Engineering Tripos Part IIB **Module 4F2**

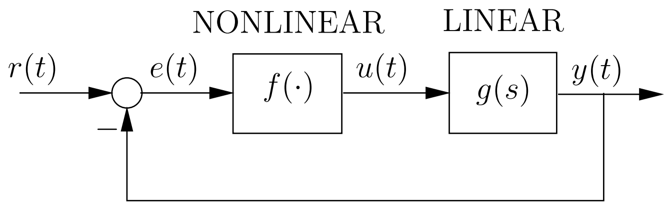
Nonlinear Systems and Control

Handout 3: Describing Functions

1 Harmonic balance

The *describing function method* (also called the method of harmonic balance) uses frequency domain (Fourier series) techniques to investigate limit cycle behaviour in nonlinear systems. The method involves an approximation, but nevertheless often gives a reliable prediction concerning limit cycle behaviour.

The usual context for the method is the following feedback system.



There is a linear system with transfer function $g(s)$ and a memoryless nonlinearity $f(\cdot)$ (e.g. saturation, hysteresis, backlash). For simplicity we will take the case of $r = 0$. We are looking for possible limit cycle (i.e. periodic) behaviour in the feedback system. Therefore let us take a trial solution:

$$e(t) = E\sin(\omega t).$$

If the output of the non-linearity is periodic with frequency ω there will be a Fourier series representation:

$$u(t) = U_0 + \sum_{k=1}^{\infty}(U_k \sin(k\omega t) + V_k \cos(k\omega t)).$$

The Fourier coefficients corresponding to the fundamental harmonic are given by

$$\begin{aligned} U_1 &= \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(E\sin\omega t)\sin\omega t\,dt = \frac{1}{\pi}\int_0^{2\pi} f(E\sin\theta)\sin\theta\,d\theta, \\ V_1 &= \frac{1}{\pi}\int_0^{2\pi} f(E\sin\theta)\cos\theta\,d\theta. \end{aligned}$$

We now consider the approximation:

$$u(t) \approx U_1 \sin(\omega t) + V_1 \cos(\omega t)$$

as the input to the linear system. In the steady state, the output of the linear system is:

$$\begin{aligned} y(t) &= |g(j\omega)|[U_1 \sin(\omega t + \phi) + V_1 \cos(\omega t + \phi)] \\ &= \mathrm{Im}[|g(j\omega)|(U_1 + jV_1)e^{j\omega t+\phi}] \\ &= \mathrm{Im}[g(j\omega)(U_1 + jV_1)e^{j\omega t}] \end{aligned}$$

where $\phi = \arg(g(j\omega))$. (We remark that the neglect of the higher harmonics is most likely to be a valid approximation when $g(s)$ behaves like a low pass filter.) In order to satisfy the feedback equations we need to have $y(t) = -e(t)$. We thus obtain

$$\begin{aligned} 0 \equiv\ & y(t) + e(t) \\ \Rightarrow\ 0 \equiv\ & \mathrm{Im}[(g(j\omega)(U_1 + jV_1) + E)e^{j\omega t}] = 0 \\ \Rightarrow\ 0 =\ & g(j\omega)(U_1 + jV_1) + E. \end{aligned}$$

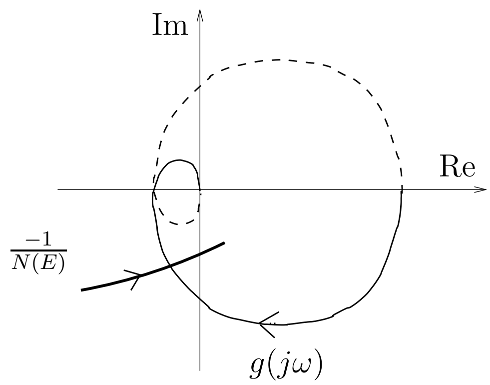
This is the condition for "harmonic balance", i.e. for the fundamental harmonic to solve the feedback equations. We now define the *describing function*

$$N(E) = \frac{U_1 + jV_1}{E}.$$

Then the condition for harmonic balance becomes

$$g(j\omega) = \frac{-1}{N(E)}.$$

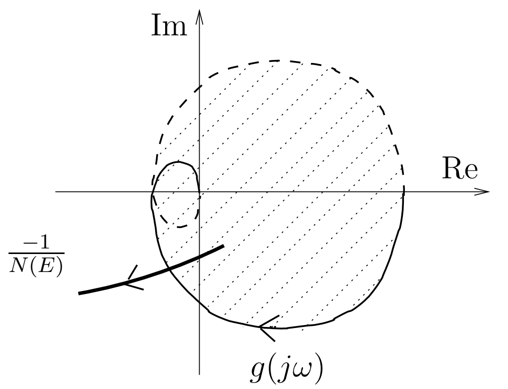
This condition has an interesting graphical interpretation. Let the Nyquist diagram $g(j\omega)$ of the linear system be plotted in the complex plane. On the same diagram plot the locus



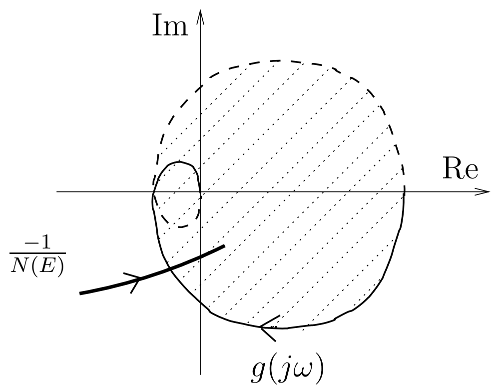
Arrows denote direction of increasing ω and E

$-1/N(E)$. Note that these loci can be plotted independently since they are respectively functions of ω and E only. A limit cycle is predicted if there is an intersection of the two loci. Moreover, the intersection point gives an estimate for the frequency ω and amplitude E of the limit cycle.

A useful prediction of the stability of limit cycles can be made from the manner of intersection of the two loci. Recall from the Nyquist stability criterion that, for an *open* loop stable $g(s)$ in the standard unity gain negative feedback configuration (non-linearity absent of course), the *closed* loop feedback system is stable providing the critical point $(-1, 0)$ lies to the left of the Nyquist locus. The point $-1/N(E)$ can be thought of as the 'critical point' with respect to the Nyquist locus. Thus, if a small increase of the amplitude E moves the value of $-1/N(E)$ *outside* of the shaded region below, then a *stabilizing* effect will be predicted, which will act to reduce the amplitude. Conversely, if an increase of E moves $-1/N(E)$ *inside* the shaded region, then a *destabilizing* effect will be predicted.



PREDICT STABLE LIMIT CYCLE



PREDICT UNSTABLE LIMIT CYCLE

2 Calculation of describing functions

2.1 Relay nonlinearity

$f(e) = \text{sign}(e)$. Since $\text{sign}(\sin(\omega t))$ is an odd function then $V_1 = 0$. Also

$$\begin{aligned} U_1 &= \frac{1}{\pi}\int_0^{2\pi} \text{sign}(E\sin\theta)\sin\theta\, d\theta, \\ &= \frac{1}{\pi}\int_0^{\pi} \sin\theta\, d\theta - \frac{1}{\pi}\int_{\pi}^{2\pi} \sin\theta\, d\theta \\ &= \frac{4}{\pi}. \end{aligned}$$

Thus $N(E) = \frac{4}{\pi E}$.

2.2 Polynomial nonlinearity

$f(e) = e^n$. If n is odd it can be shown (from De Moivre's Theorem and the binomial expansion) that

$$\sin^n \theta = \frac{2}{2^n} \sum_{k=0}^{\frac{n-1}{2}} (-1)^{(\frac{n-1}{2}-k)} \binom{n}{k} \sin([n-2k]\theta)$$

This is a Fourier series with a finite number of terms; taking the last term, $k = \frac{n-1}{2}$, gives us the fundamental component if $e = (E \sin\theta)^n$:

$$U_1 \sin\theta = E^n \frac{2}{2^n} (-1)^0 \binom{n}{\frac{n-1}{2}} \sin\theta, \qquad V_1 = 0.$$

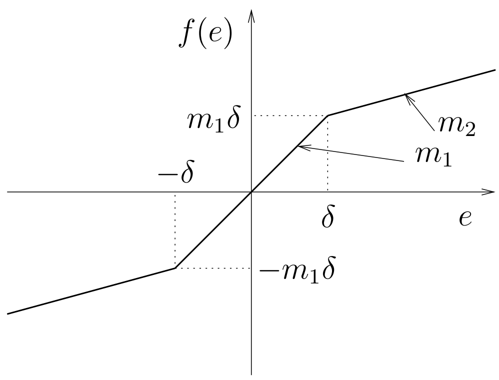
A familiar case of this is $n = 3$ (eg in the Maths Databook):

$$(E\sin\theta)^3 = E^3 \left(\frac{3}{4} \sin\theta - \frac{1}{4} \sin(3\theta) \right)$$

for which $U_1 = \frac{3E^3}{4}$, $V_1 = 0$, so $N(E) = \frac{3E^2}{4}$.

If n is even a similar formula exists, but the fundamental component is always zero (for example, $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$), so our low-pass assumption will eliminate everything. There is also a non-zero constant component — which can be handled by an extension of the harmonic balance idea.

2.3 Piecewise-linear non-linearity



$$f(e) = \begin{cases} m_1 e \text{ if } |e| < \delta \\ (m_1 - m_2)\delta + m_2 e \text{ if } e > \delta \\ (m_2 - m_1)\delta + m_2 e \text{ if } e < -\delta \end{cases}$$

Once again, since $f(e)$ is an odd function $V_1 = 0$. If $E \leq \delta$ then

$$U_1 = \frac{1}{2\pi} \int_0^\pi m_1 E \sin^2\theta \, d\theta = m_1 E.$$

If $E > \delta$ then

$$\begin{aligned} U_1 &= \frac{4}{\pi} \int_0^{\pi/2} f(E\sin\theta)\sin\theta \, d\theta, \\ &= \frac{4}{\pi} \int_0^{\sin^{-1}(\delta/E)} m_1 E \sin^2\theta \, d\theta + \frac{4}{\pi} \int_{\sin^{-1}(\delta/E)}^{\pi/2} ((m_1 - m_2)\delta + m_2 E \sin\theta)\sin\theta \, d\theta \\ &= \frac{2m_1 E}{\pi} \left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\sin^{-1}(\delta/E)} + \frac{4(m_2 - m_1)\delta}{\pi} [\cos\theta]_{\sin^{-1}(\delta/E)}^{\pi/2} \\ &\quad + \frac{2m_2 E}{\pi} \left[\theta - \frac{1}{2}\sin(2\theta)\right]_{\sin^{-1}(\delta/E)}^{\pi/2}. \end{aligned}$$

Now, since $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$ and $\sin(2\sin^{-1} x) = 2x\sqrt{1 - x^2}$ for $0 < x < \pi/2$, we get after a little algebra:

$$U_1 = \frac{2(m_1 - m_2)E}{\pi} \left[\sin^{-1}(\frac{\delta}{E}) + \frac{\delta}{E}(1 - (\frac{\delta}{E})^2)^{1/2}\right] + m_2 E.$$

Hence we get:

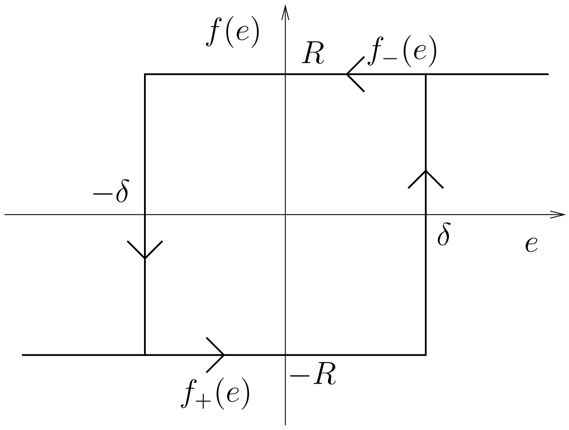
$$N(E) = \begin{cases} m_1, \text{ if } E < \delta \\ \frac{2(m_1 - m_2)}{\pi} \left[\sin^{-1}(\frac{\delta}{E}) + \frac{\delta}{E}(1 - (\frac{\delta}{E})^2)^{1/2}\right] + m_2, \text{ if } E > \delta. \end{cases}$$

2.4 Relay with hysteresis

Consider the following two-valued function:

This is a non-linearity with 'memory'. $f(e)$ takes the value $+R$ or $-R$ according to whether e was greater than δ or less than $-\delta$ on the last occasion when $|e| > \delta$. We will find the describing function for $E > \delta$ since it is not properly defined otherwise. We have

$$N(E) = \frac{1}{\pi E} \int_{-\pi/2}^{3\pi/2} f(E\sin\theta)(\sin\theta + j\cos\theta) \, d\theta$$



since we can integrate over any cycle of length 2π. Thus

$$N(E) = \frac{1}{\pi E}\int_{-\pi/2}^{\pi/2} f_+(E\sin\theta)(\sin\theta + j\cos\theta)\,d\theta + \frac{1}{\pi E}\int_{\pi/2}^{3\pi/2} f_-(E\sin\theta)(\sin\theta + j\cos\theta)\,d\theta.$$

Let $\nu = E\sin\theta$, $d\nu = E\cos\theta\,d\theta$ (and note that $\cos\theta$ has positive sign in the first integral and negative sign in the second), then

$$\begin{aligned}
\mathrm{Re}N(E) &= \frac{1}{\pi E}\int_{-\pi/2}^{\pi/2} f_+(E\sin\theta)\sin\theta\,d\theta + \frac{1}{\pi E}\int_{\pi/2}^{3\pi/2} f_-(E\sin\theta)\sin\theta\,d\theta \\
&= \frac{1}{\pi E}\int_{-E}^{E} f_+(\nu)\left(\frac{\nu}{E}\right)\frac{d\nu}{E\sqrt{1-(\nu/E)^2}} - \frac{1}{\pi E}\int_{E}^{-E} f_-(\nu)\left(\frac{\nu}{E}\right)\frac{d\nu}{E\sqrt{1-(\nu/E)^2}} \\
&= \frac{1}{\pi E}\int_{-E}^{E}[f_+(\nu)+f_-(\nu)]\left(\frac{\nu}{E}\right)\frac{d\nu}{E\sqrt{1-(\nu/E)^2}}.
\end{aligned}$$

The above formula is valid for any two-valued non-linearity. In fact, it is also valid for a single-valued non-linearity f. If $f = (f_+ + f_-)/2$ is an odd function then

$$N(E) = \frac{1}{\pi E}\int_{-E}^{E} 2f(\nu)\left(\frac{\nu}{E}\right)\frac{d\nu}{E\sqrt{1-(\nu/E)^2}}.$$

Thus, $\mathrm{Re}(N(E))$ is the same as the describing function of the 'average' non-linearity $(f_+ + f_-)/2$.

For Im$(N(E))$ there is a striking formula:

$$\begin{aligned}
\mathrm{Im}N(E) &= \frac{1}{\pi E}\int_{-E}^{E} f_+(\nu)\frac{d\nu}{E} + \frac{1}{\pi E}\int_{E}^{-E} f_-(\nu)\frac{d\nu}{E} \\
&= \frac{1}{\pi E^2}\int_{-E}^{E}[f_+(\nu) - f_-(\nu)]d\nu \\
&= -\frac{\Delta}{\pi E^2}
\end{aligned}$$

where Δ is the area enclosed by the 'loop' of the non-linear characteristic.

For the relay with hysteresis given above we obtain

$$\mathrm{Re}N(E) = \frac{4R}{\pi E}\int_{\delta}^{E} \frac{\nu\, d\nu}{E^2\sqrt{1-(\nu/E)^2}} = \frac{4R}{\pi E}\sqrt{1-(\frac{\delta}{E})^2},$$

which gives

$$N(E) = \frac{4R}{\pi E}\left(\sqrt{1-(\frac{\delta}{E})^2} - \frac{j\delta}{E}\right).$$

3 Estimates of Describing Functions

Often some idea of the describing function can be obtained without calculating it in detail. The describing function is sometimes called the *equivalent linear gain*, and this name gives a pointer as to how it can be estimated. This is best illustrated by a couple of examples.

Example: Piecewise-linear nonlinearity. In section 2.3 we considered a nonlinearity with an incremental gain of m_1 for small signals, and m_2 for large signals. For a small signal the gain is exactly m_1, because the nonlinearity does not come into play. For extremely large signals the gain is essentially m_2. Clearly the effective gain for any input signal lies between these two. Thus the describing function (being real in this case) must always lie between m_1 and m_2.

Example: Relay with dead-zone. Consider the following behaviour:

$$f(e) = \begin{cases} -1, & \text{if } e \le -\delta \\ 0, & \text{if } |e| < \delta \\ +1, & \text{if } e \ge \delta \end{cases}$$

For $|e| < \delta$ the describing function is clearly 0. As $|e|$ increases slightly above δ the output suddenly jumps to amplitude 1, so the effective gain increases quickly, and hence the describing function increases quickly (to a value which has to be calculated — see Examples Paper). As $|e|$ increases further, the output amplitude does not increase, and so the effective gain — and hence the describing function — decreases gradually to 0.

4 Rigour

The first part of the describing function method, namely the prediction of limit cycles, can be made rigorous using appropriate bounds on $g(j\omega)$ and possibly by bringing in some higher harmonics. A rather mathematical discussion along these lines and sufficient conditions for validity can be found in A.I. Mees, *Dynamics of Feedback Systems*, Wiley, 1981, Chapter 5. (This is beyond the scope of this module).

The prediction of stability/instability has so far proved difficult to treat rigorously and provide useful sufficient conditions for validity. It therefore remains as a quick method which gives a hint as to what might happen.

M.C. Smith
Revised by J.M. Maciejowski February 2011