When is a controller optimal in the sense of $\mathcal{H}_\infty$ loop-shaping?

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Abstract

In this paper we characterize the controllers which are possible solutions of a certain $\mathcal{H}_\infty$ control design problem. The problem considered is the optimal robustness problem for (weighted) normalized coprime factor/gap metric uncertainty, which is the basis for the Glover-McFarlane $\mathcal{H}_\infty$ loop-shaping design method. Given a plant P and a corresponding controller C we ask if C can be obtained from the optimization procedure for some choice of weighting function. This paper considers single-input/single-output systems and gives necessary and sufficient conditions for optimality which involve right half plane pole/zero counts and a certain winding number test based on the Nyquist diagram of PC. The results give a characterization of this class of $\mathcal{H}_\infty$-optimal designs in the language of classical control.

1 Introduction

The use of an optimization procedure for controller design implicitly defines a space of controllers which are potentially optimal. It is within this space that the search for a suitable controller is made. An effective control synthesis method requires (i) the search to be easily related to the designer's goals through the choice of weighting functions etc., and (ii) the space of possible controllers to be a "sensible" one. In this paper we will characterize the potentially optimal controllers for a popular $\mathcal{H}_\infty$ synthesis problem, namely the optimization of weighted normalized coprime factor/gap metric uncertainty which is the basis of Glover-McFarlane loop-shaping. This method appears to be particularly straightforward for the designer in the choice of weighting functions.

The problem of characterizing the controllers which are potentially optimal is a so-called inverse problem. A solution of the inverse problem of LQ control was given in [10]. The necessary and sufficient condition obtained (that the $\mathcal{H}_\infty$-norm of the sensitivity function is at most one) is very sharp, though the result has limited use due to the assumption of state feedback. This type of stability margin guarantee was shown not to hold in the general LQG set-up [2]. A special type of inverse problem in $\mathcal{H}_\infty$ control has been studied in [11]. The problem of mixed sensitivity optimization is considered with a restriction placed on the weighting functions, namely that their magnitudes are the inverses of the sensitivity and complementary sensitivity magnitudes. Necessary and sufficient conditions for optimality, involving RHP pole/zero counts, are obtained under certain assumptions, e.g., plant and controller have no poles and zeros on the imaginary axis or at infinity.

The $\mathcal{H}_\infty$ optimization problem considered in this paper is the one used in the Glover-McFarlane $\mathcal{H}_\infty$ loop-shaping method [12]. This problem is the computation of the optimally robust controller for a weighted plant subject to normalized coprime factor uncertainty, which is the same as the optimal robustness problem for uncertainty in the weighted gap metric. An earlier study of the associated inverse problem has been given in [14]. A necessary and sufficient condition for optimality is given in terms of the solution of a certain quadratic equation (frequency by frequency) and a subsequent winding number test. Although this condition can be regarded as a solution of the inverse problem, in that a definite test procedure is obtained, a more explicit procedure is desirable which is easier to relate to classical control concepts. This paper presents such conditions which involve only right half plane pole/zero counts and encirclement tests based on the Nyquist diagram. The reader is referred to [3] for the proofs.

The paper is organized as follows. In Section 3 the problem set-up is described and some characterizations of optimality are given, which are similar to the results of [14]. That is, a solution of a quadratic equation is required to find the modulus of a potential weighting function, followed by a winding number test (see Theorem 3). This result is developed from first principles using a result of Poreda [13] on badly approximable functions. This type of mathematical result also occurs in the work of [8], [9] and is exploited in previous work on $\mathcal{H}_\infty$ inverse problems [11], [14]. As shown in [14] (see also Theorem 4), given P and C, the possible robustness margin β cannot exceed a certain quantity $\beta = \beta_{\max} = \||S| + |T|\|_\infty^{-1}$. In Sections 4 and 5, the main new results of the paper are presented, which are conditions for optimality, not requiring the explicit computation of a potential weighting function. The cases $\beta < \beta_{\max}$ and $\beta = \beta_{\max}$ are considered separately in Sections 4 and 5. Theorem 5 shows that optimality for *some* $\beta < \beta_{\max}$ is equivalent to optimality for *any* $\beta < \beta_{\max}$, which is equivalent to having a certain pole/zero excess for P over C in the right half plane. The case $\beta = \beta_{\max}$ is a bit more involved. To test for optimality in this case a certain construction on the Nyquist diagram of PC is required along with some associated winding number counts (Theorem 6). It is worth remarking that if optimality cannot be obtained for $\beta = \beta_{\max}$ then it cannot be achieved for $\beta < \beta_{\max}$ (though the converse of this statement is not true). We point out that, if the conditions of Theorem 5 or 6 are met, then the modulus of a suitable weighting function is easily calculated. The case $\beta_{\max} = 1$ is somewhat pathological and is treated briefly in [3]. An illustrative example is presented in Section 6. The work in Sections 3 to 6 is concerned with weighting functions which are invertible in $\mathcal{H}_\infty$. In Section 7 the case of weighting functions with poles or zeros on the imaginary axis is discussed.

2 Mathematical preliminaries

$\mathbb{R}$ and $\mathbb{C}$ denote the real and complex fields. $\mathcal{H}_2$, $\mathcal{H}_\infty$ denote the standard Hardy spaces in the right half of the complex plane (RHP). Let $\mathcal{S}$ be the set of real-rational functions in $\mathcal{H}_\infty$. For a function $G(j\omega)$, $G(j\omega)^*$ denotes the complex conjugate of $G(j\omega)$, namely $\overline{G(j\omega)}$. $\mathcal{A}_0$ denotes the set of functions in $\mathcal{H}_\infty$ which are continuous on the compactified RHP. For a rational function $G(s)$, $\eta_o(G)$ (respectively $\eta_a(G)$) denotes the number of poles of G in the open RHP (respectively on $j\mathbb{R} \cup \{\infty\}$), counting multiplicities. For a closed curve Γ in $\mathbb{C}$, with $0 \notin \Gamma$, we denote the *winding number* of Γ about the origin (number of anticlockwise encirclements) by $\text{wno}(\Gamma)$. If $G(s)$ is continuous on a closed curve Θ, we denote the winding number of the image of Θ under $G(s)$ by $\text{wno}_\Theta(G(s))$. In this paper, we will

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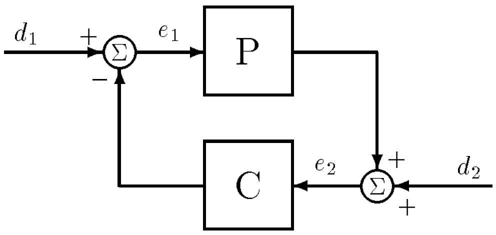

Figure 1: Standard feedback configuration.

use the symbol $\mathcal{D}$ to denote the path consisting of $j\mathbb{R} \cup \{\infty\}$ and oriented in the direction of increasing frequency. We will also use the notation $\mathrm{wno}(G(s)) = \mathrm{wno}_{\mathcal{D}}(G(s))$ if there is no ambiguity.

3 Optimal robustness problem

We consider the feedback configuration of Fig. 1, denoted by $[P, C]$, where P and C are the (scalar) transfer functions of the plant and the controller respectively. Throughout the paper we assume that P and C are real-rational functions. We say that $[P, C]$ is *stable* if the closed loop transfer functions, mapping d_i to e_j $(i, j = 1, 2)$, belong to $\mathcal{H}_\infty$. If $[P, C]$ is stable then the following quantity can be defined:

$$b(P, C) := \left\| \begin{pmatrix} 1 \\ P \end{pmatrix} (1 + CP)^{-1} (1\ ,\ C) \right\|_\infty^{-1}.$$

This quantity is the optimal robustness radius for normalized coprime factor/gap metric uncertainty (see [6], [7]). It appears that $b(P, C)$ is a useful quantity to maximize for a feedback system, especially if it is weighted appropriately. In fact, maximization of $b(WP, C)$ over stabilizing compensators, for some weighting function W chosen by the designer, is the basis of the Glover-McFarlane $\mathcal{H}_\infty$ loop shaping method [12]. This problem is also the same as the optimal robustness problem in the weighted gap metric (see [1], [5]).

Let $P = N/M$ $(M, N \in \mathcal{S})$ be a coprime factorization which is normalized, i.e. $MM^* + NN^* = 1$. Since C stabilizes P, then we can find a factorization $C = U/V$ $(U, V \in \mathcal{S})$ such that $MV + NU = 1$. With such a factorization

$$C_1 = \frac{U + MQ}{V - NQ},$$

for $Q \in \mathcal{H}_\infty$, parametrizes all stabilizing controllers. Moreover [6]

$$b(P, C_1) = (1 + \|M^*U - N^*V + Q\|_\infty^2)^{-1/2} \qquad (1)$$

We now introduce the following terminology. A function $F \in \mathcal{L}_\infty$ is said to be *badly approximable* if $\inf_{Q \in \mathcal{H}_\infty} \|F + Q\|_\infty = \|F\|_\infty$.

Theorem 1. Let $[P, C]$ be stable. The following statements are equivalent:

(a) $b(P, C) = \sup_{C_1 stblz} b(P, C_1)$.

(b) $M^*U - N^*V$ is *badly approximable.*

(c) Either $M^*U - N^*V \equiv 0$, or the following conditions hold

$$|(M^*U - N^*V)(j\omega)| = \alpha \text{ (for all } \omega), \qquad (2)$$

$$\mathrm{wno}(M^*U - N^*V) > 0, \qquad (3)$$

for some constant $\alpha > 0$.

Proof. (a) $\Leftrightarrow$ (b). We have:

$$\begin{aligned} & \sup_{C_1 stblz} b(P, C_1) = b(P, C) \\ \Leftrightarrow\ & \inf_{Q \in \mathcal{H}_\infty} \|M^*U - N^*V + Q\|_\infty = \|M^*U - N^*V\|_\infty \\ \Leftrightarrow\ & M^*U - N^*V \text{ is badly approximable.} \end{aligned}$$

(b) $\Leftrightarrow$ (c). This equivalence follows directly from [13, Theorem 1]. (The difference in sign in the winding number condition is due to the fact that $\mathcal{D}$ is oriented in the direction of increasing frequency, i.e., the boundary curve has negative orientation with respect to the right half plane.) □

Let $[P, C]$ be stable and $W, W^{-1} \in \mathcal{A}_0$. Define

$$b(P, C, W) := \left\| \begin{pmatrix} 1 \\ WP \end{pmatrix} (1 + CP)^{-1} \left(1\ ,\ W^{-1}C\right) \right\|_\infty^{-1}, \qquad (4)$$

and note that $b(P, C, W) = b(WP, W^{-1}C)$. This quantity is the stability margin for uncertainty in the weighted gap metric (see [6]). We also remark that, in the Glover-McFarlane $\mathcal{H}_\infty$ loop-shaping method, the final compensator to be implemented is $C = W\hat{C}$ where $\hat{C}$ maximizes $b(WP, \hat{C})$. From (4), $b(WP, \hat{C}) = b(P, C, W)$, so in fact C also maximizes $b(P, C, W)$. Given the weighting function $W, W^{-1} \in \mathcal{A}_0$, we say that C is *β-optimal for P* (with weight W) if

$$\beta = b(P, C, W) = \sup_{C_1 stblz} b(P, C_1, W).$$

Theorem 2 [3]. Let $[P, C]$ be stable, let $W, W^{-1} \in \mathcal{A}_0$ and suppose that WP has a normalized coprime factorization over $\mathcal{A}_0$. Then C is β-optimal for P if and only if, either $\beta = 1$ and $W^{-1}C = W^*P^*$, or $\beta < 1$ and the following two conditions hold:

$$\left| \left(\frac{W^{-1}C - W^*P^*}{1 + PC} \right)(j\omega) \right| = \alpha \text{ (for all } \omega), \qquad (5)$$

$$\mathrm{wno}\left(\frac{W^{-1}C - W^*P^*}{1 + PC} \right) + 2\eta_o(P) + \eta_a(P) > 0, \qquad (6)$$

where $\alpha = (\beta^{-2} - 1)^{1/2}$ is constant. □

From Theorem 2, optimality of C for the weighting function W requires the two conditions (5), (6) to be satisfied which will be called the *modulus condition* and the *winding number condition* respectively. We will first consider the modulus condition. Let $WW^* = W_R$. Then

$$\left| \frac{W^{-1}C - W^*P^*}{1 + PC} \right| = \alpha$$

is equivalent to

$$PP^*W_R^2 - (PC + C^*P^* + \alpha^2|1 + PC|^2)W_R + C^*C = 0, \quad (7)$$

which is the quadratic equation obtained in [14]. It is evident that a first step in finding a weighting function $W, W^{-1} \in \mathcal{A}_0$ so that Theorem 2 holds is to solve (7), frequency by frequency, for a real, positive, continuous W_R. In fact, such a W_R can always be spectrally factorized as $W_R = WW^*$ with $W, W^{-1} \in \mathcal{H}_\infty$. The procedure to find W is to take the Hilbert transform of $\log W_R^{1/2}$ to find the phase. In [3] it is shown that the Hilbert transform of $\log W_R^{1/2}$ is continuous which proves that $W, W^{-1} \in \mathcal{A}_0$. It is necessary to prove this since not all positive continuous functions have continuous Hilbert transforms (see [4, p.377]).

Lemma 1 [3]. Let P and C be rational. Let $W_R(j\omega)$ be a continuous, non-negative solution of (7) which is bounded and bounded away from zero. Then there exists a factorization $W_R = WW^*$ such that $W, W^{-1} \in \mathcal{A}_0$. Furthermore, WP has a normalized coprime factorization over $\mathcal{A}_0$. $\square$

Theorem 3 [3]. Let $[P, C]$ be stable and let $\beta = (1+\alpha^2)^{-1/2} < 1$. Then there exists a weighting function $W, W^{-1} \in \mathcal{A}_0$ with magnitude $W_R(j\omega) = |W(j\omega)|^2$ such that C is β-optimal for P if and only if $W_R, W_R^{-1} \in \mathcal{L}_\infty$ is a continuous solution of (7) and

$$\text{wno}\left(\frac{C - W_R P^*}{1 + PC}\right) + 2\eta_o(P) + \eta_a(P) > 0. \quad \square \tag{8}$$

Theorem 3 gives a straightforward procedure to determine if a compensator C is β-optimal. Namely, we need to check (8) for all the continuous solutions $W_R, W_R^{-1} \in \mathcal{L}_\infty$ of (7), of which there are finitely many. It should be noted that $(C - W_R P^*)/(1+PC)$ always has a well-defined winding number for continuous solution $W_R, W_R^{-1} \in \mathcal{L}_\infty$ of (7). We remark that Theorem 3 has the same spirit as the result of Sefton and Ober [14, Theorem 9.7], though it is more easily tested since coprime factorizations of P and C are not needed and the phase of $W_R^{1/2}$ is not needed. In the next two sections we present conditions for β-optimality, directly in terms of P and C, which do not require solutions of (7) to be found.

Observe that equation (7) has real coefficients for $s \in j\mathbb{R} \cup \{\infty\}$ and we are seeking solutions such that W_R is real and positive. First note that there is a solution $W_R \geq 0$ to (7) if and only if

$$\alpha^2 |1 + PC|^2 + PC + C^* P^* \geq 2|PC|. \tag{9}$$

This is equivalent to

$$\beta^{-1} = (1+\alpha^2)^{1/2} \geq \frac{1 + |PC|}{|1 + PC|} = |S| + |T| \tag{10}$$

where $S = 1/(1+PC)$ and $T = PC/(1+PC)$ are the classical sensitivity and complementary sensitivity functions. Defining

$$\beta_{\max} = \left\| \frac{1 + |PC|}{|1 + PC|} \right\|_\infty^{-1} = \| |S| + |T| \|_\infty^{-1}, \tag{11}$$

we have the following result.

Theorem 4 [14]. Let $[P, C]$ be stable. If $\beta > \beta_{\max}$, then no weighting function $W, W^{-1} \in \mathcal{A}_0$ exists so that C is β-optimal for P. $\square$

Theorem 4 shows that $\beta_{\max}$ is an upper bound on the possible stability margin $b(P, C, W)$ for fixed P and C and arbitrary W. In case $\beta \leq \beta_{\max}$ there is no simple violation of the modulus condition. To check for optimality we need to bring in the winding number condition as well. Note that the frequencies where equality is achieved in (10) are the frequencies at which a repeated solution W_R of (7) is obtained. Thus, if $\beta < \beta_{\max}$ there are at most two continuous choices for W_R consisting of the largest solution of (7) frequency by frequency, or the smallest one. This situation will be considered in the next section. In case $\beta = \beta_{\max}$, there will be several frequencies where the solutions W_R of (7) are repeated. This allows the possibility of swapping between the largest and smallest solutions in making a continuous choice for W_R. This situation will be considered in Section 5.

4 Solution for $\beta < \beta_{\max} < 1$

Define

$X = \{$poles of P and zeros of C on $j\mathbb{R} \cup \{\infty\}\}$,

$Y = \{$zeros of P and poles of C on $j\mathbb{R} \cup \{\infty\}\}$.

Theorem 5 [3]. Let $[P, C]$ be stable and $\beta_{\max} < 1$. The following statements are equivalent.

(a) There exists a $W, W^{-1} \in \mathcal{A}_0$ such that C is β-optimal for P for *some* $0 < \beta < \beta_{\max}$.

(b) There exists a $W, W^{-1} \in \mathcal{A}_0$ such that C is β-optimal for P for *any* $0 < \beta \leq \beta_{\max}$.

(c) At least one of the following two conditions holds:
   (i) $X = \emptyset$ and $\eta_o(P) > \eta_o(C^{-1})$,
   (ii) $Y = \emptyset$ and $\eta_o(P^{-1}) > \eta_o(C)$.

If (c)(i) (respectively (c)(ii)) holds, W is determined, as in Lemma 1, from the smallest (respectively largest) solution of (7). $\square$

The proof of Theorem 5 makes use of the following lemmas.

Lemma 2. Let $[P, C]$ be stable and let $\beta \leq \beta_{\max} < 1$. If $X = \emptyset$ (respectively $Y = \emptyset$), then the smallest (respectively largest) solution of (7) is continuous with $W_R, W_R^{-1} \in \mathcal{L}_\infty$, and satisfies $\|W_R\|_\infty \to 0$ as $\beta \to 0$ (respectively $\|W_R^{-1}\|_\infty \to 0$ as $\beta \to 0$). If $X \neq \emptyset$ and $Y \neq \emptyset$, then (7) has no solution with $W_R, W_R^{-1} \in \mathcal{L}_\infty$ for any $\beta < \beta_{\max}$. $\square$

Lemma 3. Let P and C be rational and suppose either $X = \emptyset$ or $Y = \emptyset$. For $0 < \beta \leq \beta_{\max} < 1$, let $W_R(j\omega)$ be chosen frequency by frequency, as the smallest solution of (7) if $X = \emptyset$ or the largest solution if $Y = \emptyset$. Then

$$\text{wno}\left(\frac{C - W_R P^*}{1 + PC}\right)$$

is independent of β for $0 < \beta \leq \beta_{\max}$. $\square$

5 Solution for $\beta = \beta_{\max} < 1$

Take the points $\omega_1 < \omega_2 < \cdots < \omega_r$ on $j\mathbb{R} \cup \{\infty\}$ where $|S| + |T|$ achieves its supremum. Throughout this section we assume $\beta_{\max} < 1$. This means that PC cannot have poles or zeros at ω_i and $(PC)(j\omega_i)$ cannot lie on the positive real axis. Thus we can define a curve Γ_i as follows: take the path $(PC)(j\omega)$ for $\omega_i \leq \omega \leq \omega_{i+1}$, indent around the poles and zeros of P and C into the *right* half plane, and complete to a closed curve by joining $(PC)(j\omega_{i+1})$ to $(PC)(j\omega_i)$ by any path which doesn't go through the origin or intersect the positive real axis. Observe that $\text{wno}(\Gamma_i)$ is well-defined for $i = 1, 2, \cdots, r-1$. We define Γ_r by considering the path traced out by $(PC)(j\omega)$ as ω goes from ω_r to ∞ and then from $-\infty$ to ω_1. For notational convenience, we identify $\omega_{r+1} = \omega_1$ and define the set $[\omega_r, \omega_{r+1}]$ to be $(-\infty, \omega_1] \cup [\omega_r, \infty) \cup \{\infty\}$. On each interval $[\omega_i, \omega_{i+1}]$ define the following quantities (for $i = 1, 2, \cdots, r$):

$$n_i^- = \begin{cases} -\infty & \text{if } X \cap [\omega_i, \omega_{i+1}] \neq \emptyset \\ 0 & \text{if } X \cap [\omega_i, \omega_{i+1}] = \emptyset \end{cases}$$

$$n_i^+ = \begin{cases} -\infty & \text{if } Y \cap [\omega_i, \omega_{i+1}] \neq \emptyset \\ -\text{wno}(\Gamma_i) & \text{if } Y \cap [\omega_i, \omega_{i+1}] = \emptyset \end{cases}$$

and let $n_i = \max(n_i^-, n_i^+)$. Also define:

$$m_i^- = \begin{cases} -\infty & \text{if } X \cap [\omega_i, \omega_{i+1}] \neq \emptyset \\ \text{wno}(\Gamma_i) & \text{if } X \cap [\omega_i, \omega_{i+1}] = \emptyset \end{cases}$$

$$m_i^+ = \begin{cases} -\infty & \text{if } Y \cap [\omega_i, \omega_{i+1}] \neq \emptyset \\ 0 & \text{if } Y \cap [\omega_i, \omega_{i+1}] = \emptyset \end{cases}$$

and let $m_i = \max(m_i^-, m_i^+)$.

Theorem 6 [3]. Let $[P, C]$ be stable and $\beta_{\max} < 1$. Then there exists a $W, W^{-1} \in \mathcal{A}_0$ such that C is β-optimal for P with $\beta = \beta_{\max}$ if and only if $\Delta > 0$ where Δ is given by the following two alternative expressions:

$$\Delta = \eta_o(P) - \eta_o(C^{-1}) + \sum_{i=1}^{r} n_i \tag{12}$$

$$= \eta_o(P^{-1}) - \eta_o(C) + \sum_{i=1}^{r} m_i. \quad \square \tag{13}$$

Remarks.

1. If Theorem 5(c) holds, C is β-optimal for P for all $0 < \beta \leq \beta_{\max} < 1$. This means that the number of solutions for $W, W^{-1} \in \mathcal{A}_0$ is infinite since there is at least one solution for each β. If Theorem 5(c) fails, there is still the possibility that C is $\beta_{\max}$-optimal for P. This needs to be checked using Theorem 6. In this case there are finitely many possible choices of $W, W^{-1} \in \mathcal{A}_0$ obtained by swapping between the largest and smallest solutions of (7), at the frequencies ω_i, in making a continuous choice of W_R. The procedure to choose between the largest and smallest solutions is as follows. If $X \cap [\omega_i, \omega_{i+1}] = \emptyset$ we may choose the smallest solution and if $Y \cap [\omega_i, \omega_{i+1}] = \emptyset$ we may choose the largest one, which gives 2, 1 or 0 admissible choices of $|W|$ in each interval. (Clearly if some interval has no admissible choice of $|W|$ we cannot have optimality, as was pointed out in [14].) For any admissible choice, we calculate

$$\eta_o(P) - \eta_o(C^{-1}) - \sum_{i \in \chi} \mathrm{wno}(\Gamma_i), \tag{14}$$

where χ corresponds to the segments where the largest solution is chosen. If (14) is positive then the corresponding weighting function gives $\beta_{\max}$-optimality (otherwise not). The set of admissible choices of W for which (14) is positive is the set of W for which C is $\beta_{\max}$-optimal for P. The quantity Δ in Theorem 6 is simply the maximum of (14) over all admissible W. Note, as in the statement of Theorem 6, (14) is equal to the alternative expression

$$\eta_o(P^{-1}) - \eta_o(C) + \sum_{i \in \bar{\chi}} \mathrm{wno}(\Gamma_i),$$

where $\bar{\chi}$ corresponds to the segments where the smallest solution is chosen.

2. The conditions of Theorem 6 can be checked from a knowledge of the right half plane poles and zeros of P, C and the Nyquist diagram of PC when the maximizing frequencies ω_i of $|S| + |T|$ are known. These points on the Nyquist diagram can be estimated from the loci $(1 + |z|)/|1 + z| = c$ (constant) in the complex plane. In particular we find the largest c which gives an intersection with the Nyquist diagram and select the corresponding points of intersection. The largest such c equals $\beta_{\max}^{-1}$.

3. Given P and C, a typical situation is that $|S| + |T|$ achieves its supremum at two frequencies $\pm\omega_0 \neq 0$. In this case, a small perturbation to P and C (say in the gap metric) will not alter the possibility of achieving $\beta_{\max}$-optimality. However, in the (non-generic) case of a higher number of supremal points of $|S| + |T|$, a small perturbation to P and/or C may cause $\beta_{\max}$-optimality to be lost (see Example 2 in [3]). This means that $\beta_{\max}$-optimality is not a robust property in general.

6 Example

Example 1. Consider
$$P(s) = \frac{1}{(s+1)^3} \text{ and } C(s) = k.$$

For closed loop stability we require $-1 < k < 8$ from the Nyquist stability criterion. Fig. 2 and Fig. 3 show the Nyquist diagram of PC and the modulus of the weighting functions when $k = 1$. As $\eta_o(P) = \eta_o(P^{-1}) = 0$ and $\eta_o(C) = \eta_o(C^{-1}) = 0$, then from Theorem 5, C is not β-optimal for P if $\beta < \beta_{\max} < 1$. To check if C is $\beta_{\max}$-optimal we need to use Theorem 6 and consider the positive real axis crossings of $(PC)(j\omega)$ to determine whether $\Delta > 0$. In Fig. 2 we see that there is an intersection point (marked by a "$*$") between the path of PC (shown by the solid line) and the graph of $(1 + |z|)/|1 + z| = \beta_{\max}^{-1}$ (shown by the dashed line for

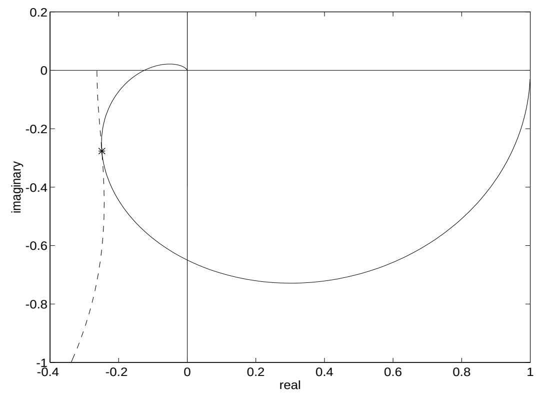

Figure 2: The Nyquist diagram of PC for $k = 1$.

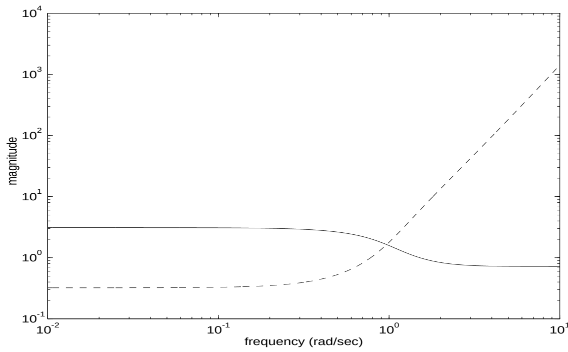

Figure 3: The modulus of the candidate weights for $k = 1$.

$\beta_{\max} = 0.5832$). So it is possible for us to switch between upper and lower solutions of (7) at the frequency corresponding to the "$*$" point, which is the frequency where the two weighting functions have the same modulus in Fig. 3. Referring to Fig. 3, we choose the upper solution in the lower frequency range and the lower solution in the higher frequency range, which is shown by the solid line. Calculation of the appropriate winding number shows that $\Delta = 1$, which means that C is optimal for P with this weighting function when $k = 1$. The only other possible choice which gives $W, W^{-1} \in \mathcal{A}_0$ is to select the lower solution in the lower frequency range. However, this gives $\Delta = 0$, so this weighting function does not give optimality.

It can also be seen from the Nyquist diagram (see [3] for a full explanation) that C is optimal for some weighting function if and only if $\beta = \beta_{\max} < 1$ and $0 < k < 8$. Fig. 4 shows the form of the optimal weighting function as a function of k. Observe that, for k close to 8, the weighting function has very large (small) modulus for $\omega < \sqrt{3}$ ($\omega > \sqrt{3}$) and that the value of $\beta_{\max}$ is very small. For k close to zero, the weighting function has very small modulus for all ω and $\beta_{\max} \to 1$.

It is worth making a remark on the fact that values of k which are arbitrarily close to 8 are still potentially optimal for some weighting function. On the surface, this would seem to indicate an unsatisfactory feature of the $\mathcal{H}_\infty$ loop shaping method. However, to produce such a k it must be noted that the weighting functions need to have an extremely rapid attenuation rate close to the frequency $\sqrt{3}$, which corresponds to a rather unusual performance demand. Moreover, the value of β which results from this weighting function is very small, indicating that the performance goals have not been met. Thus, the designer's approach needs to be quite unreasonable in order to obtain such a controller.

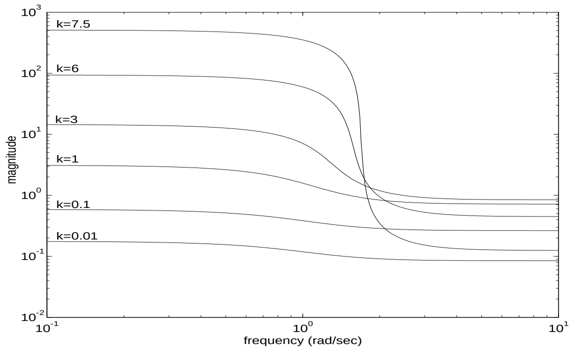

Figure 4: The optimal weighting function with respect to k.

7 Non-invertible weighting functions

In the $\mathcal{H}_\infty$ loop-shaping method it is often desirable to use a weighting function which has poles or zeros on the imaginary axis, e.g. if integral action is desired in the controller. In this section we investigate if C is β-optimal for P for weighting functions of this kind. We first need to make precise how the optimization problem is defined. Let

$$H = \begin{pmatrix} 1 \\ WP \end{pmatrix} (1 + CP)^{-1} \begin{pmatrix} 1 \ , \ W^{-1}C \end{pmatrix}.$$

We consider the following minimization problem:

$$\inf\{\|H\|_\infty : C \text{ such that } [P, C] \text{ is stable and } H \in \mathcal{H}_\infty\}. \quad (15)$$

As before we denote the value of (15) by $b(P, C, W)^{-1}$. We will now make a further restriction on the weighting functions. We will assume that P and W do not have any "pole-zero cancellations" between them on $j\mathbb{R} \cup \{\infty\}$. This assumption is made precise below. The assumption is introduced since the minimization problem does not appear to have a useful meaning otherwise. An additional benefit is that (15) can be solved using the existing theory after establishing Lemma 4.

Assumption 1. Suppose that W takes the form $W = W_1 W_2$ where $W_1, W_1^{-1} \in \mathcal{A}_0$ and W_2 is rational with no poles or zeros in the open right half plane. Let $W = N_0/M_0$ be a coprime factorization over $\mathcal{A}_0$, and let $P = N/M$ be a coprime factorization over $\mathcal{S}$. We assume that

$$\begin{pmatrix} N N_0 \\ M M_0 \end{pmatrix} \quad (16)$$

is left invertible over $\mathcal{A}_0$, i.e. there exists $X_1, X_2 \in \mathcal{A}_0$ such that

$$X_1 N N_0 + X_2 M M_0 = 1.$$

Lemma 4 [3]. Under Assumption 1, $[WP, W^{-1}C]$ is stable $\Longrightarrow [P, C]$ is stable. $\Box$

Since $H \in \mathcal{H}_\infty \Leftrightarrow [WP, W^{-1}C]$ is stable, it follows from Lemma 4 that (15) is the same as

$$\inf\{\|H\|_\infty : C \text{ such that } [WP, W^{-1}C] \text{ is stable}\}$$

under Assumption 1. Therefore, under this assumption, C is β-optimal for P with weighting function W if and only if $W^{-1}C$ is β-optimal for WP with weighting function equal to 1. Assuming for the moment that there is a normalized coprime factorization $WP = N_W/M_W$ over $\mathcal{A}_0$ (and $W^{-1}C = U_W/V_W$ satisfies $N_W U_W + M_W V_W = 1$ with $U_W, V_W \in \mathcal{A}_0$) then C is β-optimal for P if and only if

$$M_W^* U_W - N_W^* V_W$$

is badly approximable (with modulus $(\beta^{-2} - 1)^{1/2}$). This implies that

$$\left| \frac{W^{-1}C - W^* P^*}{1 + PC} \right|$$

has constant modulus, i.e. $W_R = |W|^2$ satisfies (7). Therefore we still need to look at solutions of (7) in order to find candidate $|W|$'s. Under Assumption 1, we now have new possibilities, namely, solutions where $W_R \to 0$ or $W_R \to \infty$ (corresponding to zeros or poles of C on $j\mathbb{R} \cup \{\infty\}$). We introduce the notation

$$\hat{X} = \{\text{poles of } P \text{ on } j\mathbb{R} \cup \{\infty\}\},$$

$$\hat{Y} = \{\text{zeros of } P \text{ on } j\mathbb{R} \cup \{\infty\}\}.$$

Then the following results hold.

Theorem 5' [3]. Let $[P, C]$ be stable and $\beta_{\max} < 1$. The following statements are equivalent:

(a) There exists W satisfying Assumption 1 such that C is β-optimal for P for *some* $0 < \beta < \beta_{\max}$.

(b) There exists W satisfying Assumption 1 such that C is β-optimal for P for *any* $0 < \beta \le \beta_{\max}$.

(c) At least one of the following two conditions holds:

(i) $\hat{X} = \emptyset$ and $\eta_o(P) > \eta_o(C^{-1})$,

(ii) $\hat{Y} = \emptyset$ and $\eta_o(P^{-1}) > \eta_o(C)$.

If (c)(i) (respectively (c)(ii)) holds, W is determined from the smallest (respectively largest) solution of (7). $\Box$

We can proceed in a similar way with the case of $\beta = \beta_{\max} < 1$. As in Section 5 (Remark 1) we select a candidate (continuous) choice of W_R. Under Assumption 1 we are now allowed to select the smallest solution if $\hat{X} \cap [\omega_i, \omega_{i+1}] = \emptyset$ and the largest solution if $\hat{Y} \cap [\omega_i, \omega_{i+1}] = \emptyset$. Let $\hat{n}_i$ and $\hat{m}_i$ be defined in the same way as n_i and m_i except that X is replaced by $\hat{X}$ and Y is replaced by $\hat{Y}$.

Theorem 6' [3]. Let $[P, C]$ be stable and $\beta_{\max} < 1$. Then there exists W satisfying Assumption 1 such that C is β-optimal for P with $\beta = \beta_{\max}$ if and only if $\Delta > 0$ where Δ is given by the following two alternative expressions:

$$\begin{aligned} \Delta &= \eta_o(P) - \eta_o(C^{-1}) + \sum_{i=1}^{r} \hat{n}_i \\ &= \eta_o(P^{-1}) - \eta_o(C) + \sum_{i=1}^{r} \hat{m}_i. \qquad \Box \end{aligned}$$

Example 2. Consider

$$P(s) = \frac{1}{s+1} \text{ and } C(s) = \frac{s+2}{s}.$$

The plot of the modulus of the possible weighting functions is shown in Fig. 5. With a restriction to invertible weighting functions, the smallest solution has to be chosen at all frequencies, which means that $\beta_{\max}$-optimality cannot be achieved (Theorem 6). If non-invertible weighting functions satisfying Assumption 1 are allowed then it is also possible to choose the upper solution in the frequency range $[-1.42, 1.42]$. A simple sketch of the Nyquist diagram of PC shows that one (net) clockwise crossing of the positive real axis is obtained in the interval $[-1.42, 1.42]$, which gives $\text{wno}(\Gamma_1) = -1$. Hence, from Theorem 6, C is $\beta_{\max}$-optimal for P with this choice of weighting function (shown by the solid line in Fig. 5). A 4th-order approximation for W computed in [3] is:

$$W(s) = 0.7072 \times \frac{(s+2)(s+1.9635)(s^2+2.9688s+2.7795)}{s(s+1.4765)(s^2+2.5246s+1.8481)}.$$

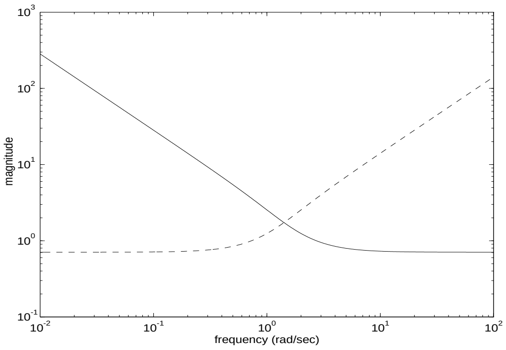

Figure 5: The modulus of the candidate weighting functions.

To conclude this section we make some brief comments for the case of weighting functions which do not satisfy Assumption 1. Such weighting functions do not have a natural interpretation in the framework of $\mathcal{H}_\infty$ loop-shaping. However, (15) is still a well-defined problem which can be solved from first principles as follows. Using the Youla parametrization, $C = (U + MQ)/(V - NQ)$, we obtain

$$H = \begin{pmatrix} M \\ WN \end{pmatrix} \left(V - NQ\ ,\ W^{-1}(U + MQ)\right).$$

If $WN, W^{-1}M \in \mathcal{H}_\infty$ then the minimization can proceed directly using standard reduction procedures (otherwise some further restrictions on C are needed to ensure $H \in \mathcal{H}_\infty$). We illustrate this procedure in a simple example.

Example 3. Consider $P(s) = 1/s$ and $W(s) = s$. We will find a unique optimal controller which solves (15). Let $N = 1/(s+1)$, $M = s/(s+1)$, $U = 1$, $V = 1$. Then

$$H = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left(\frac{s}{s+1}(V - NQ)\ ,\ \frac{1}{s+1}(U + MQ)\right).$$

Thus

$$\begin{aligned} \|H\|_\infty^2 &= 2\left\| \left[\frac{s}{s+1}\ ,\ \frac{1}{s+1}\right] + \frac{s}{(s+1)^2}Q[-1\ ,\ 1]\right\|_\infty^2 \\ &= \left\| \left[\frac{-s+1}{s+1}\ ,\ 1\right] + \frac{s}{(s+1)^2}Q[2\ ,\ 0]\right\|_\infty^2 \\ &= 1 + \left\| \frac{-s+1}{s+1} + \frac{2s}{(s+1)^2}Q\right\|_\infty^2 . \end{aligned}$$

Since $(-s+1)/(s+1)$ is badly approximable, $Q = 0$ is the optimal solution which gives $C = 1$ as the optimal controller. It is interesting to observe that $W = s$ is a solution of (7) with $P = 1/s$ and $C = 1$. However, it does not seem possible to predict this type of optimality using a modification of Theorem 6.

8 Concluding remarks

The main contribution of this paper has been to provide some easily testable necessary and sufficient conditions for a controller to be potentially optimal (i.e. optimal for some choice of weighting function) in the $\mathcal{H}_\infty$ loop shaping/weighted gap optimization procedure. The conditions, given in Theorems 5 (5') and 6 (6'), involve only the Nyquist diagram of PC and knowledge of the plant and controller right half plane poles and zeros. In simple cases, optimality can be determined more or less by inspection. For example, if

$$P = \frac{1}{s-1},\ C = k > 1, \tag{17}$$

then it is immediate from Theorem 5(c)(i) that C is β-optimal for any $\beta \le \beta_{\max}$. If

$$P = \frac{1}{s+1},\ C = \frac{2(s+1)}{s-1}, \tag{18}$$

then it is immediate from Theorem 6 that C cannot be optimal since Δ cannot exceed zero. If

$$P = \frac{s+1}{s+2},\ C = k, \tag{19}$$

then it is immediate from Theorem 6 that C is optimal for $k > 0$ and not optimal for any stabilizing $k < 0$.

It is evident from Example 1 and (17) that many simple, classical controllers are optimal in the $\mathcal{H}_\infty$ sense. Also from (18) and (19) it can be seen that some "obviously silly" controllers are excluded, e.g. as in (18) where "seemingly unnecessary" right half plane poles or zeros are introduced into C, or where an unsuitable choice of the sign of the feedback gain is made as in (19). On the other hand, the space of potentially optimal C's does contain controllers which appear "arbitrarily poor" in the classical sense, e.g. the choice of k close to 8 in Example 1. However, it must be remarked that one has to make quite an unreasonable performance demand via the weighting function to get such a k, and in fact the resulting poor value of β shows that the demands are not attained.

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