4F3 - Predictive Control

Lecture 5 - Setpoint Tracking and Disturbance Rejection

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Setpoint Tracking Problem

- So far, we have only considered the problem of regulating the states and inputs around the origin.
- In practice, we want to follow some nonzero, time-varying setpoint or reference signal.
  - Aircraft landing by autopilot.
  - Robot arm with pre-specified trajectory.
  - Radar tracking problems.
- We will consider the tracking of piecewise constant reference signals.

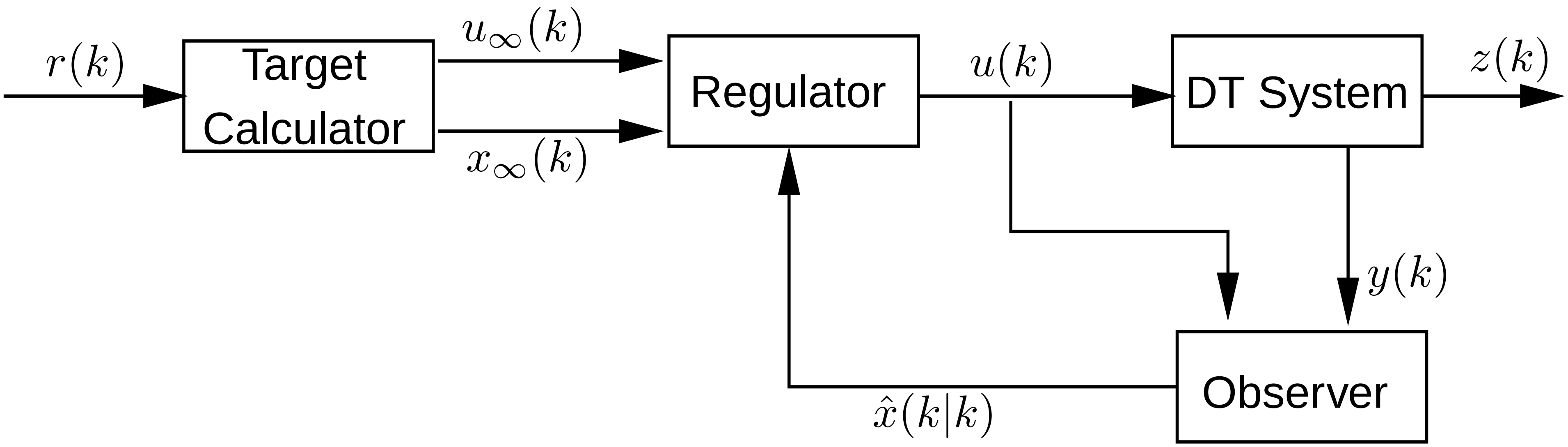
Discrete-Time Linear Systems

A linear discrete time (DT) state-space system:

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) \\
y(k) &= Cx(k) + Du(k) \\
z(k) &= Hx(k)
\end{aligned}
$$

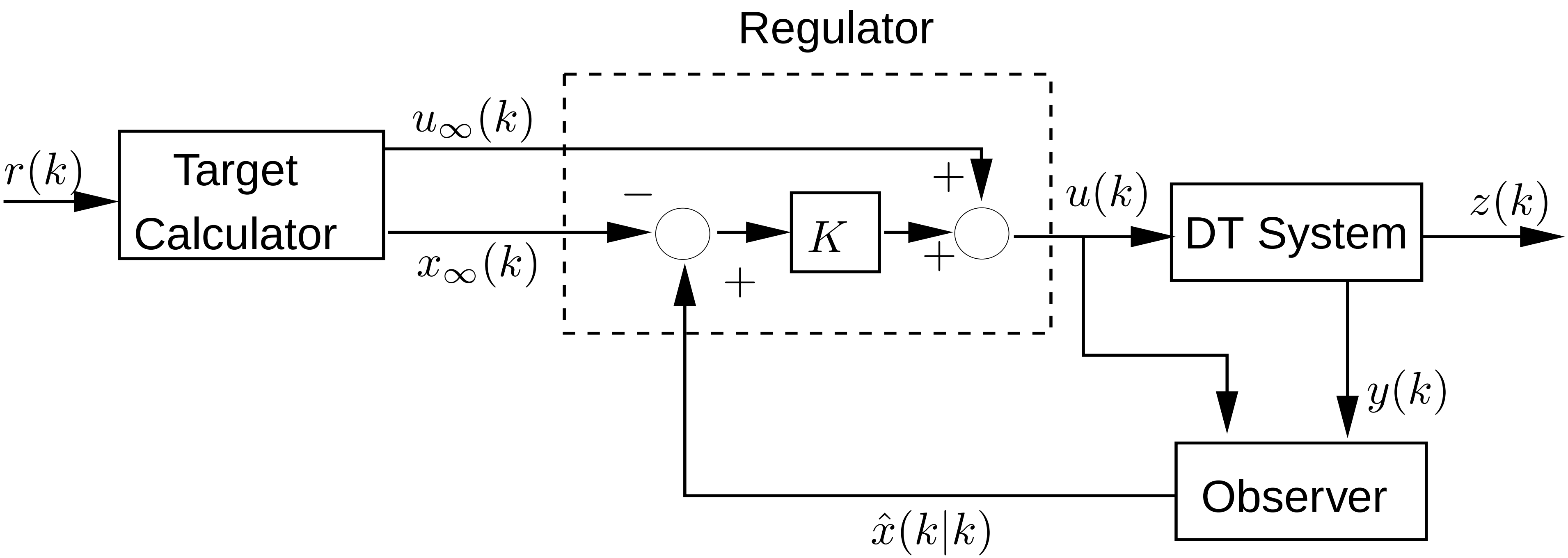
- $x \in \mathbb{R}^n \rightarrow$ State vector.
- $u \in \mathbb{R}^m \rightarrow$ Input vector {a.k.a. 'Manipulated Variables' (MV)}.
- $y \in \mathbb{R}^p \rightarrow$ Output vector {'Measured Variables'}.
- $z \in \mathbb{R}^q \rightarrow$ Controlled variables (CV).

Setpoint Tracking – General Framework



- Control the system so that $z(k) \to r(k)$ as $k \to \infty$ if the reference signal tends to a constant value (assume future values of $r(\cdot)$ are unknown).
- At each time, given the current reference value $r(k)$:
  - The Target Calculator computes target input $u_\infty(k)$ and target state $x_\infty(k)$.
  - The Regulator controls the system around the target $(x_\infty(k), u_\infty(k))$

Setpoint Tracking – General Framework



- Given a linear state feedback control gain K (either from unconstrained RHC or LQR) with $\rho(A+BK) < 1$, implement the control law:

$$u(k) = u_\infty(k) + K(\hat{x}(k|k) - x_\infty(k))$$

- Closed-loop system is stable if $u_\infty(\cdot)$ and $x_\infty(\cdot)$ are constant
- Want to design the target calculator such that $z(k) \rightarrow r(k)$ as $k \rightarrow \infty$.

Choice of Controlled Variables

- We do not always have all of the outputs as controlled variables (i.e. $H \neq C$ so that $z(k) \neq y(k)$).

- It is generally not possible to control all of the outputs or states to an arbitrary setpoint.
  - Example: it is not possible to maintain a car in a fixed position while also maintaining a nonzero velocity.

- Generally the controlled variables $z(k)$ are a linear combination or subset of the states $x(k)$ or outputs $y(k)$

Problem: Which choices are H are allowed?

Target Equilibrium Pairs

Definition (Target Equilibrium Pair) *Given a reference input r for a linear DT system, the pair (x_∞, u_∞) is called an <u>(offset-free) target equilibrium pair</u> if*

$$\begin{aligned} x_\infty &= Ax_\infty + Bu_\infty \\ Hx_\infty &= r \end{aligned}$$

- Rearranging the above gives:

$$\begin{pmatrix} (I-A) & -B \\ H & 0 \end{pmatrix} \begin{pmatrix} x_\infty \\ u_\infty \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix}$$

- Depending on the values of H and r, target equilibrium pair may or may not exist

Existence of Target Equilibrium Pairs

Proposition *A sufficient condition for guaranteeing the existence of a target equilibrium pair for any reference r is that*

$$\begin{pmatrix} (I - A) & -B \\ H & 0 \end{pmatrix}$$

is full row rank.

The above condition implies that

- H has to be full row rank
- Number of inputs $\geq$ number of CVs ($m \geq q$)

Target Calculation - No Constraints

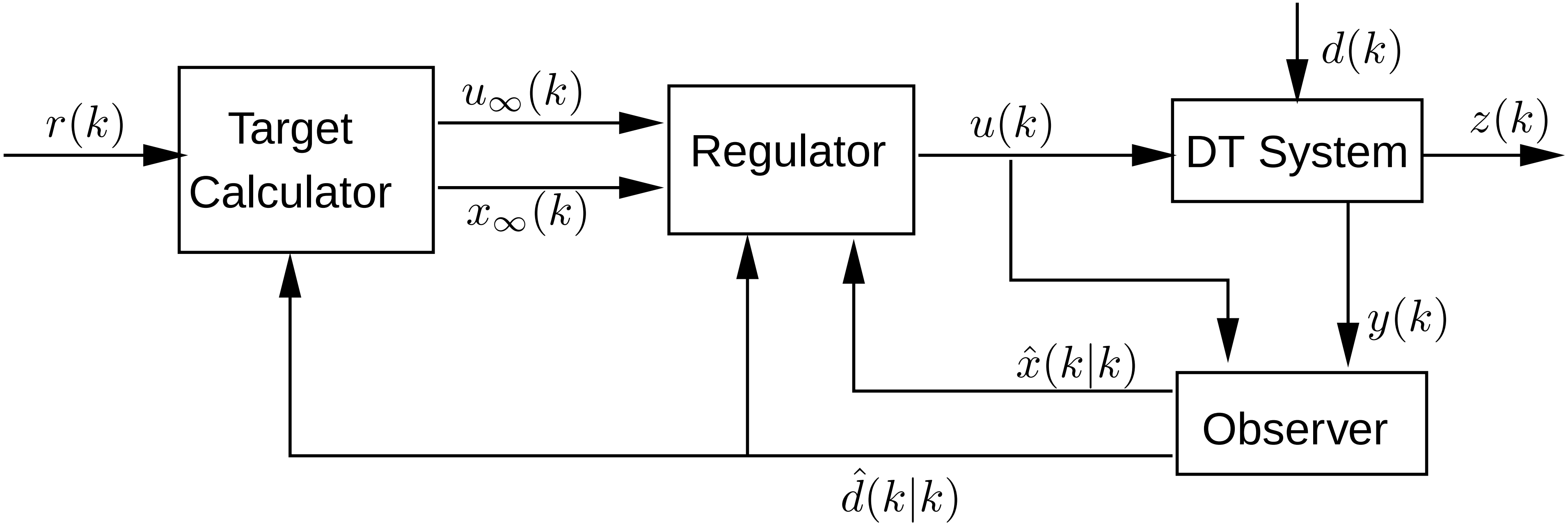
- At each time instant the target calculator solves the following set of linear equalities

$$\begin{pmatrix} (I - A) & -B \\ H & 0 \end{pmatrix} \begin{pmatrix} x_\infty \\ u_\infty \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix}$$

  Note that if a solution exists, it may not be unique.

- If (x_∞, u_∞) are calculated as above, then it is possible to show that $z(k) \to r(k)$ as $k \to \infty$ if:
  - The sequence $r(\cdot)$ converges to a constant value
  - The controller is $u(k) = u_\infty(k) + K(\hat{x}(k|k) - x_\infty(k))$
  - The matrix $(A + BK)$ and the observer are stable.

Offset-free Control – General Framework



- Future values of $r(\cdot)$ and $d(\cdot)$ are assumed unknown.
- The observer estimates the current state $\hat{x}(k|k)$ and disturbance $\hat{d}(k|k)$.
- **Problem:** Control the system so that $z(k) \to r(k)$ as $k \to \infty$ if the reference signal and the disturbance signal $d(\cdot)$ tend to constant values.

Constant Disturbance Model

We will assume a linear model with a constant disturbance acting on the states and outputs:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + B_d d(k) \\ d(k+1) &= d(k) \\ y(k) &= Cx(k) + C_d d(k) \\ z(k) &= Hx(k) + H_d d(k) \end{aligned}$$

- Disturbance inputs $d(k) \in \mathbb{R}^l$, with $B_d \in \mathbb{R}^{n\times l}$, $C_d \in \mathbb{R}^{n\times l}$ and $H_d \in \mathbb{R}^{n\times l}$.
- We will estimate $d(k)$ by forming an augmented system for the states $x(k)$ and disturbances $d(k)$.

Augmented System with Disturbance

- Augment the state of the system with the disturbance:

$$\begin{pmatrix} x(k+1) \\ d(k+1) \end{pmatrix} = \begin{pmatrix} A & B_d \\ 0 & I \end{pmatrix} \begin{pmatrix} x(k) \\ d(k) \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} C & C_d \end{pmatrix} \begin{pmatrix} x(k) \\ d(k) \end{pmatrix}$$

$$z(k) = \begin{pmatrix} H & H_d \end{pmatrix} \begin{pmatrix} x(k) \\ d(k) \end{pmatrix}$$

- We will use an observer to provide state and disturbance estimates $\hat{x}(k|k)$ and $\hat{d}(k|k)$

State and Disturbance Estimation

- The augmented system is not guaranteed to be detectable for every choice of B_d and C_d.

Proposition (Detectability) *The augmented system is detectable if and only if* (C, A) *is detectable and the matrix*

$$\begin{pmatrix} (I - A) & -B_d \\ C & C_d \end{pmatrix}$$

is full column rank.

Proof: Prove as an exercise.

- The above implies that #disturbances $\leq$ #outputs is required for detectability of the disturbance.

Target Calculation with Disturbances

- At each time instant the target calculator solves the following set of linear equalities

$$x_\infty = Ax_\infty + Bu_\infty + B_d\hat{d}$$
$$r = Hx_\infty + H_d\hat{d}$$

or, in matrix form

$$\begin{pmatrix} (I-A) & -B \\ H & 0 \end{pmatrix} \begin{pmatrix} x_\infty \\ u_\infty \end{pmatrix} = \begin{pmatrix} B_d\hat{d} \\ r - H_d\hat{d} \end{pmatrix}$$

- As in the undisturbed case, this is solvable given an appropriate rank condition.

Offset Free Control – Main Results

- Let the following conditions hold:
  - The sequence $r(\cdot)$ and $d(\cdot)$ tend to constant values
  - The rank conditions on pages 8 and 13 are satisfied.
  - The state/disturbance observer is stable.
  - The control input is given by

    $$u(k) = u_\infty(k) + K(\hat{x}(k|k) - x_\infty(k))$$

    where $(x_\infty(k), u_\infty(k))$ are chosen as on page 14.
  - The matrix K is such that $(A + BK)$ is stable.
  - Number of disturbances $=$ number of outputs.
- If all of the above holds, then $z(k) \to r(k)$ as $k \to \infty$.

Target Calculation with Constraints

- Suppose we have a constraint on the states and inputs:

$$u_{low} \leq u_i \leq u_{hi}, \qquad i = 0, 1, \ldots, N-1$$
$$y_{low} \leq y_i \leq y_{hi}, \qquad i = 1, \ldots, N$$

- As before, assume that
  - The rank conditions on pages 8 and 13 are satisfied.
  - The state/disturbance observer is stable.
  - Number of disturbances $=$ number of outputs.

- **Additional condition:** $r(k)$ and $\hat{d}(k|k)$ are such that target equilibrium pairs satisfying the above constraint always exist.

Target Calculation with Constraints

- At each time instant, the target calculator is given the current reference r and the current disturbance estimate $\hat{d}$
- The target calculator computes the target equilibrium pair (x_∞, u_∞) by solving the following Quadratic Program (QP):

$$\min_{x_\infty, u_\infty} \frac{1}{2}(u_\infty - \bar{u})^T(u_\infty - \bar{u})$$

subject to:

$$\begin{pmatrix} (I-A) & -B \\ H & 0 \end{pmatrix} \begin{pmatrix} x_\infty \\ u_\infty \end{pmatrix} = \begin{pmatrix} B_d \hat{d} \\ r - H_d \hat{d} \end{pmatrix}$$

$$u_{low} \leq u_\infty \leq u_{hi}$$

$$y_{low} \leq C x_\infty + C_d \hat{d} \leq y_{hi}$$

- The ideal steady state for the inputs is $\bar{u}$

Offset-free Predictive Control

- **Problem:** Given x_∞, u_∞ $\hat{x}$ and $\hat{d}$, compute an optimal finite horizon sequence of inputs $\{u_0, u_1, \ldots, u_{N-1}\}$ that minimize:

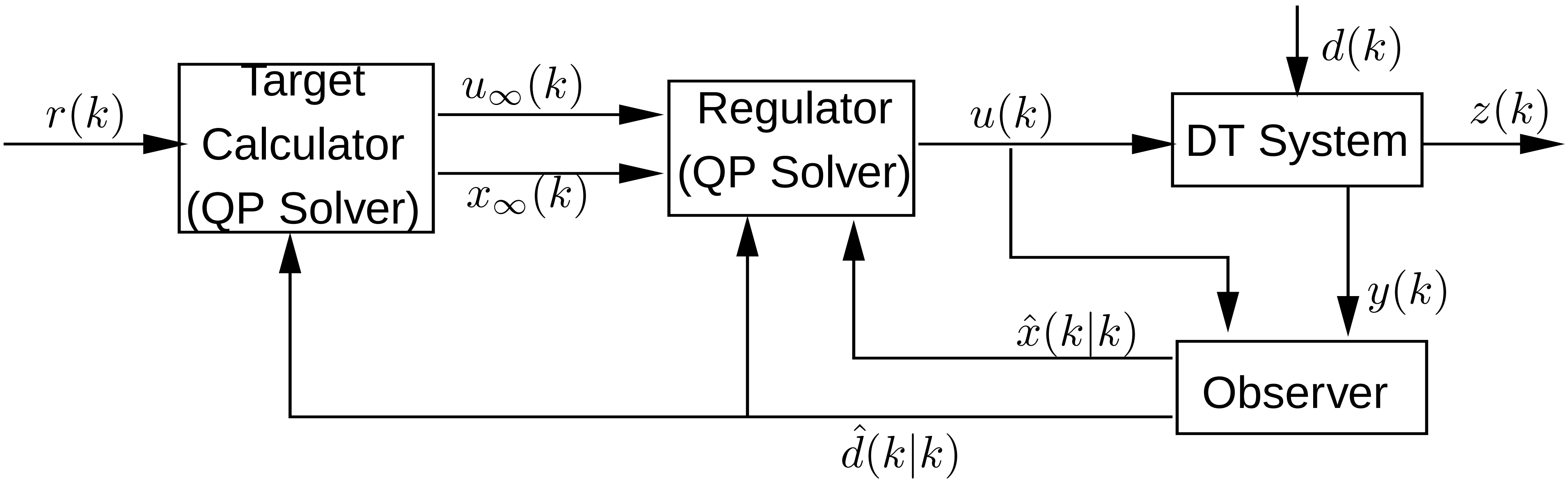
$$\sum_{i=0}^{N-1} \left[(x_i - x_\infty)^T Q (x_i - x_\infty) + (u_i - u_\infty)^T R (u_i - u_\infty)\right] + (x_N - x_\infty)^T P (x_N - x_\infty)$$

subject to:

$$\begin{aligned}
&x_0 = \hat{x} \\
&x_{i+1} = A x_i + B u_i + B_d \hat{d}, && i = 0, 1, \ldots, N-1 \\
&u_{low} \le u_i \le u_{hi} && i = 0, 1, \ldots, N-1 \\
&y_{low} \le C x_\infty + C_d \hat{d} \le y_{hi} && i = 0, 1, \ldots, N
\end{aligned}$$

- Can be written as a Quadratic Program (QP) as in Lecture 3.

Offset-free Predictive Control



At each time instant k:

- Observer takes measurement $y(k)$ and computes estimates $\hat{x}(k|k)$ and $\hat{d}(k|k)$
- Target Calculator computes $x_\infty(k)$ and $u_\infty(k)$ by solving QP on page 17
- Regulator computes an input by
  - solving the finite horizon problem on page 18 as a QP
  - implementing the first input in the sequence, i.e. $u(k) = u_0^*(x(k))$
- If QPs are always feasible and system is stable, then $z(k) \to r(k)$ as $k \to \infty$

Offset-free MPC — alternative

$$\sum_{i=0}^{N-1} \left[(y_i - y_\infty)^T Q (y_i - y_\infty) + \Delta u_i^T R \Delta u_i \right] \\ + (x_N - x_\infty)^T P (x_N - x_\infty)$$

where $\Delta u_i = u_i - u_{i-1}$
Let $y(k) - y_0 = d_0$ — assume error is due to disturbance.
Assume $d_i = d_0$ — future disturbance will remain the same.
Use this to make future predictions.

This is industrial practice — can be shown to be equivalent to use of observer and targets.