Engineering Tripos Part IIB

Module 4F2: Nonlinear Systems and Control

Examples Paper 4F2/3

Solutions

Question 1: Let $x_1 = y$, $x_2 = \dot{y}$, $x_3 = \ddot{y}$, $x_4 = u$, $x_5 = \dot{u}$. By definition

$$\dot{x}_1 = x_2,\ \dot{x}_2 = x_3,\ \dot{x}_4 = x_5.$$

From the system equations we have

$$\dot{x}_3 = g(x_1, x_2, x_3, x_4, t),\ \dot{x}_5 = h(x_4, x_5, x_1, t).$$

So, $\dot{x} = f(x,t)$ where

$$f(x,t) = \begin{bmatrix} x_2 \\ x_3 \\ g(x_1, x_2, x_3, x_4, t) \\ x_4 \\ h(x_4, x_5, x_1, t) \end{bmatrix}.$$

The dimension of the state x is 5.

Question 2: Let $x_{1_k} = y_{k-n}$, $x_{2_k} = y_{k-n+1}$, $\ldots$, $x_{n_k} = y_{k-1}$. Then

$$\begin{aligned} x_{1_{k+1}} &= x_{2_k} \\ x_{2_{k+1}} &= x_{3_k} \\ &\vdots \\ x_{n_{k+1}} &= h(x_{n_k}, x_{n-1_k}, \ldots, x_{1_k}, u_k, k) \end{aligned}$$

which is in the form

$$x_{k+1} = f(x_k, u_k, k).$$

For the system

$$y_k = y_{k-1} y_{k-2} + k \sin(y_{k-1} u_k),$$

let $x_{1_k} = y_{k-2}$, $x_{2_k} = y_{k-1}$. Then

$$\begin{aligned} x_{1_{k+1}} &= x_{2_k} \\ x_{2_{k+1}} &= x_{2_k} x_{1_k} + k \sin(x_{2_k} u_k). \end{aligned}$$

Question 3: The equilibria can be found directly, by considering points where $\dot{y} = \ddot{y} = 0$. Let us, however, reduce the system to the standard state space form first and work from there. Let $x_1 = y$, $x_2 = \dot{y}$. Then the system equations become

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(a + b\cos(x_1))x_2 - c\sin(x_1). \end{aligned}$$

Therefore, the equilibria are points where $x_2 = 0$ and $-(a + b\cos(x_1))x_2 - c\sin(x_1) = 0$, or in other words $(x_1, x_2) = (n\pi, 0)$.
To study the stability, let us consider the linearisation about an equilibrium point $(n\pi, 0)$. Taking partial derivatives in the dynamics and substituting $(x_1, x_2) = (n\pi, 0)$ leads to

$$\begin{bmatrix} 0 & 1 \\ b\sin(x_1)x_2 - c\cos(x_1) & -(a + b\cos(x_1)) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c(-1)^n & -(a + b(-1)^n) \end{bmatrix}.$$

The characteristic polynomial is

$$\det \begin{bmatrix} \lambda & -1 \\ c(-1)^n & \lambda + (a + b(-1)^n) \end{bmatrix} = \lambda^2 + \lambda(a + b(-1)^n) + c(-1)^n.$$

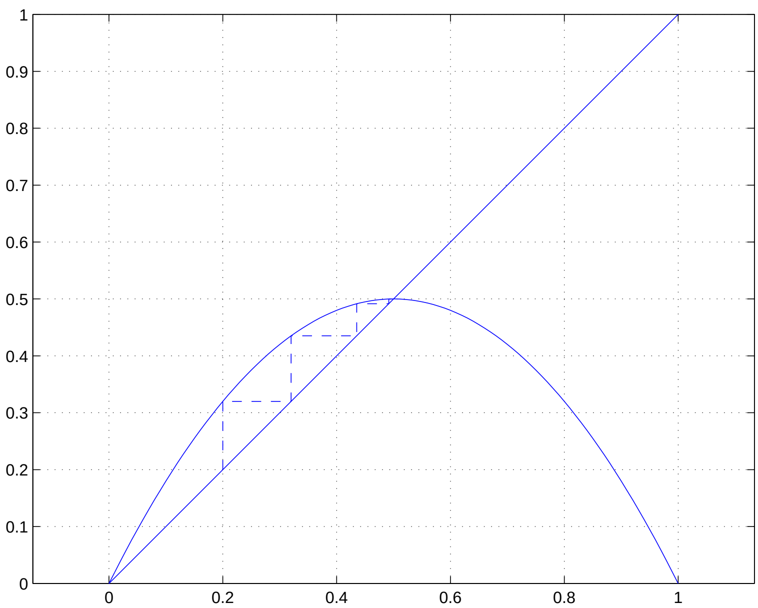
To ensure that the roots have negative real parts, it is necessary and sufficient that all coefficients have the same sign (*Routh-Hurwitz* test). Therefore, since $a > b \geq 0$ and $c > 0$ we have that

$n = 0, \pm 2, \pm 4, \ldots$ are stable equilibria

$n = \pm 1, \pm 3, \ldots$ are unstable equilibria.

For the unstable equilibria, one eigenvalue is positive and the other is negative. Therefore, they are saddle points. (See also Question 10).

Question 4: Iterations of the logistic map can be thought of as alternately projecting onto the curves $y = x$ and $y = ax(1 - x)$.
For $a = 2$ the diagram looks something like this

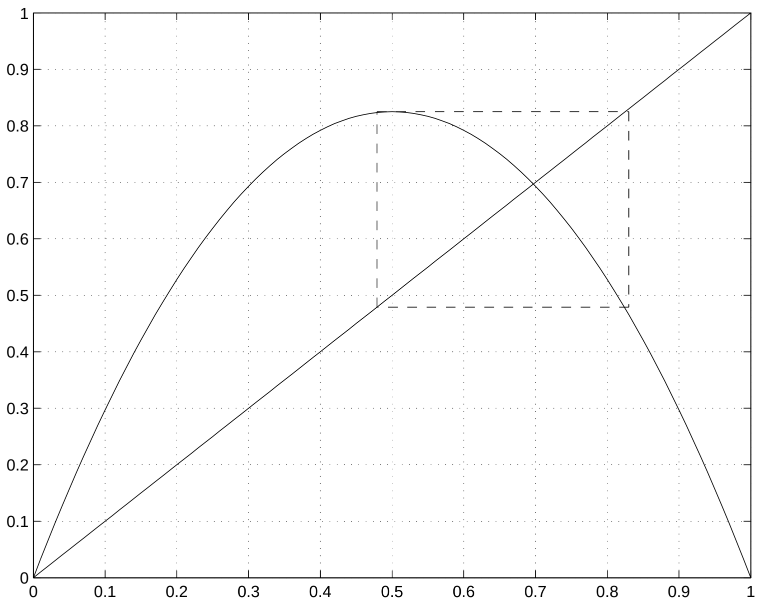


The trajectories converge to an equilibrium. The equilibrium is the solution to the equation

$$x = ax(1 - x).$$

Notice that there is always an equilibrium at $x = 0$. Depending on the value of a there may also be another equilibrium; for $a = 2$ this equilibrium is $x = 0.5$.

For $a = 3.3$ the diagram looks like this:



The trajectories oscillate between two states, roughly 0.48 and 0.83. The exact values can be computed by solving for equilibria of the two step iteration

$$x = a(ax(1-x))(1-(ax(1-x))).$$

Question 5: This just involves iterating

$$x_{k+1} = x_k + r(1-x_k)x_k$$

for each value of r, until the desired phenomena are observed. The only difficulty is that quite a few iterations may be needed for oscillations to settle down or to demonstrate chaotic phenomena. Therefore, a computer running matlab is preferable to a calculator.
With $r = 2.83$ a period-3 oscillation is observed between the values

$$0.21132575 \to 0.68299387 \to 1.29572635$$

(to 8 decimal places).
With $r = 2.845$ these values bifurcate to

$$0.1936 \to 0.6377 \to 1.2950 \to 0.2081 \to 0.6770 \to 1.2991.$$

Notice that each of the points of the period-3 oscillation has "split" to two points, giving rise to a period-6 oscillation.

Question 6: (a) The equation reduces to two scalar differential equations, $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$. The general solutions of these are $x_1(t) = c_1$ and $x_2(t) = c_2$, where c_1 and c_2 are determined by the initial conditions ($x(0) = (c_1, c_2)$).
Notice that all $x \in \mathbb{R}^2$ are equilibria in this case! Moreover, all of them are stable equilibria, since they satisfy the definition of stability given in the notes (Definition 3, Handout 2). For example, the equilibrium $\hat{x} = 0$ is stable because for all $\epsilon > 0$ there exists $\delta > 0$ such that $\|x(0)\| < \delta$ implies that $\|x(t)\| < \epsilon$ for all $t \geq 0$; since $x(t) = x(0)$ for all $t \geq 0$, simply take $\delta = \epsilon$.
Both eigenvalues of A are 0.

(b) This time the equation reduces to $\dot{x}_1 = x_2$ and $\dot{x}_2 = 0$. This gives $\ddot{x}_1 = 0$, whose general solution is

$$x_1(t) = c_1 + c_2 t \quad \text{and} \quad x_2 = c_2$$

(c_1 and c_2 again depend on the initial conditions, $x(0) = (c_1, c_2)$).
In this case all points with $x_2 = 0$ and x_1 arbitrary (i.e. everything on the x_1-axis) are equilibria of the system. This time, however, they are all unstable. For example, the equilibrium $\hat{x} = 0$ is unstable because there exists an $\epsilon > 0$ such that for all $\delta > 0$ there exists an $x(0)$ with $\|x(0)\| < \delta$ such that $\|x(t)\| > \epsilon$ for some $t \geq 0$. To see this, for any $\delta > 0$ take $x(0) = (0, \delta/2)$; clearly $\|x(0)\| < \delta$. The solution in this case is

$$x_1(t) = \delta t/2 \quad \text{and} \quad x_2 = \delta/2 \Rightarrow \|x(t)\| = \sqrt{\frac{\delta^2}{4}(t^2+1)}.$$

For t large enough this is greater than any ϵ (in particular, for $t > \sqrt{4\epsilon^2/\delta^2 - 1}$).
Again, both eigen-values of A are equal to 0. Notice that repeated eigenvalues on the imaginary axis can give rise to either stable or unstable equilibria (cf. Theorem 3, Handout 2).

Question 7: Equilibrium conditions

$$\begin{aligned} x_1 - x_1 x_2 - x_3 &= 0 && (1) \\ x_1^2 - a x_2 &= 0 && (2) \\ b x_1 - c x_3 &= 0. && (3) \end{aligned}$$

Substituting (2) and (3) into (1) leads to

$$x_1 - \frac{x_1^3}{a} - \frac{b x_1}{c} = 0. \qquad (4)$$

$x_1 = 0$ is a solution, therefore, by equations (2) and (3)

$$(0, 0, 0)$$

is an equilibrium of the system. If $x_1 \neq 0$, (4) simplifies to

$$x_1^2 = a\left(1 - \frac{b}{c}\right).$$

Therefore, if

$$d = a\left(1 - \frac{b}{c}\right) > 0$$

then there are two further equilibria

$$\left(\sqrt{d}, 1 - \frac{b}{c}, \frac{b\sqrt{d}}{c}\right) \quad \text{and} \quad \left(-\sqrt{d}, 1 - \frac{b}{c}, -\frac{b\sqrt{d}}{c}\right)$$

Linearising about $(0, 0, 0)$ gives

$$\dot{x} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -a & 0 \\ b & 0 & -c \end{bmatrix} x = Ax.$$

The characteristic equation is

$$det(\lambda I - A) = (\lambda + a)(\lambda^2 + (c-1)\lambda + (b-c)) = 0.$$

By the Routh-Hurwitz test, the roots of this polynomial have negative real parts if and only if $a > 0$, $b > c > 1$. Therefore, by Lyapunov's indirect method, the equilibrium $(0, 0, 0)$ is stable if $a > 0$, $b > c > 1$.
Linearising about $\left(\sqrt{d}, 1 - \frac{b}{c}, \frac{b\sqrt{d}}{c}\right)$ gives

$$\dot{\delta x} = \begin{bmatrix} \frac{b}{c} & -\sqrt{d} & -1 \\ 2\sqrt{d} & -a & 0 \\ b & 0 & -c \end{bmatrix} \delta x.$$

The characteristic equation is

$$\lambda^3 + \left(a + c - \frac{b}{c}\right)\lambda^2 + a\left(c + 2 - \frac{3b}{c}\right)\lambda + 2a(c - b) = 0.$$

Using Routh-Hurwitz, stability requires that

$$a + c - \frac{b}{c} > 0 \tag{5}$$

$$a\left(c + 2 - \frac{3b}{c}\right) > 0 \tag{6}$$

$$2a(c - b) > 0 \tag{7}$$

$$a\left[\left(a + c - \frac{b}{c}\right)\left(c + 2 - \frac{3b}{c}\right) - 2(c - b)\right] > 0. \tag{8}$$

Under the assumption that $d > 0$ (needed for $\left(\sqrt{d}, 1 - \frac{b}{c}, \frac{b\sqrt{d}}{c}\right)$ to be an equilibrium) we have that (7) is equivalent to $c > 0$. Hence we require that

$$\begin{aligned} c &> 0 \\ a + c &> \frac{b}{c} \\ a\left(c + 2 - \frac{3b}{c}\right) &> \frac{2(c - b)a}{a + c - \frac{b}{c}} \end{aligned}$$

(these imply (6) automatically). By Lyapunov's indirect method the equilibrium $\left(\sqrt{d}, 1 - \frac{b}{c}, \frac{b\sqrt{d}}{c}\right)$ is stable if these conditions are satisfied.
The analysis for the third equilibrium is similar.

Question 8: The system equations can be extracted directly from the block diagram.

$$\begin{aligned} u &= J\ddot{y} \\ u &= Ug(z) \qquad \text{where } g(z) = \begin{cases} 1 & \text{if } z > 1 \\ 0 & \text{if } |z| \leq 1 \\ -1 & \text{if } z < -1 \end{cases} \\ z &= k_p(r - y) - k_v\dot{y} \end{aligned}$$

Let $x_1 = y$, $x_2 = \dot{y}$. The state equations (for $r = 0$, $k_p = 2$, $k_v = 1$, and $U = 4J$) are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -4g(2x_1 + x_2) \end{aligned}$$

or, in vector field notation,

$$\dot{x} = f(x) = \begin{bmatrix} x_2 \\ -4g(2x_1 + x_2) \end{bmatrix}$$

The equilibria are the states where $x_2 = 0$ and $g(2x_1 + x_2) = 0$. Therefore all states of the form

$$|x_1| \leq \frac{1}{2}, \;\; x_2 = 0$$

are equilibria of this system.
Discontinuities occur whenever $g(z)$ changes value, i.e. whenever $2x_1 + x_2 = \pm 1$. These are two parallel lines with slope -2 passing through the points $\pm 1/2$.
To get an idea of what the trajectories look like in the x_1-x_2 plane (*phase plane*), notice that above the switching line $2x_1 + x_2 = 1$ the dynamics are $f(x) = (x_2, -4)$ while below the line $2x_1 + x_2 = -1$ the dynamics are $f(x) = (x_2, 4)$. In particular, for points $x_2 = 0$ and $x_1 < -1/2$, $f(x) = (0, 4)$ (i.e. solution moves up), whereas for points $x_2 = 0$ and $x_1 > 1/2$, $f(x) = (0, -4)$ (i.e. solution moves down). Between the switching lines $f(x) = (x_2, 0)$, therefore x_2 remains constant and x_1 changes at constant rate (either increases for $x_2 > 0$ and decreases for $x_2 < 0$). So, roughly speaking, the trajectories have a tendency to rotate clockwise around the origin.
What happens exactly on the discontinuity line? Since f is discontinuous, a solution in the sense discussed in Handout 1 does not exist. We will try to construct a plausible argument about what types of "relaxed" solutions the system may exhibit in this case. In particular, the solutions will not be differentiable functions (since f is discontinuous).
Let us investigate how the "distance" to the switching line changes along the trajectories.

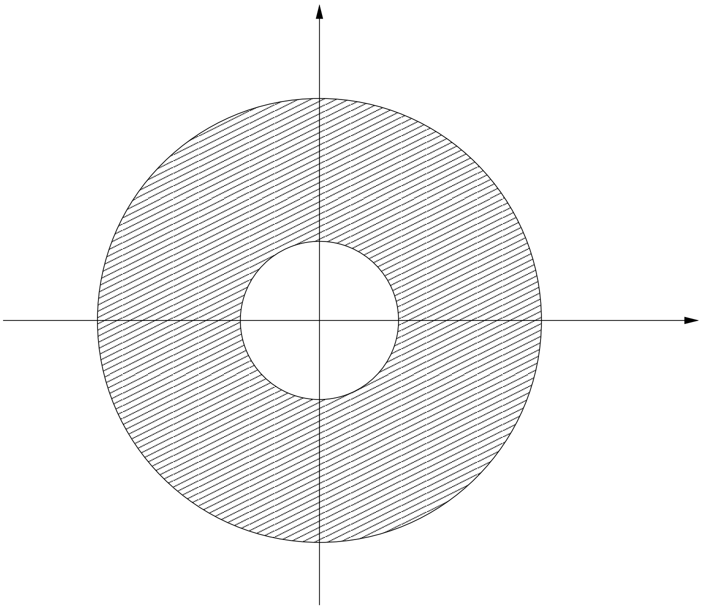
$$\frac{d}{dt}(2x_1 + x_2) = 2\dot{x}_1 + \dot{x}_2 = 2x_2 - 4g(2x_1 + x_2) = \begin{cases} 2x_2 - 4 & \text{if } 2x_1 + x_2 > 1 \\ 2x_2 & \text{if } |2x_1 + x_2| \leq 1 \\ 2x_2 + 4 & \text{if } 2x_1 + x_2 < -1 \end{cases}$$

Notice that if $x_2 \geq 2$ or $x_2 \leq 0$ the sign of $2\dot{x}_1 + \dot{x}_2$ does not change when we cross the switching line $2x_1 + x_2 = 1$. In other words the vector field "pushes" the solution towards the switching line on one side of the line and "pulls" it away from the switching line on the other. It is reasonable therefore to assume that in this case the solution of the system will hit the switching line, cross it (with a discontinuity of its derivative) and continue on the other side. A similar situation arises for the discontinuity $2x_1 + x_2 = -1$ if $x_2 \leq -2$ or $x_2 \geq 0$.
If $0 < x_2 < 2$ on the other hand, the sign of $2\dot{x}_1 + \dot{x}_2$ changes as we cross the discontinuity $2x_1 + x_2 = 1$. In this case, the vector field pushes solutions towards the discontinuity line on both sides. This will cause trajectories that cross the surface to be forced immediately back across the surface. Practically, we would expect trajectories like these to "slide" along the switching lines towards the points $(1/2, 0)$. A similar situation arises for the discontinuity $2x_1 + x_2 = -1$ if $-2 < x_2 < 0$.

Question 9: For $\epsilon \in (0,1)$ and $\delta > 0$, consider the set

$$S = \{(x,y) \in \mathbb{R}^2 \mid \delta > x^2 + y^2 - 1 \geq -\epsilon\}.$$

The set S is compact (i.e. bounded and closed).



Consider the Lyapunov function

$$V(x,y) = x^2 + y^2 - 1.$$

Differentiating along system trajectories leads to

$$\begin{aligned}
\dot{V}(x,y) &= 2x\dot{x} + 2y\dot{y} \\
&= 2x\left(y + \frac{x}{\sqrt{x^2+y^2}}[1-(x^2+y^2)]\right) + 2y\left(-x + \frac{y}{\sqrt{x^2+y^2}}[1-(x^2+y^2)]\right) \\
&= 2\sqrt{x^2+y^2}\left(1-(x^2+y^2)\right) \\
\Rightarrow \dot{V}(x,y) &\begin{cases} > 0 & \text{if } x^2+y^2 < 1 \\ = 0 & \text{if } x^2+y^2 = 1 \\ < 0 & \text{if } x^2+y^2 > 1 \end{cases}
\end{aligned}$$

($\dot{V}(x,y) = 0$ also at the point $x = y = 0$, but this is outside the region S of interest). Since $V(x,y)$ increases when $x^2 + y^2 = 1 - \epsilon$ and decreases when $x^2 + y^2 = 1 + \delta$, S is invariant.
Overall, S is a compact invariant set that contains no equilibria. By the Poincare-Bendixson Theorem it must contain a limit cycle. Moreover, the set

$$\{(x,y) \in S \mid \dot{V}(x,y) = 0\} = \{(x,y) \in S \mid x^2 + y^2 = 1\}$$

is also invariant. By LaSalle's Theorem, all system trajectories that start in S converge to this set, which is a stable limit cycle.
Since the above argument holds for any $\epsilon \in (0,1)$ and any $\delta > 0$, this suggests that the domain of attraction of the limit cycle is the entire $\mathbb{R}^2$, except the point $(0,0)$ (which is in fact an equilibrium).
A similar (and maybe somewhat easier) argument can be given by writing the system in polar coordinates.

Question 10: (i) The state equations are given in the answer to Question 3. For

$$V(x) = c(1 - \cos(x_1)) + \frac{1}{2}x_2^2$$

we have that $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ with $|x_1| < 2\pi$. Moreover,

$$\begin{aligned}\dot{V}(x) &= c\sin(x_1)x_2 + x_2[-(a + b\cos(x_1))x_2 - c\sin(x_1)] \\ &= -(a + b\cos(x_1))x_2^2 \\ &\leq 0\end{aligned}$$

(recall that $a > b \geq 0$ is assumed in Question 3). Applying Lyapunov's Stability Theorem (Theorem 4, Handout 2) to the open set

$$S = \{x \in \mathbb{R}^2 \mid |x_1| < 2\pi\}$$

we conclude that the equilibrium $x = 0$ is stable.

(ii) Consider now the compact set

$$S = \{x \in \mathbb{R}^2 \mid V(x) \leq c \text{ and } |x_1| \leq \pi/2\}.$$

Since $\dot{V}(x) \leq 0$ for all $x \in S$, S is a compact invariant set. The set

$$\{x \in S \mid \dot{V}(x) = 0\} = \{x \in S \mid x_2 = 0\}$$

(recall that $a > b$) contains no trajectories other than $x(t) = 0$ for all $t \geq 0$. This is because to remain in $\{x \in S \mid x_2 = 0\}$ we must have $x_2(t) = 0$ for all $t \geq 0$ and therefore $\dot{x}_2(t) = 0$ for all $t \geq 0$. But

$$\dot{x}_2 = -(a + b\cos(x_1))x_2 - c\sin(x_1).$$

Therefore, to remain in $\{x \in S \mid x_2 = 0\}$ we must also have $\sin(x_1(t)) = 0$ for all $t \geq 0$, or, in other words, $x_1(t) = 0$ for all $t \geq 0$, since $|x_1| \leq \pi/2$.
By LaSalle's Theorem (in particular, Corollary 2 of Handout 2) $x = 0$ is asymptotically stable.

(iii) Recall that the equilibria are $x_1 = n\pi$ and $x_2 = 0$. The equilibria $x_1 = 2n\pi$, $x_2 = 0$ are local minima of $V(x)$, while the equilibria $x_1 = (2n + 1)\pi$, $x_2 = 0$ are saddle points of $V(x)$. Since $\dot{V}(x) \leq 0$, trajectories of the system move "downhill" (i.e. from larger to smaller values of V). Therefore, we would expect $x_1 = 2n\pi$, $x_2 = 0$ to be stable equilibria and $x_1 = (2n+1)\pi$, $x_2 = 0$ to be unstable.
Rigorous proofs of the stability of the former family of equilibria can be constructed along the lines of part (i), or by using linearisation as in Question 3. The instability of the latter equilibria was also established using linearisation in Question 3.

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Revised by J. Maciejowski March 2011