SUPERVISORS' VERSION

Cambridge University Engineering Dept. Third year

Module 3F2: Systems and Control

LECTURE NOTES 4: CONTROLLABILITY AND STATE FEEDBACK

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1 Controllability

1.1 Controllability Gramian, Controllability matrix

A system:

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

is said to be **controllable** if for all initial conditions $\underline{x}(0) = \underline{x}_0$, terminal conditions $\underline{x}_1$, and $t_1 > 0$ there exists an input $u(t)$, $0 \le t \le t_1$ such that

$$\underline{x}(t_1) = \underline{x}_1.$$

That is, given $\underline{x}_0$, $\underline{x}_1$ and $t_1 > 0$, we wish to find $\underline{u}(t), 0 < t < t_1$, such that

$$\underline{x}_1 = \underline{x}(t_1) = e^{At_1}\underline{x}_0 + \int_0^{t_1} e^{A(t_1-t)}B\underline{u}(t)\,dt$$ (FROM LECTURE NOTES 1)

Note that this equation can be solved for all $\underline{x}_o$ and $\underline{x}_1$ if and only if it can be solved for all $\underline{x}_1$ with $\underline{x}_o = \underline{0}$. So we will now just consider the zero initial condition response.

Define the **controllability Gramian**, $W_c(t_1) \stackrel{def}{=} \int_0^{t_1} e^{A\tau}BB^T e^{A^T\tau}\, d\tau$.

Now assume that $W_c(t_1)$ has an inverse and let $\underline{u}(t) = \underline{u}_o(t) = B^T e^{A^T(t_1-t)} W_c(t_1)^{-1}\underline{x}_1$ when

$$
\begin{aligned}
\underline{x}(t_1) &= \int_0^{t_1} e^{A(t_1-t)}B\overbrace{B^T e^{A^T(t_1-t)} W_c(t_1)^{-1}\underline{x}_1}^{\underline{u}_o(t)}dt = \underbrace{\int e^{A(t_1-t)} BB^T e^{A^T(t_1-t)} dt}_{W_c(t_1)}\, W_c^{-1}\underline{x}_1 \\
&= W_c(t_1)W_c(t_1)^{-1}\underline{x}_1 = \underline{x}_1 \text{ as desired.}
\end{aligned}
$$

Hence if $\det W_c(t_1) \neq 0$ then we can reach any $\underline{x}(t_1)$ from $\underline{x}(0) = \underline{0}$ (and hence there exists $u(t)$ to go from any $\underline{x}(0)$ to any $\underline{x}(t_1)$).

$\det(W_c(t_1)) \neq 0 \Rightarrow$ CONTROLLABLE

(IF TRUE FOR ALL t_1)

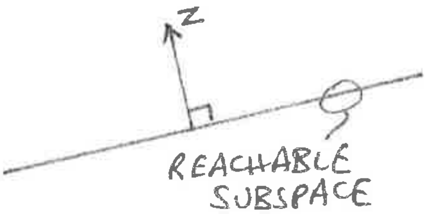
(Recall from section 3.1 of Lecture Notes 3:
$W_o(t_1) = \int_0^{t_1} e^{A^T\tau}C^T C e^{A\tau} d\tau$ ← *Observability Gramian*)

If $W_c(t_1)$ is a singular matrix there exists $\underline{z} \neq \underline{0}$ such that 'SQUARE ROOT'

$$\underline{z}^T W_c(t_1) = \underline{0} \Rightarrow \underline{z}^T W_c(t_1)\underline{z} = 0 \Rightarrow \underline{z}^T e^{At}B = \underline{0} \text{ for all } t$$

and hence $\quad = \underbrace{\int (B^T e^{A^T\tau} z)^T (B^T e^{A^T\tau} z) d\tau}_{\geq 0}$

$$\underline{z}^T\underline{x}(t_1) = \int_0^{t_1} \underbrace{\underline{z}^T e^{A(t_1-t)}B}_{=0}\underline{u}(t)\, dt = 0 \text{ for all } \underline{u}(t).$$



$\Rightarrow \underline{x}(t_1) \perp \underline{z}$ and the system is not controllable.

// Hence: System is controllable if and only if $\det W_c(t_1) \neq 0$. //

In section 3.1 of Lecture Notes 3 we showed

$$\text{Null space of } W_o(t_1) = \text{Null space of } Q.$$

Similarly we can show:

$$\text{Null space of } W_c(t_1) = \text{Null space of } P^T$$

where the **controllability matrix** P is given by

$$P \stackrel{def}{=} \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

Hence $\boxed{\text{The system is controllable if and only if rank } P = n}$

Example: E4 Exam question 3, 1997

3 Figure 2 shows a design for a hydraulically actuated table for simulating earthquakes. The table is denoted as ABC, with the point C constrained to move horizontally. DA and EB denote hydraulic rams which are pin-jointed at each end and can produce forces F_1 and F_2, respectively. The equations of motion (which should *not* be verified) are:

$$M\ddot{z} = F_1 \cos\phi_1 + F_2 \sin\phi_2$$

and

$$\begin{aligned}\frac{2}{a}\left(I + \frac{1}{4}Ma^2\cos^2\theta\right)\ddot{\theta} &= M\cos\theta\left(\frac{1}{2}a\dot{\theta}^2\sin\theta - g\right) + \\ &+ \left[\sin(\theta+\phi_1) + \sin\phi_1\cos\theta\right]F_1 + \\ &+ (\cos\theta\cos\phi_2)F_2\end{aligned}$$

where

$$\tan\phi_1 = \frac{a\sin\theta}{a + z - \frac{1}{2}a\cos\theta}, \quad \tan\phi_2 = \frac{z}{a + \frac{1}{2}a\sin\theta},$$

M and I are constants, a and z are the lengths shown in Fig. 2, and θ, ϕ_1, ϕ_2 are the angles shown in the figure.

(a) What conditions are satisfied at an equilibrium? Determine values F_{1e} and F_{2e} of the forces F_1 and F_2, which will give an equilibrium position $\theta = \theta_e$ and $z = z_e$, if $\theta_e = 0$ and $z_e = a$. [4]

(b) The linearised equations about the equilibrium $(\theta_e = 0, z_e = a)$ are: [6]

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

where

$$\underline{x} = \left[\theta, z-a, \dot{\theta}, \dot{z}\right]^T, \ \underline{u} = [F_1 - F_{1e}, F_2 - F_{2e}]^T, \ A = \begin{bmatrix} 0 & I_2 \\ P & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ Q \end{bmatrix},$$

$$P = \begin{bmatrix} \frac{-13}{12\tau^2} & -\frac{1}{2a\tau^2} \\ -\frac{g}{4} & \frac{g}{2a} \end{bmatrix}, \ Q = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}Mg\tau^2} \\ \frac{1}{M} & \frac{1}{\sqrt{2}M} \end{bmatrix}, \ \tau^2 = \frac{2I}{Mag} + \frac{a}{2g},$$ and I_2 is the 2×2 identity matrix.

(cont.

Verify that the term $-\frac{g}{4}$, which appears in P, is correct. (Do *not* verify any other terms. Assume that the nonlinear equations are correct.)

(c) Is the linearised system of part (b) controllable from $\underline{u}$? Is it controllable from u_1 (the first element of $\underline{u}$) alone? [6]

(d) Comment on the difference in the achievable behaviour of this system when only u_1 is available for control, and when the complete vector $\underline{u}$ is available. [4]

FOCUS ON THESE HERE.

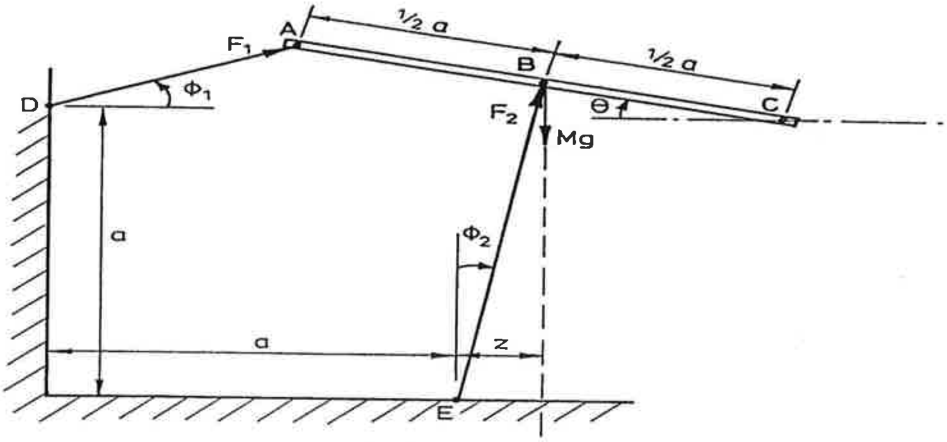


Fig. 2

(TURN OVER

Note that if the system is controllable this just implies that the state can pass through any value; it does not imply that there is an input to keep the state at this value (which depends on the equilibrium conditions):

(c)

Controllability from both inputs: $\dot{x} = Ax + Bu$,

Controllable? rank$[B, AB, A^2B, A^3B] = 4$? $(n = 4)$

Controllability from u_1 alone: $\dot{x} = Ax + b_1u_1$, $(B = [b_1, b_2])$

Controllable? rank$[b_1, Ab_1, A^2b_1, A^3b_1] = 4$?

(d)

Achievable steady-state (equilibrium) behaviour:

With both inputs:

$\dot{x} = Ax + Bu$, $\dot{x}_e = 0$ at equilibrium, so: $0 = Ax_e + Bu_e$ possible for some u_e?

With input u_1 only: $0 = Ax_e + b_1u_{1e}$ possible for some u_{1e}?

1.2 Minimum Energy Input

Theorem 1.1 *The input,*$\underline{u}(t) = \underline{u}_o(t) = B^T e^{A^T(t_1-t)} W_c(t_1)^{-1}\underline{x}_1$, *takes the state from* $\underline{x}(0) = \underline{0}$ *to* $\underline{x}(t_1) = \underline{x}_1$ *and in addition is the input with minimum energy that achieves this.*

Proof:

Let $\underline{u}(t) = \underline{u}_o(t) + \underline{u}_1(t)$ then $\underline{x}(t_1) = \underline{x}_1 + \int_0^{t_1} e^{A(t_1-t)}B\underline{u}_1(t)\,dt$ and hence $\underline{x}(t_1) = \underline{x}_1$ implies,

$$\int_0^{t_1} e^{A(t_1-t)}B\underline{u}_1(t)\,dt = \underline{0}$$

Energy in $\underline{u}(t)$ for $0 < t < t_1$ is defined as:

$$\int_0^{t_1} \|\underline{u}(t)\|^2 dt = \int_0^{t_1} \underline{u}(t)^T\underline{u}(t)\,dt = \int (\underline{u}_o + \underline{u}_1)^T(\underline{u}_o + \underline{u}_1)\,dt$$

$$= \int_0^{t_1} \left(\underline{u}_o(t)^T\underline{u}_o(t) + \underline{u}_o(t)^T\underline{u}_1(t) + \underline{u}_1(t)^T\underline{u}_o(t) + \underline{u}_1(t)^T\underline{u}_1(t)\right) dt$$

Now

$$\int_0^{t_1} \underline{u}_o(t)^T\underline{u}_1(t)\,dt = \int_0^{t_1} \underline{x}_1^T W_c(t_1)^{-1} e^{A(t_1-t)}B\underline{u}_1(t)\,dt = 0 = \int_0^{t_1} \underline{u}_1(t)^T\underline{u}_o(t)\,dt$$

and

$$\int_0^{t_1} \underline{u}_o(t)^T\underline{u}_o(t)\,dt = \underline{x}_1^T W_c(t_1)^{-1}\int_0^{t_1} e^{A(t_1-t)}BB^Te^{A^T(t_1-t)}\,dt\,W_c(t_1)^{-1}\underline{x}_1 = \underline{x}_1^T W_c(t_1)^{-1}\underline{x}_1$$

Hence

$$\int_0^{t_1} \underline{u}(t)^T\underline{u}(t)\,dt = \underline{x}_1^T W_c(t_1)^{-1}\underline{x}_1 + \int_0^{t_1} \underline{u}_1(t)^T\underline{u}_1(t)dt$$

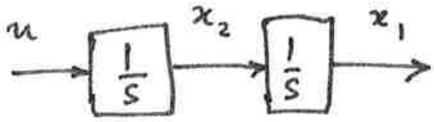
Since both terms are ≥ 0 the minimum energy is achieved when $\underline{u}_1(t) = \underline{0}$ and hence $\underline{u}(t) = \underline{u}_o(t)$ when

$$\min \int_0^{t_1} \underline{u}(t)^T\underline{u}(t)\,dt = \underline{x}_1^T W_c(t_1)^{-1}\underline{x}_1.$$

Note that if $W_c(t_1)$ is nearly singular then a very large energy input is required to reach certain states.

NOTE ANALOGY WITH LARGE $\underline{x}(0)$ GIVING SMALL $\int_0^{t_1} \|y(t)\|^2 dt$ IF $W_o(t_1)$ IS NEARLY SINGULAR.

— LECTURE NOTES 3.



Example

DOUBLE-INTEGRATOR

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad \Rightarrow e^{At}B = \begin{bmatrix} t \\ 1 \end{bmatrix},$$

$$\Rightarrow W_c(t_1) = \int_0^{t_1} \begin{bmatrix} t^2 & t \\ t & 1 \end{bmatrix} dt = \begin{bmatrix} \frac{1}{3}t_1^3 & \frac{1}{2}t_1^2 \\ \frac{1}{2}t_1^2 & t_1 \end{bmatrix}$$

$$W_c(t_1)^{-1} = \begin{bmatrix} 12/t_1^3 & -6/t_1^2 \\ -6/t_1^2 & 4/t_1 \end{bmatrix}$$

Minimum energy to go from $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is hence $12/t_1^3$ achieved when

$$\underline{u}(t) = \left(e^{A(t_1-t)}B\right)^T W_c(t_1)^{-1}\underline{x}_1 = \begin{bmatrix} t_1 - t & 1 \end{bmatrix} \begin{bmatrix} 12/t_1^3 \\ -6/t_1^2 \end{bmatrix} = \frac{6}{t_1^3}(t_1 - 2t)$$

Note in this example: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

so $A^2 = 0$, $A^3 = 0$, etc.

$$\therefore e^{At} = I + At + \frac{(At)^2}{2!} + \cdots$$

$$= I + At$$

$$= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

MOSTLY SKIPPED IN LECTURES. SUITABLE FOR DISCUSSION IN SUPERVISION (IF TIME ALLOWS).

1.3 Example: Pendulum Control

In the pendulum laboratory experiment we have the following linearized equations:

$$\underline{\dot{x}} = \frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \\ L\theta \\ L\dot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \omega_1^2 - \omega_0^2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_0^2 & 0 \end{bmatrix}}_{A} \underline{x} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{B} u$$

where x is the position of the carriage, θ is the angle of the pendulum, u is proportional to the force on the carriage, L is the length of the pendulum, ω_1 is the natural frequency of the pendulum with the carriage fixed and ω_0 is the natural frequency of the pendulum with the carriage free to move.

It is not too difficult to verify that,

$$F(t) \stackrel{def}{=} e^{At}B = \begin{bmatrix} t - \frac{(\omega_0^2-\omega_1^2)}{\omega_0^3}(\omega_0 t - \sin(\omega_0 t)) \\ 1 + \frac{(\omega_0^2-\omega_1^2)}{\omega_0^2}(\cos(\omega_0 t) - 1) \\ \frac{1}{\omega_0}\sin(\omega_0 t) \\ \cos(\omega_0 t) \end{bmatrix}$$

It becomes rather tedious to then calculate,

$$W_c(t_1) \stackrel{def}{=} \int_0^{t_1} F(t)F(t)^T\, dt$$

and even more challenging to calculate $W_c(t_1)^{-1}$!!

However given values for the parameters this can be solved numerically or even using symbolic algebra packages (although you are likely to get several pages of output!).

A sample numerical calculation is given in Figs. 1 and 3 for $t_1 = 0.75$ s and $t_1 = 0.5$ s, and final state is at rest 0.4 m along the carriage. For an 'animation' see Figs. 2 and 4.

A Taylor series expansion of $W_c(t_1)^{-1}$ can be obtained using symbolic algebra and gives the minimum energy to reach $\underline{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is approximately $\frac{100800}{\omega_1^4} t_1^{-7}$ for small t_1 — showing that the required input increases dramatically as t_1 becomes small.

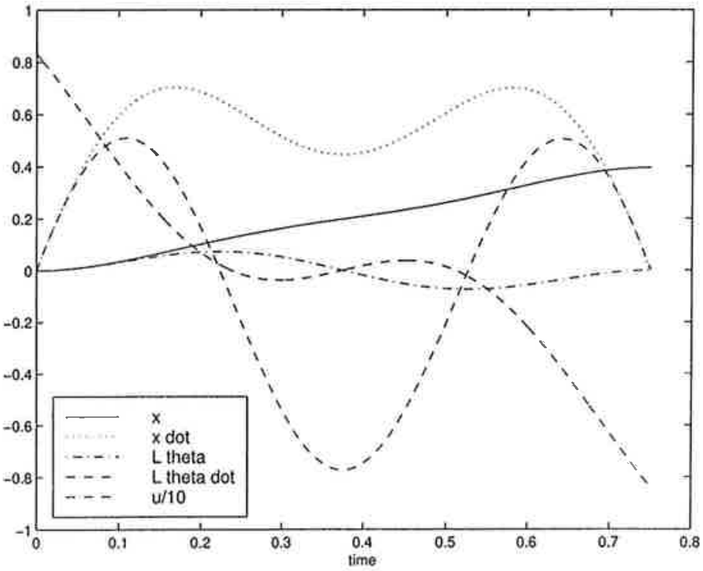


Figure 1: Pendulum trajectories

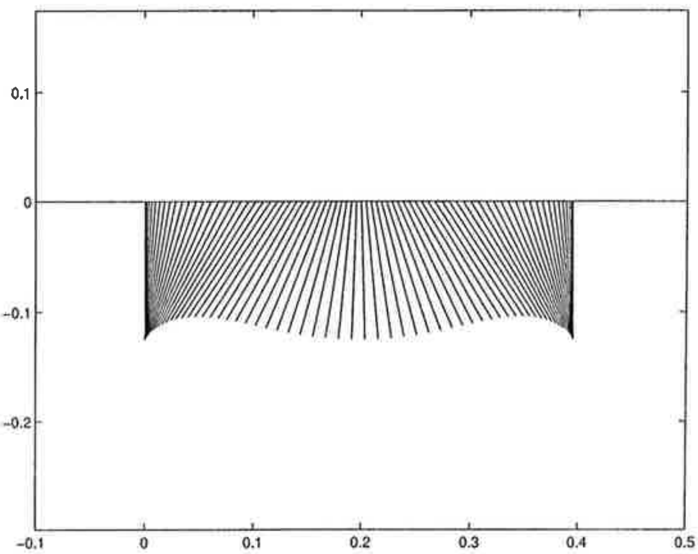


Figure 2: Animation of pendulum

(time between frames is 0.01 seconds)

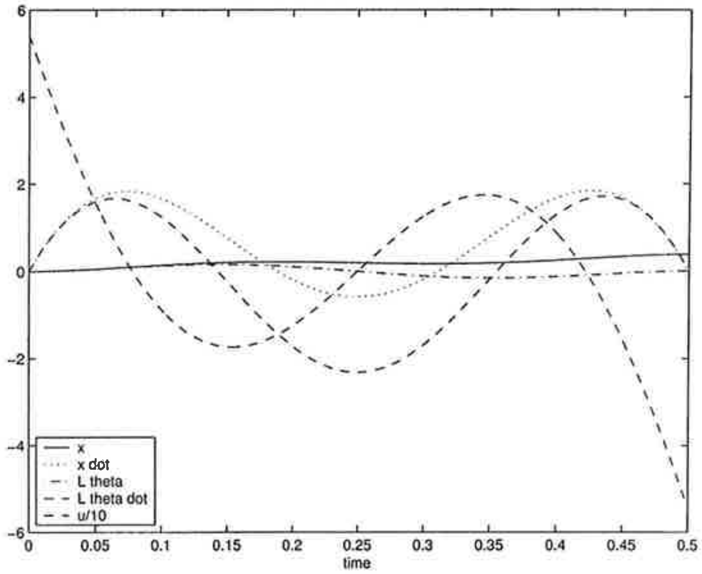


Figure 3: Pendulum trajectories

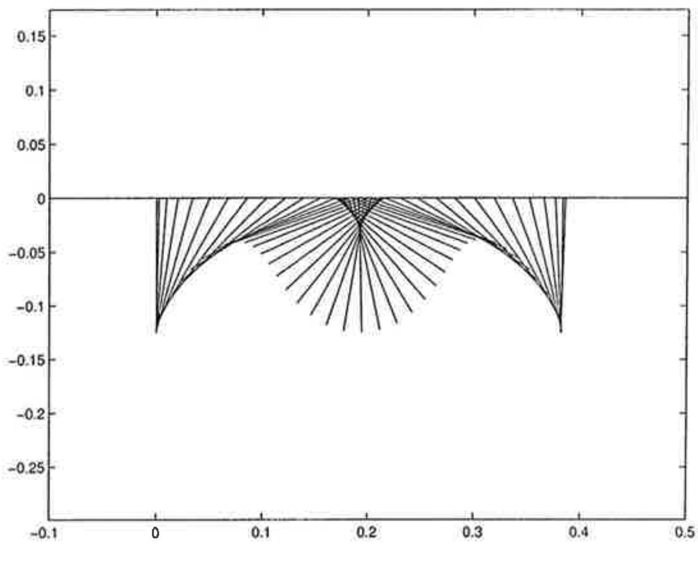


Figure 4: Animation of pendulum

DOESN'T WORK ANY MORE
BUT SHOWS THE IDEA. ASK A KEEN STUDENT TO FIX IT?

The MATLAB code to draw these figures is:

```
%  this file calculates the minimum 'energy' input to move the pendulum from
% zero state to state x1 at time t1.
% Instructions for use:
% set t1 and x1, e.g. t1=0.8 .7 .6 .55, x1=[.4;0;0;0]
% then calculates the min energy trajectory.
% then run crane_ol.mdl  to simulate the linear model with this open loop u.
% For demo purposes the animation at the end is best.  Paste this into
% the matlab window.

load pend  % loads the state space matrices
% calculate the optimal input u.
del=t1/1000; t_u=0:del:t1;
exp_del=expm(a*del);
exp_at=eye(4);
Wc=b*b'*del;
for t_i=t_u,
    exp_at=exp_at*exp_del;
    Wc=Wc+exp_at*b*b'*exp_at'*del;
end




x=Wc\x1;
u=zeros(length(t_u),1);
for i=length(t_u):-1:1,
  u(i)=b'*x;
  x=exp_del'*x;
end
return

% this signal is then input to the simulation in crane_ol.mdl
% can then plot results with
plot(t_sim,x(:,1),'b-',t_sim,x(:,2),'g:',t_sim,x(:,3),'r-.',t_sim,x(:,4),'c--',t_u,u/10,'m--
legend('x ','x dot','L theta','L theta dot','u/10',3);
xlabel('time');grid on; title('Pendulum control trajectories')

% now let's animate the results
L=0.125;
plot([-.1;.5],[0;0]), axis([-.1 .5 -.2 .05]); axis equal;  hold on
for i=1:length(t_sim),
  plot([x(i,1); x(i,1)-L*sin(x(i,3)/L)],[0;-L*cos(x(i,3)/L)],'c','EraseMode','none')
  drawnow; tic; while toc<.15, end
end
hold off
```

1.4 Reachable States and Minimal Realizations

We have seen in the previous section that if $W_c(t_1)$ is nearly singular then some directions in the state space are very difficult to reach, and if $W_c(t_1)$ is singular then some states cannot be reached and that $\underline{x}(t_1)$ is necessarily perpendicular to the null space of $W_c(t_1)$. It can in fact be shown *(details are omitted)* that the states that can be reached at time t_1 from $\underline{x}(0) = \underline{0}$ are precisely of the form:

= COLUMN SPAN

$= \{\underline{x} : \underline{x} = W_c(t_1)\underline{v}$ FOR SOME $\underline{v}\}$

$$
\begin{aligned}
\text{Reachable states} \quad &= \quad \text{Range space of } W_c(t_1) \\
&= \quad \text{Range space of } P = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}
\end{aligned}
$$

(since null spaces of P^T and $W_c(t_1)$ are the same, and using Fact 2.2 from Lecture Notes 3.)

<u>EXAMPLE</u> $(n=3)$:

$$P = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \left.\begin{matrix}\text{COLUMN SPAN} \\ \text{OR} \\ \text{RANGE SPACE}\end{matrix}\right\} = \alpha \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Definition 1.2 *A set of state equations given by (A, B, C, D) is called a minimal realization of its transfer function, $G(s) = D + C(sI - A)^{-1}B$, if there does not exist a state space realization of $G(s)$ with a lower state dimension.*

In section 3.2 of Lecture Notes 3 we saw that if a system was not observable then there was a change of state coordinates that gave an observable realisation of the transfer function with r states where $r = \text{rank}(Q)$.

If this system with r states is not controllable its state dimension could be further reduced in a similar manner and we are left with a state-space realisation of the transfer function that is both controllable and observable. It turns out that *(proof omitted)*:

Theorem 1.3 *A realization is <u>minimal</u> if and only if it is both controllable and observable.*

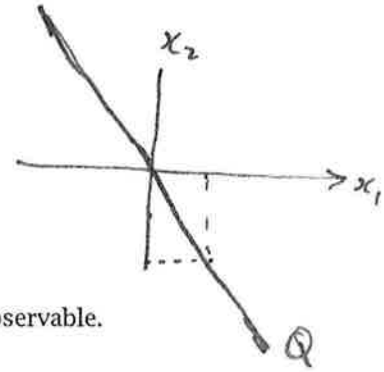
RECALL: $A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$, $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $C = [C_1, 0]$

$\Rightarrow (A_{11}, C_1)$ OBSERVABLE

In single-input/single-output systems this means that if a system is either not controllable or not observable then there are pole/zero cancellations in the transfer function.

Example:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1, 1], \quad D = 0$$



Observability:

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \Rightarrow \text{rank}(Q) = 1 \Rightarrow \text{NOT observable}$$

$$Qx_0 = 0? \quad \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ so } x = \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix} \text{ is not observable.}$$

Controllability:

$$P = [B, AB] = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \Rightarrow \text{rank}(P) = 2 \Rightarrow \text{controllable}$$

Example continued

Transfer function:

$$G(s) = C(sI - A)^{-1}B + D = [1, 1] \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$= \frac{[1 \quad 1] \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{s(s+3) + 2}$$

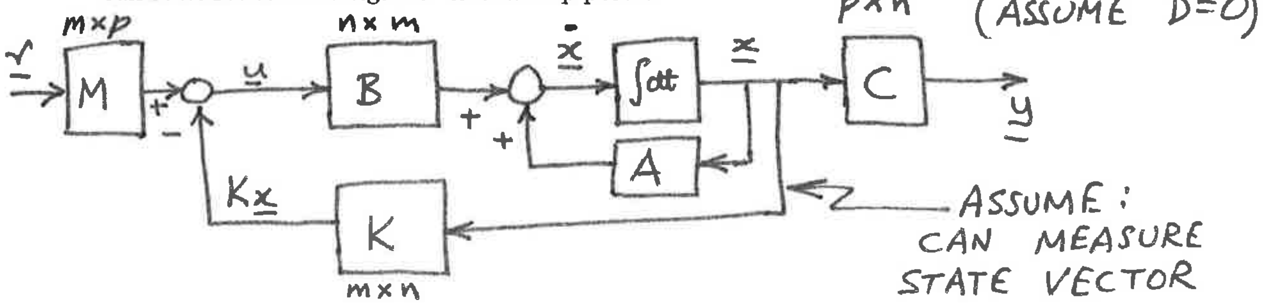
$$= \frac{[1 \quad 1] \begin{bmatrix} 1 \\ s \end{bmatrix}}{(s+1)(s+2)}$$

$$= \frac{\cancel{s+1}}{\cancel{(s+1)}(s+2)} = \frac{1}{s+2} \quad \text{pole-zero cancellation}$$

PROJECTS!

2 State Feedback

The response of a system is largely determined by the location of its closed loop poles. Can state feedback assign the closed loop poles?



System: $\dot{\underline{x}} = A\underline{x} + B\underline{u}$, with state feedback: $\underline{u} = -K\underline{x} + M\underline{r}$, giving closed loop:

$$\dot{\underline{x}} = (A - BK)\underline{x} + BM\underline{r}.$$

Theorem 2.1 *The closed loop poles will be the eigenvalues of $(A - BK)$ which can be placed arbitrarily by choice of K if and only if (A, B) is controllable.*

(The derivation of this is entirely analogous to the result in section 4.2 of Lecture Notes 3, that the eigenvalues of $(A - LC)$ can be arbitrarily assigned by choice of L — if (A, C) is observable).

Where to place the poles?

- stable
- fast enough
- but not too fast since this might
  - saturate actuators
  - give poor stability margins.

$(A+BK)^{-1}$ EXISTS IF ASYMPTOTICALLY STABLE. (NO EIGENVALUES AT 0 ∴ NOT SINGULAR.)

2.1 Steady-State Gain

Servo-system. Suppose we want $\underline{y}(t) \to \underline{r}$.

Two approaches to obtain the correct DC gain:

(a) **Choice of** M

$$\begin{cases} \dot{\underline{x}} &= (A-BK)\underline{x} + BM\underline{r} \\ \underline{y} &= C\underline{x} \end{cases}$$

$$\dot{\underline{x}} = 0 \Rightarrow \underline{x} = -(A-BK)^{-1}\underbrace{B}_{n\times m}\underbrace{M}_{m\times p}\underline{r}$$

RANK$(BM) \leq \min(m,p)$

In steady-state: $\dot{\underline{x}} = \underline{0} \Rightarrow \underline{y} = C(-A+BK)^{-1}BM\underline{r}$. NEED RANK $= p$

Choose M such that $C(-A+BK)^{-1}BM = I$ and $\underline{y}(t) \to \underline{r}$ after a step change with speed given by eigenvalues of $(A-BK)$. [Such an M usually exists if $\dim(\underline{u}) \geq \dim(\underline{y})$ but not otherwise].

This requires *exact knowledge* of the system matrices. The steady-state error being zero is not *robust* to small changes in the system. Also need to know an equilibrium condition.

(b) **Integral Action**

Integral action can be incorporated by augmenting the state by the integral of the error, i.e.

$$\dot{\underline{e}} = \underline{r} - \underline{y} = \underline{r} - C\underline{x} \qquad \underline{e}(t) = \int_0^t [\underline{r}(\tau) - \underline{y}(\tau)]\,d\tau$$

which gives

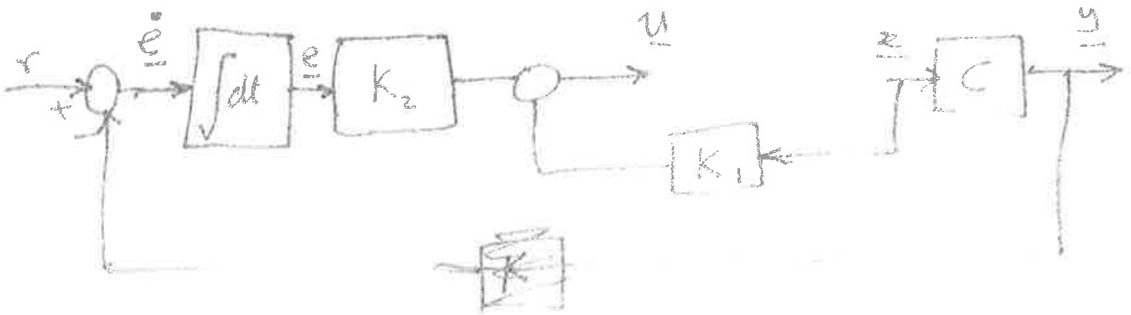
$$\begin{bmatrix} \dot{\underline{x}} \\ \dot{\underline{e}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{e} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \underline{u} + \begin{bmatrix} 0 \\ I \end{bmatrix} \underline{r}$$

with state feedback: $\underline{u} = -K_1\underline{x} - K_2\underline{e}$. $= -\begin{bmatrix} K_1 & K_2 \end{bmatrix}\begin{bmatrix} \underline{x} \\ \underline{e} \end{bmatrix}$

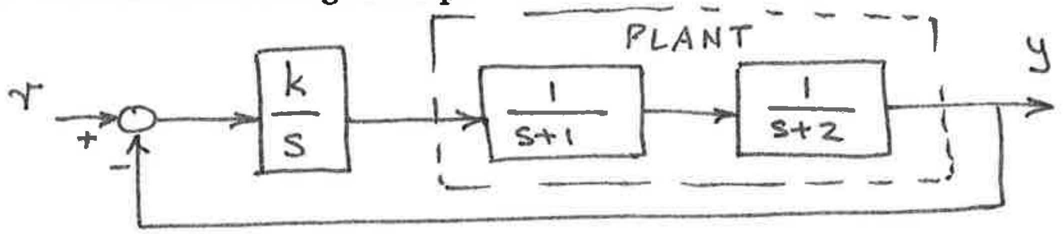
Choose K_1, K_2 to assign the closed-loop poles (possible if augmented system controllable) and then $\underline{e}(t) \to 0 \Rightarrow \underline{y}(t) \to \underline{r}$ after a step change.

Robust to small changes in A, B, C, K.

Does not require knowledge of the equilibrium condition.



2.2 State Feedback Design Example



Plant $G(s) = \frac{1}{s+2} \times \frac{1}{s+1}$

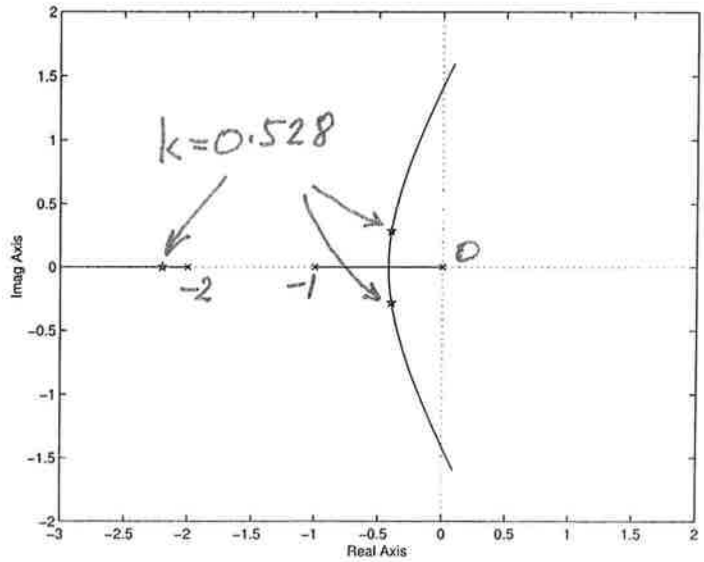
Design Spec

Response in y to a step command on r to have zero offset and small overshoot.

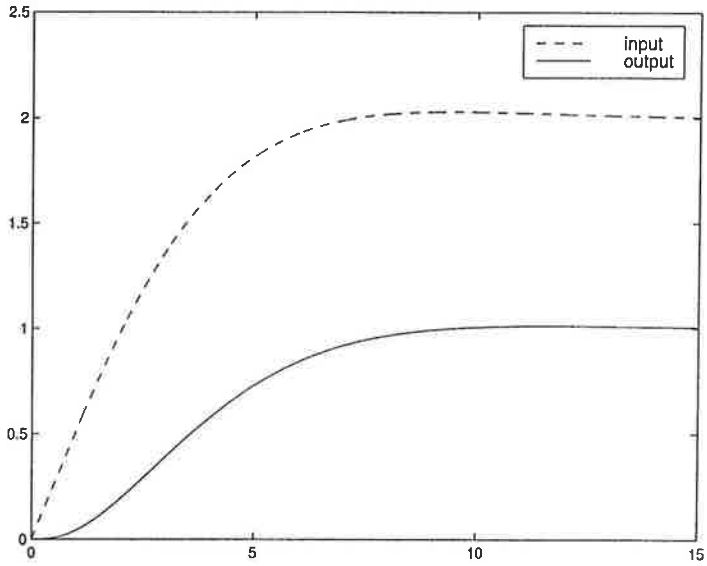
Integral Action Controller

To attain zero offset need to insert an integrator in the open loop. Assume only the output is used for feedback. CHOOSE $\frac{k}{s}$ FOR SIMPLICITY. P+I $\left(k_1 + \frac{k_2}{s}\right)$ WOULD BE MORE USUAL.

Open loop transfer function = $\frac{k}{s(s+1)(s+2)}$, with closed loop poles roots of $s^3 + 3s^2 + 2s + k = 0$

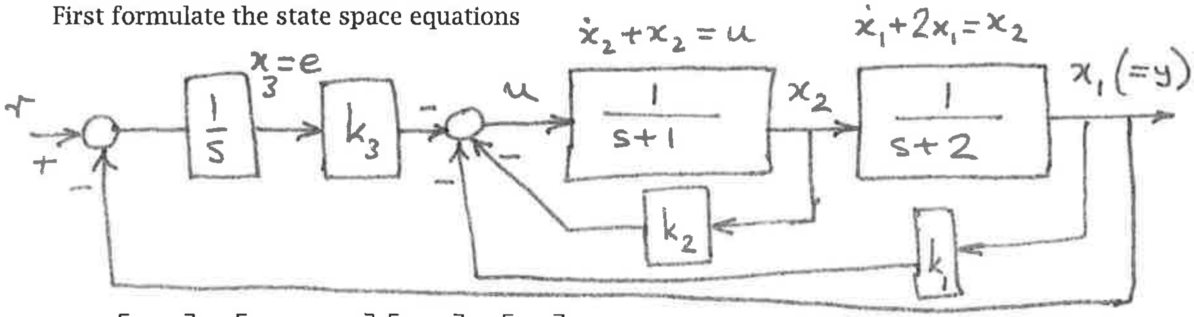


A passable design would be $k = 0.528$ which gives closed-loop poles at $-2.2, -0.4 \pm j0.283$. The closed loop response will be:



State Feedback Design

First formulate the state space equations



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \text{PLANT}$$

We will again need to add an integrator to ensure a zero offset. The state feedback formulation will now be

$$u = -k_1 x_1 - k_2 x_2 - k_3 x_3$$

$$= -\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{OR} = -\underbrace{\begin{bmatrix} k_1 & k_2 \end{bmatrix}}_{K_1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \underbrace{k_3}_{K_2} e$$

(SEE SEC. 2.1(b))

The extra state variable x_3 has been added to integrate the output error -

$$\dot{x}_3 = -x_1 + r$$

this gives an augmented set of state equations

$$\frac{d}{dt}\underline{x} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

and the proposed feedback scheme is given by $u = -\underline{k}^T\underline{x} = -\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}\underline{x}$, so the closed loop state equations become

$$\underline{\dot{x}} = (A - B\underline{k}^T)\underline{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r = \begin{bmatrix} -2 & 1 & 0 \\ -k_1 & -1-k_2 & -k_3 \\ -1 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

The closed loop characteristic equation becomes

$$\begin{aligned} \det[\lambda I - (A - B\underline{k}^T)] &= \det \begin{bmatrix} \lambda + 2 & -1 & 0 \\ k_1 & \lambda + 1 + k_2 & +k_3 \\ 1 & 0 & \lambda \end{bmatrix} = (\lambda + 2)(\lambda + 1 + k_2)\lambda + k_1\lambda - k_3 \\ &= \lambda^3 + (3 + k_2)\lambda^2 + (2 + k_1 + 2k_2)\lambda - k_3 \end{aligned}$$

Suppose we desired all the closed loop poles to be at -5, then the required characteristic equation would be:

$$(\lambda + 5)^3 = \lambda^3 + 15\lambda^2 + 75\lambda + 125$$

Equating coefficients now gives

$$3 + k_2 = 15 \Rightarrow \underline{k_2 = 12}, \quad 2 + k_1 + 2k_2 = 75 \Rightarrow \underline{k_1 = 49}, \quad \underline{k_3 = -125}$$

The transfer function from r to y can now be computed as:-

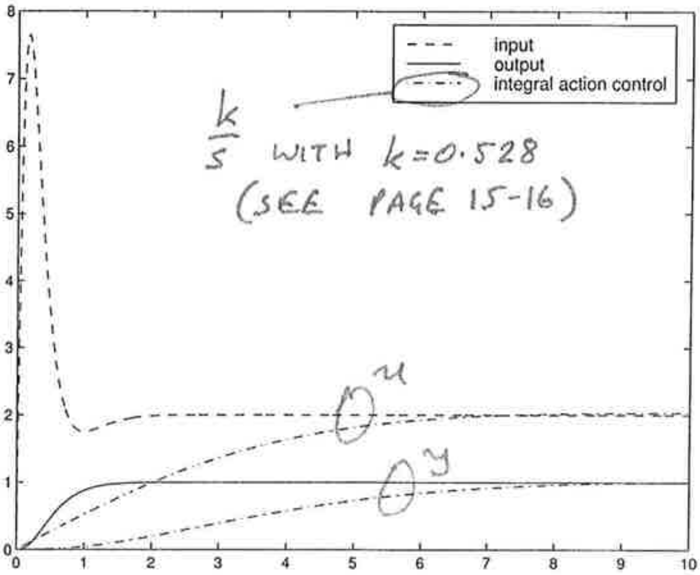
$$T(s) = \frac{Y(s)}{R(s)} = C[sI - (A - Bk^T)]^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix} = [1\ 0\ 0]\begin{bmatrix}s+2 & -1 & 0\\ 49 & s+13 & -125\\ 1 & 0 & s\end{bmatrix}^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix} = \frac{125}{(s+5)^3}$$

Also the transfer function from r to u can be computed as:-

$$\frac{U(s)}{R(s)} = -\frac{\underline{k}^T \underline{X}(s)}{R(s)} = [49\ 12\ -125]\begin{bmatrix}s+2 & -1 & 0\\ 49 & s+13 & -125\\ 1 & 0 & s\end{bmatrix}^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix} = \frac{125\underbrace{(s+1)(s+2)}_{\text{CANCELS PLANT POLES}}}{(s+5)^3}$$

(IN THIS CASE. NOT GENERALLY.)

The step responses are thus:-



Very similar to using a 3-term controller which could also give arbitrarily fast response.

3 Observers with State Feedback

$$\text{SYSTEM} \quad \begin{cases} \dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} \end{cases}$$

$$\text{OBSERVER} \quad \begin{cases} \dot{\hat{\underline{x}}} &= A\hat{\underline{x}} + B\underline{u} + L(\underline{y} - \hat{\underline{y}}) = (A - LC)\hat{\underline{x}} + B\underline{u} + L\underline{y} \\ \hat{\underline{y}} &= C\hat{\underline{x}} \end{cases}$$

$$\text{CONTROLLER} \quad \left\{ \underline{u} = -K\hat{\underline{x}} + M\underline{r} \right.$$

$$\begin{aligned} \text{Error:} \quad \underline{e} &= \underline{x} - \hat{\underline{x}} \\ \dot{\underline{e}} &= (A - LC)\underline{e} \\ \underline{u} &= -K(\underline{x} - \underline{e}) + M\underline{r} \\ \dot{\underline{x}} &= (A - BK)\underline{x} + BK\underline{e} + BM\underline{r} \end{aligned}$$

$$\Rightarrow \begin{cases} \begin{bmatrix} \dot{\underline{x}} \\ \dot{\underline{e}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{e} \end{bmatrix} + \begin{bmatrix} BM \\ 0 \end{bmatrix} \underline{r} \\ \underline{y} = [C \;\; 0] \begin{bmatrix} \underline{x} \\ \underline{e} \end{bmatrix} \end{cases}$$

NB: Eigenvalues of $\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$ = {Eigenvalues of X} ∪ {Eigenvalues of Z}

So closed-loop poles are at the eigenvalues of $(A - BK)$ and those of $(A - LC)$.
STATE FEEDBACK OBSERVER

$\underline{e}$ is not affected by $\underline{r}$ so that $\underline{e}(t) \to \underline{0}$.

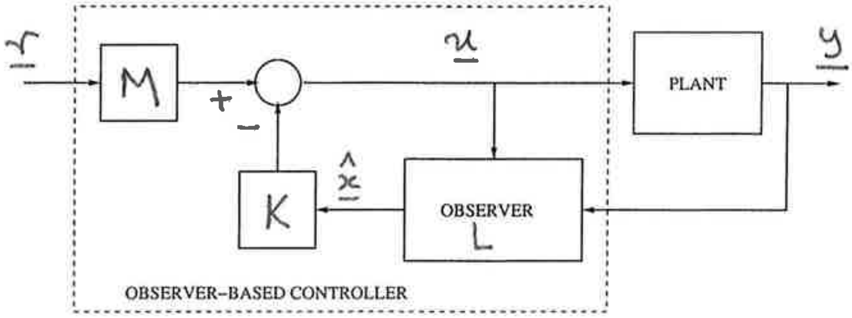
Separation of estimation and control.

Can this always be done?

If (A, B) is controllable and (A, C) is observable, then no problems —
we can place all eigenvalues anywhere we want.
If all uncontrollable and unobservable modes (states) are stable, may still be OK.
If any uncontrollable or unobservable modes are unstable, then NOT OK,
since they will remain in the closed-loop system.

$$\det\left[\lambda I - \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}\right] = \det\begin{bmatrix} \lambda I - X & -Y \\ 0 & \lambda I - Z \end{bmatrix} = \det(\lambda I - X)\det(\lambda I - Z)$$

Block diagram:



If $\underline{r} \equiv 0$ then this structure is the same as for a dynamic precompensator.

For $\underline{r} \not\equiv 0$ the structures are different.