# When is a controller optimal in the sense of $\mathcal{H}_\infty$ loop-shaping?

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## Abstract

In this paper we characterize the controllers which are possible solutions of a certain $\mathcal{H}_\infty$ control design problem. The problem considered is the optimal robustness problem for (weighted) normalized coprime factor/gap metric uncertainty, which is the basis for the Glover-McFarlane $\mathcal{H}_\infty$ loop-shaping design method. Given a plant $P$ and a corresponding controller $C$ we ask if $C$ can be obtained from the optimization procedure for some choice of weighting function. This paper considers single-input/single-output systems and gives necessary and sufficient conditions for optimality which involve right half plane pole/zero counts and a certain winding number test based on the Nyquist diagram of $PC$. The results give a characterization of this class of $\mathcal{H}_\infty$-optimal designs in the language of classical control.

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## 1 Introduction

The use of an optimization procedure for controller design implicitly defines a space of controllers which are potentially optimal. It is within this space that the search for a suitable controller is made. An effective control synthesis method requires (i) the search to be easily related to the designer's goals through the choice of weighting functions etc., and (ii) the space of possible controllers to be a "sensible" one. In this paper we will characterize the potentially optimal controllers for a popular $\mathcal{H}_\infty$ synthesis problem, namely the optimization of weighted normalized coprime factor/gap metric uncertainty which is the basis of Glover-McFarlane loop-shaping. This method appears to be particularly straightforward for the designer in the choice of weighting functions.

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The problem of characterizing the controllers which are potentially optimal is a so-called inverse problem. A solution of the inverse problem of LQ control was given in [12]. The necessary and sufficient condition obtained (that the $\mathcal{H}_\infty$-norm of the sensitivity function is at most one) is very sharp, though the result has limited use due to the assumption of state feedback. This type of stability margin guarantee was shown not to hold in the general LQG set-up [4]. A special type of inverse problem in $\mathcal{H}_\infty$ control has been studied in [13]. The problem of mixed sensitivity optimization is considered with a restriction placed on the weighting functions, namely that their magnitudes are the inverses of the sensitivity and complementary sensitivity magnitudes. Necessary and sufficient conditions for optimality, involving RHP pole/zero counts, are obtained under certain assumptions, e.g., plant and controller have no poles and zeros on the imaginary axis or at infinity.

The $\mathcal{H}_\infty$ optimization problem considered in this paper is the one used in the Glover-McFarlane $\mathcal{H}_\infty$ loop-shaping method [14]. This problem is the computation of the optimally robust controller for a weighted plant subject to normalized coprime factor uncertainty, which is the same as the optimal robustness problem for uncertainty in the weighted gap metric. An earlier study of the associated inverse problem has been given in [17]. A necessary and sufficient condition for optimality is given in terms of the solution of a certain quadratic equation (frequency by frequency) and a subsequent winding number test. Although this condition can be regarded as a solution of the inverse problem, in that a definite test procedure is obtained, a more explicit procedure is desirable which is easier to relate to classical control concepts. The present paper derives such conditions which involve only right half plane pole/zero counts and encirclement tests based on the Nyquist diagram.

The main new results of the paper are developed in Sections 4 and 5, which are conditions for optimality, not requiring the explicit computation of a potential weighting function (Theorem 5 and 6). It turns out that the test for optimality is rather easier when the potential robustness margin $\beta$ is strictly less than $\beta_{\max} = \||S| + |T|\|_\infty^{-1}$, which is the case considered in Section 4. It is shown using a homotopy argument that potential optimality for *some* $\beta < \beta_{\max}$ is equivalent to potential optimality for *any* $\beta < \beta_{\max}$, which is equivalent to having a certain pole/zero excess for $P$ over $C$ in the right half plane (Theorem 5). The case $\beta = \beta_{\max}$ is considered in Section 5. Here the test (Theorem 6) is a bit more involved (due to the fact that there is a richer class of potential weighting functions available). To test for optimality in this case a certain construction on the Nyquist diagram of $PC$ is required along with some associated winding number counts. It is worth remarking that if optimality cannot be obtained for $\beta = \beta_{\max}$ then it cannot be achieved for $\beta < \beta_{\max}$ (though the converse of this statement is not true). The remainder of the paper is organised as follows. In Section 3 the problem set-up is described and some characterizations of optimality are given, which are similar to the results of [17]. That is, a solution of a quadratic equation is required to find the modulus of a potential weighting function, followed by a winding number test (see Theorem 3). The results are developed from first principles using a result of Poreda [16] on badly approximable functions. This type of mathematical result also occurs in the work of [10], [11] and is exploited in previous work on $\mathcal{H}_\infty$ inverse problems [13], [17]. As shown in [17] (see also Section 3 Theorem 4), given $P$ and $C$, no solution of the inverse weighting problem exists if $\beta > \beta_{\max}$. The case $\beta_{\max} = 1$ is somewhat pathological and is treated briefly in Section 6. Two illustrative examples are presented in Section 7. The work in Section 3 to 7 is concerned with weighting functions which are invertible in $\mathcal{H}_\infty$. In Section 8 the analysis is extended to cover weighting functions with poles or zeros on the imaginary axis.

## 2 Mathematical preliminaries

$\mathbb{R}$ and $\mathbb{C}$ denote the real and complex fields. $\mathcal{H}_2$, $\mathcal{H}_\infty$ denote the standard Hardy spaces in the right half of the complex plane (RHP). Let $\mathcal{S}$ be the set of real-rational functions in $\mathcal{H}_\infty$. For a function $G(j\omega)$, $G(j\omega)^*$ denotes the complex conjugate of $G(j\omega)$, namely $\overline{G(j\omega)}$. The meaning of $G(s)^*$, for $s$ not on the *imaginary* axis, will be taken to be $\overline{G(-\overline{s})}$ unless otherwise stated. $\mathcal{A}_0$ denotes the set of functions in $\mathcal{H}_\infty$ which are continuous on the compactified RHP. For a rational function $G(s)$, $\eta_o(G)$ (respectively $\eta_a(G)$) denotes the number of poles of $G$ in the open RHP (respectively on $j\mathbb{R}\cup\{\infty\}$), counting multiplicities. For a closed curve $\Gamma$ in $\mathbb{C}$, with $0 \notin \Gamma$, we denote the *winding number* of $\Gamma$ about the origin (number of anticlockwise encirclements) by $\mathrm{wno}(\Gamma)$. If $G(s)$ is continuous on a closed curve $\Theta$, we denote the winding number of the image of $\Theta$ under $G(s)$ by $\mathrm{wno}_\Theta(G(s))$. In this paper, we will use the symbol $\mathcal{D}$ to denote the path consisting of $j\mathbb{R}\cup\{\infty\}$ and oriented in the direction of increasing frequency. We will also use the notation $\mathrm{wno}(G(s)) = \mathrm{wno}_{\mathcal{D}}(G(s))$ if there is no ambiguity.

## 3 Optimal robustness problem

We consider the feedback configuration of Fig. 1, denoted by $[P, C]$, where $P$ and $C$ are the (scalar) transfer functions of the plant and the controller respectively. Throughout the paper we assume that $P$ and $C$ are real-rational functions. We say that $[P, C]$ is *stable* if the closed loop transfer functions, mapping $d_i$ to $e_j$ $(i, j = 1, 2)$, belong to $\mathcal{H}_\infty$. If $[P, C]$ is stable then the following quantity can be defined:

$$b(P, C) := \left\| \begin{pmatrix} 1 \\ P \end{pmatrix} (1 + CP)^{-1} (1\ ,\ C) \right\|_\infty^{-1}.$$

This quantity is the optimal robustness radius for normalized coprime factor/gap metric uncertainty. That is, $b(P, C)$ is the radius of the largest ball of uncertainty about $P$, measured in the gap metric, which is stabilized by $C$. For the definition of the gap metric, connections with normalized coprime factorizations, and associated results on robust stabilization the reader is referred to [7], [9]. It appears that $b(P, C)$ is a useful quantity to maximize for a feedback system, especially if it is weighted appropriately. In fact, maximization of $b(WP, C)$ over stabilizing compensators, for some weighting function $W$ chosen by the designer, is the basis of the Glover-McFarlane $\mathcal{H}_\infty$ loop shaping method [14]. This problem is also the same as the optimal robustness problem in the weighted gap metric (see [2], [6]).

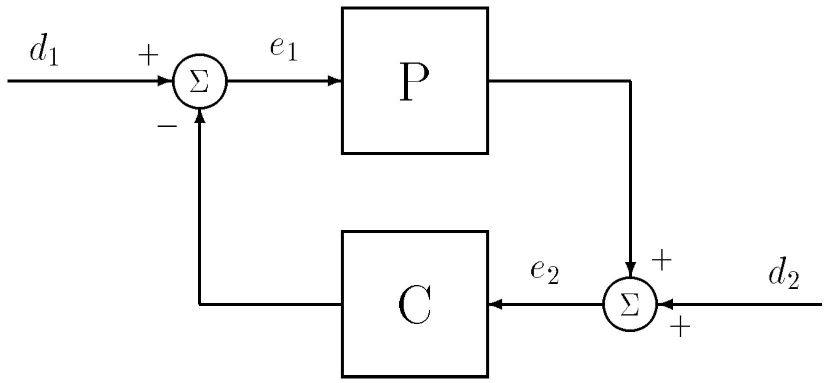


Figure 1: Standard feedback configuration.

Let $P = N/M$ ($M, N \in \mathcal{S}$) be a coprime factorization which is normalized, i.e. $MM^* + NN^* = 1$. Since $C$ stabilizes $P$, then we can find a factorization $C = U/V$ ($U, V \in \mathcal{S}$) such that $MV + NU = 1$. With such a factorization

$$C_1 = \frac{U + MQ}{V - NQ},$$

for $Q \in \mathcal{H}_\infty$, parametrizes all stabilizing controllers. Moreover [7]

$$b(P, C_1) = (1 + \|M^*U - N^*V + Q\|_\infty^2)^{-1/2}. \tag{1}$$

We now introduce the following terminology. A function $F \in \mathcal{L}_\infty$ is said to be *badly approximable* if $\inf_{Q \in \mathcal{H}_\infty} \|F + Q\|_\infty = \|F\|_\infty$ .

**Theorem 1.** Let $[P, C]$ be stable. The following statements are equivalent:

(a) $b(P, C) = \sup_{C_1 stblz} b(P, C_1)$.

(b) $M^*U - N^*V$ is *badly approximable.*

(c) Either $M^*U - N^*V \equiv 0$, or the following conditions hold

$$|(M^*U - N^*V)(j\omega)| = \alpha \text{ (for all } \omega), \tag{2}$$

$$\text{wno}(M^*U - N^*V) > 0, \tag{3}$$

for some constant $\alpha > 0$.

**Proof.** (a) $\Leftrightarrow$ (b). We have:

$$\begin{aligned} & \sup_{C_1 stblz} b(P, C_1) = b(P, C) \\ \Leftrightarrow\ & \inf_{Q \in \mathcal{H}_\infty} \|M^*U - N^*V + Q\|_\infty = \|M^*U - N^*V\|_\infty \\ \Leftrightarrow\ & M^*U - N^*V \text{ is badly approximable.} \end{aligned}$$

(b) $\Leftrightarrow$ (c). This equivalence follows directly from [16, Theorem 1]. (The difference in sign in the winding number condition is due to the fact that $\mathcal{D}$ is oriented in the direction of increasing frequency, i.e., the boundary curve has negative orientation with respect to the right half plane.) $\square$

Let $[P, C]$ be stable and $W, W^{-1} \in \mathcal{A}_0$. Define

$$b(P,C,W) := \left\| \begin{pmatrix} 1 \\ WP \end{pmatrix} (1+CP)^{-1} \begin{pmatrix} 1 \ , \ W^{-1}C \end{pmatrix} \right\|_\infty^{-1}, \tag{4}$$

and note that $b(P, C, W) = b(WP, W^{-1}C)$. This quantity is the stability margin for uncertainty in the weighted gap metric (see [7]). We also remark that, in the Glover-McFarlane $\mathcal{H}_\infty$ loop-shaping method, the final compensator to be implemented is $C = W\hat{C}$ where $\hat{C}$ maximizes $b(WP, \hat{C})$. From (4), $b(WP, \hat{C}) = b(P, C, W)$, so in fact $C$ also maximizes $b(P, C, W)$. Given the weighting function $W, W^{-1} \in \mathcal{A}_0$, we say that $C$ is *$\beta$-optimal for $P$* (with weight $W$) if

$$\beta = b(P,C,W) = \sup_{C_1 \, stblz} b(P, C_1, W).$$

**Theorem 2.** Let $[P, C]$ be stable, let $W, W^{-1} \in \mathcal{A}_0$ and suppose that $WP$ has a normalized coprime factorization over $\mathcal{A}_0$. Then $C$ is $\beta$-optimal for $P$ if and only if, either $\beta = 1$ and $W^{-1}C = W^*P^*$, or $\beta < 1$ and the following two conditions hold:

$$\left| \left( \frac{W^{-1}C - W^*P^*}{1+PC} \right)(j\omega) \right| = \alpha \text{ (for all } \omega), \tag{5}$$

$$\operatorname{wno}\left( \frac{W^{-1}C - W^*P^*}{1+PC} \right) + 2\eta_o(P) + \eta_a(P) > 0, \tag{6}$$

where $\alpha = (\beta^{-2} - 1)^{1/2}$ is constant.

**Proof.** First note from (4) that $C$ is $\beta$-optimal for $P$ (with weight $W$) if and only if $W^{-1}C$ is $\beta$-optimal for $WP$ (with weight 1). Under the assumption that there is a normalized coprime factorization $WP = N_W/M_W$ over $\mathcal{A}_0$, we can find a factorization $W^{-1}C = U_W/V_W$ over $\mathcal{A}_0$ so that $N_W U_W + M_W V_W = 1$.

With the above factorization Theorem 1 applies. In particular, on noting from (1) that

$$b(P,C,W) = (1 + \|M_W^* U_W - N_W^* V_W\|_\infty^2)^{-1/2},$$

we have: $W^{-1}C$ is $\beta$-optimal for $WP$ (with weight 1) if and only if $\beta = (1+\alpha^2)^{-1/2}$ and Theorem 1 (c) holds. Now note that

$$M_W^* U_W - N_W^* V_W = \frac{W^{-1}C - W^*P^*}{1+PC} \cdot \frac{M_W^*}{M_W}. \tag{7}$$

In case $\beta = 1$, $C$ is $\beta$-optimal if and only if $M_W^* U_W - N_W^* V_W \equiv 0$, which holds if and only if $W^{-1}C = W^*P^*$. We now consider the case of $\beta < 1$. Since $|M_W^*/M_W| = 1$ for all $\omega$, equation (5) is the same as (2). We will now establish that (6) is the same as (3). From (7) we see that

$$\mathrm{wno}(M_W^* U_W - N_W^* V_W) = \mathrm{wno}\left(\frac{W^{-1}C - W^*P^*}{1+PC}\right) + \mathrm{wno}\left(\frac{M_W^*}{M_W}\right). \tag{8}$$

We will now explore the relationship between $M_W$ and $M$. First consider a spectral factorization:

$$WW^*NN^* + MM^* = W_1 W_1^*$$

where $W_1 \in \mathcal{H}_\infty$ is outer. Since $W$ is invertible, then $W_1$ is bounded below on the imaginary axis, so $W_1^{-1} \in \mathcal{H}_\infty$. We therefore can choose $M_W = MW_1^{-1}$, hence

$$\mathrm{wno}\left(\frac{M_W^*}{M_W}\right) = \mathrm{wno}\left(\frac{M^*}{M}\right). \tag{9}$$

(This follows immediately if $W_1, W_1^{-1} \in \mathcal{A}_0$, which will always be true for the weighting functions which arise in the context of this paper. In fact, (9) follows for a general invertible $W_1$ in $\mathcal{H}_\infty$ using the properties of Fredholm index, which coincides with winding number for continuous $\mathcal{L}_\infty$ functions. In particular the symbol $W_1$ defines an invertible Toeplitz operator with zero Fredholm index (see [15, p.315,318]).) We now write $M = p_+ p_a p_- / d$ where $p_-, p_+^*$ and $d$ are Hurwitz and $p_a$ has all its roots on $j\mathbb{R}$. Then $p_a^* = \pm p_a$, so

$$\begin{aligned}
\mathrm{wno}\left(\frac{M^*}{M}\right) &= \mathrm{wno}\left(\frac{p_+^* p_-^* d}{d^* p_+ p_-}\right) \\
&= \deg(d) + \deg(p_+) - \deg(p_-) \\
&= 2\deg(p_+) + (\deg(d) - \deg(p_+ p_-)) \\
&= 2\eta_o(P) + \eta_a(P).
\end{aligned}$$

Hence, combining with (8) and (9), we have shown that (6) and (3) are the same. □

From Theorem 2, optimality of $C$ for the weighting function $W$ requires the two conditions (5), (6) to be satisfied which will be called the *modulus condition* and the *winding number condition* respectively. We will first consider the modulus condition. Let $WW^* = W_R$. Then

$$\left|\frac{W^{-1}C - W^*P^*}{1+PC}\right| = \alpha$$

is equivalent to

$$PP^*W_R^2 - (PC + C^*P^* + \alpha^2|1+PC|^2)W_R + C^*C = 0, \tag{10}$$

which is the quadratic equation obtained in [17]. It is evident that a first step in finding a weighting function $W, W^{-1} \in \mathcal{A}_0$ so that Theorem 2 holds is to solve (10), frequency by frequency, for a real, positive, continuous $W_R(j\omega)$. In fact, such a $W_R$ can always be spectrally factorized as $W_R = WW^*$ with $W, W^{-1} \in \mathcal{H}_\infty$. The procedure to find $W$ is to take the Hilbert transform of the function $\log W_R^{1/2}$, defined on the imaginary axis, to find the phase. We will show below that the Hilbert transform of $\log W_R^{1/2}$ is continuous which proves that $W, W^{-1} \in \mathcal{A}_0$. It is necessary to prove this since not all positive continuous functions have continuous Hilbert transforms (see [5, p.377]).

**Lemma 1.** Let $P$ and $C$ be rational. Let $W_R(j\omega)$ be a continuous, non-negative solution of (10) which is bounded and bounded away from zero. Then there exists a factorization $W_R = WW^*$ such that $W, W^{-1} \in \mathcal{A}_0$. Furthermore, $WP$ has a normalized coprime factorization over $\mathcal{A}_0$.

**Proof.** We can rearrange (10) in the form

$$p_1(s)W_R^2 + p_2(s)W_R + p_3(s) = 0 \tag{11}$$

where $p_1, p_2$ and $p_3$ are polynomials in $s$. (11) is a polynomial in two variables which defines an algebraic function according to standard procedures (see [1]). Hence we can find a power series expansion of $W_R$ (possible involving fraction exponents which are multiples of $s^{1/2}$) about any point [1, p.32]. It now follows from [5, Theorem 4.4, p.66] that a phase can be defined so that $W \in \mathcal{H}_\infty$ and $|W(j\omega)| = W_R^{1/2}(j\omega)$ and moreover, $W$ can be taken to be outer. To check continuity we will transform the RHP to the unit disc using the bilinear transformation $z = (1-s)/(1+s)$. By an abuse of notation, we let $W_R(z)$ denote the transformed solution of (11). We now introduce the *modulus of continuity* (see [5, p.105]) defined as follows:

$$\mu(f,\delta) = \sup_{|\theta_1-\theta_2|<\delta} |f(e^{j\theta_1}) - f(e^{j\theta_2})|$$

for some $f$ defined on the unit circle. From the facts quoted above on the power series expansions for $W_R$ we can verify that $\mu(W_R,\delta) = O(\delta^{1/2})$. Since $W_R$ is bounded away from zero on the imaginary axis then $\mu(\log W_R^{1/2},\delta) = O(\delta^{1/2})$. It therefore follows that

$$\int_0^\epsilon \frac{\mu(\log W_R^{1/2}, t)}{t} dt < \infty$$

which means that $\log W_R^{1/2}$ is Dini continuous [5, p.105]. This means that $W_R^{1/2}$ defines a continuous phase function [5, Theorem 1.3, p.106]. Thus $W \in \mathcal{A}_0$, and since $W$ is outer and bounded away from zero on the imaginary axis, $W^{-1} \in \mathcal{A}_0$.

To find the normalized coprime factors for $WP$ we need to find a spectral factorization $W_1W_1^* = WW^*NN^* + MM^*$. Since $\mu(WW^*,\delta) = O(\delta^{1/2})$ we see that $\mu(W_1W_1^*,\delta) = O(\delta^{1/2})$. Since $|W|$ is bounded away from zero on the imaginary axis then so is $|W_1|$ which means that $\mu(\log|W_1|,\delta) = O(\delta^{1/2})$. This shows, as above, the existence of a spectral factor $W_1, W_1^{-1} \in \mathcal{A}_0$. The normalized coprime factors for $WP$ are now given by $WW_1^{-1}N$ and $W_1^{-1}M$. $\square$

**Theorem 3.** Let $[P, C]$ be stable and let $\beta = (1+\alpha^2)^{-1/2} < 1$. Then there exists a weighting function $W, W^{-1} \in \mathcal{A}_0$ with magnitude $W_R(j\omega) = |W(j\omega)|^2$ such that $C$ is $\beta$-optimal for $P$ if and only if $W_R, W_R^{-1} \in \mathcal{L}_\infty$ is a continuous solution of (10) and

$$\operatorname{wno}\left(\frac{C - W_R P^*}{1+PC}\right) + 2\eta_o(P) + \eta_a(P) > 0. \tag{12}$$

**Proof.** Suppose that $C$ is $\beta$-optimal for $P$ with some $W, W^{-1} \in \mathcal{A}_0$. Then $W_R = W^*W$ satisfies (10) and

$$\operatorname{wno}\left(\frac{W^{-1}C - W^*P^*}{1+PC}\right) = \operatorname{wno}\left(\frac{C - W_R P^*}{1+PC}\right)$$

since $\operatorname{wno}(W) = 0$.

Conversely, if $W_R, W_R^{-1} \in \mathcal{L}_\infty$ is a continuous solution of (10) then we can find a $W, W^{-1} \in \mathcal{A}_0$ satisfying $W_R = W^*W$ and such that $WP$ has a normalized coprime factorization over $\mathcal{A}_0$. Moreover, conditions (5) and (6) hold. □

Theorem 3 gives a straightforward procedure to determine if a compensator $C$ is $\beta$-optimal. Namely, we need to check (12) for all the continuous solutions $W_R, W_R^{-1} \in \mathcal{L}_\infty$ of (10), of which there are finitely many. It should be noted that $(C - W_R P^*)/(1+PC)$ always has a well-defined winding number for continuous solution $W_R, W_R^{-1} \in \mathcal{L}_\infty$ of (10). We remark that Theorem 3 has the same spirit as the result of Sefton and Ober [17, Theorem 9.7], though it is more easily tested since coprime factorizations of $P$ and $C$ are not needed and the phase of $W_R^{1/2}$ is not needed. In the next two sections we develop conditions for $\beta$-optimality, directly in terms of $P$ and $C$, which do not require solutions of (10) to be found.

Observe that equation (10) has real coefficients for $s \in j\mathbb{R} \cup \{\infty\}$ and we are seeking solutions such that $W_R$ is real and positive. First note that there is a solution $W_R \geq 0$ to (10) if and only if

$$\alpha^2 |1+PC|^2 + PC + C^*P^* \geq 2|PC|. \tag{13}$$

This is equivalent to

$$\beta^{-1} = (1+\alpha^2)^{1/2} \geq \frac{1+|PC|}{|1+PC|} = |S| + |T| \tag{14}$$

where $S = 1/(1+PC)$ and $T = PC/(1+PC)$ are the classical sensitivity and complementary sensitivity functions. Defining

$$\beta_{\max} = \left\| \frac{1+|PC|}{|1+PC|} \right\|_\infty^{-1} = \| |S| + |T| \|_\infty^{-1}, \tag{15}$$

we have the following result [17].

**Theorem 4.** Let $[P, C]$ be stable. If $\beta > \beta_{\max}$, then no weighting function $W, W^{-1} \in \mathcal{A}_0$ exists so that $C$ is $\beta$-optimal for $P$.

**Proof.** If $\beta > \beta_{\max}$, there exists a frequency $\omega_0$, which is not a pole or zero of either $P$ or $C$, such that (13) is violated. For this frequency no positive solutions to (10) exist and so the modulus condition cannot be satisfied for any admissible $W$. $\square$

Theorem 4 shows that $\beta_{\max}$ is an upper bound on the possible stability margin $b(P, C, W)$ for fixed $P$ and $C$ and arbitrary $W$. Consequently, if we optimize over both $C$ and $W$ we have the bound:

$$\sup_{W} \sup_{C\, stblz} b(P, C, W) \leq (\inf_{C\, stblz} \||S| + |T|\|_\infty)^{-1}.$$

In case $\beta \leq \beta_{\max}$ there is no simple violation of the modulus condition. To check for optimality we need to bring in the winding number condition as well. Note that the frequencies where equality is achieved in (14) are the frequencies at which a repeated solution $W_R$ of (10) is obtained. Thus, if $\beta < \beta_{\max}$ there are at most two continuous choices for $W_R$ consisting of the largest solution of (10) frequency by frequency, or the smallest one. This situation will be considered in the next section. In case $\beta = \beta_{\max}$, there will be several frequencies where the solutions $W_R$ of (10) are repeated. This allows the possibility of swapping between the largest and smallest solutions in making a continuous choice for $W_R$. This situation will be considered in Section 5.

To conclude this section, we state an interesting consequence, along the lines of [17], of the need to solve (10) in finding a suitable weighting function. If we denote the largest and smallest solutions of (10) by $W_R = |W_+|^2$, $|W_-|^2$, we see that the product of roots must always satisfy:

$$|W_+| \cdot |W_-| = \frac{|C|}{|P|}. \tag{16}$$

This allows the following easy test, on (unweighted) optimally robust controllers, to distinguish between the cases of $\beta < \beta_{\max}$ and $\beta = \beta_{\max}$.

**Proposition.** Let $C$ be the optimally robust controller for $P$ with $W = 1$ (i.e. $\beta = b(P, C) = \sup_{C_1\, stblz} b(P, C_1)$). Then $\beta = \beta_{\max}$ if and only if $|C| = |P|$ for at least one frequency in $j\mathbb{R} \cup \{\infty\}$.

**Proof.** If $\beta = \beta_{\max}$ then there is at least one frequency where (10) has equal roots. Since one solution is equal to one automatically then we must have $|C| = |P|$ from (16). Conversely, if $|C| = |P|$ at some frequency, then (16) implies that $|W_+| = |W_-| = 1$ at that frequency which means that $\beta = \beta_{\max}$. $\square$

# 4 Solution of inverse problem for $\beta < \beta_{\max} < 1$

The main result of this section is stated in Theorem 5 below. We define

$X = \{$poles of $P$ and zeros of $C$ on $j\mathbb{R} \cup \{\infty\}\}$,

$Y = \{$zeros of $P$ and poles of $C$ on $j\mathbb{R} \cup \{\infty\}\}$.

**Theorem 5.** Let $[P, C]$ be stable and $\beta_{\max} < 1$. The following statements are equivalent.

(a) For *some* $\beta$, $0 < \beta < \beta_{\max}$, there exists a $W, W^{-1} \in \mathcal{A}_0$ such that $C$ is $\beta$-optimal for $P$.

(b) For *every* $\beta$, $0 < \beta \leq \beta_{\max}$, there exists a $W, W^{-1} \in \mathcal{A}_0$ (depending on $\beta$) such that $C$ is $\beta$-optimal for $P$.

(c) At least one of the following two conditions holds:

 (i) $X = \emptyset$ and $\eta_o(P) > \eta_o(C^{-1})$,

 (ii) $Y = \emptyset$ and $\eta_o(P^{-1}) > \eta_o(C)$.

If (c)(i) (respectively (c)(ii)) holds, $W$ is determined, as in Lemma 1, from the smallest (respectively largest) solution of (10).

To prove Theorem 5 we need to establish the following lemmas.

**Lemma 2.** Let $[P, C]$ be stable and let $\beta \leq \beta_{\max} < 1$. If $X = \emptyset$ (respectively $Y = \emptyset$), then the smallest (respectively largest) solution of (10) is continuous with $W_R, W_R^{-1} \in \mathcal{L}_\infty$, and satisfies $\|W_R\|_\infty \to 0$ as $\beta \to 0$ (respectively $\|W_R^{-1}\|_\infty \to 0$ as $\beta \to 0$). If $X \neq \emptyset$ and $Y \neq \emptyset$, then (10) has no solution with $W_R, W_R^{-1} \in \mathcal{L}_\infty$ for any $\beta < \beta_{\max}$.

**Proof.** If $\beta \leq \beta_{\max}$, then there are two (strictly) positive solutions to (10) for all points in $j\mathbb{R} \cup \{\infty\}$ which are not poles or zeros of $P$ and $C$. In case $C$ has a zero (respectively $P$ has a pole) at some point in $j\mathbb{R} \cup \{\infty\}$, then (10) has the solutions $W_R = 0, \alpha^2/|P|^2$ (respectively $W_R = 0, \alpha^2|C|^2$) at that point. Notice that the second solution is non-zero in each case because $[P, C]$ is stable (which means that there are no pole-zero cancellations between $P$ and $C$ on $j\mathbb{R} \cup \{\infty\}$) and $\alpha = (\beta^{-2} - 1)^{1/2} \neq 0$ since $\beta < 1$. Similarly, if $C$ has a pole (respectively $P$ has a zero) at some point in $j\mathbb{R} \cup \{\infty\}$, then (10) has the solutions $W_R = \infty, 1/\alpha^2|P|^2$ (respectively $W_R = \infty, |C|^2/\alpha^2$) at that point, and the second solution is not equal to $\infty$.

Now, if $X = \emptyset$, we can take the smallest solution $W_R$ to (10), frequency by frequency. This solution can never be infinite (since it is the smallest solution) nor can it be zero (since $X = \emptyset$). In case $Y = \emptyset$, we can take the largest solution of (10), frequency by frequency, which is again never zero or infinite. For these choices $W_R$ is continuous and $W_R, W_R^{-1} \in \mathcal{L}_\infty$.

Write $a = PP^*, b_1 = PC + P^*C^*, b_2 = |1 + PC|^2$ and $c = C^*C$. Then, solving (10), we get

$$\begin{aligned} W_R &= \frac{b_1 + \alpha^2 b_2 \pm \sqrt{(b_1 + \alpha^2 b_2)^2 - 4ac}}{2a} \\ &= \frac{\alpha^2}{2a}\left\{ b_2 + \frac{b_1}{\alpha^2} \pm b_2 \left(1 + \frac{2b_1}{b_2\alpha^2} + \frac{b_1^2 - 4ac}{b_2^2\alpha^4}\right)^{1/2}\right\} \\ &= \frac{\alpha^2}{2a}\left\{ b_2 + \frac{b_1}{\alpha^2} \pm b_2 \left(1 + \frac{b_1}{b_2\alpha^2} - \frac{2ac}{b_2^2\alpha^4} + O(\alpha^{-6})\right)\right\}. \end{aligned}$$

If we denote by $W_{R+}$ and $W_{R-}$ the largest and smallest solutions of (10), we obtain

$$
\begin{aligned}
W_{R+} &= \frac{b_2}{a}\alpha^2 + \frac{b_1}{a} - \frac{c}{b_2}\alpha^{-2} + O(\alpha^{-4}), && (17) \\
W_{R-} &= \frac{c}{b_2}\alpha^{-2} + O(\alpha^{-4}). && (18)
\end{aligned}
$$

If $X = \emptyset$, we set $W_R = W_{R-}$. From (18) and the expressions for $W_{R-}$ at poles of $C$ and zeros of $P$, we see that $W_{R-}(j\omega) \to 0$ as $\alpha \to \infty$ for all $\omega$ including $\infty$. Thus $\|W_R\|_\infty \to 0$ as $\alpha \to \infty$. Now observe that $\alpha \to \infty$ when $\beta \to 0$ since $\alpha = (\beta^{-2} - 1)^{1/2}$. Hence, if $Y = \emptyset$, we set $W_R = W_{R+}$ and we get $W_R(j\omega) \to \infty$ as $\beta \to 0$ for all $\omega$ including $\infty$. Thus $\|W_R^{-1}\|_\infty \to 0$ as $\beta \to 0$.

Finally, observe that if $X \neq \emptyset$ and $Y \neq \emptyset$, the largest solution of (10) is infinite at some frequencies and the smallest solution is zero at some frequencies. Since $\beta < \beta_{\max}$, the two solutions are always distinct (and continuous). So there is no possibility to switch between the solutions. Hence no admissible $W_R$ exists in this case. $\square$

**Lemma 3.** Let $P$ and $C$ be rational and suppose either $X = \emptyset$ or $Y = \emptyset$. For $0 < \beta \leq \beta_{\max} < 1$, let $W_R(j\omega)$ be chosen frequency by frequency, as the smallest solution of (10) if $X = \emptyset$ or the largest solution if $Y = \emptyset$. Then

$$
\operatorname{wno}\left(\frac{C - W_R P^*}{1 + PC}\right)
$$

is independent of $\beta$ for $0 < \beta \leq \beta_{\max}$.

**Proof.** Observe that $(C - W_R P^*)/(1 + PC)$ is never zero on $j\mathbb{R} \cup \{\infty\}$. Furthermore, $W_R(j\omega)$ depends continuously on both $\beta$ and $\omega$. Thus, the closed curve defined by $(C - W_R P^*)/(1 + PC)$ defines a homotopy as $\beta$ varies. Hence, the above winding number is independent of $\beta$ [3, Theorem 7.12]. $\square$

**Proof of Theorem 5.** (a) $\Leftrightarrow$ (b). Suppose (a) is holds. Then, from Lemma 2, either $X = \emptyset$ or $Y = \emptyset$ and the modulus condition can be satisfied for any $0 < \beta \leq \beta_{\max}$. From Lemma 3, the winding number of $(C - W_R P^*)/(1 + PC)$ is independent of $\beta$. Hence, the winding number condition must be satisfied for all $0 < \beta \leq \beta_{\max}$. Thus (b) follows from Theorem 3. The reverse implication is trivial.

(b) $\Leftrightarrow$ (c). Suppose (b) holds. Then either $X = \emptyset$ or $Y = \emptyset$ from Lemma 2. Let us assume that $X = \emptyset$. Then, from Lemma 2, we can choose $W_R$ to satisfy the modulus condition so that $\|W_R\|_\infty \to 0$ as $\beta \to 0$. Since $[P, C]$ is stable, this means that $\|W_R P^*/(1 + PC)\|_\infty \to 0$ as $\beta \to 0$. Hence, from Lemma 3, we see that

$$
\operatorname{wno}\left(\frac{C - W_R P^*}{1 + PC}\right) = \operatorname{wno}\left(\frac{C}{1 + PC}\right) = -\eta_o(P) - \eta_o(C^{-1}).
$$

(Note that $C/(1 + PC) \in \mathcal{H}_\infty$, since $[P, C]$ is stable.) Thus the winding number condition becomes $\eta_o(P) - \eta_o(C^{-1}) > 0$ from Theorem 3. Hence (c)(i) holds. Conversely, if (c)(i) holds, the modulus condition can be satisfied and the above implications can be reversed to show that the winding number condition is satisfied for all $0 < \beta \leq \beta_{\max}$.

Now let us assume that (b) holds and $Y = \emptyset$. Then we can choose $W_R$ so that $\|W_R^{-1}\| \to 0$ as $\beta \to 0$. Again, from Lemma 3,

$$\begin{aligned}
\text{wno}\left(\frac{C - W_R P^*}{1 + PC}\right) &= \text{wno}\left(\frac{W_R^{-1} C - P^*}{1 + PC}\right) \\
&= \text{wno}\left(\frac{-P^*}{1 + PC}\right) \\
&= \text{wno}\left(\frac{P}{1 + PC}\right) + \text{wno}\left(\frac{P^*}{P}\right) \\
&= -2\eta_o(P) - \eta_a(P) + \eta_o(P^{-1}) - \eta_o(C).
\end{aligned}$$

(The last step uses the fact that $\text{wno}(P/(1 + PC)) = -\eta_o(P^{-1}) - \eta_o(C)$ and $\text{wno}(P^*/P) = 2\eta_o(P^{-1}) + \eta_a(P^{-1}) - 2\eta_o(P) - \eta_a(P)$, which can be proved in a manner similar to that of the final part of Theorem 2.) Thus the winding number condition becomes $\eta_o(P^{-1}) > \eta_o(C)$ from Theorem 3. Similarly, the above implications can be reversed. This completes the proof of (b) $\Leftrightarrow$ (c). $\square$

# 5 Solution of inverse problem for $\beta = \beta_{\max} < 1$

The purpose of this section is to prove the necessary and sufficient condition for $\beta_{\max}$-optimality given below in Theorem 6. Take the points $\omega_1 < \omega_2 < \cdots < \omega_r$ on $j\mathbb{R} \cup \{\infty\}$ where $|S| + |T|$ achieves its supremum. Throughout this section we assume $\beta_{\max} < 1$. This means that $PC$ cannot have poles or zeros at $\omega_i$ and $(PC)(j\omega_i)$ cannot lie on the positive real axis. Thus we can define a curve $\Gamma_i$ as follows: take the path $(PC)(j\omega)$ for $\omega_i \le \omega \le \omega_{i+1}$, indent around the poles and zeros of $P$ and $C$ into the *right* half plane, and complete to a closed curve by joining $(PC)(j\omega_{i+1})$ to $(PC)(j\omega_i)$ by any path which doesn't go through the origin or intersect the positive real axis. Observe that $\text{wno}(\Gamma_i)$ is well-defined for $i = 1, 2, \cdots, r-1$. We define $\Gamma_r$ by considering the path traced out by $(PC)(j\omega)$ as $\omega$ goes from $\omega_r$ to $\infty$ and then from $-\infty$ to $\omega_1$. For notational convenience, we identify $\omega_{r+1} = \omega_1$ and define the set $[\omega_r, \omega_{r+1}]$ to be $(-\infty, \omega_1] \cup [\omega_r, \infty) \cup \{\infty\}$. On each interval $[\omega_i, \omega_{i+1}]$ define the following quantities (for $i = 1, 2, \cdots, r$):

$$n_i^- = \begin{cases} -\infty & \text{if } X \cap [\omega_i, \omega_{i+1}] \ne \emptyset \\ 0 & \text{if } X \cap [\omega_i, \omega_{i+1}] = \emptyset \end{cases}$$

$$n_i^+ = \begin{cases} -\infty & \text{if } Y \cap [\omega_i, \omega_{i+1}] \ne \emptyset \\ -\text{wno}(\Gamma_i) & \text{if } Y \cap [\omega_i, \omega_{i+1}] = \emptyset \end{cases}$$

and let $n_i = \max(n_i^-, n_i^+)$. Also define:

$$m_i^- = \begin{cases} -\infty & \text{if } X \cap [\omega_i, \omega_{i+1}] \ne \emptyset \\ \text{wno}(\Gamma_i) & \text{if } X \cap [\omega_i, \omega_{i+1}] = \emptyset \end{cases}$$

$$m_i^+ = \begin{cases} -\infty & \text{if } Y \cap [\omega_i, \omega_{i+1}] \ne \emptyset \\ 0 & \text{if } Y \cap [\omega_i, \omega_{i+1}] = \emptyset \end{cases}$$

and let $m_i = \max(m_i^-, m_i^+)$.

**Theorem 6.** Let $[P, C]$ be stable and $\beta_{\max} < 1$. Then there exists a $W, W^{-1} \in \mathcal{A}_0$ such that $C$ is $\beta$-optimal for $P$ with $\beta = \beta_{\max}$ if and only if $\Delta > 0$ where $\Delta$ is given by the following two alternative expressions:

$$\begin{aligned}\Delta &= \eta_o(P) - \eta_o(C^{-1}) + \sum_{i=1}^{r} n_i &(19)\\ &= \eta_o(P^{-1}) - \eta_o(C) + \sum_{i=1}^{r} m_i. &(20)\end{aligned}$$

**Proof.** Observe that a necessary condition for $\beta_{\max}$-optimal is that, for each $i$, at least one of $X$ and $Y$ has an empty intersection with $[\omega_i, \omega_{i+1}]$. Otherwise $W_{R+}$ is unbounded and $W_{R-}$ goes to zero at the poles/zeros of $P$ or $C$ in the interval $[\omega_i, \omega_{i+1}]$ (see equations (17) and (18)) and so it will not be possible to find a continuous weighting function $W_R, W_R^{-1} \in \mathcal{L}_\infty$. In such a case $n_i = -\infty$ for some $i$, so $\Delta = -\infty$. Thus the condition of the theorem is not satisfied. Hence we only need to prove the theorem in the case that there exists a continuous choice of $W_R$ such that $W_R, W_R^{-1} \in \mathcal{L}_\infty$. We will make this assumption for the remainder of the proof.

Let $W, W^{-1} \in \mathcal{A}_0$ be such that $W_R = |W|^2$ is *some* solution of (10). Let $W_{R-}$ be the smallest solution of (10), frequency by frequency. As in the proof of Lemma 1 we can find a $W_- \in \mathcal{H}_\infty$ (outer) such that $W_{R-} = |W_-|^2$. We will see below that $W_- \in \mathcal{A}_0$ (although it may not be the case that $W_-^{-1} \in \mathcal{A}_0$). We can now write

$$\text{wno}\left(\frac{W^{-1}C - W^*P^*}{1+PC}\right) = \text{wno}\left(\frac{W_-^{-1}C - W_-^*P^*}{1+PC}\right) + \text{wno}\left(\frac{W^{-1}C - W^*P^*}{W_-^{-1}C - W_-^*P^*}\right), \qquad (21)$$

since each of the three functions in (21) is continuous and has constant modulus so that the winding numbers are well-defined. We will now consider each of the terms on the right hand side of (21) separately. To facilitate the computation we denote the three winding numbers in (21) by $\kappa$, $\kappa_1$ and $\kappa_2$ respectively, so (21) becomes $\kappa = \kappa_1 + \kappa_2$.

To show that $W_- \in \mathcal{A}_0$ and to calculate $\kappa_1$ we introduce a transformation on $P$, $C$, $W_-$, $W_{R-}$ as follows. Let $W_P$ (respectively $W_C$) be a rational function in $\mathcal{H}_\infty$ whose zeros are exactly the zeros of $P^{-1}$ (respectively $C$) on $j\mathbb{R} \cup \{\infty\}$ (counting multiplicity). Define $\hat{P} = W_P P W_C, \hat{C} = W_P^{-1} C W_C^{-1}, \hat{W}_- = W_P^{-1} W_- W_C^{-1}, \hat{W}_{R-} = |\hat{W}_-|^2$. Clearly $\hat{P}$ (respectively $\hat{C}$) has no poles (respectively zeros) on $j\mathbb{R} \cup \{\infty\}$, and (10) becomes

$$|\hat{P}|^2 \hat{W}_{R-}^2 - (\hat{P}\hat{C} + \hat{P}^*\hat{C}^* + \alpha^2|1 + \hat{P}\hat{C}|^2)\hat{W}_{R-} + |\hat{C}|^2 = 0 \qquad (22)$$

where $\hat{W}_{R-}$ is the minimum solution of (22), frequency by frequency. Thus $\hat{W}_{R-}, \hat{W}_{R-}^{-1} \in \mathcal{L}_\infty$ and so $\hat{W}_-, \hat{W}_-^{-1} \in \mathcal{A}_0$ from Lemma 1. This proves that $W_- \in \mathcal{A}_0$. Furthermore, we are reduced to a situation where the conditions of Theorem 5(c)(i) apply. In particular,

$$\begin{aligned}\kappa_1 &= \text{wno}\left(\frac{W_-^{-1}C - W_-^*P^*}{1+PC}\right) = \text{wno}\left(\frac{\hat{W}_-^{-1}\hat{C} - \hat{W}_-^*\hat{P}^*}{1+\hat{P}\hat{C}}\right)\\ &= \text{wno}\left(\frac{\hat{C} - \hat{W}_{R-}\hat{P}^*}{1+\hat{P}\hat{C}}\right) = \text{wno}\left(\frac{\hat{C}}{1+\hat{P}\hat{C}}\right)\\ &= -\eta_o(\hat{P}) - \eta_o(\hat{C}^{-1}) = -\eta_o(P) - \eta_o(C^{-1}). &(23)\end{aligned}$$

To calculate $\kappa_2$, we introduce some small semi-circular indentations from the imaginary axis into the right half plane around the poles of $P$ and zeros of $C$. Throughout the rest of this proof we will define the conjugates of all weighting functions $W, W_-$ etc. by

$$W(s)^* = \overline{W(s)}$$

for locations in the RHP. (This does not require the weighting functions to have continuations into the LHP.) We also define the conjugates of $P$ and $C$, off-axis, by:

$$P(s)^* = \overline{P(s)},\ C(s)^* = \overline{C(s)}.$$

(Note that this is not the usual way of extending the conjugation.) Let $\mathcal{D} = j\mathbb{R} \cup \{\infty\}$ be oriented upwards and let $\mathcal{D}_\epsilon$ be the contour $\mathcal{D}$ modified with semi-circular indentations into the right half plane of radius $\epsilon$ about the points of $X$ (and radius $1/\epsilon$ about 0 if $\infty \in X$). Then we have

$$\begin{aligned} \kappa_2 &= \operatorname*{wno}_{\mathcal{D}} \left( \frac{W^{-1}C - W^*P^*}{W_-^{-1}C - W_-^*P^*} \right) \\ &= \operatorname*{wno}_{\mathcal{D}_\epsilon} \left( \frac{W^{-1}C - W^*P^*}{W_-^{-1}C - W_-^*P^*} \right) - \eta_a(P) \end{aligned} \tag{24}$$

for sufficiently small $\epsilon$. To see the latter step we first claim that

$$\frac{W^{-1}C - W^*P^*}{W_-^{-1}C - W_-^*P^*} \tag{25}$$

is continuous at zeros of $C$ on $\mathcal{D}$ when the limit is taken from the right half plane. This is true since, if $C$ has a zero of order $\ell$ at $z_0 = j\omega_0 \neq 0$, then we can write

$$W_-(s) = \frac{(s^2+\omega_0^2)^\ell}{(s+1)^{2\ell}} \hat{W}_-(s) \tag{26}$$

with $\hat{W}_-(j\omega_0)$ finite and non-zero (by a substitution similar to that used to obtain (22)). Thus the numerator and denominator in (25) are separately continuous. (Note that $P$ cannot have a pole at $z_0$.) Since $(W_-^{-1}C - W_-^*P^*)/(1+PC)$ has constant non-zero modulus and $1+PC$ cannot be zero on $\mathcal{D}$ then the denominator of (25) is non-zero, which implies the stated continuity property. Thus the small indentations around the zeros of $C$ are mapped into small excursions which do not change the winding number $\kappa_2$ for small enough $\epsilon$. Zeros of $C$ at $s = 0$ or $\infty$ are treated similarly. Next consider an indentation around a pole of $P$ of order $\ell$ at $z_0 = j\omega_0 \neq 0$. Once again we can write $W_-(s)$ as in (26) with $\hat{W}_-(j\omega_0)$ finite and non-zero. Since $C$ cannot have a zero or pole at $z_0$ (due to the consideration of stability and the assumption we made at the beginning of this proof), then

$$\frac{W^{-1}C - W^*P^*}{W_-^{-1}C - W_-^*P^*} \sim c\left(\frac{s-z_0}{s^*-z_0^*}\right)^\ell = ce^{2j\theta\ell}$$

near $z_0$, where $c$ is a non-zero constant and $s = z_0 + \epsilon e^{j\theta}$ $(-\pi/2 \leq \theta \leq \pi/2)$. Therefore, the small indentation about the pole of $P$ is mapped into a path which encircles the origin $\ell$ times anticlockwise. Adding together the contribution from all the imaginary axis poles of $P$ proves the equality (24). Next note that $W(s)$ and $W_-(s)$ are analytic, continuous and bounded away from zero on the right half plane minus the small semi-discs, which means that $\underset{\mathcal{D}_\epsilon}{\mathrm{wno}}(W(s)) = \underset{\mathcal{D}_\epsilon}{\mathrm{wno}}(W_-(s)) = 0$. Therefore

$$\kappa_2 = \underset{\mathcal{D}_\epsilon}{\mathrm{wno}} \left( \frac{C(s) - |W(s)|^2 P(s)^*}{C(s) - |W_-(s)|^2 P(s)^*} \right) - \eta_a(P). \tag{27}$$

Let $[\omega_i, \omega_{i+1}]$ denote the segment of $\mathcal{D}_\epsilon$ joining $\omega_i$ to $\omega_{i+1}$. Now observe that

$$\frac{C - |W|^2 P^*}{C - |W_-|^2 P^*} = 1$$

at all $\omega_i$, and on all segments $[\omega_i, \omega_{i+1}]$ where $|W| = |W_-|$. Thus, the curve obtained in (27) by evaluating $(C - |W|^2 P^*)/(C - |W_-|^2 P^*)$ around $\mathcal{D}_\epsilon$ comprises $r$ separate closed curves corresponding to each of the segments $[\omega_i, \omega_{i+1}]$ individually. On the intervals where $|W| = |W_-|$ the curve reduces to a single point (at 1) and makes no contribution to the overall winding number $\kappa_2$. Therefore

$$\kappa_2 = \sum_{i \in \chi} \underset{[\omega_i, \omega_{i+1}]}{\mathrm{wno}} \left( \frac{C - |W|^2 P^*}{C - |W_-|^2 P^*} \right) - \eta_a(P) \tag{28}$$

where $\chi =$ the set of indices $i$ such that $|W| = |W_+|$ on $[\omega_i, \omega_{i+1}]$.

We will now consider an individual summand in (28). We will calculate the winding number of the curve by finding all of its negative real axis crossings and relating these to the positive real axis crossing of $PC$. First write

$$\Lambda(s) := \frac{C - |W(s)|^2 P^*}{C - |W_-(s)|^2 P^*} = \frac{PC - |W(s)|^2 |P|^2}{PC - |W_-(s)|^2 |P|^2}. \tag{29}$$

From (10) we note that $|W_+(j\omega)|^2 |W_-(j\omega)|^2 = |(C/P)(j\omega)|^2$. Thus, on (open) intervals of the imaginary axis where $|W| = |W_+|$ we have

$$|W_-(s)|^2 |P(s)|^2 < |(PC)(s)| < |W(s)|^2 |P(s)|^2. \tag{30}$$

If we write $(PC)(s) = z = x + j\delta$, $|W_-(s)|^2 |P(s)|^2 = a$, $|W(s)|^2 |P(s)|^2 = b$, then from (30) $a < (x^2 + \delta^2)^{1/2} < b$ and

$$\Lambda(s) = \frac{z - b}{z - a} = \frac{(x-a)(x-b) + \delta^2 + j\delta(b-a)}{|z-a|^2}.$$

Thus, on open intervals of the imaginary axis where $|W| = |W_+|$, we have the following conclusions: $\Lambda(s)$ is real if and only if $(PC)(s)$ is real; $\Lambda(s)$ and $(PC)(s)$ are non-zero and have opposite signs at their real axis crossings; real axis crossings of $(PC)(s)$ in the clockwise sense correspond to real axis crossing of $\Lambda(s)$ in the anticlockwise sense. We will justify that the same conclusions hold for the small semi-circular indentations. For this, it is sufficient to prove that (30) is valid on the indentations, providing they are sufficiently small. The lower inequality can be seen to hold by noting that

$$\left|\left(\frac{W_-P}{C}\right)(s)\right|^2 < \left|\left(\frac{P}{C}\right)(s)\right|,$$

which follows since the left hand side remains bounded on the indentation, while the right hand side tends to infinity. The upper inequality can be seen to hold by noting that

$$\left|\left(\frac{C}{P}\right)(s)\right| < |W(s)|^2$$

since the left hand side tends to zero and the right hand side is bounded away from zero. Thus, in summary, the positive (respectively negative) real axis crossing of $\Lambda(j\omega)$ in the *anticlockwise* sense are in one-to-one correspondence with negative (respectively positive) real axis crossings of $(PC)(j\omega)$ in the *clockwise* sense. We have therefore shown that

$$\sum_{i\in\chi} \underset{[\omega_i,\omega_{i+1}]}{\mathrm{wno}} \left(\frac{C-|W|^2P^*}{C-|W_-|^2P^*}\right) = -\sum_{i\in\chi} \mathrm{wno}(\Gamma_i).$$

Thus, from (21), (23) and (28), we have

$$\mathrm{wno}\left(\frac{W^{-1}C - W^*P^*}{1+PC}\right) + 2\eta_o(P) + \eta_a(P) = \eta_o(P) - \eta_o(C^{-1}) - \sum_{i\in\chi}\mathrm{wno}(\Gamma_i). \qquad (31)$$

From Theorem 2, $[P,C]$ is $\beta_{\max}$-optimal with the weighting function $W$ if and only if the expression in (31) is positive. Therefore, $[P,C]$ is $\beta_{\max}$-optimal for *some* weighting function if and only if the maximum possible value of (31) is positive among all admissible weighting functions. The question of maximizing (31) amounts to checking, in each interval $[\omega_i,\omega_{i+1}]$, which of the choices $|W_+|$ and $|W_-|$ are admissible. In the former case we have a contribution of $-\mathrm{wno}(\Gamma_i)$ to (31) and no contribution in the latter case. If both choices are admissible we can select a contribution of $-\mathrm{wno}(\Gamma_i)$ or zero, depending on which is maximal. Therefore the maximum value of (31) is given by the expression (19).

To verify the alternative expression for $\Delta$, we proceed as follows. If we make indentations into the right half plane around all imaginary axis poles and zeros of $P$ and $C$, we have

$$\mathrm{wno}(PC) = \eta_o(P) + \eta_o(C) - \eta_o(P^{-1}) - \eta_o(C^{-1}). \qquad (32)$$

By considering the positive real axis crossings, we see that

$$\mathrm{wno}(PC) = \sum_{i=1}^{r}\mathrm{wno}(\Gamma_i). \qquad (33)$$

Therefore, from (19), (32) and (33)

$$\begin{aligned}\Delta &= \eta_o(P) - \eta_o(C^{-1}) + \sum_{i=1}^{r} n_i \\ &= \eta_o(P^{-1}) - \eta_o(C) + \sum_{i=1}^{r}(n_i + \mathrm{wno}(\Gamma_i)).\end{aligned}$$

This gives the alternative expression (20) since, from the definitions of $n_i$ and $m_i$, $n_i + \mathrm{wno}(\Gamma_i) = m_i$ for all $i$. □

**Remarks:**

(1) If Theorem 5(c) holds, $C$ is $\beta$-optimal for $P$ for all $0 < \beta \leq \beta_{\max} < 1$. This means that the number of solutions for $W, W^{-1} \in \mathcal{A}_0$ is infinite since there is at least one solution for each $\beta$. If Theorem 5(c) fails, there is still the possibility that $C$ is $\beta_{\max}$-optimal for $P$. This needs to be checked using Theorem 6. In this case there are finitely many possible choices of $W, W^{-1} \in \mathcal{A}_0$ obtained by swapping between the largest and smallest solutions of (10), at the frequencies $\omega_i$, in making a continuous choice of $W_R$. The proof of Theorem 6 gives the procedure to choose between the largest and smallest solutions. If $X \cap [\omega_i, \omega_{i+1}] = \emptyset$ we may choose the smallest solution and if $Y \cap [\omega_i, \omega_{i+1}] = \emptyset$ we may choose the largest one, which gives 2, 1 or 0 admissible choices of $|W|$ in each interval. (Clearly if some interval has no admissible choice of $|W|$ we cannot have optimality, as was pointed out in [17], and we do not need to proceed further.) For any admissible choice, we calculate

$$\eta_o(P) - \eta_o(C^{-1}) - \sum_{i \in \chi} \mathrm{wno}(\Gamma_i), \tag{34}$$

where $\chi$ corresponds to the segments where the largest solution is chosen. If (34) is positive then the corresponding weighting function gives $\beta_{\max}$-optimality (otherwise not). The set of admissible choices of $W$ for which (34) is positive is the set of $W$ for which $C$ is $\beta_{\max}$-optimal for $P$. The quantity $\Delta$ in Theorem 6 is simply the maximum of (34) over all admissible $W$. Note, as in the statement of Theorem 6, (34) is equal to the alternative expression

$$\eta_o(P^{-1}) - \eta_o(C) + \sum_{i \in \bar{\chi}} \mathrm{wno}(\Gamma_i),$$

where $\bar{\chi}$ corresponds to the segments where the smallest solution is chosen.

(2) The conditions of Theorem 6 can be checked from a knowledge of the right half plane poles and zeros of $P$, $C$ and the Nyquist diagram of $PC$ when the maximizing frequencies $\omega_i$ of $|S| + |T|$ are known. These points on the Nyquist diagram can be estimated from the loci $(1 + |z|)/|1 + z| = c$ (constant) in the complex plane (see Fig. 2). In particular we find the largest $c$ which gives an intersection with the Nyquist diagram and select the corresponding points of intersection. The largest such $c$ equals $\beta_{\max}^{-1}$.

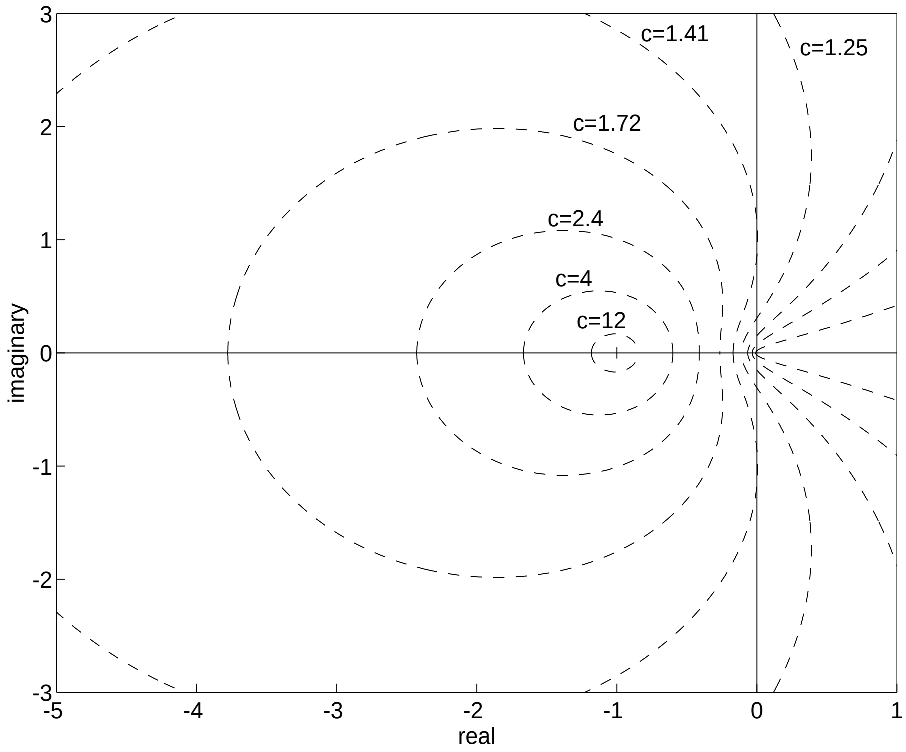


Figure 2: The loci $(1+|z|)/|1+z| = c > 1$.

(3) Given $P$ and $C$, a typical situation is that $|S|+|T|$ achieves its supremum at two frequencies $\pm\omega_0 \neq 0$. In this case, a small perturbation to $P$ and $C$ (say in the gap metric) will not alter the possibility of achieving $\beta_{\max}$-optimality. However, in the (non-generic) case of a higher number of supremal points of $|S|+|T|$, a small perturbation to $P$ and/or $C$ may cause $\beta_{\max}$-optimality to be lost (see Example 2 in Section 7). This means that $\beta_{\max}$-optimality is not a robust property in general.

# 6 Solution of inverse problem for $\beta_{\max} = 1$

If $[P, C]$ is stable, then $\beta_{\max} = 1$ is equivalent to: $P$ and $C$ are stable, $PC = \gamma$ (constant) and $\gamma \geq 0$ (see proof below). The solution of the inverse problem in this case is given by the following result.

**Theorem 7.** Let $[P, C]$ be stable with $\beta_{\max} = 1$. Then the following statements are equivalent:

(a) There exists a $W, W^{-1} \in \mathcal{A}_0$ such that $C$ is $\beta$-optimal for $P$ with $\beta = 1$;

(b) Either (i) $P = C = 0$ (in which case $W, W^{-1} \in \mathcal{A}_0$ is arbitrary), or (ii) $P$ and $C$ are units in $\mathcal{H}_\infty$ and $PC = \gamma > 0$ (in which case $W = P^{-1}\sqrt{\gamma} = C/\sqrt{\gamma}$).

The following statements are also equivalent:

(c) For *some* $\beta$, $0 < \beta < 1$, there exists a $W, W^{-1} \in \mathcal{A}_0$ such that $C$ is $\beta$-optimal for $P$;

(d) For *every* $\beta$, $0 < \beta < 1$, there exists a $W, W^{-1} \in \mathcal{A}_0$ (depending on $\beta$) such that $C$ is $\beta$-optimal for $P$;

(e) $C = 0$, $P \in \mathcal{H}_\infty$, $P$ has no zeros on $j\mathbb{R} \cup \{\infty\}$, $\eta_o(P^{-1}) > 0$ (in which case $W = (\beta^{-2} - 1)^{1/2}(P_{outer})^{-1}$, where $P_{outer}$ is the minimum phase/outer factor of $P$).

**Proof.** The condition $\beta_{\max} = 1$ implies that $1 + |PC| = |1 + PC|$ on the imaginary axis. This means that $PC$ is real and non-negative on the imaginary axis. Since $[P, C]$ is stable, $P$ and $C$ must also be stable from the Nyquist stability criterion. Since $PC$ is real and non-negative on the imaginary axis, we can spectrally factorise: $PC = Z_+ Z_-$ where $Z_+ = Z_-^*$ is stable, minimal phase. But $Z_-$ cannot have any poles (except possible at infinity) since $P$ and $C$ are stable. This means that $Z_+$ and $Z_-$ are polynomials. Closed loop stability requires that $1 + Z_+ Z_+^*$ is Hurwitz, but this is only possible if $Z_+$ is a constant since the roots of this polynomial are symmetric with respect to the imaginary axis. Thus $PC = \gamma \geq 0$ where $\gamma$ is constant.

Consider first the case $PC = \gamma > 0$. Then $P$ and $C$ are both units in $\mathcal{H}_\infty$ because of closed loop stability. This establishes the necessity of (b)(ii). Conversely, if $PC = \gamma > 0$ with $P, C$ units in $\mathcal{H}_\infty$ then the choice of $W = P^{-1}\sqrt{\gamma} = C/\sqrt{\gamma}$ implies that $W^{-1}C = W^* P^*$, which establishes (a) using Theorem 2. Now suppose that $PC = 0$. From Theorem 2,

$C$ is 1-optimal for $P$ for some $W$

$\Leftrightarrow W^{-1}C - W^* P^* = 0$, for some $W$

$\Leftrightarrow W^{-1}C = W^* P^* =$ constant, for some $W$

$\Leftrightarrow C = P = 0$.

This completes the proof of (a) $\Leftrightarrow$ (b).

We now consider the case where $\beta < 1$. It can be checked that Lemma 2 holds when the statement $\beta_{\max} < 1$ is replaced by $\beta < \beta_{\max} = 1$. Moreover Lemma 3 holds but with $\beta$ in the range of $0 < \beta < \beta_{\max} = 1$. This shows the equivalence of (c) and (d). Furthermore (c) is equivalent to condition (c) of Theorem 5. Therefore no $W$ exists to make $C$ $\beta$-optimal for $P$ in the case that $PC = \gamma > 0$ since $\eta_o(P) = \eta_o(P^{-1}) = \eta_o(C) = \eta_o(C^{-1}) = 0$. Thus, we only need to consider the case where $PC = 0$. If $P = 0$, then it follows from Theorem 2 that $C$ is $\beta$-optimal for $P$ for some $W$ if and only if (i) $W^{-1}C$ has modulus $(\beta^{-2} - 1)^{1/2}$, (ii) $\mathrm{wno}(C) > 0$. But, since $C$ is stable, $\mathrm{wno}(C) \leq 0$ so there is no solution. If $C = 0$, then Theorem 2 gives: $C$ is $\beta$-optimal for $P$ for some $W$ if and only if (iii) $W^* P^*$ has modulus $(\beta^{-2} - 1)^{1/2}$ for some $W$, (iv) $\mathrm{wno}(P^*) + 2\eta_o(P) > 0$. Condition (iv) implies that $P$ has no zeros on $j\mathbb{R} \cup \{\infty\}$ and $\eta_o(P^{-1}) > 0$. Conversely, if (e) holds then so do (iii) and (iv) with $W$ as defined in (e). $\square$

# 7 Examples

**Example 1.** Consider

$$P(s) = \frac{1}{(s+1)^3} \text{ and } C(s) = k.$$

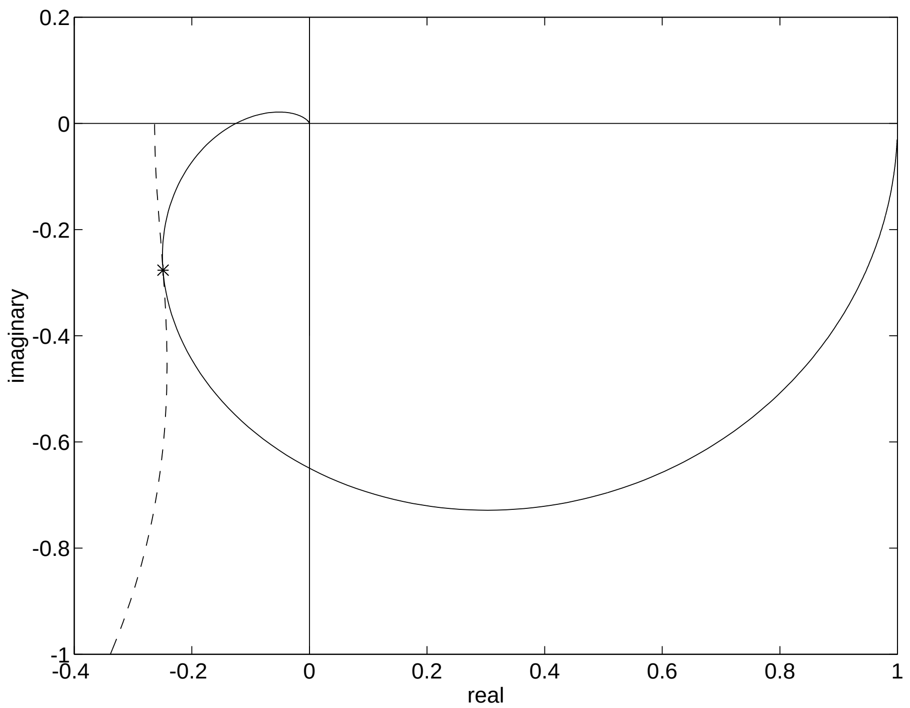


Figure 3: The Nyquist diagram of $PC$ for $k = 1$.

For closed loop stability we require $-1 < k < 8$ from the Nyquist stability criterion. Fig. 3 and Fig. 4 show the Nyquist diagram of $PC$ and the modulus of the weighting functions when $k = 1$. As $\eta_o(P) = \eta_o(P^{-1}) = 0$ and $\eta_o(C) = \eta_o(C^{-1}) = 0$, then from Theorem 5, $C$ is not $\beta$-optimal for $P$ if $\beta < \beta_{\max} < 1$. To check if $C$ is $\beta_{\max}$-optimal we need to use Theorem 6 and consider the positive real axis crossings of $(PC)(j\omega)$ to determine whether $\Delta > 0$. In Fig. 3 we see that there is an intersection point (marked by a "$*$") between the path of $PC$ (shown by the solid line) and the graph of $(1+|z|)/|1+z| = \beta_{\max}^{-1}$ (shown by the dashed line for $\beta_{\max} = 0.5832$). So it is possible for us to switch between upper and lower solutions of (10) at the frequency corresponding to the "$*$" point, which is the frequency where the two weighting functions have the same modulus in Fig. 4. Referring to Fig. 4, we choose the upper solution in the lower frequency range and the lower solution in the higher frequency range, which is shown by the solid line. Calculation of the appropriate winding number shows that $\Delta = 1$, which means that $C$ is optimal for $P$ with this weighting function when $k = 1$. The only other possible choice which gives $W, W^{-1} \in \mathcal{A}_0$ is to select the lower solution in the lower frequency range. However, this gives $\Delta = 0$, so this weighting function does not give optimality.

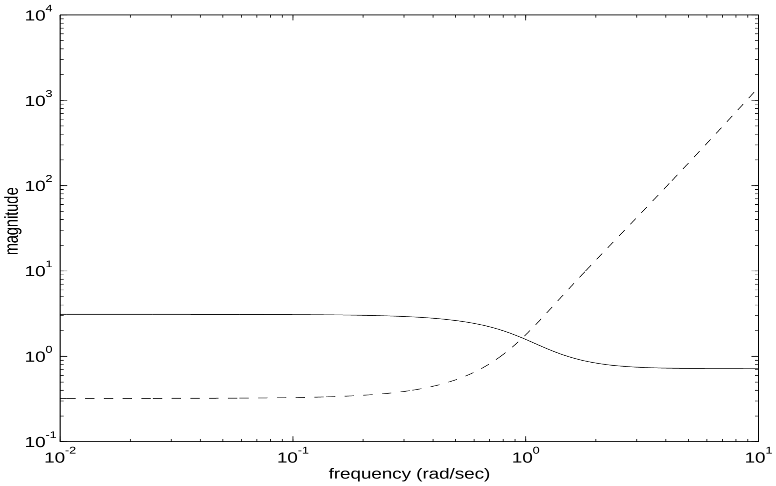


Figure 4: The modulus of the candidate weighting functions for $k = 1$.

It can also be seen from the Nyquist diagram (see below for a full explanation) that $C$ is optimal for some weighting function if and only if $\beta = \beta_{\max} < 1$ and $0 < k < 8$. A plot of $\beta_{\max}$ versus $k$ is given in Fig. 5. Fig. 6 shows the form of the optimal weighting function as a function of $k$. Observe that, for $k$ close to 8, the weighting function has very large (small) modulus for $\omega < \sqrt{3}$ ($\omega > \sqrt{3}$) and that the value of $\beta_{\max}$ is very small. For $k$ close to zero, the weighting function has very small modulus for all $\omega$ and $\beta_{\max} \to 1$.

To see that $C$ is not optimal for $k < 0$, take $\omega \neq 0$ and note that

$$|S(j\omega)| + |T(j\omega)| < \frac{1 + |k|}{|1 + k|} = |S(0)| + |T(0)|.$$

Thus $r = 1$ and $\omega_1 = 0$. Since $Y \neq \emptyset$ we have $n_1 = 0$ and hence $\Delta = 0$ which means, from Theorem 6, that $C$ is not optimal. In the case that $k = 0$, $C$ is not optimal because

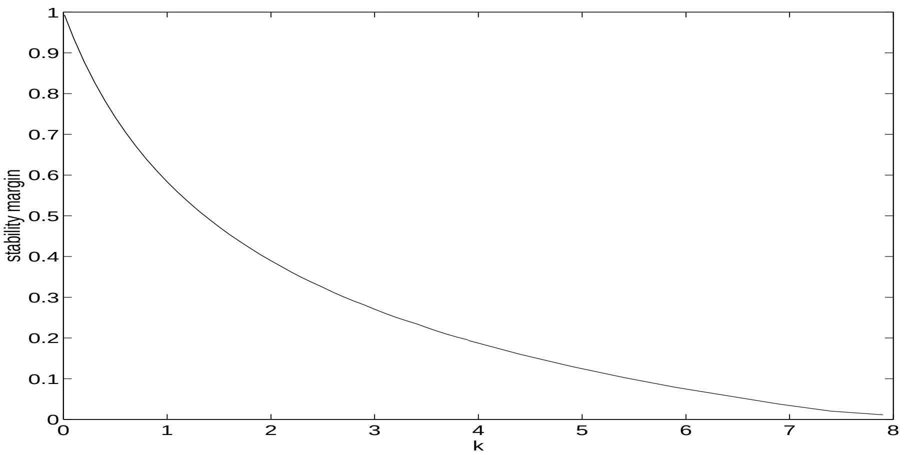


Figure 5: $\beta_{\max}$ versus $k$.

of Theorem 7. To see that $C$ is optimal for any $0 < k < 8$, observe that $|S| + |T|$ takes the value zero when $\omega = 0$ or $\infty$. Hence $|S| + |T|$ must achieve a maximum for some non-zero frequencies. Therefore, there exist frequencies $\omega_i < 0 < \omega_{i+1}$ where $|S| + |T|$ achieves its maximum. Inspection of the Nyquist diagram shows that $n_i = 1$ corresponding to a choice of the upper solution of (10) in this interval. We choose the lower solution in the interval containing the point at infinity. This gives $\Delta = 1$, which implies optimality using Theorem 6.

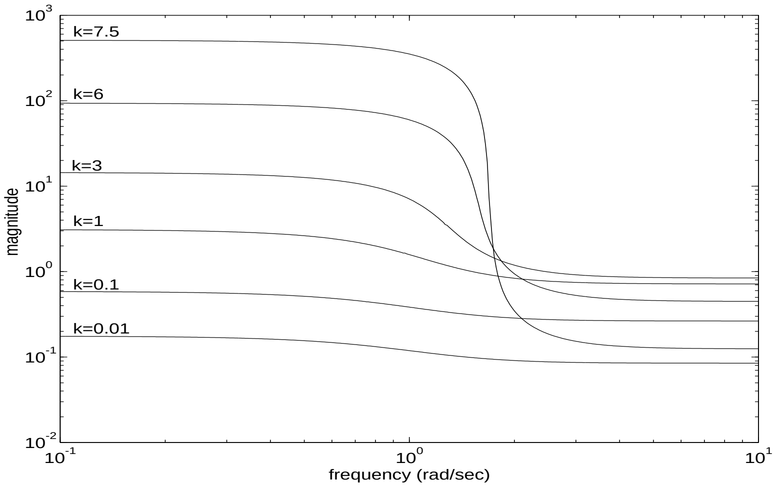


Figure 6: The optimal weighting function with respect to $k$.

It is worth making a remark on the fact that values of $k$ which are arbitrarily close to 8 are still potentially optimal for some weighting function. On the surface, this would seem to indicate an unsatisfactory feature of the $\mathcal{H}_\infty$ loop shaping method. However, to produce such a $k$ it must be noted that the weighting functions need to have an extremely rapid attenuation rate close to the frequency $\sqrt{3}$, which corresponds to a rather unusual performance demand. Moreover, the value of $\beta$ which results from this weighting function is very small, indicating that the performance goals have not been met. Thus, the designer's approach needs to be quite unreasonable in order to obtain such a controller. (See [14, especially Section 6.4.2] for a discussion of the fact that small values of $\beta_{\max}$ correspond to poor performance. Typically values of $\beta_{\max}$ in excess of 0.2 or 0.3 are sought in practical designs.)

**Example 2.** Consider

$$P(s) = \frac{s + 0.1}{(s+1)(s+2)}.$$

Calculating the $\mathcal{H}_\infty$-optimal controller $C$ for $P$ with unit weighting function, we obtain

$$C(s) = 0.1627 \times \frac{s - 0.0374}{s + 0.0544}.$$

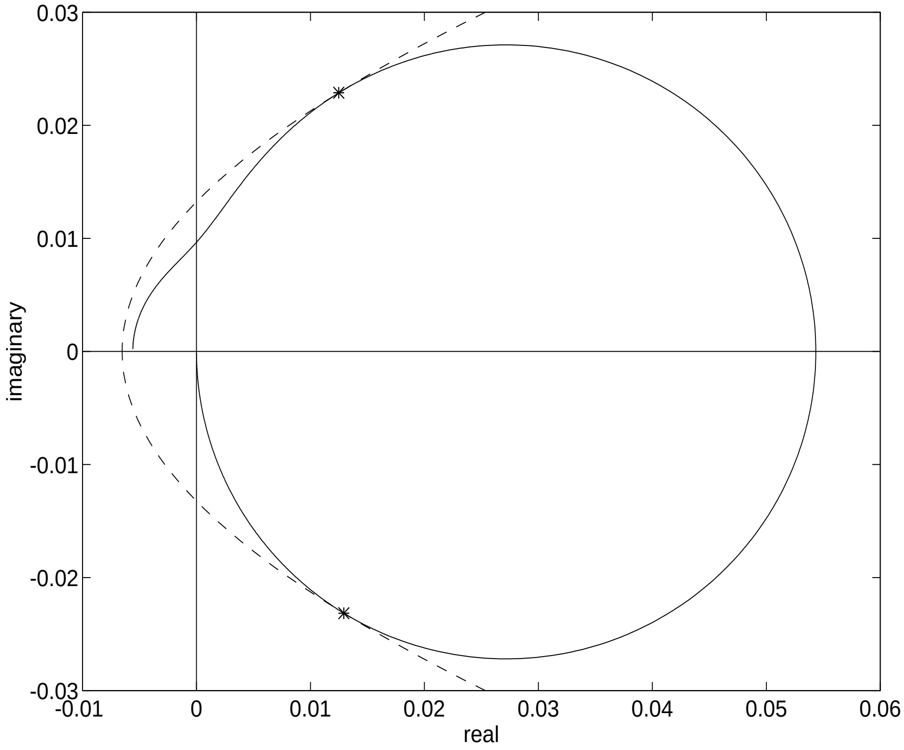


Figure 7: The Nyquist diagram of $PC$.

Fig. 7 shows the Nyquist diagram of $PC$. We now illustrate our results by solving the inverse weighting problem for this $P$ and $C$. As $\eta_o(P) = \eta_o(P^{-1}) = 0$, Theorem 5 cannot apply and it is only possible to make $C$ $\beta$-optimal for $P$ with some weighting function when $\beta = \beta_{\max} < 1$ by using Theorem 6. From (15) we can calculate that $\beta_{\max} = 0.987$ and there are four frequencies at which $|S| + |T|$ achieves its supremum ($\omega_1 < \omega_2 < \omega_3 < \omega_4$ and $\omega_1 = -\omega_4, \omega_2 = -\omega_3$). This is illustrated in Fig. 7 where the locus $(1 + |z|)/|1 + z| = \beta_{\max}^{-1}$ (shown in dashed lines) intersects the Nyquist diagram of $PC$ at two distinct positive frequencies. The intersection points are shown by the "$*$" points in Fig. 7. In Fig. 8, the solutions $W_R^{1/2}$ of (10) are plotted. The frequencies $\omega_3$ and $\omega_4$ are again in evidence as the points where (10) has repeated roots. In this example there are several ways to switch between upper and lower solutions of (10) in the frequency intervals $[\omega_i, \omega_{i+1}]$. We now investigate which possible choices give weighting function which makes $C$ $\beta_{\max}$-optimal for $P$.

From Theorem 6, in order to make $C$ optimal for some weighting function, we are looking for *clockwise* crossings of $(PC)(j\omega)$ with the positive real axis in Fig. 7 in order to make $\Delta > 0$. Corresponding to Fig. 8, this means that in the middle frequency interval (when $(PC)(j\omega)$ has a clockwise crossing in Fig. 7) the upper solution should be chosen. In the higher frequency range, we have to choose the lower solution (since the upper one goes to infinity). In the lower frequency range, we can choose either solution (since $(PC)(j\omega)$ does not have crossings of the positive real axis). If the lower solution is chosen in this range, the unity weighting function results (which we clearly knew to be an optimal choice in advance). An alternative weight to make $C$ $\beta_{\max}$-optimal is given by choosing instead the upper solution

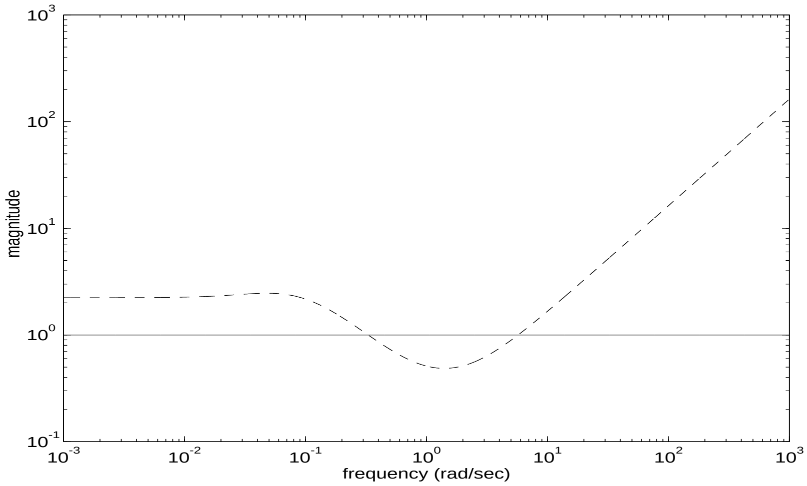


Figure 8: The modulus of the candidate weighting functions.

in the lower frequency range.

An interesting point in this example is that when the parameters of $P$ and $C$ change the optimality may remain or may be lost. The reason for this is that, typically, small perturbations in $P$ and $C$ will decrease the number of intersection points $\omega_i$ and so reduce the possibilities for swapping between upper and lower solutions. For example, if we change the parameter 0.1 of $P$, the optimality will remain when this parameter is increased and the optimality will be lost when this parameter is decreased. In the former case (corresponding to an increase in the parameter) the intersection points $(\omega_1, \omega_4)$ remain and it is still possible to switch to the upper solution in the interval $[\omega_1, \omega_4]$ to get optimality. In the latter case (corresponding to a decrease in the parameter) the intersection points $(\omega_2, \omega_3)$ remain and it is no longer possible to satisfy Theorem 6.

As an aside, we remark that the top graph Hankel singular value of $P$ (i.e. the Hankel singular value of $\binom{M}{N}$) is *not* repeated in this example, so this does not correspond to a case where the optimal controller is a discontinuous function of $P$ in the gap metric [8].

# 8 Non-invertible weighting functions

In the $\mathcal{H}_\infty$ loop-shaping method it is often desirable to use a weighting function which has poles or zeros on the imaginary axis, e.g. if integral action is desired in the controller. In this section we investigate if $C$ is $\beta$-optimal for $P$ for weighting functions of this kind. We first need to make precise how the optimization problem is defined. Let

$$H = \begin{pmatrix} 1 \\ WP \end{pmatrix} (1 + CP)^{-1} \begin{pmatrix} 1 \ , \ W^{-1}C \end{pmatrix}.$$

We consider the following minimization problem:

$$\inf\{\|H\|_\infty : C \text{ such that } [P, C] \text{ is stable and } H \in \mathcal{H}_\infty\}. \tag{35}$$

As before we denote the value of (35) by $b(P, C, W)^{-1}$. We will now make a further restriction on the weighting functions. We will assume that $P$ and $W$ do not have any "pole-zero cancellations" between them on $j\mathbb{R} \cup \{\infty\}$. This assumption is made precise below. The assumption is introduced since the minimization problem does not appear to have a useful meaning otherwise. An additional benefit is that (35) can be solved using the existing theory after establishing Lemma 4.

**Assumption 1.** Suppose that $W$ takes the form $W = W_1 W_2$ where $W_1, W_1^{-1} \in \mathcal{A}_0$ and $W_2$ is rational with no poles or zeros in the open right half plane. Let $W = N_0/M_0$ be a coprime factorization over $\mathcal{A}_0$, and let $P = N/M$ be a coprime factorization over $\mathcal{S}$. We assume that

$$\begin{pmatrix} NN_0 \\ MM_0 \end{pmatrix} \tag{36}$$

is left invertible over $\mathcal{A}_0$, i.e. there exists $X_1, X_2 \in \mathcal{A}_0$ such that

$$X_1 N N_0 + X_2 M M_0 = 1.$$

**Lemma 4.** Under Assumption 1, $[WP, W^{-1}C]$ is stable $\Longrightarrow$ $[P, C]$ is stable.

**Proof.** If $[WP, W^{-1}C]$ is stable then $1/(1+PC) = M/(M+NC) \in \mathcal{H}_\infty$ and $WP/(1+PC) = N_0N/M_0(M+NC) \in \mathcal{H}_\infty$. Hence

$$\frac{X_2 M + X_1 \frac{N_0 N}{M_0}}{M + NC} = \frac{\frac{1}{M_0}}{M + NC} \in \mathcal{H}_\infty$$

which implies that

$$\frac{N}{M + NC} = \frac{P}{1 + PC} \in \mathcal{H}_\infty. \tag{37}$$

Similarly, $PC/(1+PC) = NC/(M+NC) \in \mathcal{H}_\infty$ and $W^{-1}C/(1+PC) = M_0MC/N_0(M+NC) \in \mathcal{H}_\infty$. This means that

$$\frac{X_1 NC + X_2 \frac{M_0 M}{N_0} C}{M + NC} = \frac{\frac{1}{N_0} C}{M + NC} \in \mathcal{H}_\infty$$

which implies that

$$\frac{MC}{M + NC} = \frac{C}{1 + PC} \in \mathcal{H}_\infty. \tag{38}$$

(37) and (38) together now imply that $[P, C]$ is stable. $\square$

Since $H \in \mathcal{H}_\infty \Leftrightarrow [WP, W^{-1}C]$ is stable, it follows from Lemma 4 that (35) is the same as

$$\inf\{\|H\|_\infty : C \text{ such that } [WP, W^{-1}C] \text{ is stable}\}$$

under Assumption 1. Therefore, under this assumption, $C$ is $\beta$-optimal for $P$ with weighting function $W$ if and only if $W^{-1}C$ is $\beta$-optimal for $WP$ with weighting function equal to 1. Assuming for the moment that there is a normalized coprime factorization $WP = N_W/M_W$ over $\mathcal{A}_0$ (and $W^{-1}C = U_W/V_W$ satisfies $N_W U_W + M_W V_W = 1$ with $U_W, V_W \in \mathcal{A}_0$) then $C$ is $\beta$-optimal for $P$ if and only if

$$M_W^* U_W - N_W^* V_W$$

is badly approximable (with modulus $(\beta^{-2} - 1)^{1/2}$). As before this implies that

$$\left| \frac{W^{-1}C - W^* P^*}{1 + PC} \right|$$

has constant modulus, i.e. $W_R = |W|^2$ satisfies (10). Therefore we still need to look at solutions of (10) in order to find candidate $|W|$'s. Under Assumption 1, we now have new possibilities, namely, solutions where $W_R \to 0$ or $W_R \to \infty$ (corresponding to zeros or poles of $C$ on $j\mathbb{R} \cup \{\infty\}$).

We first consider the case of $\beta < \beta_{\max} < 1$. We introduce the notation

$\hat{X} = \{\text{poles of } P \text{ on } j\mathbb{R} \cup \{\infty\}\}$,

$\hat{Y} = \{\text{zeros of } P \text{ on } j\mathbb{R} \cup \{\infty\}\}$.

Since the solutions of (10) are distinct when $\beta < \beta_{\max}$, we have: $C$ is $\beta$-optimal for $P$ with some $W$ if and only if, either (I) $C$ is $\beta$-optimal for $P$ with $|W|^2 = W_R =$ smallest solution of (10), or (II) $C$ is $\beta$-optimal for $P$ with $|W|^2 = W_R =$ largest solution of (10).

Now note that (I) holds if and only if the smallest solution of (10) vanishes only at the zeros of $C$ on $j\mathbb{R} \cup \{\infty\}$ (i.e. $\hat{X} = \emptyset$) and the winding number condition holds for the corresponding weighting function. (If the weighting function vanishes due to poles of $P$ on $j\mathbb{R} \cup \{\infty\}$ then Assumption 1 cannot be satisfied, since then (36) will go to zero at the relevant frequency. Otherwise $NN_0$ and $MM_0$ cannot vanish simultaneously in the compactified right half plane which means that (36) is left invertible using the Corona Theorem [5].) We now consider the following transformation: let

$$C = W_C \hat{C}, \ P = W_C^{-1} \hat{P}, \ W = W_C \hat{W} \tag{39}$$

where $W_C$ is a rational function in $\mathcal{S}$ whose only zeros are the zeros of $C$ on $j\mathbb{R} \cup \{\infty\}$, counting multiplicities. Then (I) holds if and only if $\hat{X} = \emptyset$ and $\hat{C}$ is $\beta$-optimal for $\hat{P}$ with weighting function $\hat{W}$ (cf. (22)). Using Theorem 5(c)(i), (I) holds if and only if $\hat{X} = \emptyset$ and $\eta_o(\hat{P}) > \eta_o(\hat{C}^{-1})$. The latter inequality is equivalent to $\eta_o(P) > \eta_o(C^{-1})$. Similar reasoning is possible for case (II) above. It should be noted that (39) imposes a natural factorization of $W$ in accordance with Assumption 1. Moreover, all complex functions belong to the appropriate Lipschitz class as in Lemma 1, so the previous assumption that normalized coprime factors exist for $WP$ is valid. We therefore obtain the following result.

**Theorem 5'.** Let $[P, C]$ be stable and $\beta_{\max} < 1$. The following statements are equivalent:

(a) For *some* $\beta$, $0 < \beta < \beta_{\max}$, there exists $W$ satisfying Assumption 1 such that $C$ is $\beta$-optimal for $P$.

(b) For *every* $\beta$, $0 < \beta \leq \beta_{\max}$, there exists $W$ (depending on $\beta$) satisfying Assumption 1 such that $C$ is $\beta$-optimal for $P$.

(c) At least one of the following two conditions holds:

  (i) $\hat{X} = \emptyset$ and $\eta_o(P) > \eta_o(C^{-1})$,

  (ii) $\hat{Y} = \emptyset$ and $\eta_o(P^{-1}) > \eta_o(C)$.

If (c)(i) (respectively (c)(ii)) holds, $W$ is determined from the smallest (respectively largest) solution of (10) (where the actual computation of $W$ can be carried out using (39), (22) and Lemma 1).

We can proceed in a similar way with the case of $\beta = \beta_{\max} < 1$. As in Section 5 (Remark 1) we select a candidate (continuous) choice of $W_R$. Under Assumption 1 we are now allowed to select the smallest solution if $\hat{X} \cap [\omega_i, \omega_{i+1}] = \emptyset$ and the largest solution if $\hat{Y} \cap [\omega_i, \omega_{i+1}] = \emptyset$. For this choice of weighting function we make a substitution of the type (39), where $W_C$ contains those poles and zeros of $C$ on $j\mathbb{R} \cup \{\infty\}$ which correspond to $W_R \to \infty$ or $W_R \to 0$ for the chosen $W_R$. The resulting $\hat{P}$, $\hat{C}$ and $\hat{W}$ now satisfy the conditions for the application of Theorem 6. Since $PC = \hat{P}\hat{C}$ and the open right half plane poles and zeros of $P, \hat{P}$ $(C, \hat{C})$ are the same, we obtain the following result. Let $\hat{n}_i$ and $\hat{m}_i$ be defined in the same way as $n_i$ and $m_i$ except that $X$ is replaced by $\hat{X}$ and $Y$ is replaced by $\hat{Y}$.

**Theorem 6'.** Let $[P, C]$ be stable and $\beta_{\max} < 1$. Then there exists $W$ satisfying Assumption 1 such that $C$ is $\beta$-optimal for $P$ with $\beta = \beta_{\max}$ if and only if $\Delta > 0$ where $\Delta$ is given by the following two alternative expressions:

$$\begin{aligned} \Delta &= \eta_o(P) - \eta_o(C^{-1}) + \sum_{i=1}^{r} \hat{n}_i \\ &= \eta_o(P^{-1}) - \eta_o(C) + \sum_{i=1}^{r} \hat{m}_i. \end{aligned}$$

**Example 3.** Consider

$$P(s) = \frac{1}{s+1} \text{ and } C(s) = \frac{s+2}{s}.$$

The plot of the modulus of the possible weighting functions is shown in Fig. 9. With a restriction to invertible weighting functions, the smallest solution has to be chosen at all

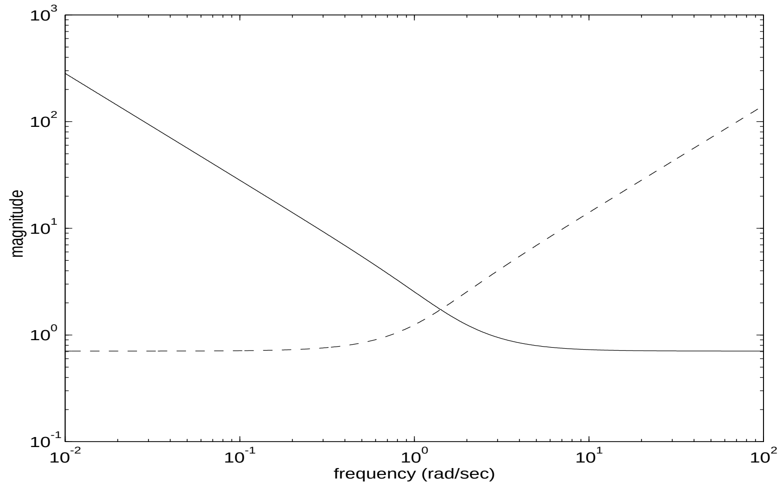


Figure 9: The modulus of the candidate weighting functions.

frequencies, which means that $\beta_{\max}$-optimality cannot be achieved (Theorem 6). If non-invertible weighting functions satisfying Assumption 1 are allowed then it is also possible to choose the upper solution in the frequency range $[-1.42, 1.42]$. A simple sketch of the Nyquist diagram of $PC$ shows that one (net) clockwise crossing of the positive real axis is obtained in the interval $[-1.42, 1.42]$, which gives $\mathrm{wno}(\Gamma_1) = -1$. Hence, from Theorem 6, $C$ is $\beta_{\max}$-optimal for $P$ with this choice of weighting function (shown by the solid line in Fig. 9). Using the substitution (39) with $W_C = C$, a rational approximation of the required weighting function can be computed via equation (22). A 4th-order approximation for $W$ is:

$$W(s) = 0.7072 \times \frac{(s+2)(s+1.9635)(s^2+2.9688s+2.7795)}{s(s+1.4765)(s^2+2.5246s+1.8481)}.$$

To conclude this section we make some brief comments for the case of weighting functions which do not satisfy Assumption 1. Such weighting functions do not have a natural interpretation in the framework of $\mathcal{H}_\infty$ loop-shaping. However, (35) is still a well-defined problem which can be solved from first principles as follows. Using the Youla parametrization, $C = (U + MQ)/(V - NQ)$, we obtain

$$H = \begin{pmatrix} M \\ WN \end{pmatrix} \Big( V - NQ \, , \; W^{-1}(U + MQ) \Big) .$$

If $WN, W^{-1}M \in \mathcal{H}_\infty$ then the minimization can proceed directly using standard reduction procedures (otherwise some further restrictions on $C$ are needed to ensure $H \in \mathcal{H}_\infty$). We illustrate this procedure in a simple example.

**Example 4.** Consider $P(s) = 1/s$ and $W(s) = s$. We will find a unique optimal controller which solves (35). Let $N = 1/(s+1)$, $M = s/(s+1)$, $U = 1$, $V = 1$. Then

$$H = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Big( \frac{s}{s+1}(V - NQ) \, , \; \frac{1}{s+1}(U + MQ) \Big) .$$

Thus

$$
\begin{aligned}
\|H\|_\infty^2 &= 2\left\| \left[ \frac{s}{s+1} \,,\, \frac{1}{s+1} \right] + \frac{s}{(s+1)^2} Q[-1 \,,\, 1] \right\|_\infty^2 \\
&= \left\| \left[ \frac{-s+1}{s+1} \,,\, 1 \right] + \frac{s}{(s+1)^2} Q[2 \,,\, 0] \right\|_\infty^2 \\
&= 1 + \left\| \frac{-s+1}{s+1} + \frac{2s}{(s+1)^2} Q \right\|_\infty^2 .
\end{aligned}
$$

Since $(-s+1)/(s+1)$ is badly approximable, $Q = 0$ is the optimal solution which gives $C = 1$ as the optimal controller. It is interesting to observe that $W = s$ is a solution of (10) with $P = 1/s$ and $C = 1$. However, it does not seem possible to predict this type of optimality using a modification of Theorem 6.

# 9 Concluding remarks

The main contribution of this paper has been to provide some easily testable necessary and sufficient conditions for a controller to be potentially optimal (i.e. optimal for some choice of weighting function) in the $\mathcal{H}_\infty$ loop shaping/weighted gap optimization procedure. The conditions, given in Theorems 5 (5') and 6 (6'), involve only the Nyquist diagram of $PC$ and knowledge of the plant and controller right half plane poles and zeros. In simple cases, optimality can be determined more or less by inspection. For example, if

$$
P = \frac{1}{s-1}, \; C = k > 1, \tag{40}
$$

then it is immediate from Theorem 5(c)(i) that $C$ is $\beta$-optimal for any $\beta \leq \beta_{\mathbf{max}}$. If

$$
P = \frac{1}{s+1}, \; C = \frac{2(s+1)}{s-1}, \tag{41}
$$

then it is immediate from Theorem 6 that $C$ cannot be optimal since $\Delta$ cannot exceed zero. If

$$
P = \frac{s+1}{s+2}, \; C = k, \tag{42}
$$

then it is immediate from Theorem 6 that $C$ is optimal for $k > 0$ and not optimal for any stabilizing $k < 0$.

It is evident from Example 1 and (40) that many simple, classical controllers are optimal in the $\mathcal{H}_\infty$ sense. Also from (41) and (42) it can be seen that some "obviously silly" controllers are excluded, e.g. as in (41) where "seemingly unnecessary" right half plane poles or zeros are introduced into $C$, or where an unsuitable choice of the sign of the feedback gain is made as in (42). On the other hand, the space of potentially optimal $C$'s does contain controllers which appear "arbitrarily poor" in the classical sense, e.g. the choice of $k$ close to 8 in Example 1. However, it must be remarked that one has to make quite an unreasonable performance demand via the weighting function to get such a $k$, and in fact the resulting poor value of $\beta$ shows that the demands are not attained. In conclusion, it appears reasonable to assert that the class of potentially optimal controllers is a sensible one.

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