Handout 2: Invariant Sets and Stability

1 Invariant Sets

Consider again the autonomous dynamical system

$$\dot{x} = f(x), \ x(0) = x_0 \tag{1}$$

with state $x \in \mathbb{R}^n$. We assume that f is Lipschitz continuous and denote the unique trajectory of (1) by $x(\cdot)$.

Definition 1 (Invariant set) *A set of states $S \subseteq \mathbb{R}^n$ of (1) is called an invariant set of (1) if for all $x_0 \in S$ and for all $t \geq 0$, $x(t) \in S$.*

Equilibria are an important special class of invariant sets.

Definition 2 (Equilibrium) *A state $\hat{x} \in \mathbb{R}^n$ is called an equilibrium of (1) if $f(\hat{x}) = 0$.*

Recall that the pendulum example treated in Handout 1 had two equilibria,

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

(Strictly speaking the pendulum has an infinite number of equilibria since all multiples of π would work for the first entry. However, all equilibria can be identified with one of the above two, by adding or subtracting an integer multiple of 2π).

It is often very convenient to "shift" an equilibrium $\hat{x}$ to the origin before analysing the behaviour of the systems near it. This involves a change of coordinates of the form

$$w = x - \hat{x}$$

In the w coordinates the dynamics of the system are

$$\dot{w} = \dot{x} = f(x) = f(w + \hat{x}) = \hat{f}(w)$$

where we have defined $\hat{f}(w) = f(w + \hat{x})$. Since

$$\hat{f}(0) = f(0 + \hat{x}) = f(\hat{x}) = 0$$

the system in the new coordinates w has an equilibrium at $\hat{w} = 0$.

Limit cycles are another important class of invariant sets that may be observed in systems of dimension 2 or higher. Roughly speaking, a limit cycle is a closed, non-trivial

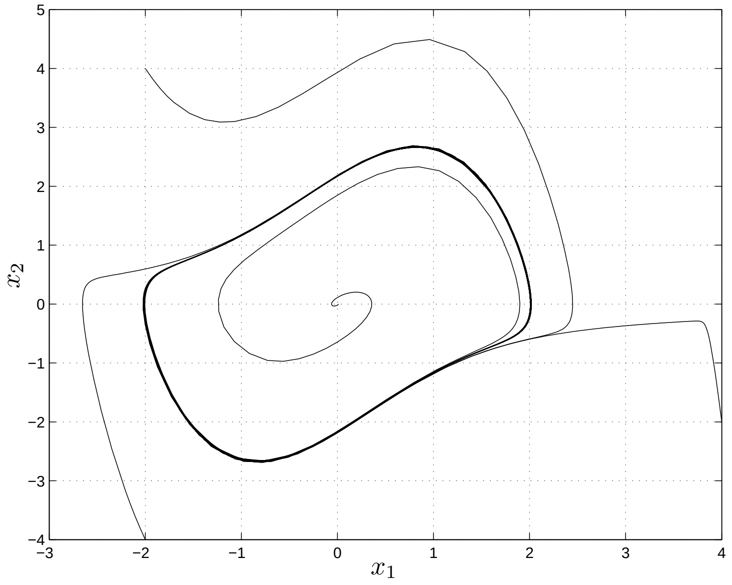

Figure 1: Simulation of the Van der Pol Oscillator

periodic trajectory, i.e. a trajectory that repeats itself after a finite (non-zero) amount of time.

Example (Van der Pol oscillator) The Van der Pol oscillator is given by the second order differential equation

$$\ddot{\theta} - \epsilon(1 - \theta^2)\dot{\theta} + \theta = 0$$

This differential equation was used by Van der Pol to study the dynamics of thermionic valve circuits. In particular, he tried to explain why nonlinear resistance phenomena occasionally cause such circuits to oscillate.

Exercise 1 Write the Van der Pol oscillator in state space form. Hence determine its equilibria.

Simulated trajectories of the Van der Pol oscillator for $\epsilon = 1$ are shown in Figure 1. Notice that there appears to be a closed, periodic trajectory (i.e. a limit cycle) in the middle of the figure, and that all neighbouring trajectories seem to converge to it. ■

Exercise 2 Are limit cycles possible in systems of dimension 1?

In higher dimensions, even more exotic types of invariant sets can be found. Examples are the **invariant torus** and the **chaotic attractor**. The easiest way to think of an invariant torus is as a two dimensional object that has the shape of the surface of a ring embedded in three dimensional space. It is possible to find dynamical systems such that all trajectories of the system that start on the torus wind around it for ever, without stopping or becoming periodic.

The following example demonstrates a chaotic attractor.

Example (Lorenz Equations) The Lorenz equations arose in the study of atmospheric phenomena, to capture complicated patterns in the weather and climate. More recently

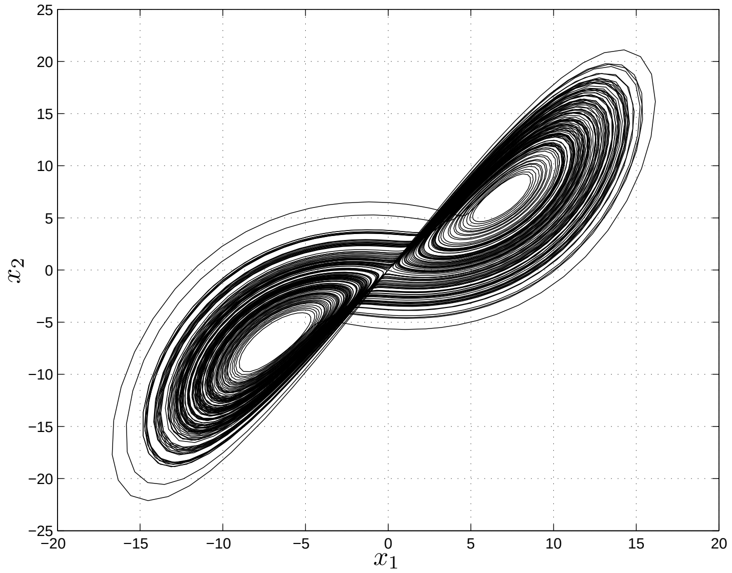

Figure 2: Simulation of the Lorenz equations

they have also found applications in models of irregular behaviour in lasers. The Lorenz equations form a three dimensional system

$$\begin{aligned}\dot{x}_1 &= a(x_2 - x_1)\\ \dot{x}_2 &= (1+b)x_1 - x_2 - x_1x_3\\ \dot{x}_3 &= x_1x_2 - cx_3\end{aligned}$$

Exercise 3 What are the equilibria of the Lorenz equations?

A simulated trajectory of the system for $a = 10$, $b = 24$, $c = 2$ and $x_0 = (-5, -6, 20)$ is shown in Figures 2 and 3. The system turns out to be chaotic. Roughly speaking, trajectories of the system that start in a particular bounded region of the state space (known as a **chaotic** or **strange attractor**) will for ever remain in that region, without exhibiting any apparent regularity or pattern (e.g. without converging to an equilibrium or becoming periodic).

This complex, irregular behaviour of the Lorenz equations is in part responsible for what is known as the "butterfly effect" in the popular science literature: the beating of the wings of a butterfly in China may cause variations in the global climate severe enough to induce tropical storms in the Caribbean. ■

Such exotic behaviour is not possible in two dimensions, as the following theorem indicates.

Theorem 1 (Poincaré-Bendixson) *Consider a two dimensional dynamical system and let $S \subseteq \mathbb{R}^2$ be a compact (i.e. bounded and closed) invariant set. If S contains no equilibrium points then all trajectories starting in S are either limit cycles themselves, or tend towards a limit cycle as $t \to \infty$.*

Besides eliminating exotic invariant sets (such as chaotic attractors) in two dimensions, this observation has some other interesting consequences.

Corollary 1 *Consider a two dimensional dynamical system.*

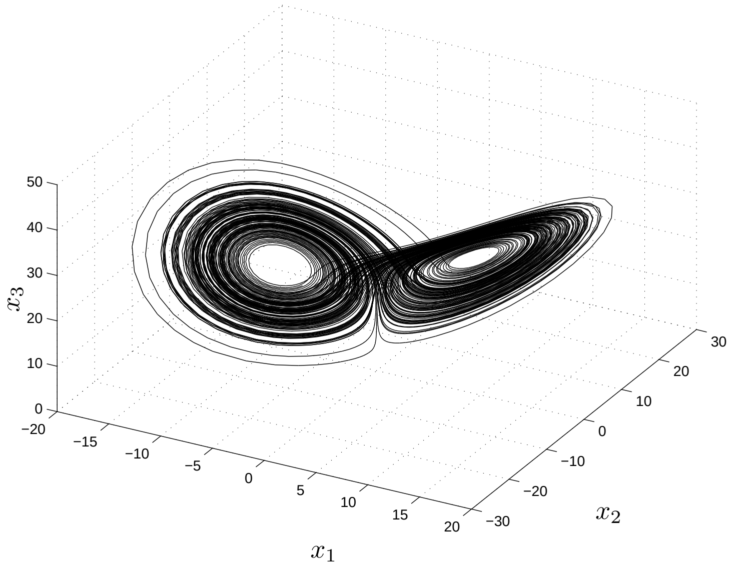

Figure 3: The Lorenz simulation in 3 dimensions

1. *A compact region $S \subseteq \mathbb{R}^2$ that contains no equilibria and on whose boundary the vector field points towards the interior of S must contain a stable (see below) limit cycle.*

2. *The region encircled by a limit cycle (in $\mathbb{R}^2$) contains either another limit cycle or an equilibrium.*

2 Stability

Stability is the most commonly studied property of invariant sets. Roughly speaking, an invariant set is called stable if trajectories starting close to it remain close to it, and unstable if they do not. An invariant set is called asymptotically stable if it is stable and in addition trajectories starting close to it converge to it as $t \to \infty$. We state these definitions formally for equilibria; similar definitions can be derived for more general types of invariant sets.

Definition 3 (Stable equilibrium) *An equilibrium, $\hat{x}$, of* (1) *is called stable if for all $\epsilon > 0$ there exists $\delta > 0$ such that $\|x_0 - \hat{x}\| < \delta$ implies that $\|x(t) - \hat{x}\| < \epsilon$ for all $t \geq 0$. Otherwise the equilibrium is said to be unstable.*

Figure 4 shows trajectories of the pendulum starting close to the two equilibria, when there is no dissipation ($d = 0$). As physical intuition suggests, trajectories starting close to $(0, 0)$ remain close to it, while trajectories starting close to $(\pi, 0)$ quickly move away from it. This suggests that $(0, 0)$ is a stable equilibrium, while $(\pi, 0)$ is unstable.

Exercise 4 How would you modify this definition to define the stability of a more general invariant set $S \subseteq \mathbb{R}^n$ (e.g. a limit cycle)? You will probably need to introduce some notion of "distance to the set S". What is an appropriate notion of "distance" between a point and a set in this context?

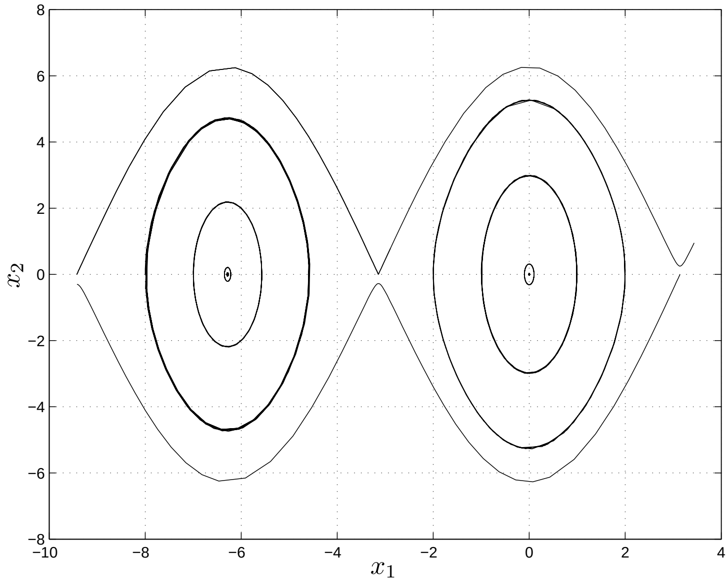

Figure 4: Pendulum motion for $d = 0$

Definition 4 (Asymptotically stable equilibrium) *An equilibrium, $\hat{x}$, of* (1) *is called locally asymptotically stable if it is stable and there exists $M > 0$ such that $\|x_0 - \hat{x}\| < M$ implies that $\lim_{t\to\infty} x(t) = \hat{x}$. The equilibrium is called globally asymptotically stable if this holds for all $M > 0$.*

Notice that an equilibrium can be called asymptotically stable only if it is stable. The **domain of attraction** of an asymptotically stable equilibrium is the set of all x_0 for which $x(t) \to \hat{x}$. By definition, the domain of attraction of a globally asymptotically stable equilibrium is the entire state space $\mathbb{R}^n$.

3 Linearisation and Lyapunov's Indirect Method

Useful information about the stability of an equilibrium, $\hat{x}$, can be deduced by studying the linearisation of the system about the equilibrium. Consider the Taylor series expansion of the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ about $x = \hat{x}$.

$$\begin{aligned} f(x) &= f(\hat{x}) + A(x - \hat{x}) + \text{ higher order terms in } (x - \hat{x}) \\ &= A(x - \hat{x}) + \text{ higher order terms in } (x - \hat{x}) \end{aligned}$$

where

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\hat{x}) & \dots & \frac{\partial f_1}{\partial x_n}(\hat{x}) \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\hat{x}) & \dots & \frac{\partial f_n}{\partial x_n}(\hat{x}) \end{bmatrix}$$

Setting $\delta x = x - \hat{x}$ and differentiating leads to

$$\dot{\delta x} = A\delta x + \text{ higher order terms in } \delta x$$

Since close to the equilibrium the linear terms will dominate the higher order ones, one would expect that close to the equilibrium the behaviour of the nonlinear system will be similar to that of the **linearisation**

$$\dot{\delta x} = A\delta x$$

This intuition turns out to be correct.

Theorem 2 (Lyapunov's indirect method) *The equilibrium $\hat{x}$ is*

1. *locally asymptotically stable if the equilibrium $\hat{\delta x} = 0$ of the linearisation is asymptotically stable;*
2. *unstable if $\hat{\delta x} = 0$ is unstable.*

Theorem 2 is useful, because the stability of linear systems is very easy to determine by computing the eigenvalues of the matrix A.

Theorem 3 (Linear Stability) *The equilibrium $\hat{\delta x} = 0$ of the linear system $\dot{\delta x} = A\delta x$ is*

1. *asymptotically stable if and only if all eigenvalues of A have negative real parts;*
2. *stable if all eigenvalues of A have non-positive real parts, and any eigenvalues on the imaginary axis are distinct;*
3. *unstable if there exists an eigenvalue with positive real part.*

Examples of typical behaviour of two dimensional linear systems around the equilibrium $\delta x = 0$ are shown in Figure 5.

Exercise 5 How many equilibria can linear systems have? Can you think of other behaviours that linear systems may exhibit near their equilibria that are not included in the figures?

Example (Pendulum Continued) Recall that the linearisation of the pendulum about the equilibrium $(0, 0)$ is

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{d}{m} \end{bmatrix} \delta x$$

(see Handout 1). We can compute the eigenvalues of the matrix by solving the **characteristic polynomial**

$$det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ \frac{g}{l} & \lambda + \frac{d}{m} \end{vmatrix} = \lambda^2 + \frac{d}{m}\lambda + \frac{g}{l} = 0$$

It is easy to verify that, if $d > 0$, then the real part of the roots of this polynomial are negative, hence the equilibrium $(0, 0)$ is locally asymptotically stable.

The linearisation of the pendulum about the equilibrium $(\pi, 0)$ on the other hand is

$$\dot{\delta x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{d}{m} \end{bmatrix} \delta x$$

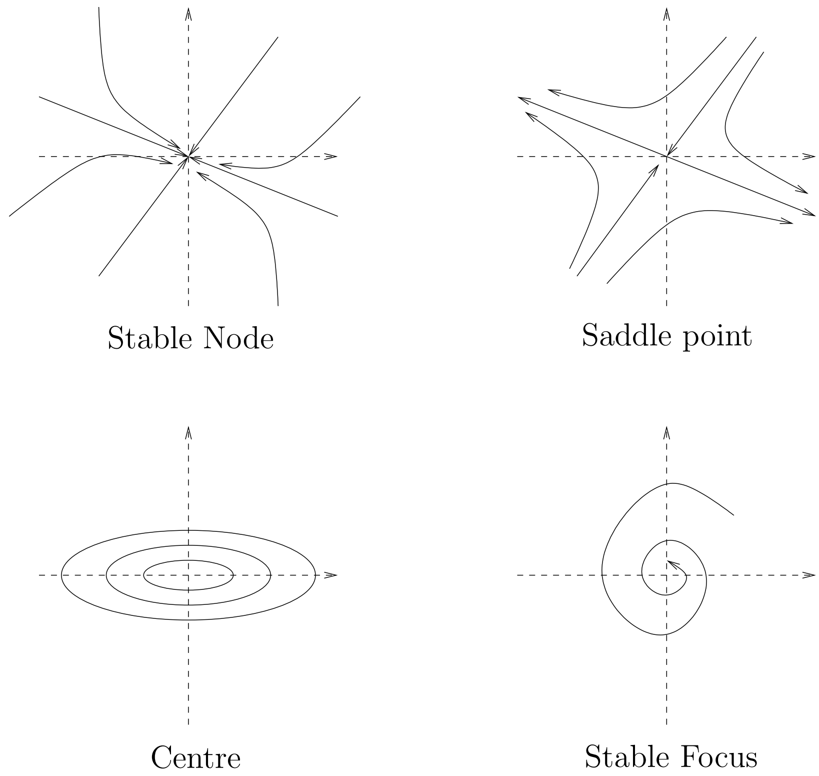

Figure 5: Typical behaviours of linear systems

The characteristic polynomial in this case is

$$\lambda^2 + \frac{d}{m}\lambda - \frac{g}{l} = 0$$

This polynomial has two real roots, one positive and one negative for all $d \geq 0$. Therefore, as expected, the equilibrium $(\pi, 0)$ is unstable. ∎

Notice that if the linearisation is only stable (e.g. has a pair of imaginary eigenvalues) then Theorem 2 is inconclusive. This is the case, for example, for the equilibrium $(0, 0)$ of the pendulum when $d = 0$. In this case the equilibrium turns out to be stable, but this is not necessarily true in general.

Example (Linearization can be inconclusive) Consider the following one dimensional dynamical systems:

$$\dot{x} = x^3 \text{ and } \dot{x} = -x^3$$

Both these systems have a unique equilibrium, at $\hat{x} = 0$. It is easy to see that $\hat{x} = 0$ is an unstable equilibrium for the first system and a stable equilibrium for the second one (see Exercise 10 below). However, both systems have the same linearisation about $\hat{x} = 0$, namely

$$\dot{\delta x} = 0$$

Notice that the linearisation is stable, but not asymptotically stable. This should come as no surprise, since Theorem 2 is expected to be inconclusive in this case. ∎

Linearisation is very useful for studying the stability of equilibria (and all sorts of other interesting local properties of nonlinear dynamical systems). However, there are some important questions that it can not answer. For example:

- It is inconclusive when the linearisation is stable but not asymptotically stable.
- It provides no information about the domain of attraction.

More powerful stability results can be obtained using Lyapunov's direct method.

4 Lyapunov Functions and the Direct Method

What makes the equilibrium $(0,0)$ of the pendulum asymptotically stable? The physical intuition is that (if $d > 0$) the system dissipates energy. Therefore, it will tend to the lowest energy state available to it, which happens to be the equilibrium $(0,0)$. With mechanical and electrical systems it is often possible to exploit such intuition about the energy of the system to determine the stability of different equilibria. Stability analysis using Lyapunov functions is a generalisation of this intuition. This method of analysing the stability of equilibria is known as Lyapunov's direct method.

Consider a dynamical system

$$\dot{x} = f(x)$$

that has an equilibrium at $\hat{x}$. Without loss of generality (as discussed earlier) we will assume that $\hat{x} = 0$, to simplify the notation.

Theorem 4 (Lyapunov Stability) *Assume there exists a differentiable function $V : S \to \mathbb{R}$ defined on some open region $S \subset \mathbb{R}^n$ containing the origin, such that*

1. $V(0) = 0$.
2. $V(x) > 0$ *for all* $x \in S$ *with* $x \neq 0$.
3. $\dot{V}(x) \leq 0$ *for all* $x \in S$.

Then $\hat{x} = 0$ is a stable equilibrium of $\dot{x} = f(x)$.

Here $\dot{V}$ denotes the derivative of V along the trajectories of the dynamical system, i.e.

$$\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x)\, \dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x)\, f_i(x) = \nabla V(x) \cdot f(x)$$

This is also called the **Lie derivative** of the function V along the vector field f. A function satisfying the conditions of Theorem 4 is called a **Lyapunov function**.

Exercise 6 Using the energy as a Lyapunov function verify that $(0,0)$ is a stable equilibrium for the pendulum even if $d = 0$. Recall that the linearisation is inconclusive in this case.

The idea of the proof of Theorem 4 is shown in Figure 6. Since V is differentiable, it has a local minimum at 0. Moreover, V can not increase along the trajectories of the system. Therefore, if the system finds itself inside a **level set** of V, say

$$\{x \in \mathbb{R}^n \mid V(x) \leq c\}$$

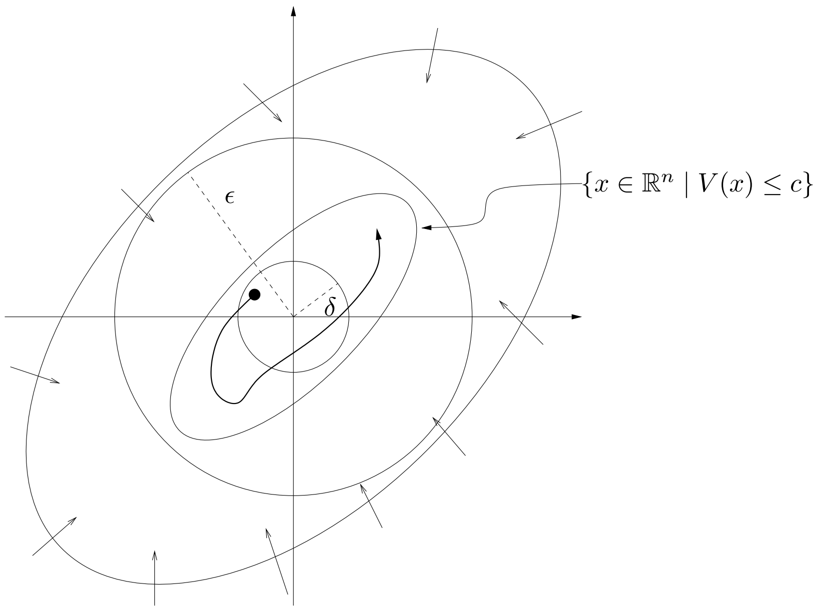

Figure 6: Proof of Theorem 4 (by picture)

(the set of all $x \in \mathbb{R}^n$ such that $V(x) \leq c$) it must remain inside for ever. If we want to remain ϵ-close to 0 (i.e. we want $\|x(t)\| < \epsilon$ for all $t \geq 0$) we can chose c such that

$$V(x) \leq c \Rightarrow \|x\| < \epsilon.$$

Then choose $\delta > 0$ such that

$$\|x\| < \delta \Rightarrow V(x) \leq c$$

(ensuring that such c and δ exist can be quite tricky, but it is possible under the conditions of the theorem). Then all trajectories starting within δ of 0 will remain in the set $\{x \in \mathbb{R}^n \mid V(x) \leq c\}$ and therefore will remain within ϵ of 0. This proves the stability of 0.

Theorem 5 (Lyapunov Asymptotic Stability) *Assume there exists a differentiable function $V : S \to \mathbb{R}$ defined on some open region $S \subset \mathbb{R}^n$ containing the origin, such that*

1. $V(0) = 0$.
2. $V(x) > 0$ *for all* $x \in S$ *with* $x \neq 0$.
3. $\dot{V}(x) < 0$ *for all* $x \in S$ *with* $x \neq 0$.

Then $\hat{x} = 0$ *is a locally asymptotically stable equilibrium of* $\dot{x} = f(x)$.

Exercise 7 Can we use the energy as a Lyapunov function to show that $(0, 0)$ is a locally asymptotically stable equilibrium for the pendulum?

The last condition can be difficult to meet. The following version often turns out to be more useful in practice.

Theorem 6 (LaSalle's Theorem) *Let $S \subseteq \mathbb{R}^n$ be a compact (i.e. closed and bounded) invariant set. Assume there exists a differentiable function $V : S \to \mathbb{R}$ such that*

$$\dot{V}(x) \leq 0 \; \forall x \in S$$

Let M be the largest invariant set contained in $\{x \in S \mid \dot{V}(x) = 0\}$ (the set of $x \in S$ for which $\dot{V}(x) = 0$). Then all trajectories starting in S approach M as $t \to \infty$.

Corollary 2 (LaSalle for Equilibria) *Let $S \subseteq \mathbb{R}^n$ be a compact (i.e. closed and bounded) invariant set. Assume there exists a differentiable function $V : S \to \mathbb{R}$ such that*

$$\dot{V}(x) \leq 0 \; \forall x \in S$$

If the set $\{x \in S \mid \dot{V}(x) = 0\}$ contains no trajectories other than $x(t) = 0$, then 0 is locally asymptotically stable. Moreover, all trajectories starting in S converge to 0.

Exercise 8 Can we use the energy and LaSalle's theorem to show that $(0, 0)$ is a locally asymptotically stable equilibrium for the pendulum?

A couple of remarks about the use of these theorems:

1. Theorems 5 and 6 can also be used to estimate the domains of attraction of equilibria. Since the system can never leave sets of the form $\{x \in \mathbb{R}^n \mid V(x) \leq c\}$ any such set contained in S has to be a part of the domain of attraction of the equilibrium.
2. The conditions of the theorems are sufficient but not necessary. If we can find a Lyapunov function that meets the conditions of the theorems we know that the equilibrium has to be stable/asymptotically stable. If we can not find such a function we can not draw any immediate conclusion.

Finding Lyapunov functions is more of an art than an exact science. For physical systems, our best bet is often to use our intuition to try to guess Lyapunov functions related to the energy of the system.

Another popular choice are quadratic functions. The simplest choice (that occasionally works) is

$$V(x) = \|x\|^2$$

Exercise 9 Use this Lyapunov function to show that the equilibrium $x = 0$ of $\dot{x} = -x^3$ is globally asymptotically stable.

More generally we can try

$$V(x) = x^T P x$$

where P is a symmetric (i.e. $P^T = P$), positive definite matrix. Recall that a matrix is called positive definite (denoted by $P > 0$) if $x^T P x > 0$ for all $x \neq 0$. (All the eigenvalues of such a P are positive.) A matrix is called positive semidefinite (denoted by $P \geq 0$) if $x^T P x \geq 0$ for all $x \neq 0$. (All the eigenvalues of such a P are non-negative.)

Quadratic Lyapunov functions always work for linear systems. Consider

$$\dot{x} = Ax$$

and the candidate Lyapunov function

$$V(x) = x^T P x$$

for some $P = P^T > 0$. Differentiating leads to

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= (Ax)^T P x + x^T P (Ax) \\ &= x^T (A^T P + PA) x \end{aligned}$$

Therefore, if we can find a $P = P^T > 0$ such that $x^T(A^T P + PA)x < 0$ for all x we are in business! This is possible, for example, if for some $Q = Q^T > 0$ the Lyapunov matrix equation

$$A^T P + PA = -Q \tag{2}$$

has a symmetric, positive definite solution $P = P^T > 0$.

Theorem 7 (Linear Lyapunov Stability) *For any matrix A the following statements are equivalent*

1. *All eigenvalues of A have negative real parts.*
2. *We can find $P = P^T > 0$ that solves* (2) *for some $Q = Q^T > 0$.*
3. *We can find $P = P^T > 0$ that solves* (2) *for any $Q = Q^T > 0$.*

Engineering systems often contain memoryless nonlinearities, $h(\cdot)$ such that $h(0) = 0$ and $zh(z) > 0$ for $z \neq 0$ (for an example see Figure 8). If the input to such a nonlinearity is a state variable, say x_i, then a positive function that is often useful is

$$V(x) = \int_0^{x_i} h(z) dz$$

In this case

$$\frac{\partial V}{\partial x_i}(x) = h(x_i)$$

Finally, a trick that often helps is to add different candidate functions. If your quadratic $x^T P x$ does not work and your system happens to have suitable memoryless nonlinearities, why not try

$$V(x) = x^T P x + \sum_{i=1}^{n} \int_0^{x_i} h_i(z) dz$$

5 The Hopfield Neural Network

As an application of LaSalle's theorem, we show how it can be used to analyse the stability of a neural network. Artificial neural networks, like their biological counterparts, take advantage of distributed information processing and parallelism to efficiently perform certain tasks. Artificial neural networks have found applications in pattern recognition, learning and generalisation, control of dynamical processes, etc.

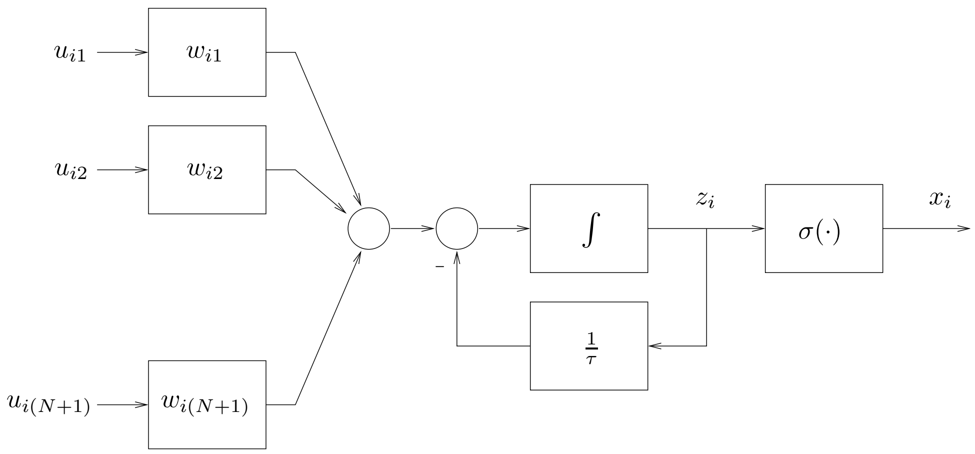

Figure 7: Model of a neuron

A neural network is an interconnection of a number of basic building blocks known as **neurons**. In the Hopfield neural network, each neuron is a simple circuit consisting of a capacitor, a resistor and two operational amplifiers. The dynamics of each neuron are governed by the simple system shown in Figure 7. $\sigma(\cdot) : \mathbb{R} \to \mathbb{R}$ is a nonlinear saturation function known as a **sigmoid**. A typical choice (Figure 8) is

$$\sigma(z) = \frac{e^{\lambda z} - e^{-\lambda z}}{e^{\lambda z} + e^{-\lambda z}} = \tanh(\lambda z)$$

Notice that the shape of σ restricts x_i to the range $(-1, 1)$. Moreover, σ is differentiable and **monotone increasing**, i.e. $\frac{d\sigma}{dz} > 0$. Finally, σ is **invertible**, i.e. for all $x_i \in (-1, 1)$ there exists a unique $z_i \in \mathbb{R}$ such that $\sigma(z_i) = x_i$. We denote this unique z_i by $z_i = \sigma^{-1}(x_i)$.

Since $x_i = \sigma(z_i)$,

$$\dot{x}_i = \frac{d\sigma}{dz}(z_i)\dot{z}_i = \frac{d\sigma}{dz}(z_i)\left(-\frac{1}{\tau}z_i + \sum_{i=1}^{N+1} w_{ik}u_{ik}\right) = h(x_i)\left(-\frac{1}{\tau}\sigma^{-1}(x_i) + \sum_{i=1}^{N+1} w_{ik}u_{ik}\right)$$

where we have defined

$$h(x_i) = \frac{d\sigma}{dz}(\sigma^{-1}(x_i)) > 0$$

Consider now a network of N neurons connected so that for all i

- $u_{ik} = x_k$, $1 \leq k \leq N$ (i.e. the k^{th} input to neuron i is the output of neuron k. The i'th input to neuron i is fed back from its own output.)
- $u_{i(N+1)} = u_i$ (a constant external input to the neuron, e.g. a current injected to the circuit)

Assume that

- $w_{ik} = w_{ki}$, $1 \leq i, k \leq N$ (i.e. the connection from neuron k to neuron i is as "strong" as the connection from neuron i to neuron k)

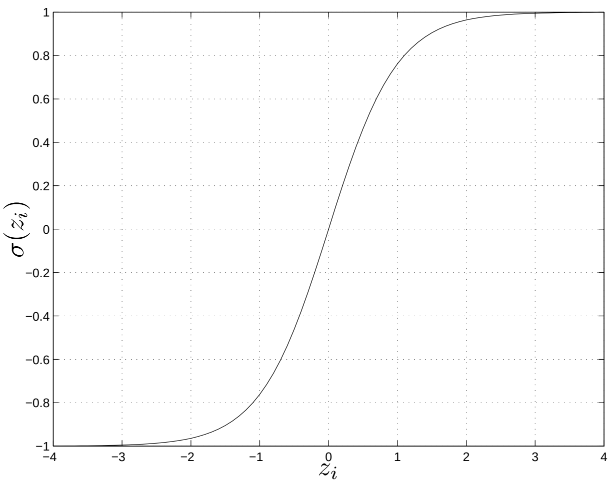

Figure 8: Sigmoid functions

- $w_{i(N+1)} = 1$ (no weight associated with external input)

The entire network can be modelled as an N dimensional system with states x_i, $i = 1, \ldots N$. The evolution of each state is governed by

$$\dot{x}_i = h(x_i)\left(-\frac{1}{\tau}\sigma^{-1}(x_i) + \sum_{i=1}^{N} w_{ik}x_k + u_i\right)$$

The network has a number of equilibria, depending on the choice of the weights w_{ik} and the external inputs u_i. Will the trajectories of this system converge to one of them or will they keep oscillating? Consider the candidate Lyapunov function

$$V(x) = -\frac{1}{2}x^T W x + \frac{1}{\tau}\sum_{i=1}^{N}\int_0^{x_i}\sigma^{-1}(s)ds - \sum_{i=1}^{N} x_i u_i$$

where W is the matrix with entries w_{ik}. The fact that $w_{ik} = w_{ki}$ implies that $W = W^T$. Notice that

$$\dot{x}_i = -h(x_i)\frac{\partial V}{\partial x_i}(x)$$

Therefore,

$$\dot{V}(x) = \sum_{i=1}^{N}\frac{\partial V}{\partial x_i}(x)\dot{x}_i = -\sum_{i=1}^{N} h(x_i)\left(\frac{\partial V}{\partial x_i}(x)\right)^2 \leq 0$$

since $h(x_i) > 0$. Moreover,

$$\dot{V}(x) = 0 \Rightarrow \frac{\partial V}{\partial x_i}(x) = 0\,\forall i \Rightarrow \dot{x}_i = 0\forall i$$

Therefore, the only points where $\dot{V}(x) = 0$ are the equilibria of the system.

Assume we could find a compact invariant region S on which this calculation is valid. Then, using Theorem 6 we would be able to deduce that our neural network will converge to one of its equilibria. A (somewhat tedious but otherwise straightforward) argument shows that in fact for some $\epsilon > 0$ small enough the set

$$S = \{x \in \mathbb{R}^n \mid |x_i| \leq 1 - \epsilon\}$$

is indeed invariant.

Comment: The set S is also compact, since

- It is bounded, because all components x_i have both an upper bound $1 - \epsilon$ and a lower bound $-1 + \epsilon$
- It contains its boundary because $x_i = \pm(1 - \epsilon)$ belongs to S.)

However, for the purposes of this course you do not need to worry about the subtleties of *compact* sets, *open* sets, *closed* sets, etc.

J.M. Maciejowski February 2011