1 Introduction

By way of introduction to the material of these lectures on state space models of dynamical systems and control the pendulum laboratory experiment will be first described and then some basic matrix manipulation results will be reviewed.

1.1 Pendulum Experiment

The dynamic equations for this laboratory experiment are (assuming that the moments of inertia of the pendulum about its centre of mass is negligible and ignoring friction):

\begin{align}
L \ddot{\theta}(t) &= \ddot{x}(t) \cos(\theta(t)) - g \sin(\theta(t)) \\
\left(1 + \frac{M}{m} + \frac{I}{(ma^2)}\right) \ddot{x}(t) &= \frac{T}{ma} + L \ddot{\theta}(t) \cos(\theta(t)) - L \sin(\theta(t)) \dot{\theta}^2(t)
\end{align}

(1.1) (1.2)

Now define the states of this system to be the vector:

\[ \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \\ L \dot{\theta} \end{bmatrix} \]

We can consider the input to be the amplifier input voltage, output current or the resulting torque, \( T \), which are just related by scaling factors. We will take the input to be

\[ u = \frac{T}{aM + I/a}, \]
and also define the natural frequencies given by
\[
\omega_1^2 = \frac{g}{L}
\]
\[
\omega_0^2 = \omega_1^2 \left(1 + \frac{M/m + I/(ma^2)}{\omega_1^2 - \omega_0^2}\right)
\]\n⇒ \[ M/m + I/(ma^2) = \frac{\omega_0^2 - \omega_1^2}{\omega_1^2 - \omega_0^2} \]

Recall that \(\omega_1\) is the natural frequency of the pendulum when the cart is fixed, and it can be shown that \(\omega_0\) is the natural frequency of the pendulum when no force acts on the cart.

With this notation equations (1.1) and (1.2) become:
\[
\begin{pmatrix}
\frac{\omega_0^2}{\omega_0^2 - \omega_1^2} - \cos(x_3/L) & 0 \\
-\cos(x_3/L) & 1
\end{pmatrix}
\begin{pmatrix}
x_2 \\
x_4
\end{pmatrix}
= \begin{pmatrix}
\frac{\omega_0^2}{\omega_0^2 - \omega_1^2} u - \frac{1}{L} \sin(x_3/L)x_4^2 \\
-L\omega_1^2 \sin(x_3/L)
\end{pmatrix}
\] (1.3)

Solving this equation for \(\dot{x}_2\) and \(\dot{x}_4\) together with the trivial identities \(\dot{x}_1 = x_2\) and \(\dot{x}_3 = x_4\) give the nonlinear first order vector differential equation:
\[
\dot{x}(t) = f(x, u) \quad (1.4)
\]

where defining \(\alpha(x_3) = \sin^2(x_3/L) + \frac{\omega_0^2}{\omega_0^2 - \omega_1^2}\) we have,
\[
f(x, u) = \begin{pmatrix}
x_2 \\
\frac{1}{\alpha(x_3)} \left(\frac{\omega_0^2}{\omega_0^2 - \omega_1^2} u - \frac{1}{L} \sin(x_3/L)x_4^2 - L\omega_1^2 \sin(x_3/L) \cos(x_3/L)\right)
\end{pmatrix}
\]
\[
\begin{pmatrix}
x_4 \\
\frac{1}{\alpha(x_3)} \left(\frac{\omega_0^2}{\omega_0^2 - \omega_1^2} \cos(x_3/L) u - \frac{1}{L} \sin(x_3/L) \cos(x_3/L)x_4^2 - \frac{\omega_0^2 \omega_1^2}{\omega_0^2 - \omega_1^2} L \sin(x_3/L)\right)
\end{pmatrix}
\]

The equation (1.4) is a standard form that is a common starting point for analysis.

Equilibrium States

A state, \(x_e\), and input \(u_e\) are said to be an equilibrium state and input if the system remains in this condition when released from this initial state, \(x(0) = x_e\) and \(u(t) = u_e\) for \(t \geq 0\). Hence in (1.4) we need \(f(x_e, u_e) = 0\), so that \(\dot{x}(t) = 0\) and hence remains zero.

Hence in this example \(x_{2e} = x_{4e} = 0\) and (1.3) immediately gives that \(\sin(x_{3e}/L) = 0\) and \(u_e = 0\). There are therefore the two obvious equilibria with the pendulum vertically down, \(x_{3e} = 0\), or vertically up, \(x_{3e} = L\pi\). In both cases the carriage position, \(x_1\), can be arbitrary.
Linearization

Given an equilibrium condition we can linearize the equations about this point and obtain a good approximation for the system behaviour near the equilibrium. Linearizing about the equilibrium with \( x_3 = 0 \) and for example \( x_1 = 0 \), gives

\[
\dot{x}(t) \approx \begin{bmatrix}
\frac{\partial x_2}{\partial x} & \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial (\omega_0^2 - \omega_1^2)} & \frac{\partial x_2}{\partial x_3} \\
\frac{\partial x_4}{\partial x} & \frac{\partial x_4}{\partial u} & \frac{\partial x_4}{\partial (\omega_0^2 - \omega_1^2)} & \frac{\partial x_4}{\partial x_3}
\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u
\]

This is the standard form for a LINEAR dynamical system that we will study in some detail using matrix techniques.

System Poles

To solve the state equations to get time histories we can take the Laplace transform of (1.6) giving:

\[
sX(s) - x(0) = AX(s) + BU(s)
\]

\[
\Rightarrow (sI - A)X(s) = BU(s)
\]

\[
\Rightarrow X(s) = (sI - A)^{-1}BU(s)
\]

\[
= \frac{N(s)}{d(s)}BU(s), \text{ where } d(s) = \det(sI - A).
\]

Therefore the transfer functions from the input to the states have a common denominator, \( d(s) \), whose roots will be the system poles. Moreover the roots of \( d(s) = 0 \) are the Eigenvalues of the matrix \( A \).

The eigenvalues for \( A \) given above are at 0, 0, ±\( j\omega_0 \).
**State Feedback**

Suppose we can measure the four states and make the input, $u$, a linear function of the measured states:

$$u = w - k_1 x_1 - k_2 x_2 - k_3 x_3 - k_4 x_4$$

$$= w - k \mathbf{x}, \text{ where } k = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix}$$

The closed loop state equations will now be:

$$\dot{\mathbf{x}} = A \mathbf{x} + B (w - k \mathbf{x}) = (A - B k) \mathbf{x} + B w$$

The closed loop poles will now be the eigenvalues of the matrix $(A - B k)$ which can in fact be made arbitrary by suitable choice of the feedback gain vector.
2 State Space Descriptions Of Dynamical Systems

2.1 States

The essence of a dynamical system is its memory, i.e. the present output, $y(t)$ depends on past inputs, $u(\tau)$, for $\tau \leq t$.

Whereas in a static system $y(t)$ is a function of $u(\tau)$ at $\tau = t$ only.

Three sets of variables define a dynamical system -

1. The inputs, $u_1(t), u_2(t), ... u_m(t)$ (input vector $u(t)$)
2. The state variables, $x_1(t), ... x_n(t)$ (state vector $x(t)$)
3. The outputs, $y_1(t), y_2(t), ... y_p(t)$, (output vector $y(t)$)

The states may depend on $u(\tau)$ and $x(\tau)$ for $\tau \leq t$. However for any $t_o < t_1, x(t_1)$ can always be determined from $x(t_o)$ and $u(t)$, $t_o \leq \tau \leq t_1$. [NB this is the definition of system state] That is $x(t_o)$ summarizes the effect on the future of inputs and states prior to $t_o$.

The output, $y$ at time $t$, is a memoryless function of $x(t)$ and $u(t)$.
The most general class of dynamical system that we will consider is described by the set of first order ordinary differential equations -

\[ S \begin{cases}
    \frac{dx_i}{dt} = f_i(x_1(t), x_2(t), \ldots x_n(t), u_1(t), \ldots u_m(t), t), & i = 1, 2, \ldots n, \\
y_j(t) = g_j(x_1(t), \ldots x_n(t), u_1(t), \ldots u_m(t), t) & j = 1, 2, \ldots p.
\end{cases} \]

\( S \) is the standard form for a state-space dynamical system model. Or a vector form -

\[ S \begin{cases}
    \dot{x}(t) = f(x(t), u(t), t) \\
y(t) = g(x(t), u(t), t)
\end{cases} \]

Note that \( S \) has only first order ode's, but we can use a standard technique to convert high order o.d.e's to first order vector o.d.e.'s, by the use of auxiliary variables.

Ex:

\[ \ddot{y} + 6\dot{y} + 5(\dot{y})^3 + 12\sin(y) = \cos(t) \]

a state variable is given by: \( x_1 = y, \ x_2 = \dot{y}, \ x_3 = \ddot{y} \), when,

\[ \begin{align*}
    \dot{x}_1 &= x_2 \\
    \dot{x}_2 &= x_3 \\
    \dot{x}_3 &= -6x_1x_3 - 5x_2^3 - 12\sin(x_1) + \cos(t)
\end{align*} \]

which is in the form

\[ \dot{x}(t) = f(x(t), t) \]

where \( f \) is a vector valued function of a vector, i.e.

\[ f(x, t) = \begin{bmatrix} x_2 \\ x_3 \\ -6x_1x_3 - 5x_2^3 - 12\sin(x_1) + \cos(t) \end{bmatrix} \]
For linear time-invariant dynamical systems we use the standard form:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

2.2 General Guidelines For State-Space Modelling

*Method 1:* Choose system states, \(x_1, x_2, \ldots, x_n\) by considering the independent ‘energy storage devices’ or ‘memory elements’,

- electrical circuits - current in L or voltage on C.
- mechanics - positions and velocities of masses (linear and angular).
- chemical engineering - temperature, pressure, volume, concentration.

Then use the basic physical laws derive expressions involving \(\dot{x}_i\), e.g.

- \(V = Ldi/dt, i = Cdv/dt\)
- \(m \times \text{acc}^n = \text{force};\) \(d(\text{pos}^n)/dt = \text{velocity}.\)
- \(d(\text{volume})/dt = \text{flow} = \text{function of pressure}.\)

Solve for \(\dot{x} = f(x, u, t)\).
Method 2: Choose state variables as successive time derivatives.

Example: Original ODE: \( J\ddot{\theta} + B\dot{\theta} = M. \)
Let \( x_1 = \theta, \ x_2 = \dot{\theta}. \) Then

In general for \( n \)'th-order ODE in \( \theta \) define

\[
x_1 = \theta, \quad x_2 = \frac{d\theta}{dt}, \quad \ldots, \quad x_n = \frac{d^{n-1}\theta}{dt^{n-1}}
\]

2.3 Ideal Operational Amplifier Circuits

In a negative feedback configuration the amplifier acts so as to make the +ve and -ve inputs have essentially equal voltages.

\[
\begin{align*}
C_1\dot{x}_1 &= (V_i - x_1)/R_1; \quad C_2\dot{x}_2 = x_1/R_2 \\
V_o &= x_1 + x_2
\end{align*}
\]

Linear operation until the amplifier saturates when quite different equations may hold.
If the amplifier output saturates at \( \pm V_s \), then for \( |x_1 + x_2| \geq V_s \)
we will have: \( C_2\dot{x}_2 = (\pm V_s - x_2)/R_2. \)
2.4 Velocity Fields In The State-Space

Consider the free motion of the time invariant dynamical system -

\[ \dot{x} = f(x) \implies x(t + \delta t) \approx x(t) + f(x(t))\delta t \]

This implies a velocity field in the state space which can give a good qualitative idea of the system’s behaviour in simple cases.

**Ex.**

The velocity field can be sketched by drawing an arrow in the direction of for many values of \( x \). Equilibrium points (or singular points) where \( f(x_e) = 0 \), may be stable or unstable.

**Ex.**

\[ -L \frac{dx_1}{dt} = x_1 R + x_2 \]
\[ C \frac{dx_2}{dt} = x_1 \]
\[ \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -R/L & -1/L \\ 1/C & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

i) Let \( R = 1/2, \ L = 1, \ C = 1 \) (in consistent units) then

\[ A = \begin{bmatrix} -\frac{1}{2} & -1 \\ 1 & 0 \end{bmatrix} \]
(ii) Let $R = 4, L = 1, C = 1 \Rightarrow A = \begin{bmatrix} -4 & -1 \\ 1 & 0 \end{bmatrix}$
The MATLAB code to produce these two sets of state-plane trajectories is as follows:

```matlab
% A=[-.5 -1; 1 0]; % for underdamped example
A=[-4 -1; 1 0]; % for overdamped example

hold off

[x,y]=meshgrid(-2:.2:2,-2:.2:2);
xdot=A(1,1)*x+A(1,2)*y;
ydot=A(2,1)*x+A(2,2)*y;

quiver(x,y,xdot,ydot); drawnow; hold on

for x0=[-2 2],
for y0=[-2 -1.5 -1 -4+2*sqrt(3) 0 .5 1 1.5];
xx=[x0;y0*sign(x0)],zeros(2,1000)];
for i=1:1000; xx(:,i+1)=(eye(2)+.02*A)*xx(:,i); end;
plot(xx(1,:)',xx(2,:)','r')
drawnow
end
end
```

3 Linearizing Nonlinear Dynamical Systems

3.1 Linearizing the standard form

Suppose we have a nonlinear (time invariant) dynamical system in state space form -

\[
S \begin{cases}
\dot{x} & = f(x, u) \\
y & = g(x)
\end{cases}
\]

Let \( \bar{x} \) be an equilibrium state for the system when \( u(t) = u_\circ \) (constant)

i.e. \( f(\bar{x}, u_\circ) = 0 \) and also let \( y_\circ = g(\bar{x}) \).

Now consider small perturbations from this equilibrium, let

\[
\bar{x}(t) = \bar{x} + \delta x(t), \quad u(t) = u_\circ + \delta u(t), \quad y(t) = y_\circ + \delta y(t)
\]
A Taylor Series expansion of the i-th equation of \( \dot{x} = f(x, u) \) gives

\[
\dot{x}_i = \dot{x}_{ei} + \delta x_i = f_i(x_e, u_e) + \frac{\partial f_i}{\partial x_1} \bigg|_{x_e, u_e} \delta x_1 + \frac{\partial f_i}{\partial x_2} \bigg|_{x_e, u_e} \delta x_2 + \ldots + \frac{\partial f_i}{\partial x_n} \bigg|_{x_e, u_e} \delta x_n
\]

\[
+ \frac{\partial f_i}{\partial u_1} \bigg|_{x_e, u_e} \delta u_1 + \ldots + \frac{\partial f_i}{\partial u_m} \bigg|_{x_e, u_e} \delta u_m + \text{Remainder}
\]

Thus for small \( \delta x \) and \( \delta u \) (\( \Rightarrow \) v. small remainder) we get

\[
\delta x_i \approx A\delta x + B\delta u
\]

where

\[
A = \frac{\partial f_i}{\partial x}(x_e, u_e) = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \ldots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \ldots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \ldots & \frac{\partial f_n}{\partial x_n}
\end{bmatrix}_{x = x_e, u = u_e}
\]

\[
B = \frac{\partial f_i}{\partial u}(x_e, u_e) = \begin{bmatrix}
\frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \ldots & \frac{\partial f_1}{\partial u_m} \\
\frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \ldots & \frac{\partial f_2}{\partial u_m} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial u_1} & \frac{\partial f_m}{\partial u_2} & \ldots & \frac{\partial f_m}{\partial u_m}
\end{bmatrix}_{x = x_e, u = u_e}
\]

Similarly \( \delta y = C\delta x \) where \( C = \frac{\partial g(x)}{\partial x} \).

The linearized system equations are thus

\[
\begin{align*}
\delta \dot{x} &= A\delta x + B\delta u \\
\delta y &= C\delta x
\end{align*}
\]

which will accurately predict the system behaviour for \( x \) and \( u \) close to \( x_e \) and \( u_e \) respectively.
3.2 Linearizing when the State Equations are Implicit

Quite often a system’s equations cannot easily be written as $\dot{x} = f(x,u)$ but can be written as

$$F(\dot{x}, x, u) = 0 \quad (n \text{ equations in the } n \text{ unknowns } \dot{x}_1...\dot{x}_n).$$

The linearized model can be derived without solving for $\dot{x}$ as follows. Since $(\bar{x}, u_e)$ gives an equilibrium we have,

$$F(0, \bar{x}, u_e) = 0$$

Now linearize $F$ about $(0, \bar{x}, u_e)$ to get

$$L \delta \dot{x} + M \delta x + N \delta u \simeq 0 \quad \Rightarrow \quad \dot{\delta x} \simeq -L^{-1}M \delta x - L^{-1}N \delta u$$

(N.B. no nonlinear equations to solve except to obtain the equilibrium point)
3.3 Behaviour of Nonlinear Systems

As mentioned above the linearized equations will accurately predict the behaviour of the nonlinear system for $x$ and $u$ close to their equilibrium values. When the states and inputs are far away from the equilibrium values then the behaviour can be quite different, e.g.

- many equilibria with some stable and some unstable (e.g. inverted pendulum).
- limit cycles.
- divergence.

**Example: Van der Pol oscillator**

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + (1 - x_1^2)x_2
\end{align*}
\]

This has a stable *Limit Cycle* and an unstable equilibrium at $x = 0$.

![Figure 3: State space trajectories for Van der Pol Oscillator](image)

Note that if we were to change the signs on the $\dot{x}_1$ and $\dot{x}_2$ terms then this will just change the directions of the arrows in the state space. Hence the origin would then be a stable equilibrium and the limit cycle would be unstable (i.e. if perturbed from the limit cycle then it would either decay to the origin or diverge to infinity).
3.3.1 Example: E4 Exam question 3, 1997

Figure 2 shows a design for a hydraulically actuated table for simulating earthquakes. The table is denoted as ABC, with the point C constrained to move horizontally. DA and EB denote hydraulic rams which are pin-jointed at each end and can produce forces $F_1$ and $F_2$, respectively. The equations of motion (which should not be verified) are:

$$ M \ddot{z} = F_1 \cos \phi_1 + F_2 \sin \phi_2 $$

and

$$ \frac{2}{a} \left( I + \frac{1}{4} Ma^2 \cos^2 \theta \right) \ddot{\theta} = M \cos \theta \left( \frac{1}{2} a \theta^2 \sin \theta - g \right) + $$

$$ + \left[ \sin (\theta + \phi_1) + \sin \phi_1 \cos \theta \right] F_1 + $$

$$ + \left( \cos \theta \cos \phi_2 \right) F_2 $$

where

$$ \tan \phi_1 = \frac{a \sin \theta}{a + z - \frac{1}{2} a \cos \theta}, \quad \tan \phi_2 = \frac{z}{a + \frac{1}{2} a \sin \theta}. $$

$M$ and $I$ are constants, $a$ and $z$ are the lengths shown in Fig. 2, and $\theta, \phi_1, \phi_2$ are the angles shown in the figure.

(a) What conditions are satisfied at an equilibrium? Determine values $F_{1e}$ and $F_{2e}$ of the forces $F_1$ and $F_2$, which will give an equilibrium position $\theta = \theta_e$ and $z = z_e$, if $\theta_e = 0$ and $z_e = a$.

(b) The linearised equations about the equilibrium $(\theta_e = 0, z_e = a)$ are:

$$ \ddot{x} = Ax + Bu $$

where

$$ x = \begin{bmatrix} \theta, z - a, \dot{\theta}, \dot{z} \end{bmatrix}^T, \quad u = [F_1 - F_{1e}, F_2 - F_{2e}]^T, \quad A = \begin{bmatrix} 0 & I_2 \\ P & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ Q \end{bmatrix}, $$

$$ P = \begin{bmatrix} -13 \frac{1}{12\tau^2} & -1 \frac{1}{2a\tau^2} \\ -\frac{g}{4} & \frac{g}{2a} \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 \sqrt{2Mg\tau^2} \\ \frac{1}{M} & \frac{1}{\sqrt{2M}} \end{bmatrix}, \quad \tau^2 = \frac{2I}{Mag} + \frac{a}{2g}, \quad \text{and} \quad I_2 \text{ is the } 2 \times 2 \text{ identity matrix.} $$

(\text{cont.})
Verify that the term \(-\frac{g}{4}\), which appears in \(P\), is correct. (Do not verify any other terms. Assume that the nonlinear equations are correct.)

(c) Is the linearised system of part (b) controllable from \(y\)? Is it controllable from \(u_1\) (the first element of \(u\)) alone?

(d) Comment on the difference in the achievable behaviour of this system when only \(u_1\) is available for control, and when the complete vector \(u\) is available.

\[\text{(TURN OVER)}\]
4 Solutions of Linear State Equations

4.1 Using Laplace Transforms

Taking Laplace Transforms of
\[
\dot{x}(t) = Ax(t) + Bu(t); \quad x(0) = x_0
\]
\[
y(t) = Cx(t) + Du(t)
\]
gives
\[
sX(s) - x_0 = AX(s) + BU(s)
\]
\[
(sI - A)X(s) = x_0 + BU(s)
\]
\[
X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s)
\]
\[
Y(s) = C(sI - A)^{-1}x_0 + (D + C(sI - A)^{-1}B)U(s)
\]

initial condition response input response

For \(x_0 = 0\), \(Y(s) = (D + C(sI - A)^{-1}B)U(s)\)

and
\[
G(s) = D + C(sI - A)^{-1}B
\]
is called the transfer function matrix.

The \(i,j^{th}\) entry of \(G(s)\) gives the transfer function from \(u_j\) to \(y_i\).
4.2 Transfer function poles

Poles are values of $s$ at which the transfer function becomes infinite:

$$\|G(p)\| = \infty \Rightarrow p \text{ is a pole of } G(s)$$

This can only happen when the matrix $(sI - A)$ becomes singular, i.e., when

$$\det(sI - A) = 0$$

namely at the eigenvalues of $A$.

Later we will see that eigenvalues of $A$ are not always poles of $G(s)$.

Hence we have the important result:

\[
\text{Poles of } G(s) \subset \text{eigenvalues of } A
\]

Analytical expression for transfer function matrix

It can be shown that, for any matrix $M$,

\[
M^{-1} = \frac{1}{\det M} \begin{bmatrix}
M_{11} & M_{21} & \ldots & M_{n1} \\
M_{12} & \vdots & \ddots & \vdots \\
M_{1n} & \ldots & & M_{nn}
\end{bmatrix}
\]

where $M_{ij}$ is called the cofactor of $m_{ij}$ given by

$$M_{ij} = (-1)^{i+j} \det(M \text{ with } i\text{-th row and } j\text{-th column deleted})$$

Hence

\[
(sI - A)^{-1} = \frac{1}{\alpha(s)} N(s) \quad \text{where} \quad \alpha(s) = \det(sI - A) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n
\]

and

\[
N(s) = N_1 s^{n-1} + N_2 s^{n-2} + \cdots + N_{n-1} s + N_n
\]

The transfer function can therefore be written as

\[
G(s) = \frac{1}{\alpha(s)} (CN(s)B + D\alpha(s)) \text{ with } (CN(s)B + D\alpha(s)) \text{ being a matrix of polynomials in } s.
\]
Cayley-Hamilton Theorem

\[ A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n I = 0 \]

since

\[ \alpha(s)I = (sI - A) \left( N_1 s^{n-1} + N_2 s^{n-2} + \cdots + N_n \right) \]

Equating coefficients of \( s^k \) and premultiplying by \( A^k \) for \( k = n, \ldots, 1, 0 \) gives

\[ \begin{align*}
  s^n & : A^n I = A^n N_1 \\
  s^{n-1} & : A^{n-1} \alpha_1 I = A^{n-1} N_2 - A^{n-1} A N_1 \\
  \vdots & \vdots \\
  s & : a_n I = A N_n - A^2 N_{n-1} \\
  s^0 & : \alpha_1 I = -A N_n
\end{align*} \]

Adding these equalities gives the result.
Hence any power of \( A \) is a linear combination of \( I, A, A^2, \ldots, A^{n-1} \) — only!

### 4.3 Initial Condition Response of the State

For \( \mathbf{u}(t) = 0 \) we have \( \dot{\mathbf{x}}(t) = A \mathbf{x}(t) \) and hence,

\[ X(s) = (sI - A)^{-1} \mathbf{x}_0 \]

\[ \implies \mathbf{x}(t) = \mathcal{L}^{-1} \left( (sI - A)^{-1} \right) \mathbf{x}_0 = \Phi(t) \mathbf{x}_0 \]

where

\[ \Phi(t) = \mathcal{L}^{-1} \left( (sI - A)^{-1} \right) \]

\[ = \mathcal{L}^{-1} \left\{ \sum_{k \geq 0} A^k s^{-(k+1)} \right\} \]

\[ = I + At + A^2 t^2 / 2! + \cdots + A^k t^k / k! + \cdots \]

\[ \underset{\text{def}}{=} e^{At} \]
Note that
\[ \frac{d}{dt} \Phi(t) = A e^{At} = e^{At} A \]
(check by differentiating series expansion of \( \Phi(t) \)).

Hence with \( z(t) = \Phi(t)z_0 \),
\[
\frac{d}{dt} z(t) = \frac{d}{dt} \{ \Phi(t)z_0 \} = A e^{At} z_0 = A \{ \Phi(t)z_0 \} = Az(t)
\]
\[
\Phi(0) = I
\]
\[
z(0) = z_0
\]
and the differential equation and initial condition are satisfied as required.

\( \Phi(t) \) is called the **state transition matrix**.

**Properties of \( e^{At} \)**

(1) **Change of state coordinates**

If
\[
A = T^{-1} \dot{A} T
\]
then
\[
A^2 = T^{-1} \dot{A} T T^{-1} \ddot{A} T = T^{-1} \dddot{A} T
\]
\[
A^k = T^{-1} \dddot{A}^k T
\]
hence
\[
e^{At} = \sum_{k=0}^{\infty} A^k t^k / k!
\]
\[
= \sum_{k=0}^{\infty} T^{-1} \dddot{A}^k T t^k / k!
\]
\[
= T^{-1} \left( \sum_{k=0}^{\infty} \dddot{A}^k t^k / k! \right) T
\]
\[
\Rightarrow e^{At} = T^{-1} e^{\dot{A}^k T} T
\]
Why is this ‘change of coordinates’?

If $\dot{z}(t) = A\dot{z}(t)$ and $z = T\bar{z}$, then

$$\dot{z}(t) = T\dot{\bar{z}}(t) = TAT^{-1}\bar{z}(t) = A\bar{z}(t)$$

$$\bar{z}(t) = e^{At}\bar{z}(0)$$

$$z(t) = T^{-1}e^{At}T\bar{z}(0)$$

Special case: Eigenvectors as coordinate axes

Recall that for $W$ the matrix of eigenvectors of $A$, then the defining relations,

$$Aw_1 = w_1\lambda_1$$
$$Aw_2 = w_2\lambda_2$$
$$\vdots$$
$$Aw_n = w_n\lambda_n$$

can be written as:

$$A\begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}_w = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}_w \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}_\Lambda$$

Hence for non-defective $A$ we have the eigenvalue/eigenvector decomposition:

$$A = W\Lambda W^{-1}$$
So that for \( T = W^{-1} \) we have,

\[
\bar{A} = TAT^{-1} = W^{-1}AW = \Lambda = \begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_n
\end{bmatrix} = \text{diag} \{ \lambda_i \}
\]

\[
e^{AT} = \sum_{k=0}^{\infty} \frac{\Lambda^k t^k}{k!} = \text{diag} \{ \sum \lambda_i^k t^k/k! \}
\]

\[
= \text{diag} \{ e^{\lambda_i t} \}.
\]

This gives one way of evaluating \( e^{At} \).

(Another way is to evaluate \( L^{-1}(sI-A)^{-1} \).)

2) **Semigroup property** — Don’t worry about the fancy name.

\[
e^{A(t_1 + t_2)} = e^{At_1}e^{At_2} = e^{At_2}e^{At_1}
\]

since \( \Phi(t_1 + t_2) \Phi(0) = \Phi(t_2) \Phi(t_1) \Phi(0) \) for all \( \Phi(0) \)

NB: This only works because \((At_1)(At_2) = (At_2)(At_1)\).
For arbitrary matrices \( A \) and \( B \), \( e^{A+B} \neq e^A e^B \).

3) **Inverse**

\[
I = e^{A0} = e^{A(t-t)} = e^{At}e^{-At} \Rightarrow \quad (e^{At})^{-1} = e^{-At}
\]
4) Derivative — Repeated here for completeness.

\[
\frac{d}{dt} e^{At} = A e^{At} = e^{At} A
\]

5) Integral

\[
\int_0^t e^{A\tau} d\tau = \int_0^t \sum_{k=0}^{\infty} \frac{A^{k+1}}{(k+1)!} d\tau = \sum_{k=0}^{\infty} \left[ \frac{A^{k+1}}{(k+1)!} \right]_0^t = A^{-1} \left\{ \sum_{k=0}^{\infty} \frac{A^{k+1}}{(k+1)!} - 0 \right\} = A^{-1} e^{At} - A^{-1} \quad \text{if} \quad \det(A) \neq 0.
\]

If \( \det(A) = 0 \) then the above formula is not valid and the integration needs to be done directly. e.g. \( A = 0 \).

### 4.4 Example: Rotating Rigid Body

Let \( I_1, I_2, I_3 \) be the moments of inertia of a rigid body rotating in free space, about its 3 principal axes, and \( w_1, w_2, w_3 \) the corresponding angular velocities. Then in the absence of externally applied torques EULER’S EQUATIONS OF MOTION are:

\[
\begin{align*}
I_1 \ddot{w}_1 &= (I_2 - I_3) w_2 w_3 \\
I_2 \ddot{w}_2 &= (I_3 - I_1) w_3 w_1 \\
I_3 \ddot{w}_3 &= (I_1 - I_2) w_1 w_2
\end{align*}
\]

nonlinear state space equations

This is a lossless system since the Kinetic Energy,

\[
V(w_1, w_2, w_3) = \frac{1}{2} I_1 w_1^2 + \frac{1}{2} I_2 w_2^2 + \frac{1}{2} I_3 w_3^2
\]

\[
\frac{dV}{dt} = I_1 w_1 \dot{w}_1 + I_2 w_2 \dot{w}_2 + I_3 w_3 \dot{w}_3 = w_1 w_2 w_3 [(I_2 - I_3) + (I_3 - I_1) + (I_1 - I_2)] = 0
\]

⇒ if the trajectory starts on an ellipsoid \( V(w_1, w_2, w_3) = \) constant, it stays on it.

In addition conservation of angular momentum implies that,

\[
\frac{d}{dt} \left\{ I_1^2 w_1^2 + I_2^2 w_2^2 + I_3^2 w_3^2 \right\} = 2 w_1 w_2 w_3 [I_1 (I_2 - I_3) + I_2 (I_3 - I_1) + I_3 (I_1 - I_2)] = 0
\]
Suppose \( I_1 = 6, I_2 = 2, I_3 = 5 \) then

\[
\begin{align*}
\dot{w}_1 &= -\frac{1}{2}w_2w_3 \\
\dot{w}_2 &= -\frac{1}{2}w_3w_1 \\
\dot{w}_3 &= \frac{4}{5}w_1w_2
\end{align*}
\]

Equilibrium Solutions satisfy: \( \dot{w}_1 = \dot{w}_2 = \dot{w}_3 = 0 \) \( \Rightarrow w_2w_3 = w_1w_3 = w_1w_2 = 0 \) \( \Rightarrow \)

*either* (a) \( w_2 = w_3 = 0 \) \& \( w_1 = \bar{w}_1 \)

*or* (b) \( w_3 = w_1 = 0 \) \& \( w_2 = \bar{w}_2 \)

*or* (c) \( w_1 = w_2 = 0 \) \& \( w_3 = \bar{w}_3 \)

*or* (d) \( w_1 = w_2 = w_3 = 0 \).

The linearized equations are

\[
\dot{\delta w} \approx \frac{\partial f}{\partial \delta w} \delta w = 
\begin{bmatrix}
0 & -\frac{1}{2}w_3 & -\frac{1}{2}w_2 \\
-\frac{1}{2}w_3 & 0 & -\frac{1}{2}w_1 \\
\frac{4}{5}w_2 & \frac{4}{5}w_1 & 0 \\
\end{bmatrix}
\delta w.
\]

case (a)

\[
\Rightarrow \delta w \approx \left[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & -\frac{1}{2}\bar{w}_1 \\
0 & \frac{4}{5}\bar{w}_1 & 0
\end{array}
\right]
\delta w.
\]

\[
\Rightarrow \delta w_1(t) \approx \delta w_1(0) & \frac{d}{dt} \left[
\begin{array}{c}
\delta w_2 \\
\delta w_3
\end{array}
\right] \approx \left[
\begin{array}{cc}
0 & -\frac{1}{2}\bar{w}_1 \\
\frac{4}{5}\bar{w}_1 & 0
\end{array}
\right]
\left[
\begin{array}{c}
\delta w_2 \\
\delta w_3
\end{array}
\right]
\]

and the state trajectories are ellipses (with the ratio of the principal axes = \( \sqrt{8/5} \approx 1.26 \).
case (b)

\[ \delta \dot{w} \simeq \begin{bmatrix}
0 & 0 & -\frac{1}{2} \bar{w}_2 \\
0 & 0 & 0 \\
\frac{4}{5} \bar{w}_2 & 0 & 0
\end{bmatrix} \delta w \]

\[ \delta w_2 \simeq \delta w_2(0) \& \frac{d}{dt} \frac{\delta w_1}{\delta w_3} \simeq \begin{bmatrix} 0 & -\frac{1}{2} \bar{w}_2 \\
\frac{4}{5} \bar{w}_2 & 0 \end{bmatrix} \begin{bmatrix} \delta w_1 \\
\delta w_2 \end{bmatrix} \]

and

\[ \begin{bmatrix} \delta w_1 \\
\delta w_2 \end{bmatrix} = \begin{bmatrix} 1 \\
1 \end{bmatrix} e^{-\frac{1}{2} \bar{w}_3 t} \left( \frac{\delta w_1(0) + \delta w_2(0)}{2} \right) \]

\[ + \begin{bmatrix} 1 \\
-1 \end{bmatrix} e^{\frac{1}{2} \bar{w}_3 t} \left( \frac{\delta w_1(0) - \delta w_2(0)}{2} \right) \]

---

case (c)

\[ \delta \dot{w} \simeq \begin{bmatrix}
0 & -\frac{1}{2} \bar{w}_3 & 0 \\
-\frac{1}{2} \bar{w}_3 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \delta w \]

\[ \delta w_3(t) \simeq \delta w_3(0) \frac{d}{dt} \begin{bmatrix} \delta w_1 \\
\delta w_2 \end{bmatrix} = -\frac{1}{2} \bar{w}_3 \begin{bmatrix} 0 & 1 \\
1 & 0 \end{bmatrix} \begin{bmatrix} \delta w_1 \\
\delta w_2 \end{bmatrix} \]

and

\[ \begin{bmatrix} \delta w_1 \\
\delta w_2 \end{bmatrix} = \begin{bmatrix} 1 \\
1 \end{bmatrix} e^{-\frac{1}{2} \bar{w}_3 t} \left( \frac{\delta w_1(0) + \delta w_2(0)}{2} \right) \]

\[ + \begin{bmatrix} 1 \\
-1 \end{bmatrix} e^{\frac{1}{2} \bar{w}_3 t} \left( \frac{\delta w_1(0) - \delta w_2(0)}{2} \right) \]
The trajectories in the 3-dimensional state space can thus be sketched as follows on a particular ellipsoid \( V(w_1, w_2, w_3) = \text{const.} \).

**Figure 4**: State space trajectories of a rotating rigid body

### 4.5 Convolution Integral

Consider

\[
\dot{\mathbf{x}} - A\mathbf{x} = B\mathbf{u} \quad (\ast)
\]

Comparing with the scalar case note

\[
\frac{d}{dt} \left( e^{-At} \mathbf{x}(t) \right) = \frac{d}{dt} \left( e^{-At} \right) \mathbf{x}(t) + e^{-At} \frac{dx}{dt} = -e^{-At} A\mathbf{x}(t) + e^{-At} \frac{dx}{dt}
\]

now premultiply \((\ast)\) by \( e^{-At} \) to give

\[
e^{-At} \frac{dx}{dt} - e^{-At} A\mathbf{x} = e^{-At} B\mathbf{u}
\]

\[
\Rightarrow \frac{d}{dt} \left( e^{-At} \mathbf{x}(t) \right) = e^{-At} B\mathbf{u}(t)
\]

\[
\Rightarrow e^{-At} \mathbf{x}(t) - \mathbf{x}_0 = \int_0^t e^{-A\tau} B\mathbf{u}(\tau) d\tau
\]

\[
\Rightarrow \mathbf{x}(t) = e^{At} \mathbf{x}_0 + e^{At} \int_0^t e^{-A\tau} B\mathbf{u}(\tau) d\tau
\]

\[
\Rightarrow \mathbf{x}(t) = e^{At} \mathbf{x}_0 + \int_0^t e^{A(t-\tau)} B\mathbf{u}(\tau) d\tau
\]
and

\[
y(t) = Ce^{At}x_0 + Du(t) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau
\]

\begin{align*}
\text{initial condition response} & \quad \text{input response} \\
D\delta(t) + Ce^{At}B & \quad t \geq 0 \\
0 & \quad t < 0
\end{align*}

Let \(H(t) = \begin{cases} 
D\delta(t) + Ce^{At}B & t \geq 0 \\
0 & t < 0
\end{cases}\) then if \(x_0 = 0\),

\[
y(t) = \int_0^t H(t-\tau)u(\tau)d\tau = H(t) * u(t)
\]

\(H(t)\) is called the impulse response matrix, since for multiple-input/multiple-output systems, if an impulse is applied at time \(0^+\) to the \(j\)-th input, with the other inputs at zero, then the \(i\)-th output will be

\[
y_i(t) = \int_0^t h_{ij}(t-\tau)u_j(\tau)d\tau = h_{ij}(t)
\]

Note that the transfer function, \(G(s) = \mathcal{L}(H(t))\).

4.6 Frequency Response

Consider a linear time-invariant system that is asymptotically stable. What is the response due to sinusoidal input at each of the inputs? Let

\[
u_j(t) = A_j \cos(\omega_o t + \theta_j), \; j = 1, 2, \ldots, m
\]

then

\[
y_i(t) \to B_i \cos(\omega_o t + \phi_i) \text{ as } t \to \infty
\]

where

\[
B_i e^{j\phi_i} = g_{i1}(j\omega_o)A_1 e^{j\theta_1} + g_{i2}(j\omega_o)A_2 e^{j\theta_2} + \cdots + g_{im}(j\omega_o)A_m e^{j\theta_m}
\]

i.e. the sum of the sinusoidal responses from each input. The rate at which the steady state is achieved depends on how quickly the impulse response tends to zero, which in turn depends on the pole positions.
4.7 Stability of $\dot{x} = Ax$

Stability of systems is concerned with whether as $t \to \infty$,

$$\dot{x}(t) \to 0$$

$$\to \infty$$

or remains bounded?

but since $\dot{x}(t) = e^{At}x_0$ the question becomes whether the elements of $e^{At} \to 0, \to \pm \infty$ or remain bounded as $t \to \infty$?

Consider

$$(sI - A)^{-1} = N(s)/\alpha(s)$$

where $\alpha(s)$ is the characteristic polynomial of $A$,

$$\alpha(s) = \det(sI - A) = s^n + \alpha_1 s^{n-1} + .. + \alpha_n$$

and

$$N(s) = N_1 s^{n-1} + N_2 s^{n-2} + .. + N_n$$

where $N_i$ are $n \times n$ constant matrices. (Recall section 4.2.)

If we factor $\alpha(s)$ as

$$\alpha(s) = (s - \lambda_1)^{n_1} (s - \lambda_2)^{n_2} .. (s - \lambda_r)^{n_r}$$

where $\lambda_i$ will be the eigenvalues of $A$ and $n_1 + n_2 + .. + n_r = n = \dim(A)$, then the partial fraction expansion gives

$$(sI - A)^{-1} = \sum_{i=1}^{r} \sum_{k=1}^{n_i} \frac{C_{i,k}}{(s - \lambda_i)^k}$$

for suitable constant matrices $C_{i,j}$.

The inverse transform then gives

$$e^{At} = \sum_{i=1}^{r} \sum_{k=1}^{n_i} C_{i,k} \frac{k!}{(k-1)!} e^{\lambda_i t}$$
Let $\lambda_i = \sigma_i + j\omega_i$ then $|e^{\lambda_i t}| = e^{\sigma_i t} |e^{j\omega_i t}| = e^{\sigma_i t}$

3 cases

(i) $\sigma_i < 0$, $|t^{k-1}e^{\lambda_i t}| \rightarrow 0, k = 1, 2, 3, ...$

(ii) $\sigma_i > 0$, $|t^{k-1}e^{\lambda_i t}| \rightarrow \infty, k = 1, 2, 3, ...$

(iii) $\sigma_i = 0$

(a) $|t^{k-1}e^{\lambda_i t}| = 1 \quad k = 1$

(b) $|t^{k-1}e^{\lambda_i t}| \rightarrow \infty \quad k = 2, 3,$

Hence $x(t) \rightarrow 0$ as $t \rightarrow \infty$ for all $x_0$ if and only if all $\lambda_i$ satisfy $Re(\lambda_i) < 0$. (i.e. case i)).

Also $x(t)$ remains bounded as $t \rightarrow \infty$ for all $x_0$ if and only if $Re(\lambda_i) \leq 0$ and in partial fraction expansion of $(sI - A)^{-1}$ there are no terms of the form $C_{i,k}/(s - j\omega_i)^k$ with $k \geq 2$. (i.e. case (i) or (iii)(a).)
5 State Space Equations for Composite Systems

5.1 Cascade of Two Systems

Let $G_1(s)$ be realized by the state equation:

\[
\begin{align*}
\dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t) \\
\dot{w}(t) &= C_1 x_1(t) + D_1 u(t)
\end{align*}
\]

and $G_2(s)$ be realized by, (input $w$, output $y$)

\[
\begin{align*}
\dot{x}_2(t) &= A_2 x_2(t) + B_2 w(t) \\
y(t) &= C_2 x_2(t) + D_2 w(t)
\end{align*}
\]

then $Y(s) = G_2(s)G_1(s)U(s)$ is realized by

\[
\frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} u(t)
\]

\[
y(t) = \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} D_2 D_1 \end{bmatrix} w(t)
\]

The cascade realization of a single-input single-output system can hence be obtained if $G(s)$ is factored into second order factors such as

\[
G(s) = \frac{s^2 + c_1 s + d_1}{(s^2 + a_1 s + b_1)} \frac{(s^2 + c_2 s + d_2)}{(s^2 + a_2 s + b_2)} \ldots
\]

then the overall system can be realized as the cascade of these factors.
5.2 Parallel combination of two systems

Let $G_1(s)$ be realized by the state equation:

\[
\begin{align*}
\dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t) \\
y_1(t) &= C_1 x_1(t) + D_1 u(t)
\end{align*}
\]

and $G_2(s)$ be realized by (input $u$, output $y_2$)

\[
\begin{align*}
\dot{x}_2(t) &= A_2 x_2(t) + B_2 u(t) \\
y_2(t) &= C_2 x_2(t) + D_2 u(t)
\end{align*}
\]

then $Y(s) = (G_1(s) + G_2(s))U(s)$ is realized by

\[
\begin{align*}
\frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x(t) \\ (D_1 + D_2) u(t) \end{bmatrix}
\end{align*}
\]