4F3 - Predictive Control

Discrete-time State Space Control Theory

For reference only

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Course Outline

- Introduction to predictive control
- Discrete-time state space control theory — handout only
- Predictive control without constraints
- Predictive control with constraints
- Stability and feasibility in predictive control
- Setpoint tracking and offset-free control
- Industrial case study – Dr Paul Austin – ***Fri. 5 March***
- Examples Class – 2 Examples Papers – ***Tue. 9 March***

Outline of Lecture 2

- Review of linear algebra
- Discrete time models and sampled data systems
- Stability of discrete time systems
- Reachability and Observability
- State feedback control design
- Observer design
- Output feedback and the separation principle

References

- Linear Algebra
  - G. Strang. *Introduction to Linear Algebra*, Wellesley-Cambridge Press. 3rd ed., 2003.
  - D. C. Lay. *Linear Algebra and Its Applications*, Adison Wesley. 3rd ed., 2003.

- Discrete Time Control
  - G. F. Franklin et. al. *Digital Control of Dynamic Systems*, Adison Wesley. 3rd ed., 2003.
  - K. J. Åstrom and B. Wittenmark. *Computer Controlled Systems*, Prentice Hall. 3rd ed., 2003.

Linear Independence and Rank

Definition (Linear Independence) *The vectors* $\{v_1, \ldots, v_n\}$ *are* linearly independent *if and only if* $c_1 v_1 + \cdots + c_n v_n = 0$ *only happens when* $c_1 = \cdots = c_n v_n = 0$*.*

- The *rank* of a matrix A is defined as the number of linearly independent columns – also known as the *column rank*.
- The column rank of A is the same as its row rank, i.e.

$$\operatorname{rank} A = \operatorname{rank} A^T$$

Solutions to $Ax = b$

Given a set a linear inequalities $Ax = b$ with $A \in \mathbb{R}^{r \times c}$

- Solution exists for a *given* $b \Leftrightarrow \operatorname{rank} A = \operatorname{rank}(A\ b)$
- Solution exists for *every* $b \Leftrightarrow \operatorname{rank} A = r$.
  - (Only possible when $r \leq c$)
- If a solution exists, it is unique $\Leftrightarrow \operatorname{rank} A = c$
  - (Only possible when $r \geq c$)
- When A is square and invertible (i.e. full rank), the unique solution is $x = A^{-1}b$

Eigenvectors and Eigenvalues

Suppose that $A \in \mathbb{R}^{n \times n}$ and there exists $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ such that $Ax = \lambda x$ and $x \neq 0$. Then:

- The vector x is an *eigenvector* of A.
- The scalar λ is an *eigenvalue* of A.
- The *spectral radius* of A is defined as:

$$\rho(A) := \max \{|\lambda_1(A)|, \ldots, |\lambda_n(A)|\}$$

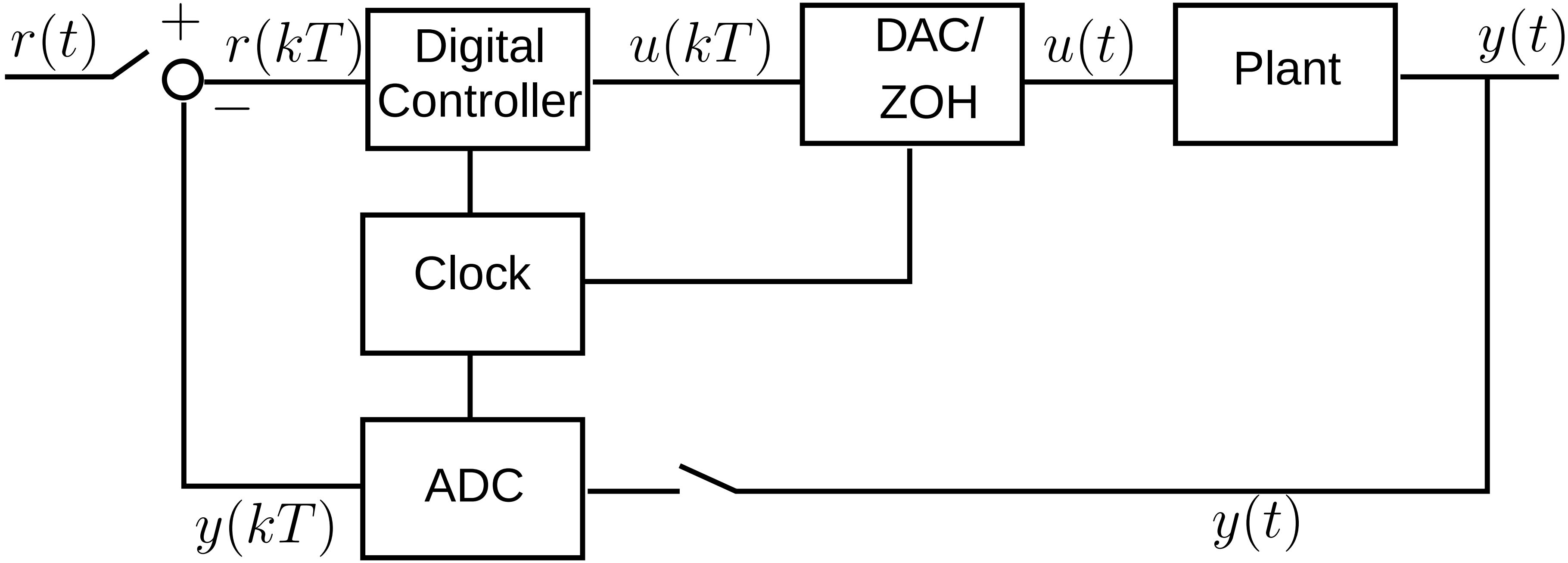
Some useful facts:

- Eigenvalues of A are the roots of its characteristic polynomial.

$$Ax = \lambda x \Leftrightarrow \det(\lambda I - A) = 0$$

- Every eigenvalue has a non-zero eigenvector.

Digital (Sampled-Data) Control



- Sample period is T. Sample number is k. Time is t.
- $s(t) \rightarrow$ continuous time signal. $s(kT) \rightarrow$ discrete time signal.
- Will generally shorten $s(kT)$ to $s(k)$.

Linear Systems (Continuous Time)

A linear continuous time (CT) state-space system:

$$\dot{x} = Fx + Gu$$
$$y = Cx + Du$$
$$z = Hx$$

- $x \in \mathbb{R}^n \rightarrow$ State vector.
- $u \in \mathbb{R}^m \rightarrow$ Input vector {a.k.a. ‘Manipulated Variables’ (MV)}.
- $y \in \mathbb{R}^p \rightarrow$ Output vector {‘Measured Variables’}.
- $z \in \mathbb{R}^q \rightarrow$ Controlled variables (CV).

In many cases, $H = C$, so that $y = z$.

Solution of Linear CT Systems

Given:

- The CT linear system $\dot{x} = Fx + Gu$
- An initial state $x(t_0)$ at time t_0
- An input trajectory $u(\cdot)$ over the interval $[t_0, t]$

the solution of the CT system at time t is given by:

$$x(t) = e^{F(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{F(t-\tau)}Gu(\tau)\mathrm{d}\tau$$

We can use this to get a DT model for our CT system.

DT models for Sampled-Data Systems

- With a ZOH at the the output of our digital controller:

$$u(t) = u(kT) \text{ for the interval } kT \leq t \leq k(T+1)$$

- The DT state-space model

$$\begin{aligned} x(kT+T) &= Ax(kT) + Bu(kT) \\ y(kT) &= Cx(kT) + Du(kT) \\ z(kT) &= Hx(kT) \end{aligned}$$

is an *exact* representation of the sampled CT system if

$$A = e^{FT}, \quad B = \left(\int_0^T e^{F\tau} \mathrm{d}\tau \right) G$$

DT models for Sampled-Data Systems

- With a ZOH at the the output of our digital controller, we can get an *exact* DT representation of our CT system if:
  - the CT system is linear, or
  - the CT system is linear with input saturation

- It is generally *not* possible to get an exact DT representation for a nonlinear CT system:

$$\dot{x} = f(x, u)$$

$$y = h(x, u)$$

- For a nonlinear model, we can use a DT approximation or use numerical integration to find $x(kT)$.

Solution of Linear DT Systems

Given:

- The DT linear system $x(k+1) = Ax(k) + Bu(k)$
- An initial state $x(k)$ at discrete time k
- An input sequence $\{u(k), \ldots, u(k+N-1)\}$

the solution of the DT system at time $k+N$ is given by:

$$x(k+N) = A^N x(k) + \sum_{i=0}^{N-1} A^i B u(k+N-1-i)$$

Try to prove this to yourself – we'll return to it in lecture 3.

Stability of Nonlinear DT Systems

Consider a general nonlinear DT system with $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$x(k+1) = f(x(k))$$

Let $x(k)$ denote the state of this system at time discrete time step k if the state at $k = 0$ is $x(0)$.

$$x(k) = \underbrace{f(f(\dots f(f(x(0)))\dots))}_{k \text{ times}} = f^k(x(0))$$

Definition (DT Equilibrium Point) *A state x is an* equilibrium point *of the DT system if $x = f(x)$.*

Important: This is a *different* definition than in the CT case.

Stability of Nonlinear DT Systems

Definition (Stability) *An equilibrium point* x *is* stable *if for all* $\epsilon > 0$*, there exists* $\delta > 0$ *such that*

$$\|x(0) - x\| \leq \delta \Rightarrow \|x(k) - x\| \leq \epsilon \text{ for all } k \geq 0$$

Definition (Asymptotic Stability) *An equilibrium point* x *is* locally asymptotically stable *if it is stable and there exists an* $M > 0$ *such that*

$$\|x(0) - x\| < M \Rightarrow \lim_{k \to \infty} x(k) = x$$

The equilibrium point x *is* globally *asymptotically stable if the above holds for* all $M > 0$.

Stability of Linear DT Systems

Consider a linear DT systems without an input:

$$x(k+1) = Ax(k)$$

A state x is an equilibrium point if $x = Ax$:

- Any eigenvector with eigenvalue $\lambda = 1$ is an equilibrium point.
- The origin $x = 0$ is an equilibrium point.

Some useful facts:

- The origin is globally asymptotically stable if and only if all of the eigenvalues of A are inside the unit disk (i.e. $\rho(A) < 1$).
- The matrix A is called *stable* if and only if $\rho(A) < 1$.
- For sampled data systems with ZOH, $\lambda_i(A) = e^{\lambda_i(F)T}$
  - CT system stable $\Rightarrow$ DT system stable.

Reachability

Given the system

$$x(k+1) = Ax(k) + Bu(k)$$

Definition (Reachability) *The matrix pair* (A, B) *is* reachable *if it is possible to drive the above system to any state (from any other state) in finite time.*

- Analogous to controllability of the CT system (F, G)
- Controllability/Reachability is *not* always preserved when the system is discretized, i.e.

(F, G) controllable $\not\Rightarrow$ (A, B) reachable

Tests for Reachability

Definition (Reachability Matrix) *The* reachability matrix *of the pair (A, B) is defined as:*

$$\mathcal{C}(A,B) := \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}$$

Proposition (Reachability Tests) *The following are equivalent:*

a) The pair (A, B) is reachable.

b) Rank of $\mathcal{C}(A,B) = n$.

c) The eigenvalues of $(A + BK)$ can be freely assigned by suitable choice of K (requires complex eigenvalues to appear in conjugate pairs).

Observability

Given the system

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) + Du(k)$$

Definition (Observability) *The matrix pair* (C, A) *is* observable *if it is possible to determine the initial system state from a finite sequence of measurements.*

- Analogous to observability of the CT system (C, F)
- Observability is *not* always preserved when the system is discretized, i.e.

(C, F) observable $\nRightarrow$ (C, A) observable

Tests for Observability

Definition (Observability Matrix) *The* observability matrix *of the pair* (C, A) *is defined as:*

$$\mathcal{O}(C, A) := \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

Proposition (Observability Tests) *The following are equivalent:*

a) The pair (C, A) *is observable.*

b) Rank of $\mathcal{O}(C, A) = n$*.*

c) The eigenvalues of $(A + LC)$ *can be freely assigned by suitable of choice of* L *(requires complex eigenvalues to appear in conjugate pairs).*

d) (A^T, C^T) *is reachable.*

Stabilizability and Detectability

Reachability and observability can be relaxed to deal with just the unstable part of the system:

Definition (Stabilizability) *The matrix pair* (A, B) *is stabilizable if there exists a matrix* K *such that* $(A + BK)$ *is stable.*

Definition (Detectability) *The matrix pair* (C, A) *is detectable if there exists a matrix* L *such that* $(A + LC)$ *is stable.*

- These are weaker definitions than those for reachability and observability.
- These definitions do *not* guarantee that the eigenvalues of $(A + BK)$ or $(A + LC)$ can be arbitrarily assigned.

Tests for Stabilizability and Detectability

Let Λ be the set of eigenvalues on or outside the unit circle:

$$\Lambda := \{\lambda_i(A) : |\lambda_i(A)| \geq 1\}$$

Proposition (Stabilizability) *The pair (A, B) is stabilizable if and only if*

$$\begin{pmatrix} (A - \lambda I) & B \end{pmatrix}$$

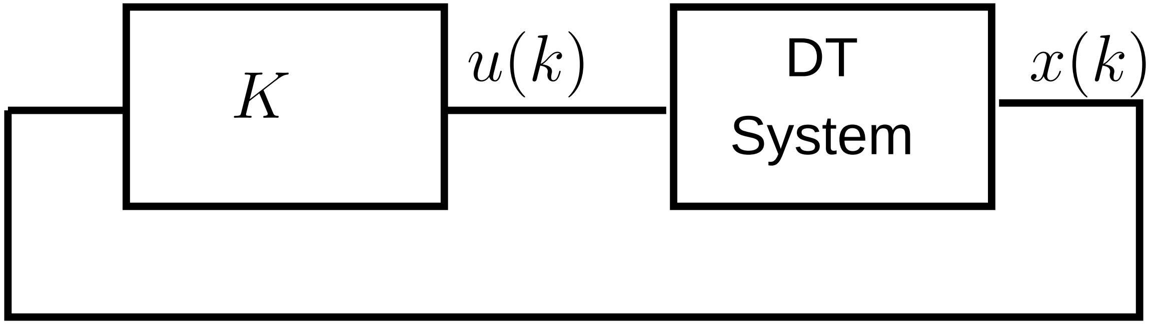
is full row rank for all $\lambda \in \Lambda$.

Proposition (Detectability) *The pair (C, A) is detectable if and only if*

$$\begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$$

is full column rank for all $\lambda \in \Lambda$.

State Feedback Control Design



Given the DT system

$$x(k+1) = Ax(k) + Bu(k)$$

want to design a state feedback control law $u(k) = Kx(k)$ so that the origin of the closed loop system

$$x(k+1) = (A+BK)x(k)$$

is globally asymptotically stable $\Rightarrow$ requires $(A+BK)$ to be stable.

State Feedback and Pole Placement

State feedback design procedure when (A, B) is reachable:

1) Specify desired eigenvalues $\{p_1, \ldots, p_n\}$, with each $|p_i| < 1$ and complex values in conjugate pairs.
2) Compute coefficients of the polynomial

$$p(\lambda) = (\lambda - p_1)(\ldots)(\lambda - p_n)$$

3) Compute characteristic polynomial $\det(\lambda I - (A + BK))$
4) Equate coefficients of the polynomials in (2) and (3).
5) Solve for the elements of the matrix K.

Note that if (A, B) is only stabilizable, then the values $\{p_1, \ldots, p_N\}$ can't be chosen arbitrarily.

Observer Design

When all the states can not be measured ($C \neq I$), then states can be estimated with an *observer* (or *estimator*).

- The DT system equations are:

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = C(k)u(k)$$

- The equations for the *observer* are:

$$\begin{aligned} \hat{x}(k|k) &= \hat{x}(k|k-1) + L[\hat{y}(k|k-1) - y(k)] \\ \hat{x}(k+1|k) &= A\hat{x}(k|k) + Bu(k) \\ \hat{y}(k|k-1) &= C\hat{x}(k|k-1) \end{aligned}$$

- Notation: $\hat{x}(i|j)$ denotes an estimate of $x(\cdot)$ at time i using all the measurements available through time j

Observer Design

Define the *state estimation error* to be

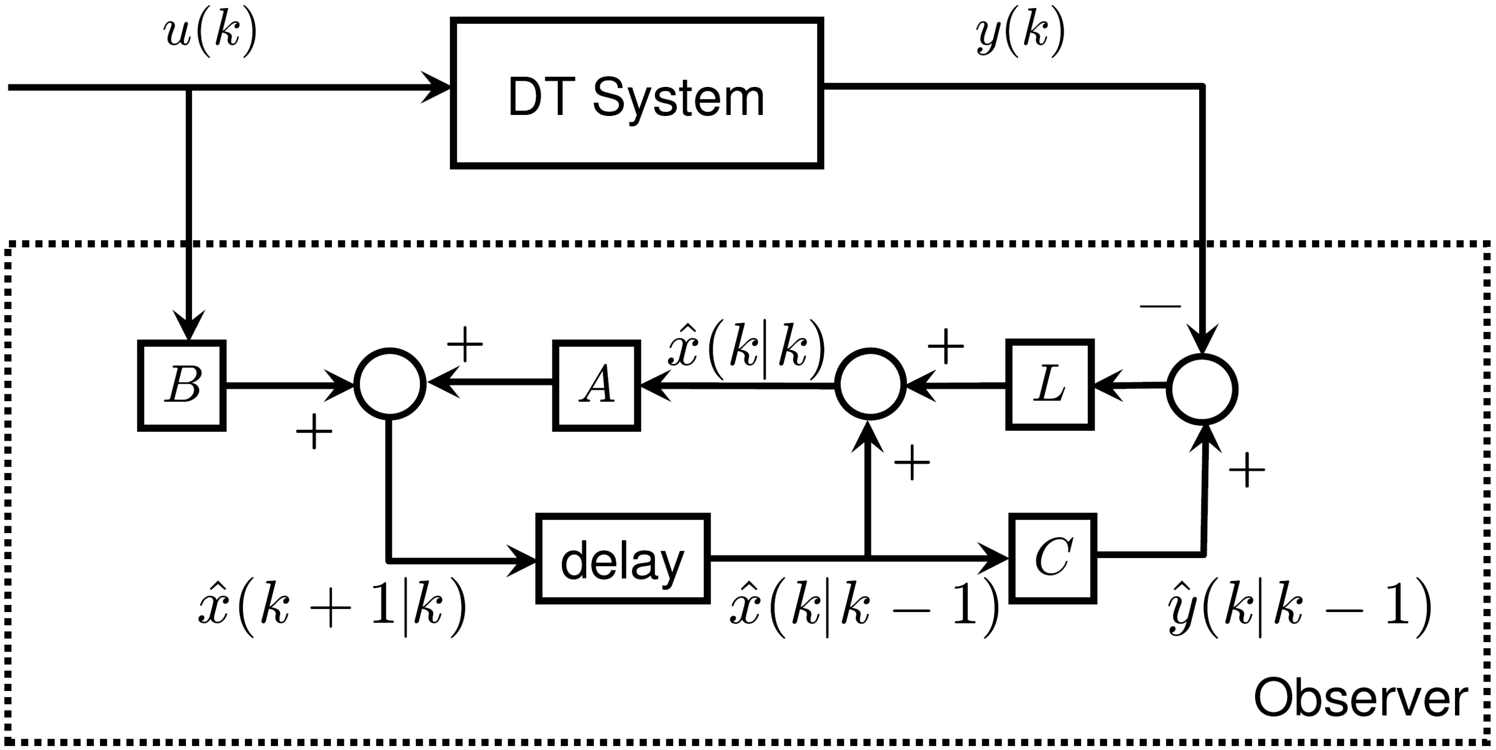
$$e(k) := x(k) - \hat{x}(k|k)$$

- Can eliminate variables in observer equations to get

$$\hat{x}(k+1|k+1) = (A+ALC)\hat{x}(k|k) + (B+LCB)u(k) - Ly(k+1)$$
$$e(k+1) = (A + LCA)e(k)$$

- When the matrix $(A + LCA)$ is stable, then
  - the observer is stable.
  - the estimation error converges to zero.

Observer Design



$$\hat{x}(k|k) = \hat{x}(k|k-1) + L[\hat{y}(k|k-1) - y(k)]$$

$$\hat{x}(k+1|k) = A\hat{x}(k|k) + Bu(k)$$

$$\hat{y}(k|k-1) = C\hat{x}(k|k-1)$$

Observer Design and Pole Placement

We can place eigenvalues of $(A + LCA)$ arbitrarily if (CA, A) is observable.

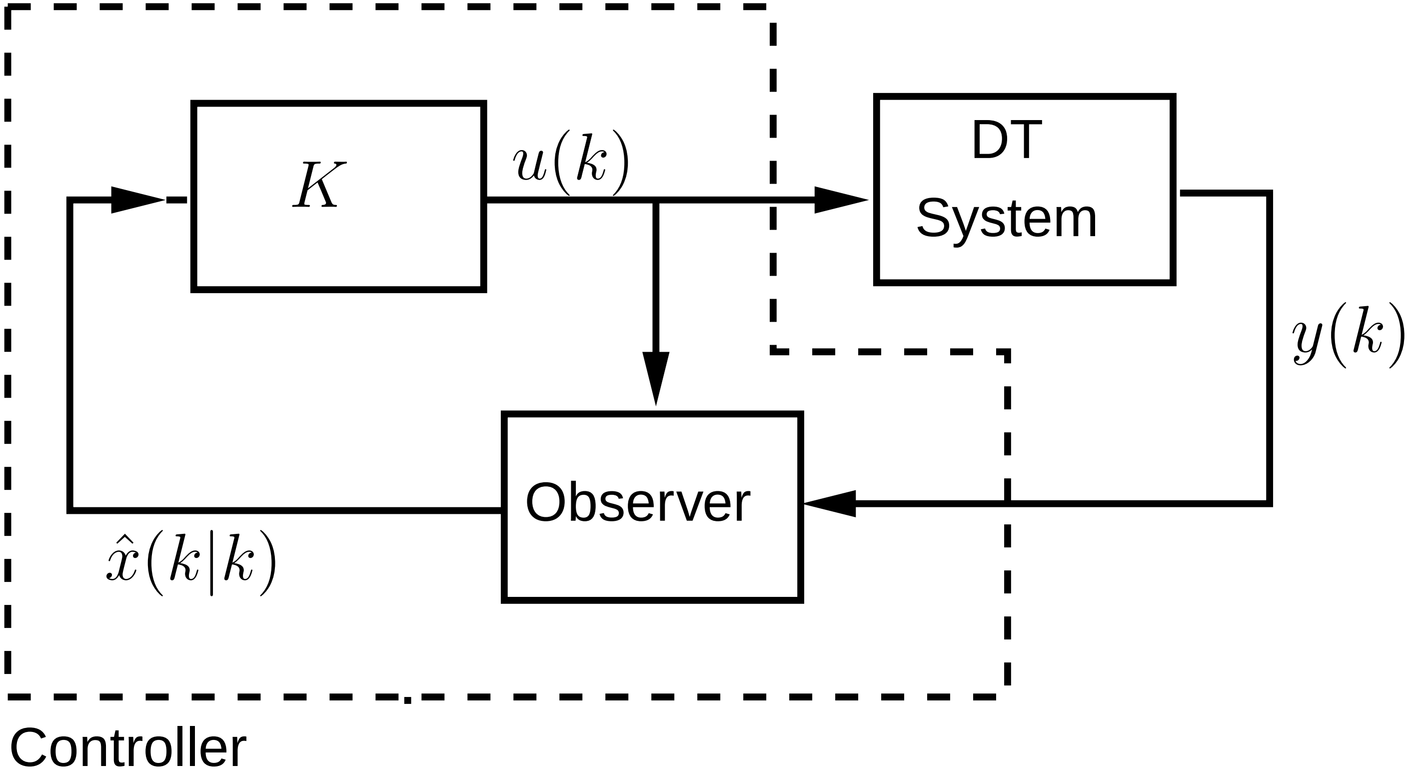
- If (C, A) is observable *and* $\lambda_i(A) \neq 0$ then (CA, A) observable.

Observer design procedure when (CA, A) is observable:

1) Specify desired eigenvalues $\{p_1, \ldots, p_n\}$, with each $|p_i| < 1$ and complex values in conjugate pairs.
2) Equate coefficients of polynomial $p(\lambda)$ with those of $\det(\lambda I - (A + LCA))$
3) Solve for the elements of the matrix L.

Note that if (CA, A) is only detectable, then the values $\{p_1, \ldots, p_N\}$ can't be chosen arbitrarily.

Output Feedback Control



The state feedback matrix K and observer L can be designed separately to form an output feedback control law.

The Separation Principle

Define the state estimation error as before:

$$e(k) := k(x) - \hat{x}(k|k)$$

The closed loop system becomes:

$$\begin{aligned} x(k+1) &= (A+BK)x(k) - BKe(k) \\ e(k+1) &= (A+LCA)e(k) \end{aligned}$$

- The eigenvalues of the closed loop system are
  - The eigenvalues of $(A+BK)$ (from control design)
  - The eigenvalues of $(A+LCA)$ (from observer design)
- This is a special case of the *separation principle*.