

Cambridge University Engineering Dept.

Third year

Module 3F2: Systems and Control**LECTURE NOTES 2: 'CLASSICAL' METHODS**

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1 Revision of Feedback Control

1.1 Why use Feedback?

- To reduce effects of uncertainty:

- Disturbances
- Model errors

Wind/waves, Friction, Impurities, ...

Approximations, Tolerances, Ageing, ...

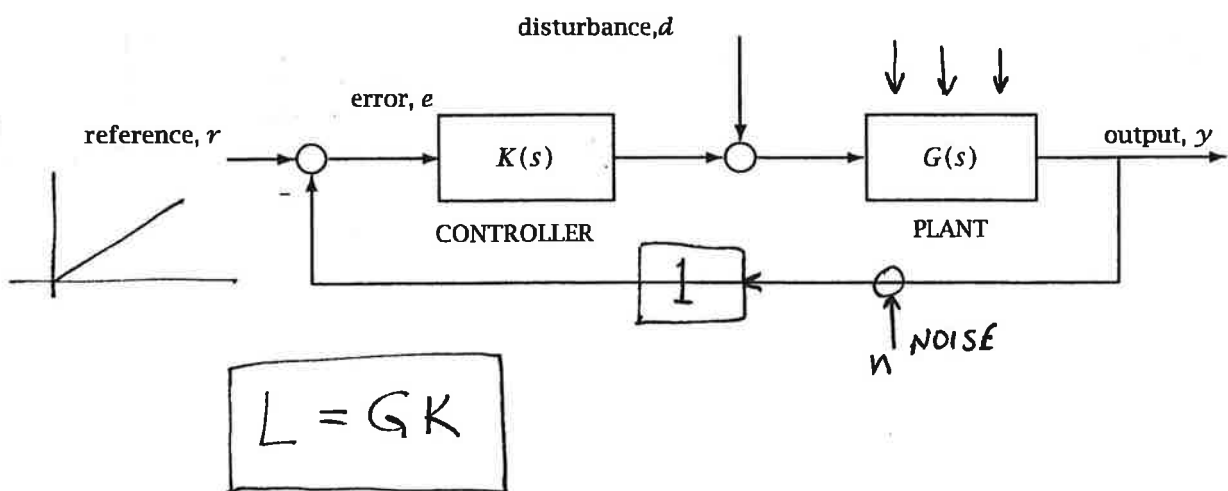
- To stabilise unstable system:

- Inverted pendulum, Bicycle
- High-performance fighter aircraft (*Fly-by-wire*)
- Helicopter, Submarine (depth)
- Exothermic chemical reactor, Nuclear reactor

Problems with feedback:

- May destabilise system
- Sensors introduce noise

1.2 The Standard Feedback Loop



↑ NB: ORDER OF MULTIPLICATION
IF MATRICES

1.3 Sensitivity and Complementary Sensitivity

Let $L(s)$ be the (open) loop transfer function: $L(s) = G(s)K(s)$

"Return-ratio"

Sensitivity: $S(s) = \frac{1}{1+L(s)}$

(Multivariable: $S(s) = [I + L(s)]^{-1}$)

$$\frac{\bar{e}}{\bar{r}} = S = \frac{dT/T}{dG/G}, \quad \frac{\bar{y}}{\bar{d}} = GS$$

Small $|S| \Rightarrow$ feedback is beneficial.

Complementary Sensitivity: $T(s) = \frac{L(s)}{1+L(s)}$

(Multivariable: $T(s) = L(s)[I + L(s)]^{-1}$)

$$\frac{\bar{y}}{\bar{r}} = -\frac{\bar{y}}{\bar{n}} = T$$

(n = sensor noise)

$T \approx 1 \Rightarrow$ good "tracking" but no noise filtering.

Fact: $S(s) + T(s) = 1$

(Multivariable: $S(s) + T(s) = I$)

Hence **trade-off**:

- Small $|S(j\omega)|$ at low frequencies
- Small $|T(j\omega)|$ at high frequencies.

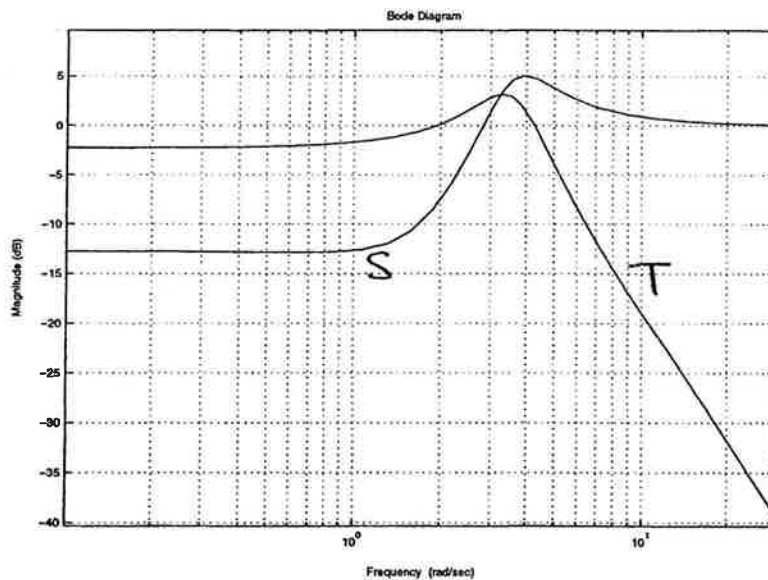


Figure 1: Sensitivity (S) and Complementary Sensitivity (T)

1.4 Steady-state Error

Constant reference: $r(t) = \alpha$, $\tilde{r}(s) = \frac{\alpha}{s}$. Assume closed-loop is *asymptotically stable*.

$$\lim_{t \rightarrow \infty} e(t) = S(0)\alpha = \frac{1}{1+L(0)}\alpha \quad (\text{Final Value Theorem or } S(j\omega) \text{ with } \omega = 0)$$

$$\lim_{t \rightarrow \infty} e(t) = 0 \text{ if } |G(0)K(0)| = \infty \Leftarrow G(s)K(s) \text{ has pole at } s = 0 \text{ — integral action}$$

Constant disturbance: $d(t) = \beta$, $\tilde{d}(s) = \frac{\beta}{s}$.

$$\frac{\tilde{e}}{\tilde{d}} = -GS. \text{ Zero steady-state error} \Leftarrow K(s) \text{ has pole at } s = 0.$$

$$G(0)S(0) = \frac{G(0)}{1+G(0)K(0)}$$

Ramp reference: $r(t) = \alpha t$, $\tilde{r}(s) = \frac{\alpha}{s^2}$.

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s\tilde{e}(s) = \lim_{s \rightarrow 0} \frac{\alpha}{s} S(s). \quad (\text{Final Value Theorem})$$

Hence:

- *Finite* steady-state error $\Leftarrow G(s)K(s)$ has a pole at $s = 0$.
- *Zero* steady-state error $\Leftarrow G(s)K(s)$ has *two* poles at $s = 0$.

Small steady-state error requires high gain at "DC".

1.5 The Nyquist Stability Theorem

Motivation:

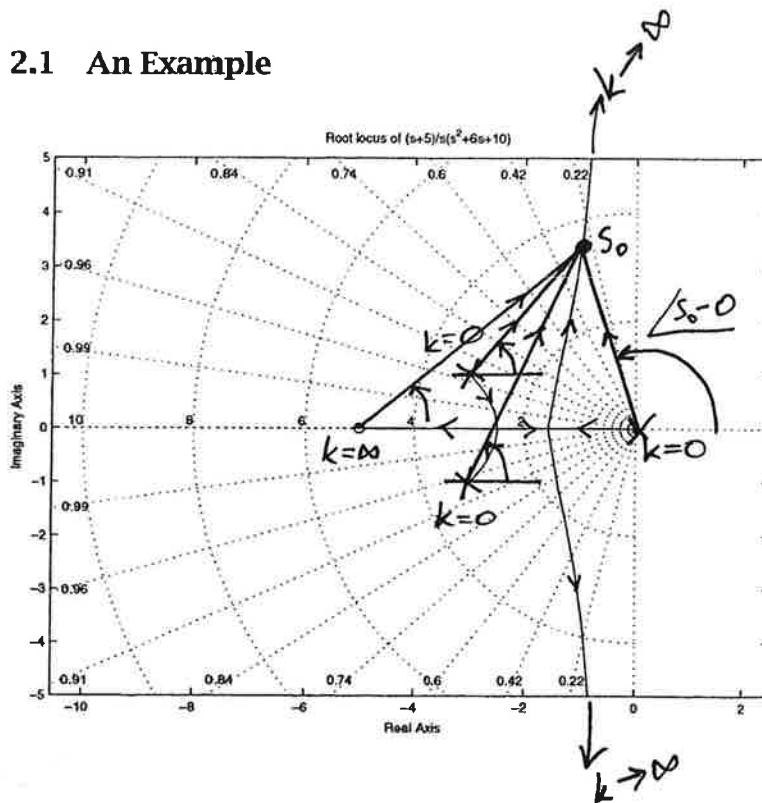
- The frequency response can be determined experimentally.
- Or from transfer function or state-space model.
- Want a test for closed-loop stability that uses *open-loop* information.

Theorem:

- Plot $L(j\omega) = G(j\omega)K(j\omega)$ on the Argand diagram, for $-\infty < \omega < +\infty$ — the *Nyquist plot*.
- The closed loop is stable if and only if the Nyquist plot encircles the point $-1 + j0$ p_u times counterclockwise, where p_u is the number of *unstable poles* of $G(s)$ and $K(s)$.

2 The Root-Locus Method

2.1 An Example



X POLES
O ZEROS

$$\frac{s+5}{s(s^2+6s+10)} \quad (c=1)$$

Root-locus diagram for

$$L(s) = \frac{s+5}{s(s^2+6s+10)}$$

This shows the locations of the roots of $1 + kL(s) = 0$ for $k > 0$.

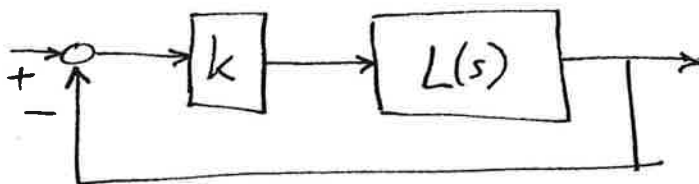
In this case $0 \leq k \leq 2 \times 10^4$.

Useful when the loop dynamics are fixed, and only the gain varies.

'rlocus' in Matlab

CLOSED-LOOP
CHARACTERISTIC EQN.

↓
CLOSED-LOOP
POLES



$$L(s) = G(s)K(s)$$

2.2 The Angle Condition

$$L(s) = \frac{n(s)}{d(s)} = \frac{c(s - z_1)(s - z_2) \dots (s - z_m)}{(s - p_1)(s - p_2) \dots (s - p_n)} = \frac{c(s^m + \dots)}{(s^n + \dots)}$$

MONIC

where any complex zeros or poles occur in conjugate pairs and $m \leq n$.

We assume for the moment that $c > 0$.

Suppose that s_0 is on the root-locus:

$$1 + kL(s_0) = 0 \Rightarrow L(s_0) = -\frac{1}{k} \quad \text{real and negative} \quad (2.1)$$

Hence *angle condition* for s_0 to be on the root-locus:

$$\angle L(s_0) = (2\ell + 1)\pi \quad (2.2)$$

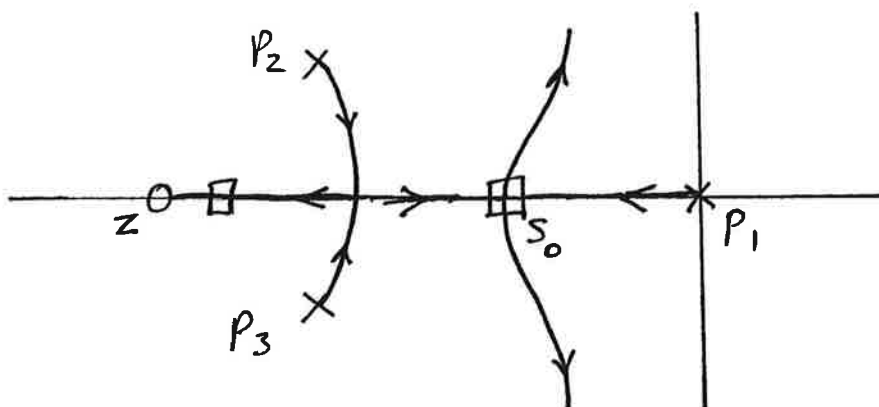
$$\underbrace{\sum_{i=1}^m \angle(s_0 - z_i)}_{\text{NUMERATOR TERMS}} - \underbrace{\sum_{i=1}^n \angle(s_0 - p_i)}_{\text{DENOMINATOR TERMS}} = (2\ell + 1)\pi \quad (2.3)$$

2.3 Finding Gain from the Root-Locus Plot

Once a root-locus plot has been obtained, it can be calibrated with k values. From (2.1) we have, at a point s_0 on the root-locus:

$$k = \frac{1}{|L(s_0)|} \quad \leftarrow 1 + kL(s_0) = 0$$

$$= \frac{1}{c} \times \frac{|s_0 - p_1| \times |s_0 - p_2| \times \dots \times |s_0 - p_n|}{|s_0 - z_1| \times |s_0 - z_2| \times \dots \times |s_0 - z_m|}$$



2.4 Constructing the Root-Locus Plot

Nowadays we can use software to draw root-locus diagrams (eg `rlocus` in *Matlab*).

But it is useful to have some understanding of how the form of the locus is determined. A set of about 15 'construction rules' has been developed. The 5 most important ones are given here. They are all consequences of (2.3) and properties of polynomials.

- Rule 1. The root-locus diagram is symmetric with respect to the real axis and consists of n branches.
- Rule 2. For $k = 0$ the n branches start at the open loop poles p_i . As $k \rightarrow \infty$, m branches tend to the zeros z_i and $n - m$ branches tend to infinity.
- Rule 3. Points on the real axis which lie to the left of an odd number of poles *and* zeros are on the root-locus.
- Rule 4. The breakaway points are those points on the root-locus for which $\frac{d}{ds}L(s) = 0$ (same as $dk/ds = 0$).
- Rule 5. As $k \rightarrow \infty$, the $n - m$ branches which tend to infinity do so along straight line asymptotes at angles $(2\ell + 1)\pi/(n - m)$ to the +ve real axis ($\ell = 0, \dots, n - m - 1$), and emanate from the point ('centre of gravity' — pole +ve mass, zero -ve mass):

$$\frac{\sum_{i=1}^n p_i - \sum_{i=1}^m z_i}{n - m}.$$

Proof of Rule 2:

$$1 + k L(s) = 1 + k \frac{n(s)}{d(s)} = 0$$

$$d(s_0) + kn(s_0) = 0 \quad \text{from (2.1)}$$

So if $k = 0$ then $d(s_0) = 0 \Rightarrow$ Branches start at poles.

Suppose $k \rightarrow \infty$. If $|s_0|$ stays bounded then $|d(s_0)| \ll |kn(s_0)|$, so $kn(s_0) \rightarrow 0 \Rightarrow$ Finite branches end at zeros. There can be at most m of them. The remaining $n - m$ branches go to ∞ .

Application to previous example:

$n = 3$, $m = 1$, so $n - m = 2$. Two asymptotes at angles $\pi/2$ and $3\pi/2$.

Asymptotes emanate from (Rule 5):

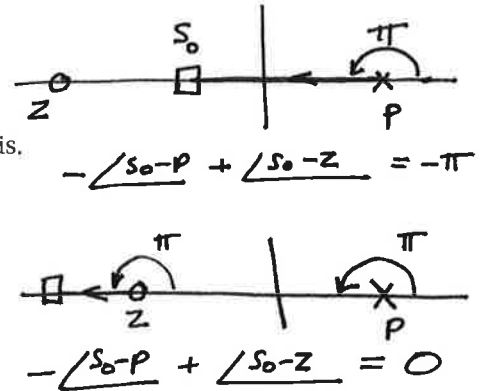
$$\frac{(0 - 3 - 3) - (-5)}{3 - 1} = -\frac{1}{2}$$

Proof of Rule 3:

Consider a point s_0 , a pole p_i , and a zero z_i , all on the real axis.

$$\angle(s_0 - p_i) = \begin{cases} 0 & \text{if } s_0 > p_i \\ \pi & \text{if } s_0 < p_i \end{cases}$$

The same holds for $\angle(s_0 - z_i)$. Rule 3 follows from (2.3).

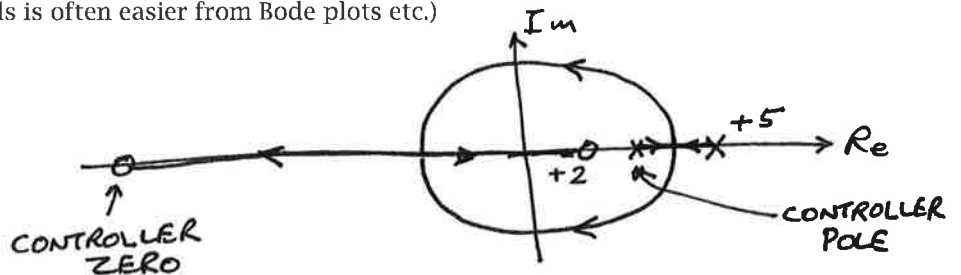
**Example of use of Rule 3:**

Suppose that $G(s)$ has one pole and one zero in the right half-plane, eg $p_1 = +5$, $z_1 = +2$.

Rule 3 shows that $K(s)$ must have a pole between +2 and +5

— the controller must be *unstable*!

(Figuring out the details is often easier from Bode plots etc.)

**Application of Rule 4 to example of Fig.2.1:**

$$\frac{d}{ds} L(s) = 0$$

$$\begin{aligned} \frac{d}{ds} \left(\frac{s+5}{s^3+6s^2+10s} \right) &= 0 \\ \Rightarrow \frac{1(s^3+6s^2+10s) - (s+5)(3s^2+12s+10)}{(s^3+6s^2+10s)^2} &= 0 \\ \Rightarrow \frac{-2s^3-21s^2-60s-50}{(s^3+6s^2+10s)^2} &= 0 \\ \Rightarrow \frac{-2(s+1.5505)(s+2.5)(s+6.4495)}{(s^3+6s^2+10s)^2} &= 0 \end{aligned}$$

$$\frac{d}{ds} \left(\frac{u}{v} \right) = \frac{u'v - uv'}{v^2}$$

From Fig.2.1 it is seen that the root at -6.4495 is not on the root-locus.

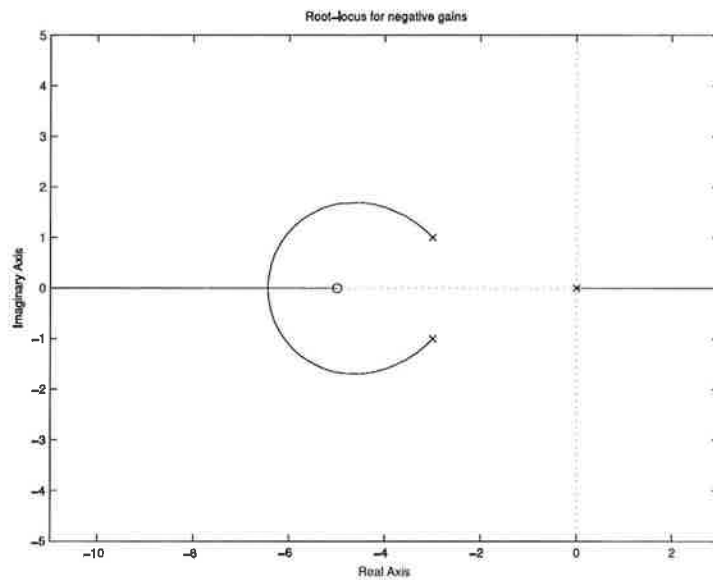
The other two roots give the *breakaway points* (ie repeated roots).

2.5 Root-locus for negative k (or negative c)

$$1 + kG(s_0) = 0 \Rightarrow G(s_0) = -\frac{1}{k} > 0$$

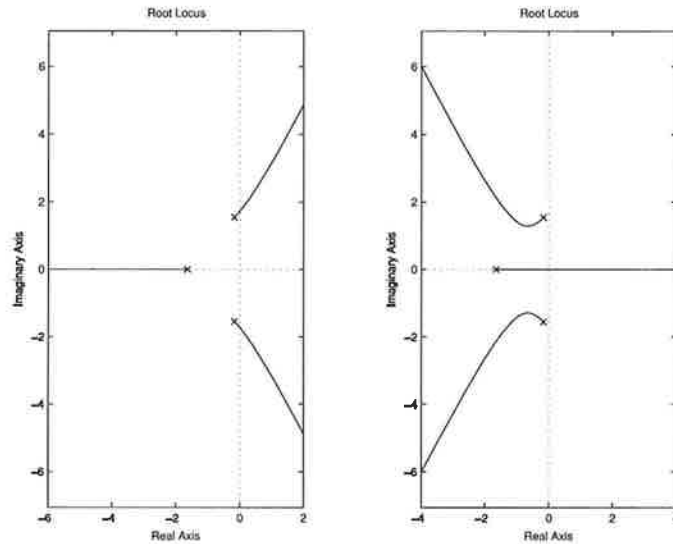
$$\Rightarrow \angle G(s_0) = 2\ell\pi$$

- Rules 1,2,4 remain unchanged.
- Rule 3: Replace 'odd' by 'even'.
- Angles of asymptotes become $2\ell\pi/(n - m)$.
(Points from which asymptotes emanate remain unchanged.)



Root-locus diagram for negative gain when

$$L(s) = \frac{s + 5}{s(s^2 + 6s + 10)}$$



Root-locus diagrams for positive and negative gains when

$$L(s) = \frac{1}{s^3 + 2s^2 + 3s + 4}$$

2.6 Studying Parameter Variations

Root-locus diagrams can be used to study the variation of closed-loop poles as other parameters vary — not just the loop-gain k .

All that is needed is to put the closed-loop characteristic equation into the form

$$1 + \lambda H(s) = 0 \tag{2.4}$$

where λ is the parameter that is varying, and $H(s)$ is a transfer function.

Example: Robot placing objects of varying mass

The 1-D equation of motion of a robot moving a mass m with viscous friction c and elastic tether is $m\ddot{x} = u - c\dot{x} - \alpha x$ where x is the mass position and u is the applied force. The use of a PI controller is proposed, with a transfer function $k(s + z)/s$.

With $m = 0.1$ kg, $c = 0.6$ N/(m/sec), $\alpha = 1$ N/m and $z = 5$ we have

$$G(s) = \frac{1}{0.1s^2 + 0.6s + 1} = \frac{10}{s^2 + 6s + 10} \quad \text{and} \quad K(s) = k \frac{s + 5}{s}$$

Letting $L(s) = G(s)K(s)/k$ the closed-loop characteristic equation is $1 + kL(s) = 0$.

Using Fig.2.1, $10k = 1.395$ places two closed-loop poles at -1.55 (one of the breakaway points) and the third pole at -2.9 .

What if the mass varies?

The closed-loop characteristic equation is

$$1 + k \frac{(s + 5)}{s(ms^2 + 0.6s + 1)} = 0 \qquad 1 + kL(s) = 0$$

which has the same roots as

$$(ms^3 + 0.6s^2 + s) + k(s + 5) = 0$$

or

$$1 + \frac{1}{m} \frac{\overbrace{0.6s^2 + [1 + k]s + 5k}^{H(s)}}{s^3} = 0 \qquad \begin{array}{l} \text{DIVIDE BY } ms^3 \\ \swarrow \\ 1 + \lambda H(s) = 0 \end{array}$$

which is in the form of (2.4) with $\lambda = 1/m$. The root-locus plot for this, with $k = 1.395/10$, is shown in Fig.2.6. The roots with $m = 0.1$ are marked.

Variations of closed-loop poles as $1/m$ varies can be clearly seen.

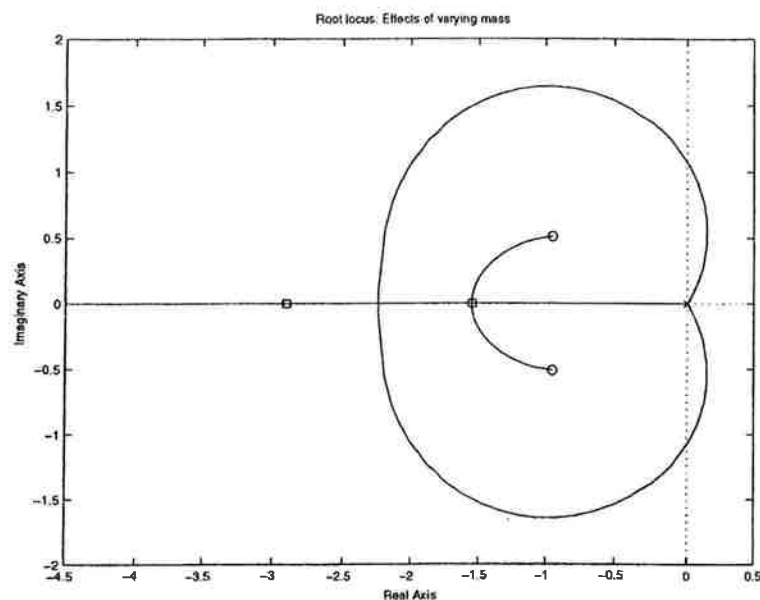


Figure 2: Root-locus diagram for variation of mass in robot problem.

3 The Routh-Hurwitz Criterion

The closed-loop characteristic equation

$$1 + G(s)K(s) = 0$$

has the same roots as

$$d_G(s)d_K(s) + n_G(s)n_K(s) = 0 \quad \text{polynomial}$$

$$G(s) = \frac{n_G(s)}{d_G(s)}, \quad K(s) = \frac{n_K(s)}{d_K(s)}$$

The Routh-Hurwitz criterion tests whether a polynomial has any roots with nonnegative real parts. So it tests for asymptotic stability.

Sometimes useful for finding value of k at which root-locus crosses imaginary axis.

Consider the polynomial (assume $a_0 > 0$):

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n \quad (3.5)$$

Easy to check that all roots have negative real parts *only if* $a_i > 0$ for each i .

A **Routh array** can be constructed for arbitrary n

— see Franklin, Powell and Emami-Naeini, 3rd edition, sec.4.4.3 (for example) for details.

For $n = 2, 3, 4$ simplifies as follows:

These are in Electrical Data Book

All the roots of (3.5) have negative real parts *if and only if*:

$n = 2$: $a_i > 0$, No other conditions

$n = 3$: $a_i > 0$, $a_1 a_2 > a_0 a_3$

$n = 4$: $a_i > 0$, $a_1 a_2 a_3 > a_0 a_3^2 + a_4 a_1^2$