1 Harmonic balance

The describing function method (also called the method of harmonic balance) uses frequency domain (Fourier series) techniques to investigate limit cycle behaviour in nonlinear systems. The method involves an approximation, but nevertheless often gives a reliable prediction concerning limit cycle behaviour.

The usual context for the method is the following feedback system.

There is a linear system with transfer function $g(s)$ and a memoryless nonlinearity $f(\cdot)$ (e.g. saturation, hysteresis, backlash). For simplicity we will take the case of $r = 0$. We are looking for possible limit cycle (i.e. periodic) behaviour in the feedback system. Therefore let us take a trial solution:

$$e(t) = E \sin(\omega t).$$

If the output of the non-linearity is periodic with frequency $\omega$ there will be a Fourier series representation:

$$u(t) = U_0 + \sum_{k=1}^{\infty} (U_k \sin(k\omega t) + V_k \cos(k\omega t)).$$

The Fourier coefficients corresponding to the fundamental harmonic are given by

$$U_1 = \frac{\omega}{\pi} \int_0^{2\pi/\omega} f(E \sin \omega t) \sin \omega t \, dt = \frac{1}{\pi} \int_0^{2\pi} f(E \sin \theta) \sin \theta \, d\theta,$$

$$V_1 = \frac{1}{\pi} \int_0^{2\pi} f(E \sin \theta) \cos \theta \, d\theta.$$
We now consider the approximation:

\[ u(t) \approx U_1 \sin(\omega t) + V_1 \cos(\omega t) \]

as the input to the linear system. In the steady state, the output of the linear system is:

\[
y(t) = |g(j\omega)||U_1 \sin(\omega t + \phi) + V_1 \cos(\omega t + \phi)|
\]
\[
= \text{Im}[|g(j\omega)|(U_1 + jV_1)e^{j(\omega t + \phi)}]
\]
\[
= \text{Im}[g(j\omega)(U_1 + jV_1)e^{j\omega t}]
\]

where \( \phi = \text{arg}(g(j\omega)) \). (We remark that the neglect of the higher harmonics is most likely to be a valid approximation when \( g(s) \) behaves like a low pass filter.) In order to satisfy the feedback equations we need to have \( y(t) = -e(t) \). We thus obtain

\[
0 \equiv y(t) + e(t)
\]
\[
\Rightarrow 0 \equiv \text{Im}[g(j\omega)(U_1 + jV_1 + E)e^{j\omega t}] = 0
\]
\[
\Rightarrow 0 = g(j\omega)(U_1 + jV_1) + E.
\]

This is the condition for “harmonic balance”, i.e. for the fundamental harmonic to solve the feedback equations. We now define the describing function

\[
N(E) = \frac{U_1 + jV_1}{E}.
\]

Then the condition for harmonic balance becomes

\[
g(j\omega) = -\frac{1}{N(E)}.
\]

This condition has an interesting graphical interpretation. Let the Nyquist diagram \( g(j\omega) \) of the linear system be plotted in the complex plane. On the same diagram plot the locus

\[
\text{Arrows denote direction of increasing } \omega \text{ and } E
\]
−1/N(E). Note that these loci can be plotted independently since they are respectively functions of ω and E only. A limit cycle is predicted if there is an intersection of the two loci. Moreover, the intersection point gives an estimate for the frequency ω and amplitude E of the limit cycle.

A useful prediction of the stability of limit cycles can be made from the manner of intersection of the two loci. Recall from the Nyquist stability criterion that, for an open loop stable g(s) in the standard unity gain negative feedback configuration (non-linearity absent of course), the closed loop feedback system is stable providing the critical point (−1, 0) lies to the left of the Nyquist locus. The point −1/N(E) can be thought of as the ‘critical point’ with respect to the Nyquist locus. Thus, if a small increase of the amplitude E moves the value of −1/N(E) outside of the shaded region below, then a stabilizing effect will be predicted, which will act to reduce the amplitude. Conversely, if an increase of E moves −1/N(E) inside the shaded region, then a destabilizing effect will be predicted.

2 Calculation of describing functions

2.1 Relay nonlinearity

f(e) = sign(e). Since sign(sin(ωt)) is an odd function then V₁ = 0. Also

\[ U₁ = \frac{1}{\pi} \int_{0}^{2\pi} \text{sign}(E \sin \theta) \sin \theta \, d\theta, \]

\[ = \frac{1}{\pi} \int_{0}^{\pi} \sin \theta \, d\theta - \frac{1}{\pi} \int_{\pi}^{2\pi} \sin \theta \, d\theta \]

\[ = \frac{4}{\pi}. \]
Thus \( N(E) = \frac{4}{\pi E} \).

### 2.2 Polynomial nonlinearity

\( f(e) = e^n \). If \( n \) is odd it can be shown (from De Moivre’s Theorem and the binomial expansion) that

\[
\sin^n \theta = \frac{2}{2^n} \sum_{k=0}^{n-1} (-1)^{\left\lfloor \frac{n-1}{2} - k \right\rfloor} \binom{n}{k} \sin([n - 2k] \theta)
\]

This is a Fourier series with a finite number of terms; taking the last term, \( k = \frac{n-1}{2} \), gives us the fundamental component if \( e = (E \sin \theta)^n \):

\[
U_1 \sin \theta = E^n \frac{2}{2^n} (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{n/2} \sin \theta, \quad V_1 = 0.
\]

A familiar case of this is \( n = 3 \) (eg in the Maths Databook):

\[
(E \sin \theta)^3 = E^3 \left( \frac{3}{4} \sin \theta - \frac{1}{4} \sin(3\theta) \right)
\]

for which \( U_1 = \frac{3E^3}{4}, V_1 = 0 \), so \( N(E) = \frac{3E^2}{4} \).

If \( n \) is even a similar formula exists, but the fundamental component is always zero (for example, \( \sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos(2\theta) \)), so our low-pass assumption will eliminate everything. There is also a non-zero constant component — which can be handled by an extension of the harmonic balance idea.

### 2.3 Piecewise-linear non-linearity
$$f(e) = \begin{cases} 
  m_1e & \text{if } |e| < \delta \\
  (m_1 - m_2)\delta + m_2e & \text{if } e > \delta \\
  (m_2 - m_1)\delta + m_2e & \text{if } e < -\delta 
\end{cases}$$

Once again, since $f(e)$ is an odd function $V_1 = 0$. If $E \leq \delta$ then

$$U_1 = \frac{1}{2\pi} \int_0^\pi m_1E \sin^2 \theta \, d\theta = m_1E.$$ 

If $E > \delta$ then

$$U_1 = \frac{4}{\pi} \int_0^{\pi/2} f(E \sin \theta) \sin \theta \, d\theta,$$

$$= \frac{4}{\pi} \int_0^{\sin^{-1}(\delta/E)} m_1E \sin^2 \theta \, d\theta + \frac{4}{\pi} \int_{\sin^{-1}(\delta/E)}^{\pi/2} ((m_1 - m_2)\delta + m_2E \sin \theta) \sin \theta \, d\theta$$

$$= \frac{2m_1E}{\pi} \left[ \theta - \frac{1}{2} \sin(2\theta) \right]_0^{\sin^{-1}(\delta/E)} + \frac{4(m_2 - m_1)\delta}{\pi} \frac{\cos \theta}{\sin \theta} \bigg|_{\sin^{-1}(\delta/E)}^{\pi/2},$$

Now, since $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$ and $\sin(2\sin^{-1} x) = 2x\sqrt{1 - x^2}$ for $0 < x < \pi/2$, we get after a little algebra:

$$U_1 = \frac{2(m_1 - m_2)E}{\pi} \left[ \sin^{-1}(\frac{\delta}{E}) + \frac{\delta}{E}(1 - (\frac{\delta}{E})^2)^{1/2} \right] + m_2E.$$

Hence we get:

$$N(E) = \begin{cases} 
  m_1, & \text{if } E < \delta \\
  \frac{2(m_1 - m_2)}{\pi} \left[ \sin^{-1}(\frac{\delta}{E}) + \frac{\delta}{E}(1 - (\frac{\delta}{E})^2)^{1/2} \right] + m_2, & \text{if } E > \delta.
\end{cases}$$

### 2.4 Relay with hysteresis

Consider the following two-valued function:

This is a non-linearity with ‘memory’. $f(e)$ takes the value $+R$ or $-R$ according to whether $e$ was greater than $\delta$ or less than $-\delta$ on the last occasion when $|e| > \delta$. We will find the describing function for $E > \delta$ since it is not properly defined otherwise. We have

$$N(E) = \frac{1}{\pi E} \int_{-\pi/2}^{3\pi/2} f(E \sin \theta)(\sin \theta + j \cos \theta) \, d\theta$$
since we can integrate over any cycle of length $2\pi$. Thus

$$N(E) = \frac{1}{\pi E} \int_{-\pi/2}^{\pi/2} f_+(E \sin \theta)(\sin \theta + j \cos \theta) d\theta + \frac{1}{\pi E} \int_{\pi/2}^{3\pi/2} f_-(E \sin \theta)(\sin \theta + j \cos \theta) d\theta.$$  

Let $\nu = E \sin \theta$, $d\nu = E \cos \theta d\theta$ (and note that $\cos \theta$ has positive sign in the first integral and negative sign in the second), then

$$\text{Re}N(E) = \frac{1}{\pi E} \int_{-E}^{E} f_+(\nu) \left( \frac{\nu}{E} \right) \frac{d\nu}{E \sqrt{1 - (\nu/E)^2}} + \frac{1}{\pi E} \int_{-E}^{E} f_-(\nu) \left( \frac{\nu}{E} \right) \frac{d\nu}{E \sqrt{1 - (\nu/E)^2}}.$$  

The above formula is valid for any two-valued non-linearity. In fact, it is also valid for a single-valued non-linearity $f$. If $f = (f_+ + f_-)/2$ is an odd function then

$$N(E) = \frac{1}{\pi E} \int_{-E}^{E} 2f(\nu) \left( \frac{\nu}{E} \right) \frac{d\nu}{E \sqrt{1 - (\nu/E)^2}}.$$  

Thus, $\text{Re}(N(E))$ is the same as the describing function of the ‘average’ non-linearity $(f_+ + f_-)/2$. 

\[6\]
For $\text{Im}(N(E))$ there is a striking formula:

$$
\text{Im}N(E) = \frac{1}{\pi E} \int_{-E}^{E} f_+(\nu) \frac{d\nu}{E} + \frac{1}{\pi E} \int_{-E}^{E} f_-(\nu) \frac{d\nu}{E}
$$

$$
= \frac{1}{\pi E^2} \int_{-E}^{E} [f_+(\nu) - f_-(\nu)] d\nu
$$

$$
= -\frac{\Delta}{\pi E^2}
$$

where $\Delta$ is the area enclosed by the ‘loop’ of the non-linear characteristic.

For the relay with hysteresis given above we obtain

$$
\text{Re}N(E) = \frac{4R}{\pi E} \int_{\delta}^{E} \frac{\nu d\nu}{E^2 \sqrt{1 - (\nu/E)^2}} = \frac{4R}{\pi E} \sqrt{1 - \left(\frac{\delta}{E}\right)^2},
$$

which gives

$$
N(E) = \frac{4R}{\pi E} \left( \sqrt{1 - \left(\frac{\delta}{E}\right)^2} - j\delta \right).
$$

3 Estimates of Describing Functions

Often some idea of the describing function can be obtained without calculating it in detail. The describing function is sometimes called the \textit{equivalent linear gain}, and this name gives a pointer as to how it can be estimated. This is best illustrated by a couple of examples.

\textit{Example: Piecewise-linear nonlinearity.} In section 2.3 we considered a nonlinearity with an incremental gain of $m_1$ for small signals, and $m_2$ for large signals. For a small signal the gain is exactly $m_1$, because the nonlinearity does not come into play. For extremely large signals the gain is essentially $m_2$. Clearly the effective gain for any input signal lies between these two. Thus the describing function (being real in this case) must always lie between $m_1$ and $m_2$.

\textit{Example: Relay with dead-zone.} Consider the following behaviour:

$$
f(e) = \begin{cases} 
-1, & \text{if } e \leq -\delta \\
0, & \text{if } |e| < \delta \\
+1, & \text{if } e \geq \delta
\end{cases}
$$

For $|e| < \delta$ the describing function is clearly 0. As $|e|$ increases slightly above $\delta$ the output suddenly jumps to amplitude 1, so the effective gain increases quickly, and hence the describing function increases quickly (to a value which has to be calculated — see Examples Paper). As $|e|$ increases further, the output amplitude does not increase, and so the effective gain — and hence the describing function — decreases gradually to 0.
4 Rigour

The first part of the describing function method, namely the prediction of limit cycles, can be made rigorous using appropriate bounds on $g(j\omega)$ and possibly by bringing in some higher harmonics. A rather mathematical discussion along these lines and sufficient conditions for validity can be found in A.I. Mees, *Dynamics of Feedback Systems*, Wiley, 1981, Chapter 5. (This is beyond the scope of this module).

The prediction of stability/instability has so far proved difficult to treat rigorously and provide useful sufficient conditions for validity. It therefore remains as a quick method which gives a hint as to what might happen.

M.C. Smith
Revised by J.M. Maciejowski February 2011