

Module 3F2: Systems and Control
EXAMPLES PAPER 2 — ROOT-LOCUS

Solutions

1. (a) **For the system**

$$L(s) = \frac{1}{(s+a)(s+b)} \quad (a, b \text{ both real})$$

show that the root-locus diagram (for positive gains k) consists of the segment of the real axis between $-a$ and $-b$, and the perpendicular bisector of that segment.

Using *Rule 3* (see Lecture Notes 2) every point on the real axis between the two poles, namely every point on the segment between $-a$ and $-b$, is on the root-locus, since every such point is to the left of one pole.

Using *Rule 5* there are 2 asymptotes, perpendicular to the real axis (angles $(2\ell + 1)\pi/2, \ell = 0, 1$). These asymptotes emanate from the point $(-a - b)/2$, namely the mid-point of the segment between $-a$ and $-b$.

We are asked to show that every point on these asymptotes is actually on the root-locus itself. Consider a point s_0 on one of these asymptotes. From Figure 1 it is clear that $\angle(s_0 + a) + \angle(s_0 + b) = \pi$ (geometry of isosceles triangles). Hence every such point satisfies the *Angle Condition* and so is on the root-locus.

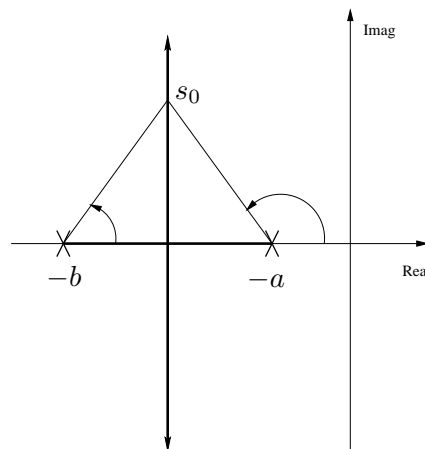


Figure 1:

Note that this question can also be answered by considering the roots of the quadratic equation $1 + kL(s) = 0$, or

$$s^2 + (a + b)s + (ab + k) = 0$$

and noting that the sum of the two roots is always $-(a + b)$.

- (b) **Sketch the root-locus diagram for positive gains k for the system**

$$L(s) = \frac{1}{s(s+1)^2}$$

From *Rule 3* every point on the negative real axis is on the root-locus — because for $-1 < s_0 < 0$, s_0 is to the left of one pole, and for $s_0 < -1$ it is to the left of 3 poles.

There are 3 poles and no zeros ($n = 3, m = 0$), so by *Rule 5* there are 3 asymptotes, making angles $(2\ell + 1)\pi/3, \ell = 0, 1, 2$ with the positive real axis, and the asymptotes emanate from $(-1 - 1 + 0)/3 = -2/3$.

Look for breakaway points, using *Rule 4*:

$$\begin{aligned} \frac{d}{ds}L(s) &= \frac{d}{ds}\{s^{-1}(s+1)^{-2}\} \\ &= -s^{-2}(s+1)^{-2} - 2s^{-1}(s+1)^{-3} \\ &= \frac{-(s+1) - 2s}{s^2(s+1)^2} = \frac{-3s-1}{s^2(s+1)^2} \\ &= 0 \quad \text{if } s = -\frac{1}{3} \end{aligned}$$

This leads to the sketch shown in Figure 2. *Note: The breakaway point need not be located accurately, since the question says 'sketch', so the use of Rule 4 is optional here.*

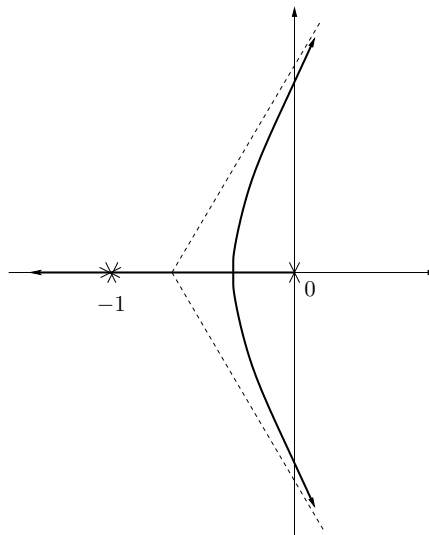


Figure 2:

Find the (positive) value of k at which closed-loop stability is lost

- (i) from your diagram, and**
(ii) using the Routh-Hurwitz criterion.

(i) *From the diagram:* At any point s_0 on the root-locus, the *Angle Condition* is satisfied: $\angle(s_0) + 2\angle(s_0 + 1) = \pi$ (since one pole is at 0, and two are at -1). Let's find exactly where the root-locus crosses the imaginary axis — that's the point at which closed-loop stability is lost. If s_0 is on the imaginary axis then $\angle s_0 = \pi/2$. Hence the angle condition reduces to $2\angle(s_0 + 1) = \pi/2$, or $\angle(s_0 + 1) = \pi/4$. Hence we must have $s_0 = j1$ — see Figure 3. Now find the corresponding gain from (see section 2.3 of Lecture Notes 2):

$$k = \frac{1}{|L(s_0)|} = |s_0| \times |s_0 + 1| \times |s_0 + 1| = 1 \times \sqrt{2} \times \sqrt{2} = 2.$$

(ii) *From Routh-Hurwitz criterion:* The closed-loop poles are the solutions of $1 + kL(s) = 0$, namely:

$$\begin{aligned} 1 + k \frac{1}{s(s+1)^2} &= 0 \\ \Rightarrow s(s+1)^2 + k &= 0 \\ \Rightarrow s^3 + 2s^2 + s + k &= 0 \end{aligned}$$

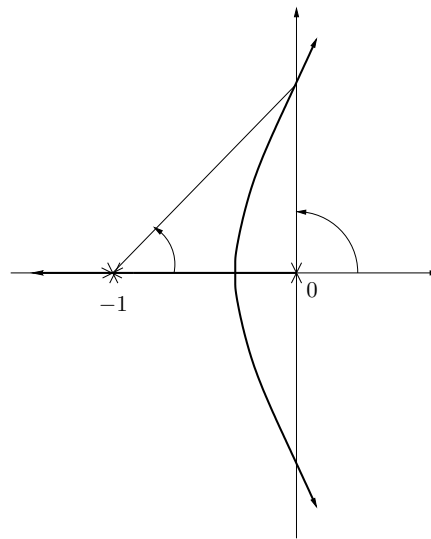


Figure 3:

Now the Routh-Hurwitz criterion for this to have all solutions with negative real parts is (see section 3 of Lecture Notes 2, with $n = 3$):

$$2 \times 1 > 1 \times k$$

which is just violated when $k = 2$.

(c) Draw the root-locus diagram for positive gains k for the system

$$L(s) = \frac{s}{(s + 0.5)(s + 1)}$$

and hence show that the closed-loop system is stable for all $k > 0$. Also sketch the root-locus diagram for negative gains, and find the value of k at which closed-loop stability is lost.

$k > 0$: Using *Rule 3*, one branch of the root-locus is the real axis between -0.5 and 0 , and the other branch is the real axis to the left of -1 . And that is the whole of the root-locus. So both roots are real and negative for all $k > 0$, and hence the closed-loop is (asymptotically) stable.

$k < 0$: Using *Rule 3* — modified for $k < 0$ — every point on the real axis to the right of 0 is on the root-locus, as is every point on the real axis between -1 and -0.5 . There is a breakaway point somewhere between -1 and -0.5 (could be calculated using *Rule 4* but not important for rough sketch) and another one somewhere to the right of 0 — see Figure 4. The negative gain at which stability is just lost can be calculated in (at least) two ways:

1. Analytically, since only second-order in this case:

$$\begin{aligned} 1 + k \frac{s}{(s+1)(s+0.5)} &= 0 \\ \Rightarrow (s+1)(s+0.5) + ks &= 0 \\ \Rightarrow s^2 + (1.5+k)s + 0.5 &= 0 \\ \Rightarrow \text{stability just lost when } 1.5+k &= 0 \text{ or } k = -1.5. \end{aligned}$$

2. Using root-locus methods: First find the value of ω_0 , where $s_0 = j\omega_0$ is the point at which the root-locus crosses the imaginary axis.

$$\begin{aligned} \text{The angle condition for } k < 0 \text{ is: } \angle(j\omega_0 + 1) + \angle(j\omega_0 + 0.5) - \angle(j\omega_0) &= 0 \\ \Rightarrow \alpha + \beta - \frac{\pi}{2} = 0, \text{ so } \alpha + \beta = \frac{\pi}{2}, \text{ where } \alpha = \angle(j\omega_0 + 1) \text{ and } \beta = \angle(j\omega_0 + 0.5). \end{aligned}$$

Now $\tan \alpha = \omega_0$, $\tan \beta = 2\omega_0$, and $\tan(\alpha + \beta) = \infty$.

$$\text{But } \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}, \text{ so } \tan \alpha \tan \beta = 1,$$

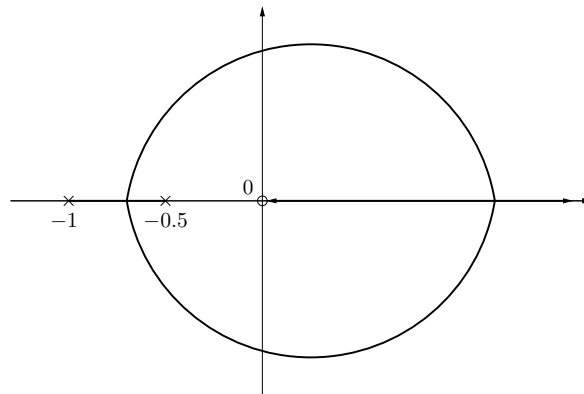


Figure 4:

hence $2\omega_0^2 = 1$ or $\omega_0 = 1/\sqrt{2}$. (This could also be estimated graphically by trial and error — in more complicated cases an analytical solution would not be possible.) Now the corresponding gain must be found, using the method shown in section 2.3 of Lecture Notes 2:

$$|k| = \frac{|j/\sqrt{2}+1| \times |j/\sqrt{2}+0.5|}{1/\sqrt{2}} = \frac{\sqrt{3/2} \times \sqrt{3/4}}{1/\sqrt{2}} = \frac{3}{2}.$$

Hence $k = -3/2$.

2. (a) **Sketch the variation of the closed-loop poles as the tacho feedback gain k_d varies (i) using root-locus construction rules**

The closed-loop poles are the solutions of

$$s^2 + 10k_d s + 25 = 0$$

$$\Rightarrow (s^2 + 25) + 10k_d s = 0$$

$$\Rightarrow 1 + k_d \frac{10s}{s^2 + 25} = 0$$

which is in the root-locus form.

So the root-locus diagram has one zero at 0 and two poles at $\pm 5j$. By *Rule 3* every point on the negative real axis is on the root-locus. By *Rule 5* there is one asymptote, along the negative real axis, and by *Rule 2* the zero attracts one branch of the root-locus. So there must be one breakaway point where the two complex roots become real, and this can (*optionally*) be calculated using *Rule 4*:

$$\frac{d}{ds} \left(\frac{s}{s^2 + 25} \right) = 0$$

$$\Rightarrow \frac{1(s^2 + 25) - s(2s)}{(s^2 + 25)^2} = 0$$

$$\Rightarrow -s^2 + 25 = 0 \Rightarrow s = \pm 5$$

so the breakaway point is at -5 . ($+5$ is not on the locus.)

The locus traced by the two complex branches is in fact a semi-circle, but this is not easy to show by this method. See Figure 5.

- Sketch the variation of the closed-loop poles as the tacho feedback gain k_d varies (ii) by finding an explicit expression for the closed-loop poles.**

The closed-loop poles are the solutions of $s^2 + 10k_d s + 25 = 0$, namely:

$$s = \frac{-10k_d \pm \sqrt{(10k_d)^2 - 4 \times 25}}{2} = -5 \left(k_d \pm \sqrt{k_d^2 - 1} \right)$$

When $k_d < 1$ these are complex conjugates, $-5 \left(k_d \pm j\sqrt{1 - k_d^2} \right)$, with modulus

$5\sqrt{k_d^2 + (1 - k_d^2)} = 5$, so the locus for $k_d < 1$ lies on a circle of radius 5. (This can also be seen by noting that $\omega_n = 5$ independently of k_d .)

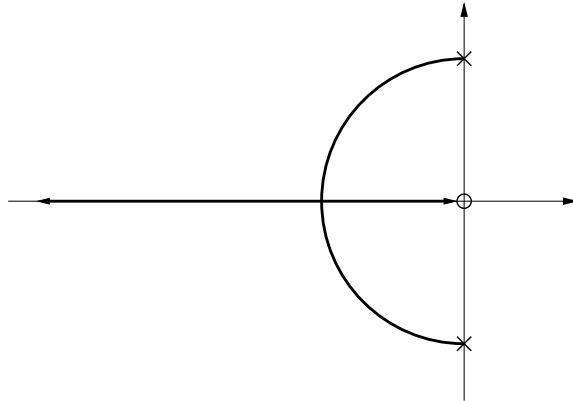


Figure 5:

When $k_d > 1$ these are two real roots, $s_1 < -5$, $-5 < s_2 < 0$ (since $\sqrt{k_d^2 - 1} < k_d$).

- (b) **What is the damping factor of the closed loop as a function of k_d ?**

The closed-loop characteristic polynomial is $s^2 + 10k_d s + 25$. Comparing this to the standard second-order form $s^2 + 2\zeta\omega_n s + \omega_n^2$ (or $\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n} + 1$) gives $\omega_n = 5$ and hence $10\zeta = 10k_d$, hence $\zeta = k_d$.

Sketch the time response of the load angular position to a step change of 1 radian in desired angular position for values of $k_d = 0.6$ and $k_d = 1.2$.

This is a second-order system with $\zeta = k_d$, and $\omega_n = 5$. From Examples Paper 1, Q.1(b) we know that the closed-loop transfer function from θ_d to θ is $G(s) = 25/(s^2 + 10k_d s + 25)$, so the steady-state gain is $G(0) = 1$, hence the final value of θ will be 1 rad. Now get the step response sketches from the Mechanics Data Book, paying attention to the correct calibration of the time axis (the Data Book shows $\omega_n t$), and showing the final value correctly. See Figure 6.

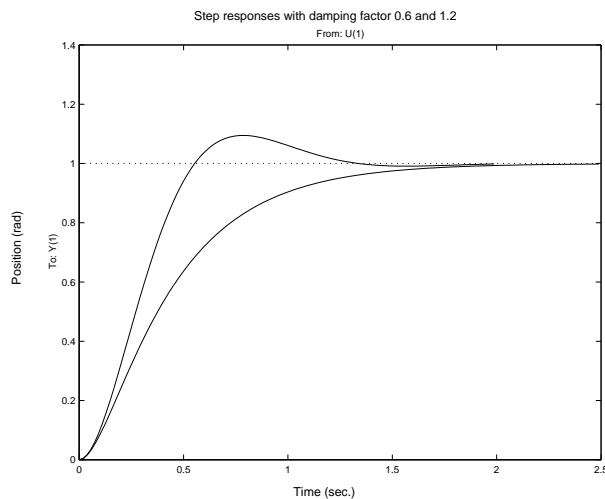


Figure 6:

3. A negative feedback system consists of a plant whose transfer function is that given in Question 1(b), and a controller which is just a positive gain k . The reference signal is a ramp $r(t) = 2t$. Suppose that k is set to that value which gives two coincident real closed-loop poles at $-1/3$. What is this value of k , and what is the steady-state error $e = r - y$ (where y is the output of the plant) obtained with this value?

Use the root-locus diagram obtained for Question 1(b).

The question tells you that the breakaway point on the root-locus diagram is at $-1/3$ (in case it has not been worked out in Question 1(b)). Working out the corresponding gain, using the method described in section 2.3 of Lecture Notes 2 gives:

$$k = \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{4}{27}$$

The transfer function from the reference input to the error is (from notes or work out from first-principles):

$$\frac{\bar{e}(s)}{\bar{r}(s)} = \frac{1}{1 + kL(s)}$$

and recall that $\bar{r}(s) = \frac{2}{s^2}$. Apply the Final-Value Theorem to get the steady-state value:

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s \bar{e}(s) = \lim_{s \rightarrow 0} s \times \frac{1}{1 + kL(s)} \times \frac{2}{s^2} = \lim_{s \rightarrow 0} \frac{2}{s \left(1 + \frac{4/27}{s(s+1)^2} \right)} = \frac{54}{4} = \frac{27}{2}$$

How should the gain be adjusted to reduce this error? What can be said about the locations of the closed-loop poles if this is done?

From the working above it can be seen that the steady-state error is, in general, $2/k$. So the gain should be increased to reduce the steady-state error. From the root-locus diagram the effect of this will be to make two of the closed-loop poles complex. If the gain is increased too far, these will become very underdamped or even unstable. [From the answer to Question 1(b) it is seen that stability is lost for $k > 2$. So the smallest steady-state error achievable is 1, but in practice rather greater than that in order to have reasonable closed-loop damping (stability margins).]