Engineering Tripos Part IIB **Module 4F2**

Nonlinear Systems and Control

Handout 4: Circle and Popov Criteria

1 Introduction

The stability criteria discussed in these notes are reminiscent of the Nyquist criterion of linear feedback control. They concern the following feedback set-up:

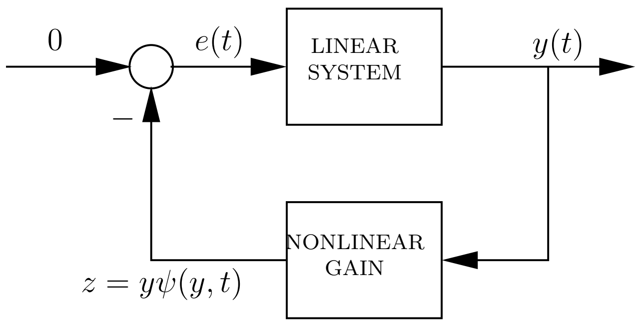


Figure 1: Feedback interconnection

The linear system has state-space description:

$$\dot{\mathbf{x}} = A\mathbf{x} + be, \quad (1)$$
$$y = c^T\mathbf{x} \quad (2)$$

and corresponding transfer function

$$G(s) = c^T(sI - A)^{-1}b.$$

The nonlinear gain $\psi(y,t)$ is assumed to satisfy: (i) $\psi(0,t) = 0$, (ii) $\psi(0,t) \geq 0$. Sometimes (ii) will be replaced by:

(iii) $k_1 \leq \psi(y,t) \leq k_2$.

Condition (iii) is sometimes referred to as $y\psi(y,t)$ belongs to the sector $[k_1, k_2]$.

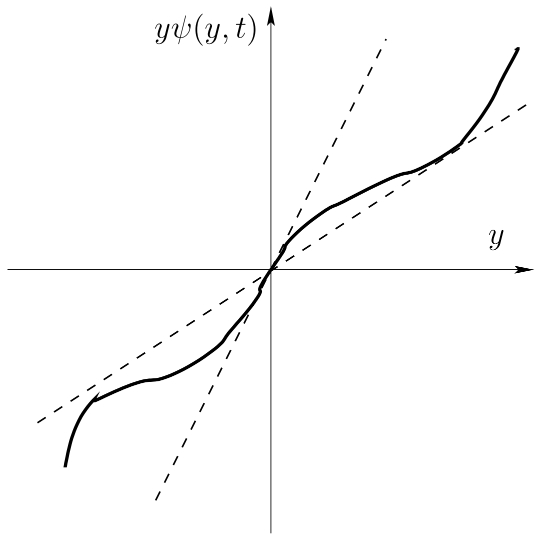


Figure 2: Sector bounded nonlinearity

Some famous conjectures were made about the stability of the above feedback set-up which turned out to be wrong. However, they acted as a starting point towards some correct conditions (Circle/Popov):

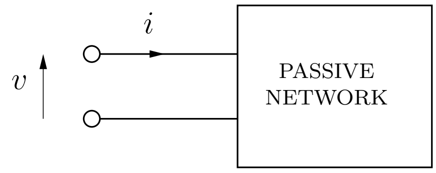
Aizerman's conjecture. If the closed loop system is stable with $\psi(y,t) = k$ for every k such that $k_1 \le k \le k_2$ then the nonlinear feedback system is globally asymptotically stable providing $k_1 \le \psi(y,t) \le k_2$.

Kalman's conjecture. Same as Aizerman's conjecture but with $k_1 \le \psi(y,t) \le k_2$ replaced by: $k_1 \le \frac{\partial(y\psi(y,t))}{\partial y} \le k_2$.

Counterexamples to each of the above are known.

2 Passivity

Consider a passive electrical network with impedance $Z(s) = V(s)/I(s)$.



Passivity of the network means that it can only absorb power. Formally: a network is passive if for all square integrable pairs $v(t)$ and $i(t)$ and all T then $\int_{-\infty}^{T} v(t)i(t)dt \geq 0$. This readily gives a necessary condition on the impedance as follows.

$$\begin{aligned}
&\text{Total energy delivered to the network} \\
&\quad = \int_{-\infty}^{\infty} v(t)i(t)dt \\
&\quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(j\omega)\overline{I(j\omega)}d\omega \\
&\quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(j\omega)|I(j\omega)|^2 d\omega \\
&\quad = \frac{1}{\pi}\int_{0}^{\infty} \Re e(Z(j\omega))|I(j\omega)|^2 d\omega
\end{aligned}$$

where the second step above uses Parseval's theorem (see Appendix) and the third step uses the fact that $i(t)$, $v(t)$ and the impulse response of the network are real signals so that: $\overline{I(j\omega)} = I(-j\omega)$ etc. Passivity of the network requires the above quantity to be non-negative for all $i(t)$. Thus, a necessary condition for passivity is that $\Re e(Z(j\omega)) \geq 0$ for all ω. This says that sinusoidal currents and voltages can never be more than 90 degrees out of phase. Thus the Nyquist diagram of Z cannot enter the left half plane. In order to give a precise necessary and sufficient condition for passivity we first give a formal definition.

Definition.
1. $G(s)$ is *positive real* if (i) $G(s)$ has no poles for $\Re e(s) > 0$ and (ii) $\Re e(G(s)) > 0$ for $\Re e(s) > 0$.
2. $G(s)$ is *strictly positive real* if (i) $G(s)$ has no poles for $\Re e(s) \geq 0$ or at ∞ and (ii) $\Re e(G(s)) > 0$ for $\Re e(s) \geq 0$.

The definition of positive realness can be expressed in an alternative way by replacing 1.(ii) by 1.(ii′): $\Re e(G(j\omega)) \geq 0$ for all ω and all the poles of $G(s)$ on the imaginary axis or at infinity are simple with positive residues. We can now state a classic result on network theory. (A proof of this result, which is beyond the scope of the course, can be found in: Newcomb, R.W., *Linear Multiport Synthesis*, 1966.)

Theorem. Consider a network with impedance $Z(s)$ (rational). The network is passive if and only if $Z(s)$ is positive real.

It is evident that a passive network is stable. In fact there is an intimate connection between passive networks and Lyapunov functions.

Lemma (Kalman-Popov-Yakubovich). Also known as the "KPY Lemma" or the "Positive Real Lemma". Given a system (1, 2) which is a minimal realisation (i.e. every eigenvalue of A is a pole of $G(s)$). Then $G(s)$ is positive real if and only if there exists a matrix $P = P^T > 0$ and a vector $\mathbf{q}$ which satisfy

$$\begin{aligned} A^T P + PA &= -\mathbf{q}\mathbf{q}^T, \\ P\mathbf{b} &= \mathbf{c}. \end{aligned} \tag{3}$$

Proof (sketch of necessity). Let us take the case where A has no imaginary axis eigenvalues. Consider $G(s) + G(-s)$. It has poles and zeros in symmetric positions with respect to the imaginary axis and is non-negative on the imaginary axis. It can be shown that this is sufficient to be able to "spectrally factor", i.e. to find an $L(s)$ such that

$$G(s) + G(-s) = L(-s)L(s)$$

where $L(s)$ has the same poles as $G(s)$, so there is some $\mathbf{q}$ such that $L(s) = \mathbf{q}^T(sI - A)^{-1}\mathbf{b}$. We can now find a solution $P = P^T > 0$ of the Lyapunov matrix equation (3) (cf. equation (2), Handout 2). Adding and subtracting sP to both sides of (3) gives

$$(-sI - A)^T P + P(sI - A) = \mathbf{q}\mathbf{q}^T.$$

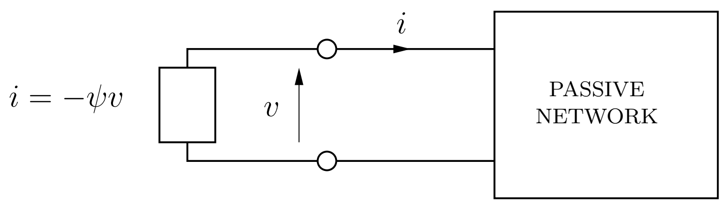
Pre-multiplying by $\mathbf{b}^T(-sI - A^T)^{-1}$ and post-multiplying by $(sI - A)^{-1}\mathbf{b}$ gives:

$$\begin{aligned} \mathbf{b}^T P(sI - A)^{-1}\mathbf{b} + \mathbf{b}^T(-sI - A^T)^{-1}P\mathbf{b} &= \left[\mathbf{b}^T(-sI - A^T)^{-1}\mathbf{q}\right]\left[\mathbf{q}^T(sI - A)^{-1}\mathbf{b}\right] \\ &= L(-s)L(s). \end{aligned}$$

Since the decomposition of $L(-s)L(s)$ into $G(s) + G(-s)$ is effectively a partial fraction decomposition, then it is a unique decomposition, so $G(s) = \mathbf{b}^T P(sI - A)^{-1}\mathbf{b}$. The minimality of the realisation of $G(s)$ means that $\mathbf{b}^T P = \mathbf{c}^T$.

3 The Circle Criterion

Consider the connection of a "nonlinear resistor" across the terminals of a passive network. It is intuitively clear that this connection should be stable since energy can only be



dissipated in both elements. The generalisation of this to feedback systems is the following.

Lemma. Let $G(s)$ be a minimal realisation. Then the origin is a stable equilibrium point for the feedback system of Figure 1 if:

1. $G(s)$ is positive real,
2. $\psi(y,t) \geq 0$.

Proof. Let P and $\mathbf{q}$ be defined as in the Kalman-Yakubovich lemma. Consider the candidate Lyapunov function $V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}$. Note that the state equations of the system are:

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x} + \mathbf{b}e, \\ &= A\mathbf{x} - \psi(\mathbf{c}^T\mathbf{x}, t)\mathbf{b}\mathbf{c}^T\mathbf{x},\end{aligned}$$

so $\mathbf{x} = \mathbf{0}$ is an equilibrium point. Then

$$\begin{aligned}\dot{V} &= \dot{\mathbf{x}}^T P\mathbf{x} + \mathbf{x}^T P\dot{\mathbf{x}} \\ &= \mathbf{x}^T[A - \psi(y,t)\mathbf{b}\mathbf{c}^T]^T P\mathbf{x} + \mathbf{x}^T P[A - \psi(y,t)\mathbf{b}\mathbf{c}^T]\mathbf{x} \\ &= \mathbf{x}^T[A^T P + PA]\mathbf{x} - \mathbf{x}^T\psi(y,t)\mathbf{c}\mathbf{b}^T P\mathbf{x} - \mathbf{x}^T P\psi(y,t)\mathbf{b}\mathbf{c}^T\mathbf{x} \\ &= -\mathbf{x}^T\mathbf{q}\mathbf{q}^T\mathbf{x} - 2\psi(y,t)\mathbf{x}^T\mathbf{c}\mathbf{c}^T\mathbf{x} \\ &= -(\mathbf{x}^T\mathbf{q})^2 - 2\psi(y,t)(\mathbf{x}^T\mathbf{c})^2 \leq 0.\end{aligned}$$

Hence $\mathbf{x} = \mathbf{0}$ is stable.

Notes.
(1) If $G(s)$ is strictly positive real, global asymptotic stability can be proved.
(2) The nonlinear gain $\psi(y,t)$ is allowed to be time-varying.

The lemma is quite restrictive in the form stated. We will therefore consider how it can be transformed to widen its applicability. This will lead to the Circle Criterion. Suppose we had a sector condition on the nonlinearity, namely:

$$\alpha \leq \psi(y,t) \leq \beta.$$

Now define

$$\tilde{\psi} = \frac{\psi - \alpha}{\beta - \psi}$$

and note that $\tilde{\psi} \geq 0$. Now we consider the block diagram of Figure 1 along with some loop transformations. Note that the additional feedforwards and feedbacks in Figure 3 are

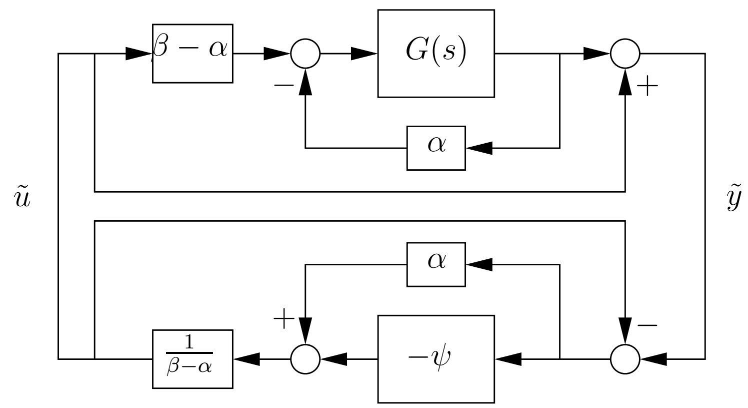


Figure 3: Loop transformation of Figure 1.

self-cancelling, which makes the closed-loop stability of Figure 1 and Figure 3 equivalent. From Figure 3 we have:

$$
\begin{aligned}
& \tilde{u} = \frac{1}{\beta - \alpha}(\alpha - \psi)(\tilde{y} - \tilde{u}) \\
\Rightarrow \quad & (\beta - \psi)\tilde{u} = (\alpha - \psi)\tilde{y} \\
\Rightarrow \quad & \tilde{u} = -\tilde{\psi}\tilde{y}.
\end{aligned}
$$

Also we can determine $\tilde{G}$ such that $\tilde{y} = \tilde{G}\tilde{u}$:

$$
\begin{aligned}
\tilde{y} &= \tilde{u} + \frac{G}{1 + \alpha G}(\beta - \alpha)\tilde{u} \\
&= \frac{1 + \beta G}{1 + \alpha G} = \tilde{G}\tilde{u}.
\end{aligned}
$$

Thus, Figure 3 is the same as Figure 4.

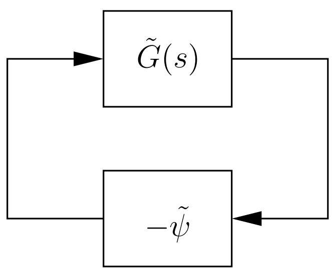


Figure 4: Loop transformation of Figure 1.

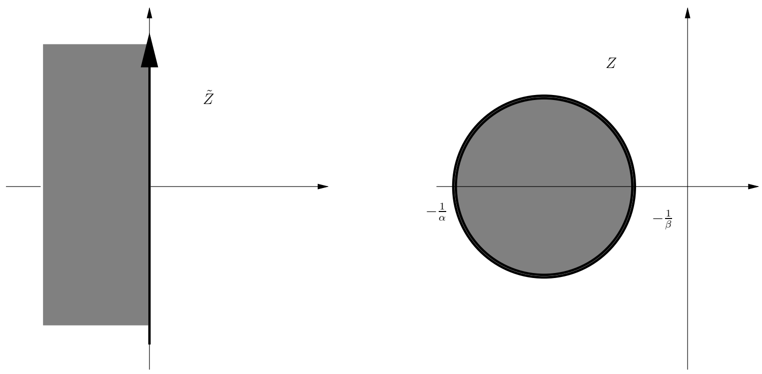
We can now apply the lemma to Figure 4 to deduce that positive realness of $\tilde{G}$ is needed for stability. What conditions does this impose on G? Solving for $G(s)$ gives

$$G(s) = \frac{\tilde{G}(s) - 1}{\beta - \alpha\tilde{G}(s)}.$$

Now we can check that the transformation

$$Z = \frac{\tilde{Z} - 1}{\beta - \alpha\tilde{Z}}$$

maps the imaginary axis onto the circle with diameter $[\frac{-1}{\alpha}, \frac{-1}{\beta}]$, and moreover, the right half plane maps to the inside of this circle. Thus the condition "$\Re e\left\{\tilde{G}(j\omega)\right\} \geq 0$" becomes "the locus of $G(j\omega)$ does not penetrate the circle with diameter $[\frac{-1}{\alpha}, \frac{-1}{\beta}]$".



For $\tilde{G}(s)$ to be (strictly) positive-real, we also need it to be asymptotically stable. But $\tilde{G}(s) = 1 + \frac{(\beta-\alpha)G(s)}{1+\alpha G(s)}$ and this transfer function is stable if the feedback combination of $G(s)$ and $-\alpha$ is stable. By the Nyquist criterion, this will be the case if the locus of $G(j\omega)$ encircles the point $-1/\alpha$ as many times anticlockwise as $G(s)$ has unstable poles. Putting all this together gives:

Theorem (Circle Criterion). The system of Figure 1 with

$$\alpha \leq \psi(y, t) \leq \beta$$

is globally asymptotically stable if the Nyquist locus $G(j\omega)$ $(-\infty < \omega < \infty)$ does not penetrate the disc with diameter $[\frac{-1}{\alpha}, \frac{-1}{\beta}]$ and encircles it as many times anticlockwise as $G(s)$ has unstable poles.

Fig.5 shows the Circle criterion predicting closed-loop stability for the case that the linear system in Fig.1 is stable.

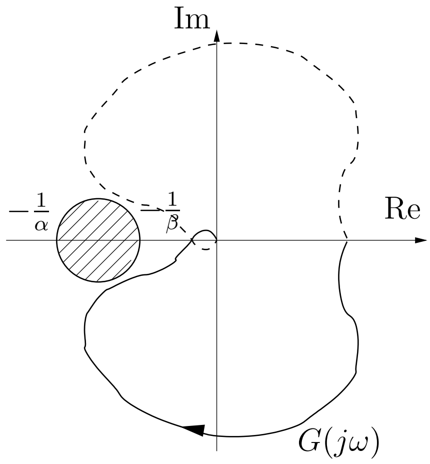


Figure 5: Circle criterion for open-loop stable system.

Consistency of Circle criterion and DF method. Note that if $\psi(y,t)$ is independent of t, $\psi(-y) = -\psi(y)$, and $\alpha \leq \psi(y) \leq \beta$, then the describing function corresponding to $\psi(.)$ satisfies

$$\alpha \leq N(E) \leq \beta$$

and hence the $-1/N(E)$ graph lies entirely within the circle of the Circle criterion.

Proof: Exercise — Q.11 on Examples Paper 2.

4 Popov Criterion

If we now restrict $\psi(y,t)$ to be independent of t and take $\alpha = 0$ so that

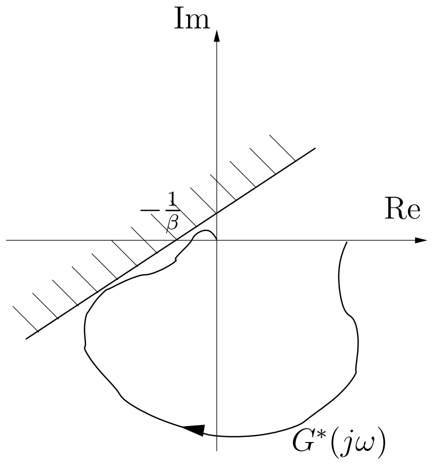
$$0 \leq \psi(y) \leq \beta$$

and if we further assume $G(s)$ is stable, then we can obtain a less conservative criterion for global asymptotic stability. Under these conditions the system shown in Figure 1 is globally asymptotically stable if the *Popov Locus*, namely the locus of

$$G^*(j\omega) = \Re e\left\{G(j\omega)\right\} + j\omega \Im m\left\{G(j\omega)\right\} \qquad (\omega \geq 0)$$

lies entirely to the right of a straight line, with positive slope, passing through $\frac{-1}{\beta}$.

Proof. Details omitted — see Khalil (3rd ed), Theorem 7.3 and its proof, which covers the multivariable case.



(1) The proof uses the Lyapunov function:

$$V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x} + 2\eta\beta \int_0^y \nu\psi(\nu)d\nu$$

(2) The graphical condition corresponds to

$$(1 + \eta s)G(s) + \frac{1}{\beta}$$

being positive real for some $\eta > 0$.

A Appendix: Parseval's Theorem

$$\begin{aligned}
\int_{-\infty}^{\infty} v(t)i(t)dt &= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} V(j\omega)e^{j\omega t}d\omega\right] i(t)dt && (4)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} V(j\omega) \int_{-\infty}^{\infty} i(t)e^{j\omega t}dt\, d\omega && (5)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} V(j\omega)I(-j\omega)d\omega && (6)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} V(j\omega)\overline{I(j\omega)}d\omega && (7)
\end{aligned}$$