. Let $x(\cdot) : [0, T] \to \mathbb{R}^n$ and $\hat{x}(\cdot) : [0, T] \to \mathbb{R}^n$ be solutions to $\dot{x} = f(x)$ with $x(0) = x_0$ and $\hat{x}(0) = \hat{x}_0$, respectively. Then for all $t \in [0, T]$*

$$\|x(t) - \hat{x}(t)\| \leq \|x_0 - \hat{x}_0\| e^{\lambda t}$$

"Solutions that start close, remain close."
This justifies **simulation**.

Pendulum simulation (*Matlab*)

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin(x_1) - \frac{d}{m}x_2 \end{bmatrix} = f(x)$$

```
function [xdot] = pendulum(t,x)
l = 1;  m=1;  d=1;  g=9.8;
xdot(1) = x(2);
xdot(2) = -sin(x(1))*g/l-x(2)*d/m;
```

```
>> x=[0.75 0];
>> [T,X]=ode45('pendulum', [0 10], x');
>> plot(T,X);
>> grid;
```

ode45 is 4'th order Runge-Kutta integration function.

Exercise: Try this at home! (or in the DPO)

Simulation tools

Simulink provides GUI front-end to *Matlab*.
Other similar products available.

Pendulum: