Handout 1: Dynamical Systems

Website: Handouts, copies of slides, and Examples papers for this course are available on my web page: `http://www-control.eng.cam.ac.uk/Homepage/officialweb.php?id=1`

1 Notation & Background

- $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. This is a **vector space** (also known as a linear space).
- If $n = 1$, we will drop the subscript and write just $\mathbb{R}$ (the set of real numbers or "the real line").
- If a and b are real numbers with $a \leq b$, $[a, b]$ will denote the **closed interval** from a to b, i.e. the set of all real numbers x such that $a \leq x \leq b$. $[a, b)$ will denote the right-open interval from a to b (i.e. $a \leq x < b$), etc.
- I will make no distinction between vectors and real numbers in the notation (no arrows over the letters, bold font, etc.). Both vectors and real numbers will be denoted by lower case letters.
- $\|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$ denotes the standard (Euclidean) **norm** in $\mathbb{R}^n$.
- $\mathbb{Z}$ denotes the set of integers, $\ldots, -2, -1, 0, 1, 2, \ldots$.
- $x \in A$ is a shorthand for "x belongs to a set A", e.g. $x \in \mathbb{R}^n$ means that x is an n-dimensional vector.
- $f(\cdot) : A \to B$ is a shorthand for a function mapping every element $x \in A$ to an element $f(x) \in B$. For example the function $\sin(\cdot) : \mathbb{R} \to \mathbb{R}$ maps a real number x to its sine, $\sin(x)$.
- $\forall$ is a shorthand for "for all", as in "$\forall x \in \mathbb{R}, x^2 \geq 0$".
- $\exists$ is a shorthand for "there exists", as in "$\exists x \in \mathbb{R}$ such that $\sin(x) = 0$".

I will assume that people are familiar with the concepts of vector space, state space etc. from 3F2. Lecture notes will be self contained but, by necessity, terse. I recommend that students regularly consult the references — see end of Handout.

2 Dynamical System Classification

Roughly speaking, a **dynamical system** describes the **evolution** of a **state** over **time**. To make this notion more precise we need to specify what we mean by the terms "evolution", "state" and "time".

Certain dynamical systems can also be influenced by external inputs, which may represent either uncontrollable disturbances (e.g. wind affecting the motion of an aircraft) or control signals (e.g. the commands of the pilot to the aircraft control surfaces and engines). Some dynamical systems may also have outputs, which may represent either quantities that can be measured, or quantities that need to be regulated.

Based on the type of their state, dynamical systems can be classified into:

1. **Continuous**, if the state takes values in Euclidean space $\mathbb{R}^n$ for some $n \geq 1$. We will use $x \in \mathbb{R}^n$ to denote the state of a continuous dynamical system. Most of the dynamical systems you are familiar with (e.g. from 3F2) are continuous dynamical systems.

2. **Discrete**, if the state takes values in a countable or finite set $\{q_1, q_2, \ldots\}$. We will use q to denote the state of a discrete system. For example, a light switch is a dynamical system whose state takes on two values, $q \in \{ON, OFF\}$. A computer is also a dynamical system whose state takes on a finite (albeit very large) number of values.

3. **Hybrid**, if part of the state takes values in $\mathbb{R}^n$ while another part takes values in a finite set. For example, the closed loop system we obtain when we use a computer to control an inverted pendulum is hybrid: part of the state (the state of the pendulum) is continuous, while another part (the state of the computer) is discrete.

Based on the set of times over which the state evolves, dynamical systems can be classified as:

1. **Continuous time**, if the set of times is a subset of the real line. We will use $t \in \mathbb{R}$ to denote continuous time. Typically, the evolution of the state of a continuous time system is described by an **ordinary differential equation** (ODE). Think of the linear, continuous time system in state space form

$$\dot{x} = Ax$$

   treated extensively in module 3F2 last year.

2. **Discrete time**, if the set of times is a subset of the integers. We will use $k \in \mathbb{Z}$ to denote discrete time. Typically, the evolution of the state of a discrete time system is described by a **difference equation**. Think of the linear discrete time system in state space form

$$x_{k+1} = Ax_k$$

   or of discrete-time filters etc in module 3F1.

3. **Hybrid time**, when the evolution is over continuous time but there are also discrete "instants" where something "special" happens.

Continuous state systems can be further classified according to the equations used to describe the evolution of their state

1. **Linear**, if the evolution is governed by a linear differential equation (continuous time) or difference equation (discrete time).
2. **Nonlinear**, if the evolution is governed by a nonlinear differential equation (continuous time) or difference equation (discrete time).

In this course we will primarily discuss the following classes of systems:

1. Nonlinear (continuous state), continuous time systems.
2. Nonlinear, discrete time systems.
3. Discrete state, discrete time systems (examples only).
4. Hybrid state, hybrid time systems (examples only).

Classes of systems that will not be treated at all:

- Infinite dimensional continuous state systems described, for example, by partial differential equations (PDE).
- Discrete state systems with an infinite number of states, e.g. Petri nets, push down automata, Turing machines.
- Stochastic systems, i.e. systems with probabilistic dynamics.

3 Examples

3.1 The Pendulum: A Nonlinear, Continuous Time System

Consider a pendulum hanging from a weight-less solid rod and moving under gravity (Figure 1). Let θ denote the angle the pendulum makes with the downward vertical, l the length of the pendulum, m its mass, and d the dissipation constant. The evolution of θ is governed by

$$ml\ddot{\theta} + dl\dot{\theta} + mg\sin(\theta) = 0$$

This is a nonlinear, second order, ordinary differential equation (ODE).

Exercise 1 Derive this equation from Newton's laws. Why is this ODE called nonlinear?

To determine how the pendulum is going to move, i.e. determine θ as a function of time, we would like to find a solution to this ODE . Assuming that at time $t = 0$ the pendulum starts at some initial position θ_0 and with some initial velocity $\dot{\theta}_0$, "solving the ODE" means finding a function of time

$$\theta(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$$

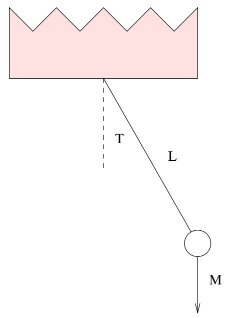

Figure 1: The pendulum

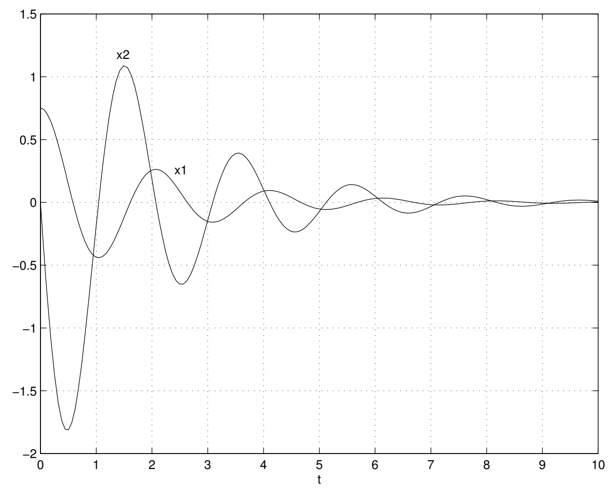

Figure 2: Trajectory of the pendulum.

such that

$$\theta(0) = \theta_0$$
$$\dot{\theta}(0) = \dot{\theta}_0$$
$$ml\ddot{\theta}(t) + dl\dot{\theta}(t) + mg\sin(\theta(t)) = 0, \ \forall t \in \mathbb{R}$$

Such a function is known as a **trajectory** (or **solution**) of the system. At this stage it is unclear if one, none or multiple trajectories exist for this initial condition. **Existence** and **uniqueness** of trajectories are both desirable properties for ODEs that are used to model physical systems.

For nonlinear systems, even if a unique trajectory exists for the given initial condition, it is usually difficult to construct it explicitly. Frequently solutions of an ODE can only be approximated by **simulation**. Figure 2 shows a simulated trajectory of the pendulum for $l = 1$, $m = 1$, $d = 1$, $g = 9.8$, $\theta(0) = 0.75$ and $\dot{\theta}(0) = 0$.

To simplify the notation we typically write dynamical system ODEs in **state space** form

$$\dot{x} = f(x)$$

where x is now a vector in $\mathbb{R}^n$ for some appropriate $n \geq 1$. The easiest way to do this for the pendulum is to set

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

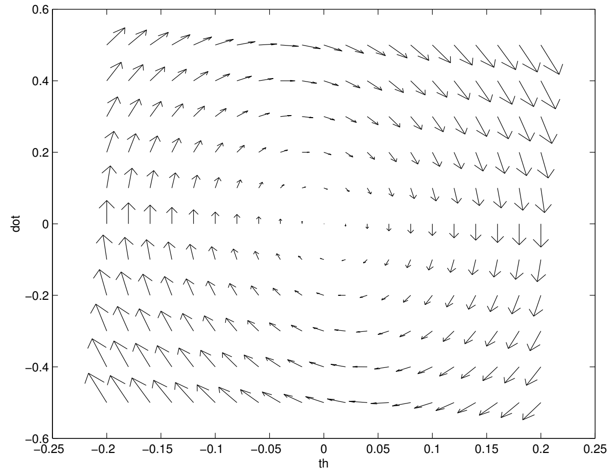

Figure 3: The pendulum vector field.

which gives rise to the state space equations

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{d}{m}x_2 \end{bmatrix} = f(x)$$

The vector

$$x \in \mathbb{R}^2$$

is called the **state** of the system. The size of the state vector (in this case $n = 2$) is called the **dimension** of the system. Notice that the dimension is the same as the order of the original ODE. The function

$$f(\cdot) : \mathbb{R}^2 \to \mathbb{R}^2$$

which describes the dynamics is called a **vector field**, because it assigns a "velocity" vector to each state vector. Figure 3 shows the vector field of the pendulum.

Exercise 2 Other choices are possible for the state vector. For example, for the pendulum one can use $x_1 = \theta^3 + \dot{\theta}$ and $x_2 = \dot{\theta}$. Verify that this gives the vector field

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 3(x_1 - x_2)^{2/3}x_2 - \frac{g}{\ell}[\sin(x_1 - x_2)^{1/3}] - \frac{d}{m}x_2 \\ -\frac{g}{\ell}[\sin(x_1 - x_2)^{1/3}] - \frac{d}{m}x_2 \end{bmatrix}$$

Not a good choice!

Solving the ODE for θ is equivalent to finding a function

$$x(\cdot) : \mathbb{R} \to \mathbb{R}^2$$

such that

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} \theta_0 \\ \dot{\theta}_0 \end{bmatrix}$$
$$\dot{x}(t) = f(x(t)), \ \forall t \in \mathbb{R}.$$

For two dimensional systems like the pendulum it is very convenient to visualise the solutions by **phase plane** plots. These are plots of $x_1(t)$ vs $x_2(t)$ parametrised by time (Figure 4).

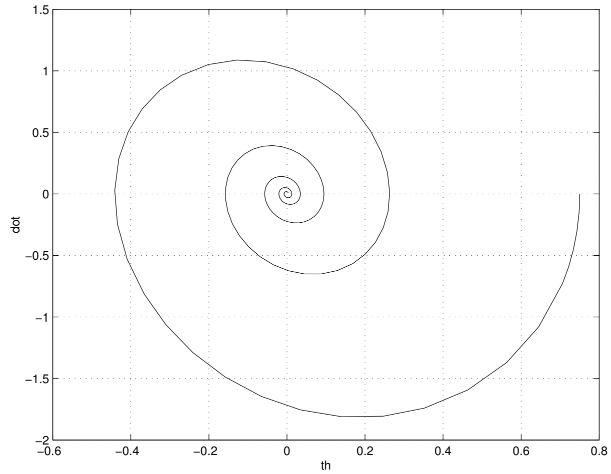

Figure 4: Phase plane plot of the trajectory of Figure 2.

Notice that for certain states $\hat{x} \in \mathbb{R}^2$,

$$f(\hat{x}) = 0$$

In particular, this is the case if

$$\hat{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \hat{x} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

The fact that $f(\hat{x}) = 0$ implies that if the system ever finds itself in state $\hat{x}$ it will never leave.

A state $\hat{x}$ with this property is called an **equilibrium** of the system. Physical intuition suggests that solutions of the system that start close to the equilibrium

$$\hat{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

remain close to it. Such an equilibrium is known as a **stable equilibrium**. In fact, if the dissipation constant d is greater than zero, then the solutions eventually converge to the equilibrium. A stable equilibrium with this property is known as an **asymptotically stable equilibrium**. On the other hand, solutions of the system starting close to (but not exactly on) the equilibrium

$$\hat{x} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

quickly move away from it. Such an equilibrium is known as an **unstable equilibrium**.

Roughly speaking, the pendulum is a nonlinear system because the vector field f is a nonlinear function. Recall that for small angles θ

$$\sin(\theta) \approx \theta$$

Therefore, for small angles the ODE is approximately equal to

$$ml\ddot{\theta} + dl\dot{\theta} + mg\theta = 0$$

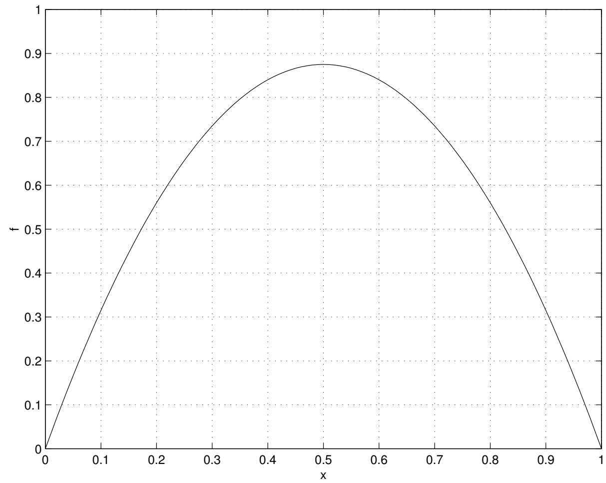

Figure 5: The logistic map.

or in state space form

$$\dot{x} = \begin{bmatrix} x_2 \\ -\frac{g}{l}x_1 - \frac{d}{m}x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{d}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = g(x)$$

Notice that the new vector field is of the form $g(x) = Ax$, i.e. it is a linear function of x. This process of transforming a nonlinear systems into a linear one is called **linearisation** (also discussed in 3F2). Notice that linearisation relies heavily on the assumption that θ is small, and is therefore only valid **locally**.

Exercise 3 Determine the linearisation about the equilibrium

$$\hat{x} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

3.2 The Logistic Map: A Nonlinear Discrete Time System

The logistic map

$$x_{k+1} = ax_k(1 - x_k) = f(x_k) \tag{1}$$

is a nonlinear, discrete time dynamical system that has been proposed as a model for the fluctuations in the population of fruit flies in a closed container with constant food supply (R. May, *Stability and Complexity of Model Ecosystems*, Princeton University Press, Princeton, NJ, 1973). We assume that the population is measured at discrete times (e.g. generations) and that it is large enough to be assumed to be a continuous variable. In the terminology of the previous section, this is a one dimensional system with state $x_k \in \mathbb{R}$, whose evolution is governed by the difference equation (1) given above.

The shape of the function f (Figure 5) reflects the fact that when the population is small it tends to increase due to abundance of food and living space, whereas when the population is large it tends to decrease, due to competition for food and the increased likelihood of epidemics. Assume that $a \leq 4$ and that the initial population is such that $0 \leq x_0 \leq 1$.

Exercise 4 Show that under these assumptions $0 \leq x_k \leq 1$ for all $k \in \mathbb{Z}$ with $k \geq 0$.

The behaviour of x_k as a function of k depends on the value of a.

1. If $0 \leq a < 1$, x_k decays to 0 for all initial conditions $x_0 \in [0, 1]$. This corresponds to a situation where there is inadequate food supply to support the population.

2. If $1 \leq a \leq 3$, x_k tends to a steady state value. In this case the population eventually stabilises.

3. If $3 < a \leq 1 + \sqrt{6} = 3.449$, x_k tends to a 2-periodic state. This corresponds to the population alternating between two values from one generation to the next.

As a increases further more and more complicated patterns are obtained: 4-periodic points, 3-periodic points, and even chaotic situations, where the trajectory of x_k is aperiodic (i.e. never meets itself).

3.3 A Manufacturing Machine: A Discrete System

Consider a machine in a manufacturing plant that processes parts of type p one at a time. The machine can be in one of three states: Idle (I), Working (W) or Down (D). The machine can transition between the states depending on certain events. For example, if the machine is idle and a part p arrives it will start working. While the machine is working it may break down. While the machine is down it may be repaired, etc.

Abstractly, such a machine can be modelled as a dynamical system with a **discrete state**, q, taking three values

$$q \in Q = \{I, W, D\}$$

The state "jumps" from one value to another whenever one of the **events**, σ occurs, where

$$\sigma \in \Sigma = \{p, c, f, r\}$$

(p for "part arrives", c for "complete processing", f for "failure" and r for "repair"). The state after the event occurs is given by a **transition relation**

$$\delta : Q \times \Sigma \to Q$$

Since both Q and Σ are finite sets, one can specify δ by enumeration.

$$\begin{aligned}
\delta(I, p) &= W \\
\delta(W, c) &= I \\
\delta(W, f) &= D \\
\delta(D, r) &= I
\end{aligned}$$

δ is undefined for the rest of the combinations of q and σ. This reflects the fact that certain events may be impossible in certain states. For example, it is impossible for the machine to start processing a part while it is down, hence $\delta(D, p)$ is undefined.

Exercise 5 If the discrete state can take n values and there are m possible events, what is the maximum number of lines one may have to write down to specify δ?

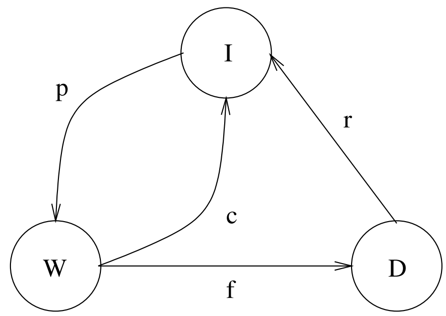

Figure 6: The directed graph of the manufacturing machine automaton.

Such a dynamical system is called an **automaton**, or a **finite state machine**. Automata are special cases of **discrete event systems**. Discrete event systems are dynamical systems whose state also jumps depending on a finite set of events but can take on an infinite number of values. The dynamics of a finite state machine can be represented compactly by a **directed graph** (Figure 6). This is a graph whose **nodes** represent the possible values of the state (in this case I, W, D). The **arcs** of the graph represent possible transitions between the state values and are labelled by the events.

Assume that the machine starts in the idle state $q_0 = I$. What are the sequences of events the machine can experience? Clearly some sequences are possible while others are not. For example, the sequence *pcp* is possible: the machine successfully processes one part and subsequently starts processing a second one. The sequence *ppc*, on the other hand is not possible: the machine can not start processing a second part before the previous one is complete. More generally, any sequence that consists of an arbitrary number of *pc*'s (possibly followed by a single *p*) is an acceptable sequence. In the discrete event literature this set of sequences is compactly denoted as

$$(pc)^*(1 + p)$$

where * denotes an arbitrary number (possibly zero) of *pc*'s, 1 denotes the empty sequence (no event takes place), and + denotes "or".

Likewise, pfr is a possible sequence of events (the machine starts processing a part, breaks down and then gets repaired) while pfp is not (the machine can not start processing a part while it is down). More generally, any sequence that consists of an arbitrary number of pfr's (possibly followed by a p or a pf) is an acceptable sequence.

Exercise 6 Write this set of sequences in the discrete event notation given above.

The set of all sequences that the automaton can experience is called the **language** of the automaton. The above discussion suggests that the language of the machine automaton is

$$(pc + pfr)^*(1 + p + pf)$$

It is important to understand the properties of these languages, for example to determine how to schedule the work in a manufacturing plant.

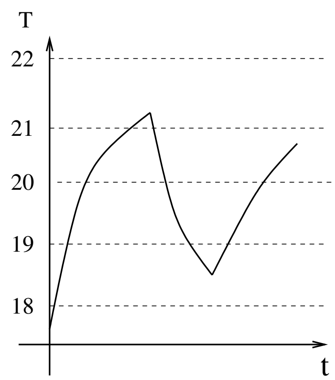

Figure 7: A trajectory of the thermostat system.

3.4 The Thermostat: A Hybrid System

Consider a room being heated by a radiator controlled by a thermostat. Assume that when the radiator is off the temperature, $x \in \mathbb{R}$, of the room decreases exponentially towards 0 degrees according to the differential equation

$$\dot{x} = -ax \tag{2}$$

for some $a > 0$.

Exercise 7 Verify that the trajectories of (2) decrease to 0 exponentially.

When the thermostat turns the heater on the temperature increases exponentially towards 30 degrees, according to the differential equation

$$\dot{x} = -a(x - 30). \tag{3}$$

Exercise 8 Verify that the trajectories of (3) increase towards 30 exponentially.

Assume that the thermostat is trying to keep the temperature at around 20 degrees. To avoid "chattering" (i.e. switching the radiator on and off all the time) the thermostat does not attempt to turn the heater on until the temperature falls below 19 degrees. Due to some thermal inertia of the room, the temperature may fall further, to 18 degrees, before the room starts getting heated. Likewise, the thermostat does not attempt to turn the heater off until the temperature rises above 21 degrees. Again the temperature may rise further, to 22 degrees, before the room starts to cool down. A trajectory of the thermostat system is shown in Figure 7. Notice that in this case multiple trajectories may be obtained for the same initial conditions, as for certain values of the temperature there is a choice between switching the radiator on/off or not. Systems for which such a choice exists are known as **non-deterministic**.

This system has both a continuous and a discrete state. The continuous state is the temperature in the room $x \in \mathbb{R}$. The discrete state, $q \in \{ON, OFF\}$ reflects whether the radiator is on or off. The evolution of x is governed by a differential equation (as was the case with the pendulum), while the evolution of q is through jumps (as was the case with the manufacturing machine). The evolution of the two types of state is coupled. When

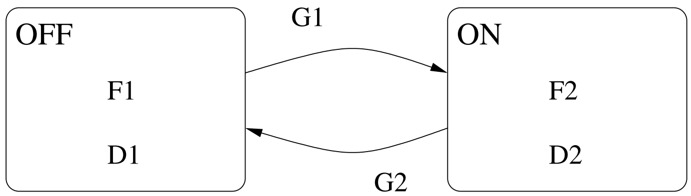

Figure 8: Directed graph notation for the thermostat system.

$q = ON$, x rises according to differential equation (3), while when $q = OFF$, x decays according to differential equation (2). Likewise, q *can not jump* from *ON* to *OFF* unless $x \geq 21$. q *must jump* from *ON* to *OFF* if $x \geq 22$, etc.

It is very convenient to compactly describe such **hybrid systems** by mixing the differential equation with the directed graph notation (Figure 8).

4 Continuous State & Time Systems: State Space Form

All continuous nonlinear systems considered in this class can be reduced to the standard **state space** form. It is usual to denote

- the **states** of the system by $x_i \in \mathbb{R}$, $i = 1, \ldots, n$,
- the **inputs** by $u_j \in \mathbb{R}$, $j = 1, \ldots, m$, and
- the **outputs** by $y_k \in \mathbb{R}$, $k = 1, \ldots, p$.

The number of states, n, is called the **dimension** (or **order**) of the system. The evolution of the states, inputs and outputs is governed by a set of functions

$$
\begin{aligned}
f_i &: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}, \text{ for } i = 1, \ldots, n \\
h_j &: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}, \text{ for } j = 1, \ldots, p
\end{aligned}
$$

Roughly speaking, at a given time $t \in \mathbb{R}$ and for given values of all the states and inputs these functions determine in what direction the state will move, and what the output is going to be.

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, \ldots, x_n, u_1, \ldots, u_m, t) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, \ldots, x_n, u_1, \ldots, u_m, t) \\
y_1 &= h_1(x_1, \ldots, x_n, u_1, \ldots, u_m, t) \\
&\vdots \\
y_p &= h_p(x_1, \ldots, x_n, u_1, \ldots, u_m, t)
\end{aligned}
$$

Exercise 9 What is the dimension of the pendulum example? What are the functions f_i?

It is usually convenient to simplify the equations somewhat by introducing vector notation. Let

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \; u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \in \mathbb{R}^m, \; y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} \in \mathbb{R}^p,$$

and define

$$\begin{aligned} f &: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n \\ h &: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^p \end{aligned}$$

by

$$f(x,u,t) = \begin{bmatrix} f_1(x_1,\ldots,x_n,u_1,\ldots,u_m,t) \\ \vdots \\ f_n(x_1,\ldots,x_n,u_1,\ldots,u_m,t) \end{bmatrix}, \; h(x,u,t) = \begin{bmatrix} h_1(x_1,\ldots,x_n,u_1,\ldots,u_m,t) \\ \vdots \\ h_p(x_1,\ldots,x_n,u_1,\ldots,u_m,t) \end{bmatrix}.$$

Then the system equations simplify to

$$\left. \begin{aligned} \dot{x} &= f(x,u,t) \\ y &= h(x,u,t) \end{aligned} \right\} \tag{4}$$

Equations (4) are known as the **state space form** of the system. The vector space $\mathbb{R}^n$ in which the state of the system takes values is known as the **state space** of the system. If the system is of dimension 2, the state space is also referred to as the **phase plane**. The function f that determines the direction in which the state will move is known as the **vector field**.

Notice that the differential equation for x is first order, i.e. involves $\dot{x}$ but no higher derivatives of x. Sometimes the system dynamics are given to us in the form of higher order differential equations, i.e. equations involving a variable $\theta \in \mathbb{R}$ and its derivatives with respect to time up to $\frac{d^r\theta}{dt^r}$ for some integer $r \geq 1$. Such systems can be easily transformed to state space form by setting $x_1 = \theta$, $x_2 = \dot{\theta}$, $\ldots$, $x_r = \frac{d^{r-1}\theta}{dt^{r-1}}$.

Exercise 10 Consider the system

$$\frac{d^r\theta}{dt^r} + g(\theta, \frac{d\theta}{dt}, \ldots, \frac{d^{r-1}\theta}{dt^{r-1}}) = 0$$

Write this system in state space form.

It may of course happen in certain examples that there are no inputs or outputs, or that there is no explicit dependence of the dynamics on time. In fact, for the next few lectures we will restrict our attention to **autonomous** systems, that is systems of the form

$$\dot{x} = f(x)$$

without inputs or outputs and with no explicit dependence on time.

Exercise 11 Is the pendulum an autonomous system?

Exercise 12 Consider a non-autonomous system of the form $\dot{x} = f(x,t)$, of dimension n. Show that it can be transformed to an autonomous system of dimension $n+1$. (Hint: append t to the state).

5 Existence and Uniqueness of Solutions

Consider an autonomous dynamical system in state space form

$$\dot{x} = f(x)$$

and assume that at time $t = 0$ the state is equal to x_0, i.e.

$$x(0) = x_0$$

We would like to "solve" the dynamics of the system to determine how the state will evolve in the future (i.e. for $t \geq 0$). More precisely, given some $T > 0$ we would like to determine a function

$$x(\cdot) : [0, T] \to \mathbb{R}^n$$

such that

$$\begin{aligned} x(0) &= x_0 \\ \dot{x}(t) &= f(x(t)), \forall t \in [0, T]. \end{aligned}$$

Such a function $x(\cdot)$ is called a **trajectory** (or **solution**) of the system. Notice that given a candidate trajectory $x(\cdot) : [0, T] \to \mathbb{R}^n$ one needs to verify both the differential condition and the initial condition to ensure that $x(\cdot)$ is indeed a solution of the differential equation.

Exercise 13 Assume that, instead of $x(0) = x_0$, it is required that $x(t_0) = x_0$ for some $t_0 \neq 0$. Show how one can construct solutions to the system

$$x(t_0) = x_0, \ \dot{x} = f(x)$$

from solutions to

$$x(0) = x_0, \ \dot{x} = f(x)$$

by appropriately redefining t. Could you do this with a non-autonomous system?

Without any additional information, it is unclear whether one can find a function $x(\cdot)$ solving the differential equation. A number of things can go wrong.

Example (No solutions) Consider the one dimensional system

$$\dot{x} = -sign(x), \ x(0) = 0$$

A solution to this differential equation does not exist for any $T \geq 0$.

Exercise 14 Assume that $x(0) = 1$. Show that solutions to the system exist for all $T \leq 1$ but not for $T > 1$.

Incidentally, something similar would happen with the radiator system if the thermostat insisted on switching the radiator on and off exactly at 20 degrees. ■

Example (Multiple Solutions) Consider the one dimensional system

$$\dot{x} = 3x^{2/3},\ x(0) = 0$$

All functions of the form

$$x(t) = \begin{cases} (t-a)^3 & t \geq a \\ 0 & t \leq a \end{cases}$$

for any $a \geq 0$ are solutions of this differential equation.

Exercise 15 Verify this.

Notice that in this case the solution is not unique. In fact there are infinitely many solutions, one for each $a \geq 0$. ■

Example (Finite Escape Time) Consider the one dimensional system

$$\dot{x} = 1 + x^2,\ x(0) = 0$$

The function

$$x(t) = \tan(t)$$

is a solution of this differential equation.

Exercise 16 Verify this. What happens at $t = \pi/2$?

Notice that the solution is defined for $T < \pi/2$ but not for $T \geq \pi/2$. ■

To eliminate such pathological cases we need to impose some assumptions on f.

Definition 1 *A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called* **Lipschitz continuous** *if there exists $\lambda > 0$ such that for all $x, \hat{x} \in \mathbb{R}^n$*

$$\|f(x) - f(\hat{x})\| < \lambda \|x - \hat{x}\|$$

λ is known as the Lipschitz constant. A Lipschitz continuous function is continuous, but not necessarily differentiable. All differentiable functions with bounded derivatives are Lipschitz continuous.

Exercise 17 Show that for $x \in \mathbb{R}$ the function $f(x) = |x|$ that returns the absolute value of x is Lipschitz continuous. What is the Lipschitz constant? Is f continuous? Is it differentiable?

Theorem 1 (Existence & Uniqueness of Solutions) *If f is Lipschitz continuous, then the differential equation*

$$\begin{aligned} \dot{x} &= f(x) \\ x(0) &= x_0 \end{aligned}$$

has a unique solution $x(\cdot) : [0, T] \to \mathbb{R}^n$ for all $T \geq 0$ and all $x_0 \in \mathbb{R}^n$.

Exercise 18 Three examples of dynamical systems that do not have unique solutions were given above. Why do these systems fail to meet the conditions of the theorem? (The details are not easy to get right.)

This theorem allows us to check whether the differential equation models we develop make sense. It also allows us to spot potential problems with proposed solutions. For example, uniqueness implies that solutions can not cross.

Exercise 19 Show that uniqueness implies that trajectories can not cross. (Hint: what would happen at the crossing point?).

6 Continuity with Respect to Initial Condition and Simulation

Theorem 2 (Continuity with Initial State) *Assume f is Lipschitz continuous with Lipschitz constant λ. Let $x(\cdot) : [0,T] \to \mathbb{R}^n$ and $\hat{x}(\cdot) : [0,T] \to \mathbb{R}^n$ be solutions to $\dot{x} = f(x)$ with $x(0) = x_0$ and $\hat{x}(0) = \hat{x}_0$ respectively. Then for all $t \in [0,T]$*

$$\|x(t) - \hat{x}(t)\| \leq \|x_0 - \hat{x}_0\| e^{\lambda t}$$

In other words, solutions that start close to one another remain close to one another.

This theorem provides another indication that dynamical systems with Lipschitz continuous vector fields are well behaved. For example, it provides theoretical justification for simulation algorithms. Most nonlinear differential equations are impossible to solve by hand. One can however approximate the solution on a computer, using numerical algorithms for computing integrals (Euler, Runge-Kutta, etc.). This is a process known as **simulation**.

Powerful computer packages, such as Matlab, make the simulation of most systems relatively straight forward. For example, the code used to generate the pendulum trajectories is based on a Matlab function:

```
function [xdot] = pendulum(t,x)
l = 1;
m=1;
d=1;
g=9.8;
xdot(1) = x(2);
xdot(2) = -sin(x(1))*g/l-x(2)*d/m;
```

The simulation code is then simply

```
>> x=[0.75 0];
>> [T,X]=ode45('pendulum', [0 10], x');
>> plot(T,X);
>> grid;
```

The continuity property ensures that the numerical approximation to the solution computed by the simulation algorithms and the actual solution remain close.

References for *Nonlinear* part of the course:

Khalil, H.K, *Nonlinear Systems*, 3rd ed, Pearson Education, 2002, QN49.

Sastry, S.S, *Nonlinear Systems: Analysis, Stability and Control*, Springer-verlag, 1999, QN47.

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