MEASURING THE ROBUSTNESS OF FEEDBACK SYSTEMS

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CHAPTER I

Introduction

This monograph is concerned with the problem of controlling uncertain dynamical systems via feedback. There are two, and only two, reasons for using feedback. The first is to reduce the effect of any unmeasured disturbances acting on the system. The second is to reduce the effect of any uncertainty about the system dynamics.[1] We shall be dealing exclusively with so-called model based control, where the starting point is a mathematical description of the system. A feedback compensator is then designed for this model, in the hope that it will be effective with the real system. Herein lies the conflict at the heart of model based control — one of the key aims of using feedback is to reduce the effect of any uncertainties in the system description, yet we need an accurate model of the system to design a feedback compensator for it in the first place. The key question we shall be addressing (although by no means answering fully) is "how much do we need to know about a system in order to design a feedback compensator that leaves the closed loop insensitive to what we don't know?"

Uncertainty arises because any mathematical model being used for design purposes can do no more than approximate to the behaviour of the real system that is to be controlled. We shall be concentrating on the use of models that are linear, time-invariant and finite dimensional, in which case their mathematical description is as a set of linear ordinary differential equations. Such descriptions are mathematically appealing; however, one of the few things we can say with any degree of certainty is that *no* physical system could possibly correspond to such a model. Our approach will therefore need some justification. Physical systems are essentially infinite dimensional, time-varying and nonlinear; that is they

[1] It is sometimes claimed that feedback is also useful for modifying the dynamic response of a system. But in the absence of uncertainty (about the plant and its initial conditions), any dynamic response that is achievable with feedback can also be achieved without feedback (provided the controller can also be implemented exactly). Of course, this is not often a practical proposition (particularly if the plant is unstable), but it is only the probability of small errors that makes it so.

are better modelled by nonlinear partial differential equations (of course, a real physical process is no more a collection of nonlinear partial differential equations than it is a collection of linear ordinary differential equations, the richer model set simply allows us to approximate to its true behaviour more closely). There exist sound techniques that enable us to approximate such nonlinear, infinite dimensional models with linear, time-invariant and finite dimensional ones. For instance, we can use modal analysis, or the various lumped parameter methods such as finite element analysis, to approximate to the behaviour of a distributed parameter, and hence infinite dimensional, model. Similarly, we can linearize a nonlinear model around various operating points — modelling it as a family of linear models, with a member at each of those operating points.

So, one of the main reasons for including uncertainty in the model is that there are aspects of the behaviour of real systems which cannot be represented by the *class* of models being used. The other main reason, of course, is that there will always be uncertainty about the actual numerical values of the various parameters of the model. Our central theme will be the development of notions of uncertainty appropriate to the feedback problem. An appropriate framework is that of the *graph* topology [ZES80] [Vid84]. In this topology, two plants are deemed "close" if any reasonable compensator for the first gives similar closed loop behaviour with the second of the plants in the standard feedback configuration of Figure I.1. Here, u and y represent the input and output of the "actual" plant (the shaded area), *ie* the actuator demands and the measurements. The signals w_1 and w_2 represent disturbances acting on the plant, whilst v_1 and v_2 represent noise on the compensator output and on the measurements respectively.

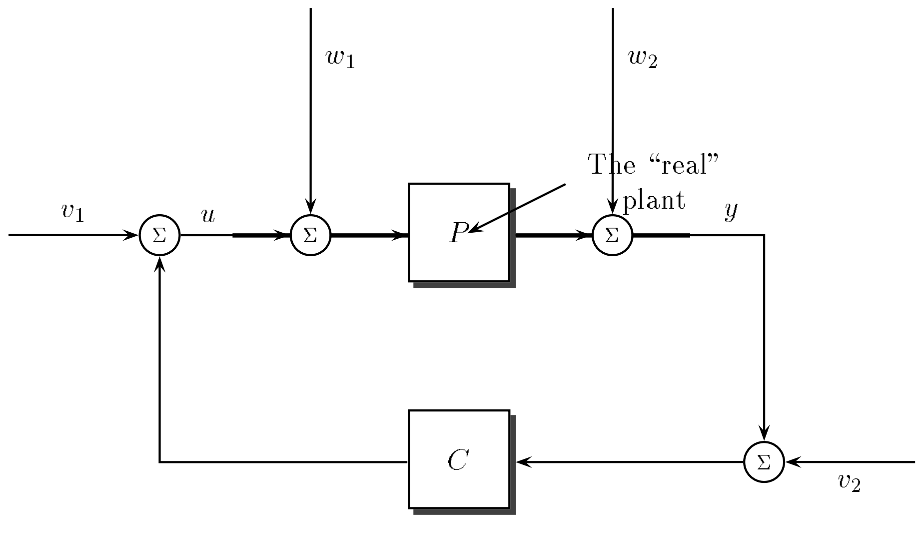


FIGURE I.1. Standard Feedback Configuration

The graph topology arises very naturally when one is attempting to design a compensator to control not only some nominal linear model of a physical plant, but also a set of perturbed models in the neighbourhood of the nominal.

Over the last decade much effort has gone into the search for metrics that induce this topology — *ie* a way of quantitatively measuring the "distance" between two systems based on the above qualitative notion of "closeness". The most successful of these to date has been the *gap metric* ([ZES80], [ES85] [Geo88]). This metric also provides a sufficient condition for the stability of a perturbed plant/compensator pair, in terms of the gain of the transfer functions from v_1 and v_2, to u and y, in Figure I.1; [GS90]

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} [-C \ \ I] \begin{bmatrix} v_2 \\ v_1 \end{bmatrix}.$$

As in [GS90] we shall define a generalized stability margin, $b_{P,C}$, in terms of the "size" of this transfer function as measured by the $\mathcal{H}_\infty$ norm. The $\mathcal{H}_\infty$ norm of a stable transfer function is the square root of its worst case energy gain *ie*

$$\|P\|_\infty := \sup_{u \neq 0} \frac{\|Pu\|_2}{\|u\|_2}.$$

For stable, finite dimensional systems P this norm can be calculated as

$$\|P\|_\infty = \max_\omega \bar{\sigma}(P(j\omega)) = \max_\omega \bar{\lambda}^{1/2}(P(j\omega)^* P(j\omega)),$$

where $\bar{\sigma}(\cdot)$ denotes the largest singular value of a matrix and $\bar{\lambda}(\cdot)$ the maximum eigenvalue. We can now define

$$b_{P,C} := \begin{cases} \left\| \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} [-C \ \ I] \right\|_\infty^{-1}, & \text{if } [P, C] \text{ is stable} \\ 0, & \text{otherwise.} \end{cases}$$

By $[P, C]$ being stable, we mean that the transfer functions from v_1 and v_2 to u and y in Figure I.1 are each stable, this is often called internal stability of the feedback loop and is equivalent to the transfer functions from w_1 and w_2 to u and y being stable. In fact a stronger result is true; the $\mathcal{H}_\infty$ norm of the transfer function from w_1 and w_2 to u and y,

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} C \\ I \end{bmatrix} (I - PC)^{-1} [-P \ \ I] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

is equal to that of the transfer function from v_1 and v_2 to u and y, *ie*

$$\left\| \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} [-C \ \ I] \right\|_\infty = \left\| \begin{bmatrix} C \\ I \end{bmatrix} (I - PC)^{-1} [-P \ \ I] \right\|_\infty,$$

or $b_{P,C} = b_{C,P}$ [GS90], as will be shown in Chapter III. The robust stability result referred to above, which was obtained by Georgiou and Smith, can now be stated as follows.

Proposition I.1. [GS90] *Given a plant P_1, a compensator C and a number β then:*

$[P_2, C]$ is stable for all P_2 satisfying $\delta_g(P_1, P_2) \leq \beta$ if, and only if, $b_{P_1,C} > \beta$, where $\delta_g(\cdot,\cdot)$ denotes the gap metric, which we shall define in Section 3.

That is, any compensator achieving the specified stability margin with the nominal plant will stabilize all plants within a certain distance of it, as measured by the gap metric.

Glover and McFarlane [GM89] (building on the framework of [VK86]) had earlier shown that, given a plant P, the problem of finding a C to maximize the stability margin $b_{P,C}$ has an explicit solution. They also showed that, even without uncertainty, maximizing this quantity is a useful means of obtaining a compensator which satisfies disturbance rejection objectives with the nominal plant [MG90]. A similar situation exists with classical gain and phase margins. These are defined as stability margins, which guarantee closed loop stability in the face of gain and phase uncertainty, though small margins also indicate poor behaviour with the nominal plant. This duality, between maximizing $b_{P,C}$ as a generalized stability margin, and the equivalent problem of minimizing its inverse as a generalized performance measure, will be another central theme to this monograph. Control system design is all about trade offs between different objectives — Glover and McFarlane have already shown that many of these trade offs are encapsulated in the generalized performance measure $1/b_{P,C}$. For the moment, we merely point out that in any practical setting, even with a perfect model of a genuinely linear, time-invariant plant, it is necessary to take account of all eight closed loop transfer functions from w_1, w_2, v_1 and v_2, to u and y, in Figure I.1. That is, if we are to control the output of the plant any realistic design strategy will take account of the amount of control available at the input of the plant, and of the possibility of disturbances and noise entering at any point of the loop.[2] Even if we have no detailed knowledge of the disturbances and noise there will always be some. In a classical framework, where compensators are designed "by hand" using graphical techniques, this is done implicitly. However, if we are to use some optimal control scheme, and if this scheme doesn't bound, either explicitly or implicitly, each of these transfer functions, then we may well end up with a closed loop that is extremely sensitive to those signals which have been omitted, or that requires arbitrarily large amounts of control effort. In an $\mathcal{H}_\infty$ framework, we must always ensure that $b_{P,C}$ is bounded from below — even if we are not explicitly maximizing it.

Proposition I.1 is a powerful and elegant result, and suggests that the gap metric is a useful measure to be used in conjunction with $b_{P,C}$. The gap does, however, have some drawbacks. The metric is defined in operator theoretic terms and it is difficult to relate the gap between two plants to their frequency responses,

[2]Disturbances entering the plant by any other means can always be represented by components at the input and the output, and the same holds for the compensator.

or variations in their physical parameters. Although the gap can be calculated ([Geo88], and see (9) below), this calculation is not particularly easy. Proposition I.1 also holds for the pointwise gap metric of Qiu and Davison [QD92b], the definition of which was motivated by the former of these points (and will be given in Chapter III). Unfortunately though, this metric cannot be calculated explicitly. More importantly, although Proposition I.1 tells us that small perturbations as measured by either metric are harmless, it says nothing about whether large perturbations are harmful in any sense. The main contribution of this monograph is to introduce a new metric, which we shall denote $\delta_\nu(\cdot,\cdot)$, that has the same qualitative properties as the gap and pointwise gap metrics but provides sharper quantitative results than both. Proposition I.1 also holds for this new metric; in fact it is the smallest metric for which it holds, as is demonstrated by the following statements (which we shall prove in Chapter III).

i) *Given a nominal plant P_1, a compensator C and a number β, then:* $[P_2, C]$ *is stable for all plants, P_2, satisfying $\delta_\nu(P_1, P_2) \leq \beta$ if, and only if, $b_{P_1,C} > \beta$.*
ii) *Given a nominal plant P_1, a perturbed plant P_2, and a number $\beta < b_{opt}(P)$ then:* $[P_2, C]$ *is stable for all compensators, C, satisfying $b_{P_1,C} > \beta$ if, and only if, $\delta_\nu(P_1, P_2) \leq \beta$,*

where $b_{opt}(P) := \sup_C b_{P,C}$, the optimal stability radius.

So, as for the gap metric, any plant at a distance less than β from the nominal will be stabilized by any compensator stabilizing the nominal with a stability margin of β. Furthermore, any plant at a distance *greater* than β from the nominal will be *destabilized* by some compensator that stabilizes the nominal with a stability margin of at least β. The only metric that has both these properties is $\delta_\nu(\cdot,\cdot)$. So, a ball in this metric is the largest set of linear, time-invariant plants that can be guaranteed to be stabilized *a priori* by a compensator solving the following $\mathcal{H}_\infty$ problem: find a C satisfying $b_{P,C} > \beta$. We shall see that, in addition to overcoming the inherent conservatism in Proposition I.1, the metric is both simple to evaluate and directly related to the frequency responses of the two systems.

Our argument may be summarised as follows. A necessary condition for a feedback compensator C to be effective with a plant P is that $b_{P,C}$ should be greater than some number β, as otherwise either the input or output of the plant (or both) will be overly sensitive to disturbances acting on the output of the plant or noise on the measurements (or, again, both). We shall argue a stronger case for this in Chapter V, and also the assertion that (under a suitable definition of the effectiveness of a compensator) this can also be regarded as a sufficient condition, perhaps for some slightly larger β. So, if P_1 is our model, P_2 is the real system and $\delta_\nu(P_1, P_2)$ is small (compared with β), then the difference between P_1 and P_2 is unimportant to the feedback problem — as any compensator that is effective with P_1 is also effective with P_2. Conversely, if $\delta_\nu(P_1, P_2)$ is large (relative to β), then some (though not necessarily all) of the compensators that look reasonable with P_1 will perform poorly with P_2. In this case it would be necessary either to get a better model of the system, or take greater account of the differences in the design process. An important point is that the distance between two plants, as measured by the metric δ_ν, has very little to do with any difference between their open loop behaviour (indeed there is no reason why it should). Consider, for example, the three systems

$$P_1 = 100/(2s+1), \ P_2 = 100/(2s-1) \text{ and } P_3 = 100/(s+1)^2.$$

P_1 and P_2 have very different step responses (see Figure I.2) — one is stable, the other divergent — yet we shall see that $\delta_\nu(P_1, P_2)$ is small (at .02 to be precise). On the other hand, P_1 and P_3 have similar step responses, yet $\delta_\nu(P_1, P_3)$ is large (at 0.8988, which we consider to be large as 1.0 is as big as both $\delta_\nu(\cdot,\cdot)$ and $b_{\cdot,\cdot}$ can get).

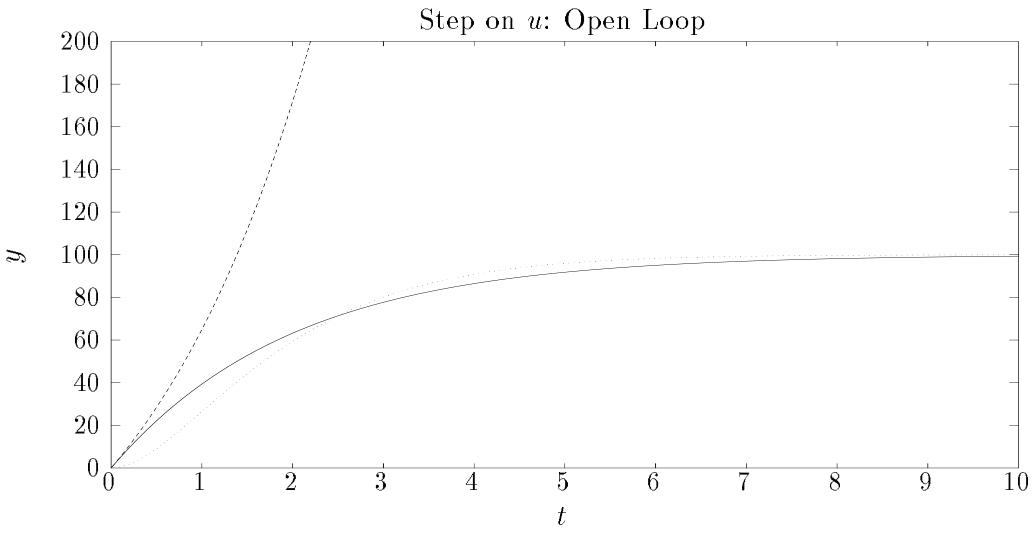


FIGURE I.2. Open Loop Step Responses

Key: —— is $P_1 := 100/(2s+1)$, ----- is $P_2 := 100/(2s-1)$, ········ is $P_3 := 100/(s+1)^2$

Figure I.3 shows the respective closed loop step responses under unity feedback to a step on v_1 of Figure I.1 (*ie* with $C = -1$, which gives $b_{P_0,C} = 1/\sqrt{2}$ and is, in fact, near optimal for P_1). This confirms the closeness of P_1 and P_2, and the dissimilarity of P_1 and P_3, in closed loop terms.

Of course, this doesn't necessarily mean that *all* reasonable feedback compensators for P_1 and P_2 (in terms of $b_{\cdot,\cdot}$) do badly with P_3, merely that some do — the unity feedback compensator being an example. If there *is* a compensator that performs well with all three plants, then we shall have to look harder to find

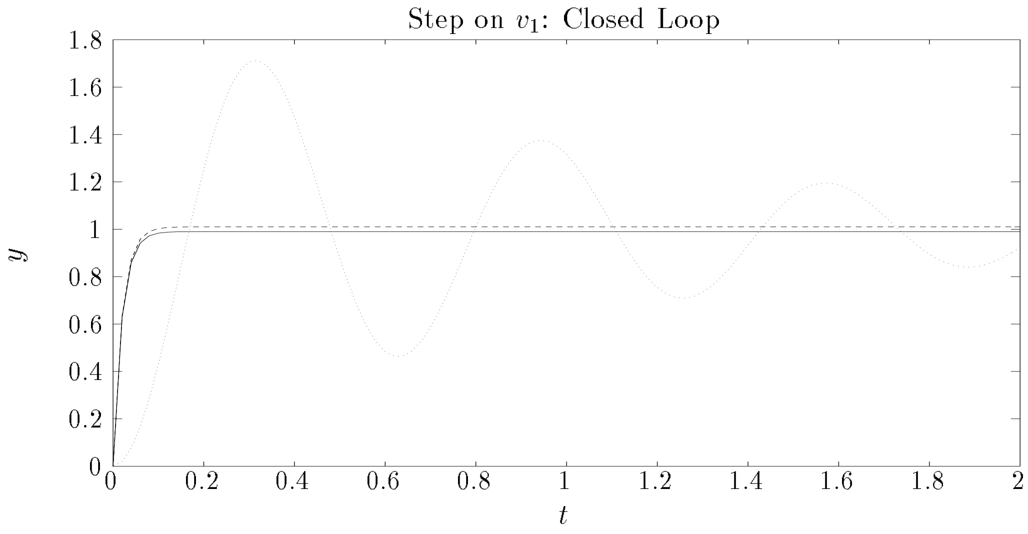


FIGURE I.3. Closed Loop Step Responses: $C = -1$
Key: —— is $P_1 := 100/(2s+1)$, ------ is $P_2 := 100/(2s-1)$, ········ is $P_3 := 100/(s+1)^2$

it. Such a compensator is $C = -(2.15s + 11.1)/(s + 23.3)$ (*ie* a phase lead compensator centred near the unity gain crossover frequency of P_3), which achieves a stability margin of approximately 0.42 with each plant (which we can be sure is in fact near the optimal solution to $\sup_C \min_{P\in\{P_1,P_2,P_3\}} b_{P,C}$, as $\sup_C b_{P_3,C}$ alone is equal to 0.4307 — in fact, this compensator was found by direct search, using the optimal compensator for P_3 as an initial guess). The resultant closed loop step responses are shown in Figure I.4.

1. Summary of contents

Chapter 2 Here we collect together some preliminary material from the literature. Section 1 is a review of the function and signal spaces that will be used, whilst Section 2 introduces coprime factorizations and graph spaces. In Section 3 we look at some common notions of uncertainty, and introduce the graph topology and the gap metric.
Chapter 3 A frequency response formula for the new metric (in the finite-dimensional case) is given here, and Section 1 contains the proof that this formula does indeed give a metric on the space of linear, time-invariant, finite dimensional systems. In Section 2 it is established that this metric has the robustness properties claimed for it above. A common criticism of the gap metric is that when the distance between two systems becomes sufficiently large the gap between them becomes identically 1, the metric then fails to further differentiate between the systems. In Section 3, making use of the frequency response intuition already

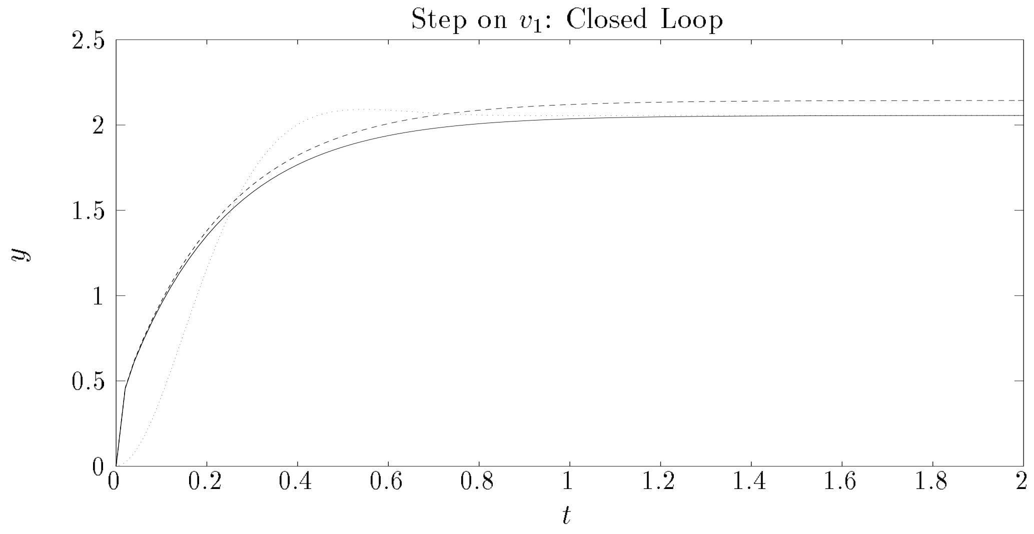


FIGURE I.4. Closed Loop Step Responses: Lead Compensator
$C = -(2.15s + 11.1)/(s + 23.3)$
Key: —— is $P_1 := 100/(2s+1)$, ----- is $P_2 := 100/(2s-1)$,
······· is $P_3 := 100/(s+1)^2$

gained, it is shown how the new metric of Section 1 is easily extended beyond 1. This extended metric continues to differentiate between systems that are arbitrarily far apart. The Chapter concludes with an examination of how we can use these ideas to characterize precisely how a frequency domain neighbourhood of a nominal plant splits under the graph topology.

Chapter 4 In Section 1 we examine the quantitative differences between the new metric and the gap metric, and establish upper and lower bounds for the gap in terms of our new metric. This allows us to prove the topological equivalence of this new metric to the gap — establishing that it does indeed induce the graph topology. There have recently appeared a number of papers on geometrical interpretations of the gap metric [FGS90] [OS91] [Sef91] [QD92a]. Section 2 considers the new metric, in the infinite dimensional case, from a similar operator theoretic viewpoint and provides further insights into the differences between the two metrics, these are demonstrated in Section 3 with a number of examples.

Chapter 5 In this chapter we develop the frequency response interpretation of the new metric. We start by taking a fresh look at the loop shaping design procedure introduced by McFarlane and Glover, giving some new results that further motivate this approach. Then, the interpretation of our metric is shown to be particularly sharp in the case of single input, single output systems. The metric is, in fact, the maximum distance between the frequency response loci of two systems (their Nyquist loci) when these are plotted on the surface of the

Riemann sphere (rather than the plane), subject to a certain Index function being zero. A generalized sensitivity function in the graph topology is then defined as the distance between the frequency response loci of a system, and the reciprocal of that of a compensator. This is an analogue of the classical sensitivity function on the plane. The ability to measure both the metric, and the generalized sensitivity function, frequency by frequency is exploited to develop an extension of the loop shaping methodology for uncertain plants that directly exploits this structure, and which we apply to a benchmark design example. The chapter concludes with an examination of the robust tracking problem in this framework.

Chapter 6 Much of the development of this chapter is in the spirit of [Glo86] and [KGP87]. The optimally robust compensator in this new metric is, of course, the same as that obtained in [MG90] for the normalized coprime factor uncertainty problem — it is shown here that a nonlinear and/or time-varying controller can do no better. Some interesting relations to approximation in this metric are also given, and it is shown that the optimal degree $n-1$ approximant can be obtained exactly. Upper and lower bounds for the achievable error using lower order approximations are also obtained, and a procedure given for obtaining an approximation that achieves the upper bound. These bounds are directly analogous to those obtained by Glover in [Glo84], which are still the best available in the additive case (*ie* on the plane).

Chapter 7 One way of regarding the difference between the gap metric and the metric introduced in this monograph is that the gap metric is based on stable perturbations to the normalized coprime factors of a system, whereas the new metric admits a certain class of admissible unstable perturbations. In this chapter we examine the subject of admissible, possibly unstable, perturbations in a general $\mathcal{H}_\infty$ control setting, including the case where the perturbations are required to be block structured. We give a precise condition for a perturbation to be admissible in this general setting, and then show how this leads to the known results for the specific cases of additive and multiplicative uncertainty, and how the new metric arises naturally when perturbations to the normalized coprime factors of a system are considered.

CHAPTER II

Mathematical Preliminaries

1. Function spaces and signal spaces

1.1. The signal spaces $\mathcal{L}_2$ and $\mathcal{H}_2$. In the context of this monograph, $\mathcal{L}_2$ is the space of all signals, or vectors of signals, with bounded energy. This is a Hilbert space with inner product

$$\langle x, y\rangle = \int_{-\infty}^{\infty} x^*(t)y(t)dt \text{ and norm } \|x\|_{\mathcal{L}_2^t} = \sqrt{\langle x, x\rangle}$$

ie the square root of the total energy. Alternatively, $\mathcal{L}_2$ may be thought of as a frequency domain space, with inner product

$$\langle \tilde{x}, \tilde{y}\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{x}^*(j\omega)\tilde{y}(j\omega)d\omega \text{ and norm } \|\tilde{x}\|_{\mathcal{L}_2^f} = \sqrt{\langle \tilde{x}, \tilde{x}\rangle}, \tag{1}$$

where $\tilde{x}$ is the Fourier transform of x. Parseval's theorem states that $\|\tilde{x}\|_{\mathcal{L}_2^f} = \|x\|_{\mathcal{L}_2^t}$. In fact, we shall think of these spaces as being one and the same and not use a separate notation for signals in the time and frequency domains. Thus, when we say that x is in $\mathcal{L}_2$ we mean that x is a signal of bounded energy, which can be thought of either as a function of time or frequency.

In the time domain, $\mathcal{L}_2$ can be decomposed into $\mathcal{L}_{2+}$ and $\mathcal{L}_{2-}$, where $\mathcal{L}_{2+}$ is the space of signals defined for positive time and zero for negative time. Similarly, $\mathcal{L}_{2-}$ is the space of signals defined for negative time, and zero for positive time. In the frequency domain, $\mathcal{L}_2$ can be similarly decomposed into $\mathcal{H}_2$ and $\mathcal{H}_2^{\perp}$, where $\mathcal{H}_2$ is the space of Fourier transforms of signals in $\mathcal{L}_{2+}$ and $\mathcal{H}_2^{\perp}$ is the space of Fourier transforms of signals in $\mathcal{L}_{2-}$. Again, in the rest of this monograph we shall not distinguish between time and frequency domain signals, and in fact we shall not use the notation $\mathcal{L}_{2+}$ and $\mathcal{L}_{2-}$ again. Thus, when we say that x is in $\mathcal{H}_2$ we mean that x is a signal of bounded energy which, when considered in the time domain, is zero for negative time. For such signals, the Fourier transform is identical to the Laplace transform with the Laplace variable s replaced by $j\omega$. $\mathcal{RH}_2$ is the subspace of $\mathcal{H}_2$ consisting of only rational functions, and may be equivalently thought of the rational functions of the complex variable s with denominator of strictly greater order than numerator and with no poles in the closed *rhp*. Similarly, the Hardy space $\mathcal{H}_2$ can be defined as the space of functions of the complex variable s that are analytic for all s in the open right half plane (*ie* $s : \Re(s) > 0$), and for which the integrals

$$\frac{1}{2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} \tilde{x}^*(s)\tilde{x}(s)ds$$

are uniformly bounded for all $\sigma > 0$, with norm as in (1).

1.2. The function spaces $\mathcal{L}_\infty$ and $\mathcal{H}_\infty$. We shall often think of systems as being operators on $\mathcal{H}_2$. A system P will be called stable if, for any input in $\mathcal{H}_2$, the output is also in $\mathcal{H}_2$. That is, a stable system maps bounded energy inputs onto bounded energy outputs. If a system is unstable then its output can have infinite energy in response to a finite energy input. $\mathcal{H}_\infty$ is the space of transfer functions of stable linear, time-invariant, continuous time, systems. This is a Hardy space with norm

$$\|P\|_{\mathcal{H}_\infty} := \sup_{u\in\mathcal{H}_2 : u\neq 0} \frac{\|Pu\|_{\mathcal{H}_2}}{\|u\|_{\mathcal{H}_2}}.$$

Thus stable systems have a finite $\mathcal{H}_\infty$ norm, and this norm measures the maximum energy gain of a system. For a constant matrix A, its maximum singular value, $\bar{\sigma}(A)$ is defined as the maximum "gain" of the matrix when the "inputs" and "outputs" are constant vectors equipped with the standard Euclidean norm $\|x\|_2 := \sqrt{\bar{x}^T x}$. Or

$$\bar{\sigma}(A) := \sup_{u\neq 0} \frac{\|Au\|_2}{\|u\|_2}.$$

The maximum singular value is more easily calculated as $\bar{\lambda}^{1/2}(A^*A)$, where $\bar{\lambda}$ denotes the maximum eigenvalue of the matrix. An input that comes arbitrarily close to attaining the $\mathcal{H}_\infty$ norm concentrates its energy at the frequency where the gain of the system, as measured by the maximum singular value of its frequency response matrix, is largest. Thus,

$$\|P\|_{\mathcal{H}_\infty} = \sup_{s:\Re(s)>0} \bar{\sigma}(P(s)). \tag{2}$$

Formally, $\mathcal{H}_\infty$ is defined as the space of functions of the complex variable s analytic for all s in the open right half plane with finite norm as defined in (2). $\mathcal{RH}_\infty$ is defined as the subspace of $\mathcal{H}_\infty$ whose elements are rational functions of s and arise as the transfer functions of finite dimensional systems, *ie* those systems described by ordinary differential equations. For such systems, the supremum is attained on the boundary $s = j\omega$, (for possibly infinite ω) *ie*

$$\|P\|_{\mathcal{RH}_\infty} = \max_{\omega\in\mathbb{R}\cup\infty} \bar{\sigma}(P(j\omega)),$$

provided $P(s)$ is analytic in the right half plane. $\mathcal{L}_\infty$ is the (Lebesgue) space of all functions essentially bounded on the imaginary axis with norm

$$\|P\|_{\mathcal{L}_\infty} := \operatorname*{ess\,sup}_{\omega} \bar{\sigma}(P(j\omega)).$$

Similarly $\mathcal{RL}_\infty$ is the subspace of $\mathcal{L}_\infty$ whose elements are rational functions of s, for which the norm simplifies to

$$\|P\|_{\mathcal{RL}_\infty} := \max_{\omega \in \mathbb{R} \cup \infty} \bar{\sigma}(P(j\omega)).$$

We shall regard $\mathcal{H}_\infty$ as a subspace of $\mathcal{L}_\infty$. Any proper rational transfer function matrix is in $\mathcal{L}_\infty$ provided it has no imaginary axis poles.

2. Coprime factors and graphs of linear systems

2.1. Graphs of linear systems. A linear system, with transfer function $P(s)$, may be regarded either as a possibly unbounded operator on $\mathcal{H}_2$ or as a multiplication operator on $\mathcal{L}_2$ that will be bounded provided P is in $\mathcal{L}_\infty$. We shall be using both notions in this monograph, and in fact the main results depend upon this duality of thinking, so it is important to clarify at this stage what we mean. We shall use $\mathcal{P}^{p\times q}$ to denote the class of rational transfer function matrices with q inputs and p outputs. Associated with an element $P \in \mathcal{P}^{p\times q}$ is a (possibly unbounded) multiplication operator on $\mathcal{H}_2$ (which we identify with $\mathcal{L}_2[0,\infty)$ in the time domain)

$$\mathbf{M}_P\colon \mathcal{H}_2 \to \mathcal{H}_2, \;\; u \mapsto Pu.$$

The *domain* of $\mathbf{M}_P$, $\mathcal{D}(\mathbf{M}_P)$, is defined as

$$\mathcal{D}(\mathbf{M}_P) := \{u \in \mathcal{H}_2 : Pu \in \mathcal{H}_2\}.$$

The domain of $\mathbf{M}_P$ is not the whole space if P is unstable, since any input that leads to an output in $\mathcal{H}_2$ must have zeros at any right half plane (*rhp*) poles of P. For example, if $P = 1/(s-1)$ then all inputs that lead to outputs in $\mathcal{H}_2$ must have the form $u = \frac{s-1}{s+1}x$, where x is any element of $\mathcal{H}_2$ (which may cancel the pole at $s=-1$). In this case we would write $\mathcal{D}(\mathbf{M}_P) = \frac{s-1}{s+1}\mathcal{H}_2$. Given an unstable plant, the action of a stabilizing feedback compensator is to restrict the inputs to the plant to those that lie in this domain.

The second time domain operator associated with the transfer function P, which we define in the frequency domain as

$$\mathbf{L}_P : \mathcal{L}_2 \to \mathcal{L}_2, \;\; u \mapsto Pu,$$

is an operator on $\mathcal{L}_2$ that is bounded (*ie* its domain is equal to the whole of $\mathcal{L}_2$) whenever $P \in \mathcal{L}_\infty$. Otherwise,

$$\mathcal{D}(\mathbf{L}_P) = \{u \in \mathcal{L}_2 : Pu \in \mathcal{L}_2\}.$$

If P is rational, then $\mathbf{M}_P$ is always causal, whereas $\mathbf{L}_P$ is only causal when $P \in \mathcal{H}_\infty$.

The *graph* of an operator is the set of all possible bounded input/output pairs. For example,

$$\mathcal{G}(\mathbf{M}_P) := \left\{ \begin{bmatrix} Pu \\ u \end{bmatrix} : u \in \mathcal{D}(\mathbf{M}_P) \right\}.$$

The graphs of $\mathbf{M}_P$ and $\mathbf{L}_P$ are actually *subspaces* of $\mathcal{H}_2$ and $\mathcal{L}_2$ respectively; as, by linearity, they are closed under addition, *ie*

$$\begin{bmatrix} y_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} y_2 \\ u_2 \end{bmatrix} \in \mathcal{G}(\mathbf{M}_P) \implies \begin{bmatrix} y_1 \\ u_1 \end{bmatrix} + \begin{bmatrix} y_2 \\ u_2 \end{bmatrix} \in \mathcal{G}(\mathbf{M}_P).$$

We shall also need the inverse graph, which we shall normally use for a compensator C,

$$\mathcal{G}^I(\mathbf{M}_C) := \left\{ \begin{bmatrix} u \\ Cu \end{bmatrix} : u \in \mathcal{D}(\mathbf{M}_C) \right\}.$$

In the context of this monograph, there are two other operators with the symbol P that are of interest, the Hankel operator, $\mathbf{H}_P$, and the Toeplitz operator, $\mathbf{T}_P$. These are defined, for $P \in \mathcal{L}_\infty$, as

$$\mathbf{H}_P : \mathcal{H}_2^\perp \to \mathcal{H}_2 \quad u \mapsto \mathbf{\Pi}_+ Pu,$$

and

$$\mathbf{T}_P : \mathcal{H}_2 \to \mathcal{H}_2 \quad u \mapsto \mathbf{\Pi}_+ Pu.$$

We use $\mathbf{\Pi}_+$ and $\mathbf{\Pi}_-$ to denote the operators of orthogonal projection onto $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$ respectively, that is, if $z \in \mathcal{L}_2$ is decomposed as $z = x + y$, $x \in \mathcal{H}_2, y \in \mathcal{H}_2^\perp$, then $\mathbf{\Pi}_+ z = x$ and $\mathbf{\Pi}_- z = y$.

2.2. Coprime factor representations. The notion of a coprime factor representation of a plant $P \in \mathcal{P}^{p\times q}$ will be used extensively throughout this monograph, and we collect here some important definitions and facts pertaining to such representations. In essence, any $P \in \mathcal{P}^{p\times q}$ can be represented as the quotient of two $\mathcal{H}_\infty$ functions, for example

$$P(s) = N(s)M(s)^{-1}, \quad M, N \in \mathcal{H}_\infty$$

is a right factorization of P. Any closed *rhp* poles of P appear as zeros of the denominator M. For any P there are many such factorizations. One way of reducing this lack of uniqueness is to remove any common factors of N and M. In particular, it is desirable to remove any common *rhp* zeros of N and M, as these would lead to hidden *rhp* pole/zero cancellations in the factorization. This leads us naturally to the concept of *coprime* factors.

DEFINITION II.1. Given $M, N \in \mathcal{H}_\infty$ with the same number of columns, then M and N are *right-coprime* if there exist $X, Y \in \mathcal{H}_\infty$ such that

$$XM + YN = I.$$

For rational N and M, this definition is equivalent to requiring that any common right divisor of N and M is invertible in $\mathcal{H}_\infty$. That is, N and M are right-coprime if, for any square Q in $\mathcal{H}_\infty$ of compatible dimensions, $NQ^{-1}, MQ^{-1} \in \mathcal{H}_\infty$ implies that $Q^{-1} \in \mathcal{H}_\infty$. We are now in a position to define a *right-coprime factorization* of P.

DEFINITION II.2. The ordered pair $\{N, M\}$ is a right-coprime factorization (*rcf*) of $P \in \mathcal{P}^{p\times q}$ if

i) M is invertible (in $\mathcal{P}$),
ii) $P = NM^{-1}$,
iii) N and M are right-coprime.

The next result may be found in [Fra87] and [Vid85]. A square $Q \in \mathcal{H}_\infty$ is called unimodular if Q^{-1} is also in $\mathcal{H}_\infty$.

PROPOSITION II.1. *Any $P \in \mathcal{P}^{p\times q}$ has an* rcf $\{N, M\}$. *Furthermore, any such representation is unique to within right multiplication by a unimodular transfer matrix.*

Thus, given $\{N, M\}$ an *rcf* of P, all possible *rcf*s may be generated as $\{NQ, MQ\}$, $Q, Q^{-1} \in \mathcal{H}_\infty$. The type of coprime factorization that will be used most often in this monograph is a *normalized* coprime factorization.

DEFINITION II.3. The ordered pair $\{N, M\}$ is a normalized *rcf* of $P \in \mathcal{P}^{p\times q}$ if $\{N, M\}$ is an *rcf* of P and

$$M^*M + N^*N = I.$$

PROPOSITION II.2. *Any $P \in \mathcal{P}^{p\times q}$ has a normalized* rcf $\{N, M\}$. *Furthermore, any such representation is unique to within right multiplication by a constant unitary matrix.*

Analogous results hold for left representations, which are now stated without discussion.

DEFINITION II.4. Given $\tilde{M}, \tilde{N} \in \mathcal{H}_\infty$ with the same number of rows, then $\tilde{M}$ and $\tilde{N}$ are *left-coprime* if there exist $\tilde{X}, \tilde{Y} \in \mathcal{H}_\infty$ such that

$$\tilde{M}\tilde{X} + \tilde{N}\tilde{Y} = I.$$

DEFINITION II.5. The ordered pair $\{\tilde{N}, \tilde{M}\}$ is a left-coprime factorization (*lcf*) of $P \in \mathcal{P}^{p\times q}$ if

i) $\tilde{M}$ is invertible (in $\mathcal{P}$),
ii) $P = \tilde{M}^{-1}\tilde{N}$,
iii) $\tilde{N}$ and $\tilde{M}$ are left-coprime.

PROPOSITION II.3. *Any $P \in \mathcal{P}^{p\times q}$ has an* lcf $\{\tilde{N}, \tilde{M}\}$. *Furthermore, any such representation is unique to within left multiplication by a unimodular transfer matrix.*

Thus, given $\{\tilde{N},\tilde{M}\}$, a *lcf* of P, all possible *lcf*s may be generated as $\{Q\tilde{N},Q\tilde{M}\}$, $Q,Q^{-1}\in\mathcal{H}_\infty$.

DEFINITION II.6. The ordered pair $\{\tilde{N},\tilde{M}\}$ is a normalized *lcf* of $P\in\mathcal{P}^{p\times q}$ if $\{\tilde{N},\tilde{M}\}$ is an *lcf* of P and

$$\tilde{M}\tilde{M}^*+\tilde{N}\tilde{N}^*=I$$

PROPOSITION II.4. *Any $P\in\mathcal{P}^{p\times q}$ has a normalized* lcf $\{\tilde{N},\tilde{M}\}$. *Furthermore, any such representation is unique to within left multiplication by a constant unitary matrix.*

2.3. Graph representations. The results of the last section allow us to give a representation of the graph of a linear system. Throughout this monograph $\{N_i,M_i\}$ will denote a normalized *rcf* of P_i, and $\{\tilde{N}_i,\tilde{M}_i\}$ a normalized left coprime factorization *lcf* of P_i. Similarly, $\{N_c,M_c\}$ and $\{\tilde{N}_c,\tilde{M}_c\}$ will denote normalized right and left coprime factorizations of a compensator C. We shall write

$$G_i:=\begin{bmatrix}N_i\\M_i\end{bmatrix},\quad \tilde{G}_i:=\begin{bmatrix}-\tilde{M}_i & \tilde{N}_i\end{bmatrix},\quad K:=\begin{bmatrix}M_c\\N_c\end{bmatrix},\quad \tilde{K}:=\begin{bmatrix}-\tilde{N}_c & \tilde{M}_c\end{bmatrix}.$$

For reasons that will become apparent, we shall refer to G_i and $\tilde{G}_i$ as normalized right and left graph symbols for P_i. Similarly, we shall refer to K and $\tilde{K}$ as normalized right and left inverse graph symbols of C. We shall use $\mathcal{G}_i$ as a shorthand for $\mathcal{G}(\mathcal{M}_{P_i})$. Note that $\tilde{G}_iG_i=\tilde{K}K=0$, and so $\begin{bmatrix}G_i & \tilde{G}_i^*\end{bmatrix}$ and $\begin{bmatrix}K & \tilde{K}^*\end{bmatrix}$ are unitary. The following proposition, as well as clarifying the theoretical relationship between these objects, will give a flavour of the functional dependence between them that we shall assume throughout the rest of this monograph for the sake of simplified notation.

PROPOSITION II.5. *Given G_0, a normalized right graph symbol for $P_0\in\mathcal{P}^{p\times q}$, then*

$$\mathcal{G}_0=G_0\mathcal{H}_2=\operatorname{Ra}\mathbf{T}_{G_0}=\operatorname{Ker}\mathbf{T}_{\tilde{G}_0},$$

where Ra *denotes the range of an operator (*ie $\operatorname{Ra}(\mathbf{A}):=\mathbf{A}\mathcal{D}(A)$*), and* Ker *the kernel (*ie $\operatorname{Ker}(\mathbf{A}):=\{y\in\mathcal{D}(\mathbf{A}):\mathbf{A}y=0\}$*).*

PROOF: Given $\left[\begin{smallmatrix}y\\u\end{smallmatrix}\right]\in\mathcal{G}_0$ then $y=N_0M_0^{-1}u$. Put $q=M_0^{-1}u$, thus $\left[\begin{smallmatrix}y\\u\end{smallmatrix}\right]=G_0q$, we need to show that $q\in\mathcal{H}_2$. As $\{N_0,M_0\}$ are right-coprime there exists an $X\in\mathcal{H}_\infty: XG_0=I$, so $q=X\left[\begin{smallmatrix}y\\u\end{smallmatrix}\right]\in\mathcal{H}_2$. Thus, any element in the graph of P can be represented as $G_0q:q\in\mathcal{H}_2$. Conversely, if $\left[\begin{smallmatrix}y\\u\end{smallmatrix}\right]=G_0q:q\in\mathcal{H}_2$ then it is clear that $y=N_0M_0^{-1}u$ and $u,y\in\mathcal{H}_2$, thus $\left[\begin{smallmatrix}y\\u\end{smallmatrix}\right]\in\mathcal{G}_0$.

In order to show the last statement, note that $\tilde{G}_0G_0=0$ implies $\mathcal{G}_0\subset\operatorname{Ker}\mathbf{T}_{\tilde{G}_0}$. Conversely, if $\left[\begin{smallmatrix}y\\u\end{smallmatrix}\right]\in\mathcal{H}_2$ and $\tilde{G}_0\left[\begin{smallmatrix}y\\u\end{smallmatrix}\right]\in\mathcal{H}_2^\perp$, then $\tilde{G}_0\left[\begin{smallmatrix}y\\u\end{smallmatrix}\right]=0$, and $y=\tilde{M}_0^{-1}\tilde{N}_0u$. □

The graph of $\mathbf{L}_{P_0}$ can be similarly represented.

PROPOSITION II.6. *Given G_0, a normalized right graph symbol for $P_0 \in \mathcal{P}^{p\times q}$, then*

$$\mathcal{G}(\mathbf{L}_{P_0}) = G_0 \mathcal{L}_2$$

PROOF: The proof of this proposition is identical to the last, replacing $\mathcal{G}_0$ with $\mathcal{G}(\mathbf{L}_{P_0})$, and $\mathcal{H}_2$ with $\mathcal{L}_2$, throughout. □

3. Topologies and metrics

In this section we shall develop some quantitative notions of uncertainty — a necessary preliminary before investigating the effects of uncertainty on feedback systems. We shall, at first, restrict our attention to stable plants — that is, plants in $\mathcal{P}^{p\times q}$ whose transfer functions are in $\mathcal{H}_\infty$. We shall denote the space of all such systems $\mathcal{P}_s^{p\times q}$ (this is a space as the sum of two stable plants is also stable). One useful, and well studied, notion goes as follows: apply the same, bounded energy, input to the nominal and perturbed plants and look at the energy of the difference between their outputs as in Figure II.5. Then look at the maximum energy gain in this setup over *all* possible inputs. That is, define a function $\delta_n : \mathcal{P}_s^{p\times q} \times \mathcal{P}_s^{p\times q} \to \mathbb{R}$ as

$$\delta_n(P_1, P_2) := \sup_{u\in\mathcal{H}_2 : u\neq 0} \frac{\|P_1 u - P_2 u\|_2}{\|u\|_2},$$

it is necessary to normalize by the norm of the input here as otherwise δ_n could be made unbounded by making the energy of the input arbitrarily large.

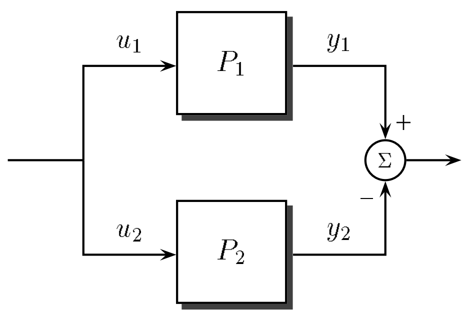


FIGURE II.5. Measuring Open Loop Uncertainty

From the definition of the $\mathcal{H}_\infty$ norm it is clear that

$$\delta_n(P_1, P_2) = \|P_1 - P_2\|_\infty = \sup_\omega \bar{\sigma}(P_1(j\omega) - P_2(j\omega)),$$

and for systems with only one input and output (*scalar* systems)

$$\delta_n(p_1, p_2) = \sup_\omega |p_1(j\omega) - p_2(j\omega)|.$$

We could call two systems P_1 and P_2 "close" if $\delta_n(P_1, P_2)$ were small. One of the main attractions of this notion of uncertainty is that it is easily measured. In the scalar case we can simply plot the Nyquist diagram of the two plants and measure the distance between them frequency by frequency. The function δ_n is then the maximum of this distance over frequency. This type of uncertainty is often known as $\mathcal{H}_\infty$ *additive* uncertainty, as if we write $P_2 = P_1 + \Delta$, with $\Delta \in \mathcal{H}_\infty$, then $\delta_n(P_1, P_2) < \beta$ if, and only if, $\|\Delta\|_\infty < \beta$. One can also weight the input and output by functions W_i, W_o in $\mathcal{H}_\infty$, with W_i^{-1} also in $\mathcal{H}_\infty$, defining a function

$$\delta_{n;W_o,W_i}(P_1, P_2) := \sup_{u \in \mathcal{H}_2 : u \neq 0} \frac{\|W_o(P_2 u - P_1 u)\|_2}{\|W_i u\|_2} = \|W_o(P_2 - P_1)W_i^{-1}\|_\infty$$

(to see this, define a new "input" $z = W_i u$ and then substitute for u — this is why it was necessary to insist that W_i was invertible in $\mathcal{H}_\infty$). So, W_i and W_o weight the difference as a function of frequency.

It is easy to see that the function δ_n defines a *metric* on the space of stable systems with p inputs and q outputs, $\mathcal{P}_s^{p\times q}$. A metric on a space $\mathcal{S}$ is a function δ that satisfies the following axioms:

M1 $\delta(x, y) = \delta(y, x)$ for all x, y in $\mathcal{S}$

M2 $\delta(x, y) \geq 0$ for all x, y in $\mathcal{S}$, and $\delta(x, y) = 0$ if, and only if, $x = y$.

M3 [Triangle inequality] $\delta(x, y) \leq \delta(x, z) + \delta(z, y)$ for all x, y, z in $\mathcal{S}$.

That δ_n satisfies the triangle inequality follows from the fact that $\|\cdot\|_\infty$ does. The metric δ_n induces a topology on $\mathcal{P}_s^{p\times q}$, the norm topology, hence the subscript. A topology on a space $\mathcal{S}$ is the collection $\mathcal{T}$ of all sets $\mathcal{T}_i \subset \mathcal{S}$ regarded as being *open* in the space, provided this collection satisfies the following axioms:

T1 Both the entire space $\mathcal{S}$, and the empty set $\emptyset$ are elements of $\mathcal{T}$.

T2 Any arbitrary union or finite intersection of sets in $\mathcal{T}$ is also an element of $\mathcal{T}$.

Any space may be equipped with a number of different topologies.

If we have a metric on a space, then we can define a set $\mathcal{U} \in \mathcal{S}$ to be open if, for every $x \in \mathcal{U}$, there exists an ϵ such that the set $\{y : \delta(x, y) < \epsilon\}$ is also contained in $\mathcal{U}$. The collection of all such sets is then guaranteed to satisfy T1 and T2, and is called the topology *induced* by the metric δ. Notice that the weighted metric $\delta_{n;W_o,W_i}$ induces the same topology irrespective of the weights, provided *both* are invertible in $\mathcal{H}_\infty$. To see this, put $\alpha = \bar{\sigma}(W_o)\bar{\sigma}(W_i^{-1}) = \bar{\sigma}(W_o)/\underline{\sigma}(W_i)$ and $\beta = \bar{\sigma}(W_i)/\underline{\sigma}(W_o)$ then, for any ϵ,

$$\delta_n(P_1, P_2) < \epsilon/\alpha \implies \delta_{n;W_o,W_i}(P_1, P_2) < \epsilon,$$

and

$$\delta_{n;W_o,W_i}(P_1, P_2) < \epsilon/\beta \implies \delta_n(P_1, P_2) < \epsilon.$$

Thus, a topology on a space provides only a loose, qualitative notion of what it means for two elements of a space to be close.

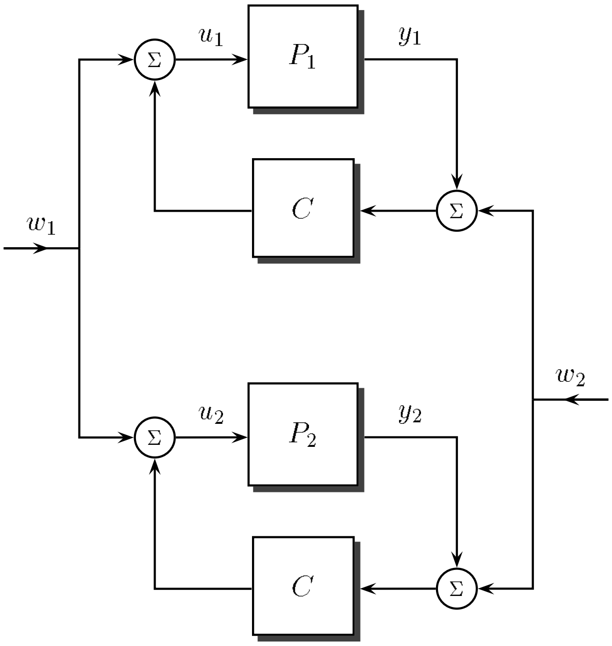


FIGURE II.6. Closed Loop Uncertainty

There are compelling arguments for the study of additive uncertainty in an open loop setting. Unfortunately, such a notion of uncertainty takes no account of the special structure of the feedback problem. We are really interested in comparing the closed loop behaviour of different plants in conjunction with the same feedback compensator, as depicted in Figure II.6. If we apply the same input to the two loops, then the signals at the plant inputs will be different; whereas uncertainty in the norm topology, as discussed above, is related to the difference in the outputs of the two plants when each receives the same input. A more appropriate notion would allow for uncertainty in both the inputs and the outputs; two systems might be deemed close if, given any input u_1 to P_1, there is input u_2 to P_2 that makes

$$\frac{\left\| \begin{bmatrix} y_1 \\ u_1 \end{bmatrix} - \begin{bmatrix} y_2 \\ u_2 \end{bmatrix} \right\|_2}{\left\| \begin{bmatrix} y_1 \\ u_1 \end{bmatrix} \right\|_2}$$

small, where $y_1 = P_1 u_1$ and $y_2 = P_2 u_2$. That is to say, if their graphs are close. This notion also makes sense for unstable plants — we just restrict u_1 to the domain of $\mathbf{M}_{P_1}$ and u_2 to the domain of $\mathbf{M}_{P_2}$.

We can define a metric on the space of all linear systems using this notion as follows. Put

$$\vec{\delta}_g(P_1, P_2) := \sup_{\begin{bmatrix} u_1 \\ y_1 \end{bmatrix} \in \mathcal{G}_1} \inf_{\begin{bmatrix} u_2 \\ y_2 \end{bmatrix} \in \mathcal{G}_2} \frac{\left\| \begin{bmatrix} y_1 \\ u_1 \end{bmatrix} - \begin{bmatrix} y_2 \\ u_2 \end{bmatrix} \right\|_2}{\left\| \begin{bmatrix} y_1 \\ u_1 \end{bmatrix} \right\|_2}. \tag{3}$$

This is called the directed gap between the graph spaces of P_1 and P_2 [Kat66], which is not in itself a metric as it is not symmetrical in P_1 and P_2. However, if we define

$$\delta_g(P_1, P_2) := \max\left\{ \vec{\delta}_g(P_1, P_2), \vec{\delta}_g(P_2, P_1) \right\}$$

then δ_g is a metric, the *gap* metric. Note that $\delta_g(P_1, P_2)$ is never greater than one (we can always put $u_2 = y_2 = 0$) and, in fact, if $\delta_g(P_1, P_2) < 1$ then $\vec{\delta}_g(P_1, P_2) = \vec{\delta}_g(P_2, P_1)$ [GS90]. These facts are standard in operator theory, and hold for the subspaces of any Hilbert space (see [Kat66]). They were introduced into control theory in [ZES80], [Geo88] and [GS90]. The gap between two subspaces $\mathcal{K}_1$, $\mathcal{K}_2$ of a Hilbert space $\mathcal{H}$ is defined as

$$\text{gap}(\mathcal{K}_1, \mathcal{K}_2) := \|\mathbf{\Pi}_{\mathcal{K}_1} - \mathbf{\Pi}_{\mathcal{K}_2}\|,$$

where $\mathbf{\Pi}_{\mathcal{K}_i}$ denotes orthogonal projection onto $\mathcal{K}_i$. Now, $\begin{bmatrix} \mathbf{\Pi}_{\mathcal{K}_1} \\ \mathbf{\Pi}_{\mathcal{K}_1^\perp} \end{bmatrix}$ is an isometry (*ie* it is norm preserving) on $\mathcal{H}$, as any $z \in \mathcal{H}$ can be decomposed as $z = x + y$, where $x \in \mathcal{K}_1$ and $y \perp \mathcal{K}_1$, with

$$\|z\|_2^2 = \langle x+y, x+y \rangle = \langle x, x \rangle + \langle y, y \rangle + \langle x, y \rangle + \langle y, x \rangle = \|x\|_2^2 + \|y\|_2^2,$$

and so,

$$\left\| \begin{bmatrix} \mathbf{\Pi}_{\mathcal{K}_1} \\ \mathbf{\Pi}_{\mathcal{K}_1^\perp} \end{bmatrix} z \right\|_2 = \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|_2 = \|z\|_2.$$

Similarly, $\begin{bmatrix} \mathbf{\Pi}_{\mathcal{K}_2} & \mathbf{\Pi}_{\mathcal{K}_2^\perp} \end{bmatrix}$ is a partial isometry onto $\mathcal{H}$; that is, an isometry on $\begin{bmatrix} \mathcal{K}_2 \\ \mathcal{K}_2^\perp \end{bmatrix}$ whose range is the whole of $\mathcal{H}$. So,

$$\begin{aligned} \text{gap}(\mathcal{K}_1, \mathcal{K}_2) &:= \left\| \begin{bmatrix} \mathbf{\Pi}_{\mathcal{K}_1} \\ \mathbf{\Pi}_{\mathcal{K}_1^\perp} \end{bmatrix} (\mathbf{\Pi}_{\mathcal{K}_1} - \mathbf{\Pi}_{\mathcal{K}_2}) \begin{bmatrix} \mathbf{\Pi}_{\mathcal{K}_2} & \mathbf{\Pi}_{\mathcal{K}_2^\perp} \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} 0 & \mathbf{\Pi}_{\mathcal{K}_1} \mathbf{\Pi}_{\mathcal{K}_2^\perp} \\ -\mathbf{\Pi}_{\mathcal{K}_1^\perp} \mathbf{\Pi}_{\mathcal{K}_2} & 0 \end{bmatrix} \right\|, \\ &= \max\left\{ \|\mathbf{\Pi}_{\mathcal{K}_1} \mathbf{\Pi}_{\mathcal{K}_2^\perp}\|, \|\mathbf{\Pi}_{\mathcal{K}_1^\perp} \mathbf{\Pi}_{\mathcal{K}_2}\| \right\}. \end{aligned}$$

These two quantities are the directed gaps, *ie*

$$\vec{\text{gap}}(\mathcal{K}_1, \mathcal{K}_2) := \|\mathbf{\Pi}_{\mathcal{K}_2^\perp} \mathbf{\Pi}_{\mathcal{K}_1}\| = \|\mathbf{\Pi}_{\mathcal{K}_1} \mathbf{\Pi}_{\mathcal{K}_2^\perp}\|$$

The last equality is a result of these operators being adjoints, *ie*

$$\langle \mathbf{\Pi}_{\mathcal{K}_2^\perp} \mathbf{\Pi}_{\mathcal{K}_1} x, y \rangle = \langle x, \mathbf{\Pi}_{\mathcal{K}_1} \mathbf{\Pi}_{\mathcal{K}_2^\perp} y \rangle.$$

An equivalent expression for the directed gap follows from

$$\begin{aligned} \|\mathbf{\Pi}_{\mathcal{K}_2^\perp} \mathbf{\Pi}_{\mathcal{K}_1}\| &= \sup_{x \in \mathcal{K}_1, x \neq 0} \frac{\|\mathbf{\Pi}_{\mathcal{K}_2^\perp} x\|}{\|x\|} \\ &= \sup_{x \in \mathcal{K}_1, x \neq 0} \inf_{y \in \mathcal{K}_2} \frac{\|\mathbf{\Pi}_{\mathcal{K}_2^\perp} x + \mathbf{\Pi}_{\mathcal{K}_2} x - y\|}{\|x\|} = \sup_{x \in \mathcal{K}_1, x \neq 0} \inf_{y \in \mathcal{K}_2} \frac{\|x - y\|_2}{\|x\|_2}, \end{aligned}$$

which is the expression we started with in (3). We shall use $\delta_g(P_1, P_2)$ to mean $\mathrm{gap}(\mathcal{G}_1, \mathcal{G}_2)$.

The gap metric was first introduced into the control literature in [ZES80]. The topology induced by the gap metric has been named the graph topology. Together with the comments already made, there are several reasons why the graph topology is much better suited to the feedback problem than the norm topology.

Any topology that is induced by a metric is characterized completely by the convergence of sequences. Given an infinite sequence of systems $\{P_i\}$ and another system P (which may, or may not, be the limit point of the sequence) then the following are equivalent (see [Vid84]):

i) $\{P_i\} \to P$ in the graph topology.
ii) $\delta_g(P, P_i) \to 0$
iii) There exists a C, which stabilizes P and all P_i for i sufficiently large, such that

$$\left\| \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} [-C \;\; I] - \begin{bmatrix} P_i \\ I \end{bmatrix} (I - CP_i)^{-1} [-C \;\; I] \right\|_\infty \to 0.$$

Furthermore, if iii) holds for any one compensator that stabilizes P then it holds for all compensators that stabilize P.

We shall prove this, and some other properties of the graph topology, in Chapter IV.

In [Geo88] Georgiou obtained a formula for the directed gap between two systems in terms of their normalized graph symbols.

$$\vec{\delta}_g(P_1, P_2) = \inf_{Q \in \mathcal{H}_\infty} \|G_1 - G_2 Q\|_\infty . \tag{4}$$

Thus, the gap metric between two plants can be calculated. The level of mathematics required to prove this result goes substantially beyond that employed in the rest of this monograph — but we can hint at how the structure of this formula arises. Note that (3) can be rewritten

$$\vec{\delta}_g(P_1, P_2) = \sup_{q_1 \in \mathcal{H}_2} \inf_{q_2 \in \mathcal{H}_2} \frac{\|G_1 q_1 - G_2 q_2\|_2}{\|q_1\|_2},$$

using the graph representations of the last section and the fact that $\|G_1 q_1\|_2 = \|q_1\|_2$ as G_1 is inner. To get from this formula to (4) it is necessary to show that, for any q_1, it is possible to make

$$\frac{\|G_1 q_1 - G_2 q_2\|_2}{\|q_1\|_2} \le \vec{\delta}_g(P_1, P_2)$$

with a q_2 of the form $q_2 = Q q_1$ for some $Q \in \mathcal{H}_\infty$, the same Q sufficing for all q_1, *ie* that

$$\sup_{q_1 \in \mathcal{H}_2} \inf_{q_2 \in \mathcal{H}_2} \frac{\|G_1 q_1 - G_2 q_2\|_2}{\|q_1\|_2} = \inf_{Q \in \mathcal{H}_\infty} \sup_{q_1 \in \mathcal{H}_2} \frac{\|G_1 q_1 - G_2 Q q_1\|_2}{\|q_1\|_2},$$

The RHS of which is equivalent to (4).

The gap between the $\mathcal{L}_2$ graph spaces of two systems is much simpler to use; although, as we shall see, it induces the wrong topology. We shall use the notation

$$\delta_{\mathcal{L}_2}(P_1, P_2) := \operatorname{gap}(\mathcal{G}(\mathbf{L}_{P_1}), \mathcal{G}(\mathbf{L}_{P_2})).$$

The projections we require to calculate the $\mathcal{L}_2$-gap have particularly simple expressions:

$$\begin{aligned} \mathbf{\Pi}_{\mathcal{G}(\mathbf{L}_{P_i})} &= \mathbf{L}_{G_i G_i^*} \\ \mathbf{\Pi}_{\mathcal{G}(\mathbf{L}_{P_i^\perp})} &= \mathbf{L}_{\tilde{G}_i^* \tilde{G}_i}, \end{aligned}$$

as $G_i G_i^* G_i q = G_i q$ and $G_i G_i^* + \tilde{G}_i^* \tilde{G}_i = I$ respectively, the last identity being a consequence of the fact that $\begin{bmatrix} G_i & \tilde{G}_i^* \end{bmatrix}$ is unitary. So,

$$\begin{aligned} \vec{\delta}_{\mathcal{L}_2}(P_1, P_2) &= \left\| \tilde{G}_2^* \tilde{G}_2 G_1 G_1^* \right\|_\infty \\ &= \left\| \tilde{G}_2 G_1 \right\|_\infty \\ &= \left\| (I + P_2 P_2^*)^{-1/2} (P_2 - P_1)(I + P_1^* P_1)^{-1/2} \right\|_\infty, \end{aligned}$$

as

$$\tilde{G}_2 G_1 = \tilde{M}_2 (P_2 - P_1) M_1, \text{ and } (I + P_2 P_2^*)^{-1} = \tilde{M}_2^* \tilde{M}_2, \ (I + P_1^* P_1)^{-1} = M_1 M_1^*.$$

In fact, $\vec{\delta}_{\mathcal{L}_2}(P_1, P_2) = \vec{\delta}_{\mathcal{L}_2}(P_2, P_1)$, as we shall show in Chapter III, and so $\delta_{\mathcal{L}_2}(P_1, P_2)$ is given by the same expression. Unfortunately, as mentioned above, this metric induces the wrong topology. The $\mathcal{L}_2$-gap metric only depends on the frequency responses of the two systems, taking no account of their right half plane pole/zero structure. For example, if

$$P = 1, \ \& \ P_\epsilon = \frac{(s - 1 - \epsilon)}{(s - 1)},$$

then $\delta_{\mathcal{L}_2}(P, P_\epsilon) = \epsilon/2 + O(\epsilon^2)$, and so $\lim_{\epsilon \to 0} \delta_{\mathcal{L}_2}(P, P_\epsilon) = 0$. However, as ϵ tends to zero, P_ϵ becomes increasingly difficult to stabilize due to the near unstable pole/zero cancellation. In fact, any stabilizing compensator for P will destabilize all P_ϵ for ϵ sufficiently small. So, on its own, the $\mathcal{L}_2$-gap metric is useless for the study of feedback systems.

In the next chapter we shall introduce a new metric, which we shall denote $\delta_\nu(\cdot,\cdot)$. This metric is made to induce the graph topology via a winding number constraint, taking the value of the $\mathcal{L}_2$-gap when the constraint is satisfied and 1 if violated. (Notice that $\|\tilde{G}_2 G_1\| \leq \|\tilde{G}_2\|\|G_1\| = 1$ always.) In the above example $\delta_\nu(P, P_\epsilon) = 1$ for all $|\epsilon| \leq 2$. It will be seen that this metric is substantially easier to work with than the gap metric, and leads to sharper quantitative results.

CHAPTER III

A New Metric for $\mathcal{P}$

In this chapter we shall give a formula for the metric δ_ν, prove that it is indeed a metric, and prove that it has the properties claimed for it in the introduction. The most important of these properties is that it exhibits the following, rather strong, form of duality with the measure $b_{P,C}$.

i) *Given a nominal plant* $P_1 \in \mathcal{P}^{p\times q}$, *a compensator* $C \in \mathcal{P}^{q\times p}$ *and a number* β, *then:*
$[P_2, C]$ *is stable for all plants,* P_2, *satisfying* $\delta_\nu(P_1, P_2) \leq \beta$ *if, and only if,* $b_{P_1,C} > \beta$.

ii) *Given a nominal plant* P_1, *a perturbed plant* P_2, $\in \mathcal{P}^{p\times q}$ *and a number* $\beta < b_{opt}(P)$ *then:*
$[P_2, C]$ *is stable for all compensators,* C, *satisfying* $b_{P_1,C} > \beta$ *if, and only if,* $\delta_\nu(P_1, P_2) \leq \beta$,

where $b_{opt}(P) := \sup_C b_{P,C}$.

So, as for the gap metric, any plant at a distance less than β from the nominal will be stabilized by any compensator stabilizing the nominal with a stability margin of β. Furthermore, any plant at a distance *greater* than β from the nominal will be *destabilized* by some compensator that stabilizes the nominal with a stability margin of at least β. The only metric that has both these properties is $\delta_\nu(\cdot,\cdot)$. So, a ball in this metric is the largest set of linear, time-invariant plants that can be guaranteed to be stabilized *a priori* by a compensator solving the following $\mathcal{H}_\infty$ problem: find a C satisfying $b_{P,C} > \beta$. In fact, we shall go further and precisely characterize the maximum amount of residual stability margin that can be guaranteed for perturbations smaller than the nominal stability margin. The necessity parts of the proofs of items i) and ii) will require somewhat different constructions; for part i) then, given a C that stabilizes the nominal with $b_{P_1,C} \leq \beta$, it is necessary to construct a P_2 satisfying $\delta_\nu(P_1, P_2) \leq \beta$ which is destabilized by C. For the necessity of part ii) then, given a P_2 satisfying $\delta_\nu(P_1, P_2) > \beta$, it is necessary to find a C from the set of all compensators achieving $b_{P_1,C} > \beta$ that destabilizes P_2.

The metric is given by $\delta_\nu(P_1, P_2) = \delta_{\mathcal{L}_2}(P_1, P_2)$, if a certain winding number condition is satisfied (see (10) below), or 1 otherwise, where

$$\delta_{\mathcal{L}_2}(P_1, P_2) = \left\| (I + P_2 P_2^*)^{-1/2}(P_2 - P_1)(I + P_1^* P_1)^{-1/2} \right\|_\infty \leq 1 \tag{5}$$

is the gap between the $\mathcal{L}_2$ graph spaces of P_1 and P_2 introduced at the end of the last chapter. This just depends on their frequency responses, whereas the standard gap, δ_g, is the gap between their $\mathcal{H}_2$ graph spaces. The winding number condition can be stated as

$$\delta_\nu(P_1, P_2) < 1 \iff \begin{cases} \det(I + P_2^* P_1)(j\omega) \neq 0 \; \forall \omega, \text{ and} \\ \operatorname{wno} \det(I + P_2^* P_1) + \eta(P_1) - \eta(P_2) = 0 \end{cases} \tag{6}$$

where $\operatorname{wno}(g)$ denotes the winding number of $g(s)$ evaluated on the standard Nyquist contour indented around any imaginary axis poles of P_1 and P_2. Compare this with the multivariable Nyquist criterion, which states that

$$[P, C] \text{ is stable } \iff \begin{cases} \det(I - PC)(j\omega) \neq 0 \forall \omega, \text{ and} \\ \operatorname{wno} \det(I - PC) + \eta(P) + \eta(C) = 0. \end{cases} \tag{7}$$

We shall see (in Theorem III.5) that (6) arises naturally as a necessary and sufficient condition for the stability of $[P_2, C]$ when $\delta_{\mathcal{L}_2}(P_1, P_2) < b_{P_1,C}$. We shall also show (in Theorem III.7) that, if $\delta_{\mathcal{L}_2}(P_1, P_2) > \beta$, then there exists a compensator satisfying $b_{P_1,C} > \beta$ which destabilizes P_2 by making $\det(I - P_2 C)(j\omega)$ singular at some frequency. It is not obvious from its definition that δ_ν actually is a metric. In particular, it will be necessary to show it satisfies the triangle inequality: *ie* that $\delta_\nu(P_1, P_2) \leq \delta_\nu(P_1, P_0) + \delta_\nu(P_0, P_2)$ for all P_0 (see Theorem III.2).

So, the metric δ_ν is the natural dual of the stability margin $b_{P,C}$, sharing with it the feature that it can be calculated directly from frequency responses, provided an encirclement condition is met. This is important, as it means that the distance between two plants can be estimated directly from frequency response measurements. Equally, the distance between a measured frequency response and a state-space realization, or between an infinite dimensional model and a finite dimensional approximation, are easily calculated. Furthermore, the metric, when less than 1, is seen to be the supremum over frequency of a pointwise distance

$$\kappa(P_1, P_2)(j\omega) := \bar{\sigma}((I + P_2 P_2^*)^{-1/2}(P_2 - P_1)(I + P_1^* P_1)^{-1/2})(j\omega),$$

the distance at other frequencies gives important additional information about the difference between two plants. In the scalar case these frequency response conditions take on a particularly sharp form when viewed on the surface of the Riemann sphere, as $\delta_\nu(P_1, P_2)$ is simply the supremum over ω of the chordal distance between the projections of the Nyquist diagrams of $P_1(j\omega)$ and $P_2(j\omega)$ onto the sphere — provided the winding number condition is satisfied. This provides an intuitive picture of uncertainty in the graph topology. In contrast, the chordal metric (introduced into control theory in [ES89] and generalized to the multivariable case, as the pointwise gap metric, in [QD92b]), is the supremum over all $s : \Re(s) > 0$ of the chordal distance between $P_1(s)$ and $P_2(s)$. In fact,

$$\delta_p(P_0, P_1) := \sup_{s:\Re(s)>0} \bar{\sigma}((I + P_2(s)P_2(s)^*)^{-1/2}(P_2(s) - P_1(s))(I + P_1(s)^*P_1(s))^{-1/2}) \tag{8}$$

This supremum is difficult to find, is generally attained in the interior of the *rhp* and is generally greater than the supremum on the imaginary axis — which would be the $\mathcal{L}_2$ gap. (Note that $P(s)^*$ and $P^*(s)$ are only the same when $s = j\omega$)

1. Definition, and properties, of the metric

Recall that the formula for the gap metric derived in [Geo88] is

$$\delta_g(P_1, P_2) = \max\left\{\vec{\delta}_g(P_1, P_2), \vec{\delta}_g(P_2, P_1)\right\},$$

where

$$\vec{\delta}_g(P_1, P_2) = \inf_{Q\in\mathcal{H}_\infty} \|G_1 - G_2Q\|_\infty = \inf_{Q\in\mathcal{H}_\infty} \left\| \begin{bmatrix} G_2^*G_1 - Q \\ \tilde{G}_2G_1 \end{bmatrix} \right\|_\infty, \tag{9}$$

(pre-multiplying by $\begin{bmatrix} G_2^* \\ \tilde{G}_2 \end{bmatrix}$, which is unitary, and hence norm preserving).

The function $\delta_\nu(\cdot, \cdot)$ is conveniently defined as follows

Definition III.1. Define a function $\delta_\nu(\cdot, \cdot) : \mathcal{P}^{p\times q} \times \mathcal{P}^{p\times q} \to \mathbb{R}$ as follows.

$$\delta_\nu(P_1, P_2) := \begin{cases} \|\tilde{G}_2G_1\|_\infty & \text{if } \det(G_2^*G_1)(j\omega) \neq 0 \forall \omega \in (-\infty, \infty) \\ & \text{and } \operatorname{wno}\det(G_2^*G_1) = 0, \\ 1 & \text{otherwise,} \end{cases} \tag{10}$$

where $\operatorname{wno}(g)$ denotes the winding number about the origin of $g(s)$, as s follows the standard Nyquist D-contour.

It is easy to see that $\|\tilde{G}_2G_1\|_\infty$ is equal to the formula (5) for $\delta_{\mathcal{L}_2}(P_1, P_2)$ given in the introduction, as

$$\tilde{G}_2G_1 = \tilde{M}_2(P_2 - P_1)M_1, \text{ and } \tilde{M}_2^*\tilde{M}_2 = (I + P_2P_2^*)^{-1}, \ M_1M_1^* = (I + P_1^*P_1)^{-1}.$$

The equivalence of the winding number condition to (6) follows from the fact that

$$G_2^*G_1 = \tilde{M}_2^*(I + P_2^*P_1)M_1$$

and $\operatorname{wno}\det(M_i) = \operatorname{wno}\det(\tilde{M}_i) = \eta(P_i)$(see [Vid85] p275). The definition given here is preferable to the combination of (5) and (6), as it generalizes more naturally to the infinite dimensional case (as we shall see in Chapter 1) and circumvents the need to use indented contours. In fact, all the functions for which we shall need to evaluate winding numbers will be both rational and invertible in $\mathcal{L}_\infty$, having no poles or zeros on the imaginary axis, including the point at infinity. In this case, $\operatorname{wno}(g)$ is simply $\frac{1}{2\pi}\times$ the net increase in the argument of $g(j\omega)$

as ω decreases from $+\infty$ to $-\infty$, choosing a continuous branch of the argument. By the principle of the argument this number may be calculated as the excess of the number of open *rhp* zeros of g over the number of open *rhp* poles of g. That is,

$$\text{for } g, g^{-1} \in \mathcal{RL}_\infty, \text{ then } \quad \operatorname{wno}(g) = \eta(g^{-1}) - \eta(g).$$

Some properties of wno, which we shall use repeatedly, are that

$$\operatorname{wno}(gh) = \operatorname{wno}(g) + \operatorname{wno}(h), \quad \operatorname{wno}(g^*) = -\operatorname{wno}(g),$$

and, for A, A^{-1}, $B \in \mathcal{RL}_\infty$, $\operatorname{wno}\det(A+B) = \operatorname{wno}\det(A)$ if $\underline{\sigma}(A)(j\omega) > \bar{\sigma}(B)(j\omega)$ for all ω. One might imagine numerical difficulties in determining the required winding number were $\det(G_2^* G_1)$ to become small at any frequency. Note, though, that

$$G_1^* G_2 G_2^* G_1 + G_1^* \tilde{G}_2^* \tilde{G}_2 G_1 = G_1^* G_1 = I,$$

(as $\begin{bmatrix} G_2 & \tilde{G}_2^* \end{bmatrix}$ is unitary, which implies $G_2 G_2^* + \tilde{G}_2^* \tilde{G}_2 = I$), so

$$\underline{\sigma}^2(G_2^* G_1)(j\omega) + \bar{\sigma}^2(\tilde{G}_2 G_1)(j\omega) = 1. \tag{11}$$

(we shall use this, and similar identities, repeatedly in the sequel). Thus, if $\det(G_2^* G_1)$ were near zero at some frequency the metric would be near 1 anyway. This also shows that the metric is defined in a continuous manner around 1, as if $\det(G_2^* G_1)$ were equal to zero at some frequency then we would have $\delta_\nu(P_1, P_2) = 1$ by both parts of the definition.

With these definitions, the claim is that $\delta_\nu(\cdot,\cdot)$ is a metric. Before proceeding to prove this it may be helpful to draw the following analogy with $\mathcal{H}_\infty$ additive uncertainty. These results are developed from first principles, and in a more general setting, in Chapter VII. It is easy to prove, using small gain arguments, that C stabilizes all P_2 of the form

$$\{P_2 : P_2 = P_1 + \Delta,\ \Delta \in \mathcal{H}_\infty,\ \|\Delta\|_\infty \le \beta\}$$

if, and only if, C stabilizes P_1 and

$$\left\| C(I - P_1 C)^{-1} \right\|_\infty < 1/\beta. \tag{12}$$

However, it is now well known [Glo86], [Vid85], [VK86]) that the *largest* class of linear, time-invariant plants that can be guaranteed to be stabilized *a priori* by a compensator known only to satisfy (12) is the set

$$\{P_2 : \|P_2 - P_1\|_\infty \le \beta,\ \eta(P_2) = \eta(P_1)\}.$$

The two sets are only equal if P_1 is stable. It is a little less easy to prove, but nonetheless analogous, that C stabilizes all P_2 of the form

$$\left\{ P_2 : P_2 = (N_1 + \Delta_N)(M_1 + \Delta_M)^{-1},\ \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \in \mathcal{H}_\infty,\ \left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \right\|_\infty \le \beta \right\}, \tag{13}$$

where G_1 is a normalized right graph symbol for P_1 if, and only if, C stabilizes P_1 and

$$\left\| \begin{bmatrix} P_1 \\ I \end{bmatrix} (I - CP_1)^{-1} [-C \;\; I] \right\|_\infty < 1/\beta. \tag{14}$$

Note that all P_2 of the form (13) are those satisfying

$$\inf_{Q \in \mathcal{H}_\infty} \|G_1 - G_2 Q\|_\infty \le \beta,$$

or, equivalently, $\vec{\delta}_g(P_1, P_2) \le \beta$ (as first shown in [GS90]). The claim here is that the *largest* class of plants that can be guaranteed to be stabilized *a priori* by a compensator satisfying (14) consists of those P_2 satisfying

$$\inf_{Q \in \mathcal{L}_\infty} \|G_1 - G_2 Q\|_{\mathcal{L}_\infty} \le \beta, \; \operatorname{wno} \det(G_2^* G_1) = 0.$$

This is precisely the set

$$\{P_2 : \delta_\nu(P_1, P_2) \le \beta\}.$$

We shall now prove that the function $\delta_\nu(\cdot, \cdot)$ defined in (10) is indeed a metric on the space of rational transfer function matrices.

LEMMA III.1.

a) $0 \le \delta_\nu(P_1, P_2) \le 1$
b) $\delta_\nu(P_1, P_2) = \delta_\nu(P_2, P_1)$
c) $\delta_\nu(P_1, P_2) = 0$ *if, and only if,* $P_1 = P_2$

PROOF: a) follows immediately from $(\tilde{G}_2 G_1)^* \tilde{G}_2 G_1 \le I$. Now, $\operatorname{wno} \det(G_2^* G_1) = -\operatorname{wno} \det(G_1^* G_2)$, so $\delta_\nu(P_1, P_2) < 1 \iff \delta_\nu(P_2, P_1) < 1$. To complete the proof of b) note that

$$\bar{\sigma}^2(\tilde{G}_2 G_1)(j\omega) = 1 - \underline{\sigma}^2(G_2^* G_1)(j\omega) = \bar{\sigma}^2(\tilde{G}_1 G_2)(j\omega) \; \forall \omega \tag{15}$$

(applying the identity (11) twice). Property c) is immediate. □

For $\delta_\nu(\cdot, \cdot)$ to be a metric it must satisfy the triangle inequality in addition to the relevant parts of properties a), b) and c) above. This we now show.

THEOREM III.2. $\delta_\nu(\cdot, \cdot)$ *is a metric on the space* $\mathcal{P}^{p \times q}$.

PROOF: In the light of Lemma III.1 it is only necessary to prove the triangle inequality:

$$\delta_\nu(P_1, P_2) \le \delta_\nu(P_1, P_0) + \delta_\nu(P_0, P_2).$$

This is trivially satisfied if either of the two distances on the right is 1, so we need only consider the following two cases:

(a) $\delta_\nu(P_1, P_2), \delta_\nu(P_1, P_0), \delta_\nu(P_0, P_2) < 1$

Notice that

$$G_1^* G_2 = G_1^* G_0 G_0^* G_2 + G_1^* \tilde{G}_0^* \tilde{G}_0 G_2 \tag{16}$$

as $\begin{bmatrix} G_0 & \tilde{G}_0^* \end{bmatrix}$ is unitary, so

$$\begin{aligned}\underline{\sigma}(G_1^*G_2)(j\omega) \geq\ & \underline{\sigma}(G_1^*G_0)(j\omega)\underline{\sigma}(G_0^*G_2)(j\omega) \\ & - \bar{\sigma}(\tilde{G}_0G_1)(j\omega)\bar{\sigma}(\tilde{G}_0G_2)(j\omega) \quad \forall\omega\end{aligned}$$

(as $|\underline{\sigma}(A+E) - \underline{\sigma}(A)| \leq \bar{\sigma}(E)$, for any $A, E \in \mathbb{C}^{n\times n}$)

$$= \cos\left(\arcsin \bar{\sigma}(\tilde{G}_0G_1)(j\omega) + \arcsin \bar{\sigma}(\tilde{G}_0G_2)(j\omega)\right)$$

(as (15) implies that $\underline{\sigma}(G_i^*G_j) = \cos\arcsin\bar{\sigma}(\tilde{G}_iG_j)$).
Thus,

$$\arcsin \bar{\sigma}(\tilde{G}_1G_2)(j\omega) \leq \arcsin \bar{\sigma}(\tilde{G}_0G_1)(j\omega) + \arcsin \bar{\sigma}(\tilde{G}_0G_2)(j\omega) \quad \forall\omega$$

(as cos is monotonically decreasing in the range $[0, \pi]$),
so

$$\arcsin \delta_\nu(P_1, P_2) \leq \arcsin \delta_\nu(P_1, P_0) + \arcsin \delta_\nu(P_0, P_2) \tag{17}$$

from which it follows that

$$\delta_\nu(P_1, P_2) \leq \delta_\nu(P_1, P_0) + \delta_\nu(P_0, P_2), \tag{18}$$

(take cos of both sides).

(b) $\delta_\nu(P_1, P_2) = 1,\ \delta_\nu(P_1, P_0), \delta_\nu(P_0, P_2) < 1$

Notice from (16) that if $\bar{\sigma}(G_1^*\tilde{G}_0^*\tilde{G}_0G_2)(j\omega) < \underline{\sigma}(G_1^*G_0G_0^*G_2)(j\omega)\forall\omega$ then we would have

$$\operatorname{wno}\det(G_1^*G_2) = \operatorname{wno}\det(G_1^*G_0) + \operatorname{wno}\det(G_0^*G_2) = 0$$

which is a contradiction. Hence, there must be some frequency ω_0 at which

$$\begin{aligned} & \bar{\sigma}(G_1^*\tilde{G}_0^*\tilde{G}_0G_2)(j\omega_0) \geq \underline{\sigma}(G_1^*G_0G_0^*G_2)(j\omega_0) \\ \Longrightarrow\quad & \frac{\bar{\sigma}(\tilde{G}_0G_1)(j\omega_0)}{\sqrt{1-\bar{\sigma}^2(\tilde{G}_0G_1)(j\omega_0)}}\frac{\bar{\sigma}(\tilde{G}_0G_2)(j\omega_0)}{\sqrt{1-\bar{\sigma}^2(\tilde{G}_0G_2)(j\omega_0)}} \geq 1\end{aligned}$$

(using (15) again)

$$\Longrightarrow \quad \delta_\nu^2(P_0, P_1) + \delta_\nu^2(P_0, P_2) \geq 1 \tag{19}$$

$$\Longrightarrow \quad \delta_\nu(P_0, P_1) + \delta_\nu(P_0, P_2) \geq 1, \tag{20}$$

which completes the proof. □

Remark III.3. The proof of the preceding theorem reveals that there is some slack in this metric, manifested in the steps from (17) to (18) and from (19) to (20). In fact, (19) is equivalent to $\arcsin \delta_\nu(P_0, P_1) + \arcsin \delta_\nu(P_0, P_2) \geq \pi/2$. So, if we now define an angle between plants as

$$\theta(P_1, P_2) := \arcsin \delta_\nu(P_1, P_2), \quad \theta(P_1, P_2) \in [0, \pi/2],$$

then

$$\theta(P_1, P_2) \leq \theta(P_1, P_0) + \theta(P_2, P_0)$$

and $\theta(\cdot, \cdot)$ is also a metric.

2. Robust stability theorems

Having shown that $\delta_\nu(\cdot, \cdot)$ is a metric, it is now necessary to show that it has the claimed robust stability properties. Our approach is, in essence, dual to that of the previous section. The first proposition is standard (see [Vid85] for b) and [GS90] for c)), we include a proof for completeness.

Proposition III.4.

a) *if* $[P_0, C]$ *is stable, then* $b_{P_0,C} = \inf_\omega \underline{\sigma}(\tilde{K}G_0)(j\omega)$

b) $[P_0, C]$ *is stable if, and only if,* $\det(\tilde{K}G_0)(j\omega) \neq 0 \forall\omega$ *and* $\mathrm{wno}\det(\tilde{K}G_0) = 0$.

c) $b_{P_0,C} = b_{C,P_0}$

Proof: Note that

$$\begin{bmatrix} P_0 \\ I \end{bmatrix} (I - CP_0)^{-1} \begin{bmatrix} -C & I \end{bmatrix} = G_0(\tilde{K}G_0)^{-1}\tilde{K}.$$

a) is then immediate. This also shows that $[P_0, C]$ is stable if $(\tilde{K}G_0)^{-1} \in \mathcal{H}_\infty$ and the converse follows from noting that G_0 has a left inverse (and $\tilde{K}$ a right inverse) in $\mathcal{H}_\infty$. This establishes b), and c) follows from the identities

$$\tilde{K}\tilde{G}_0^*\tilde{G}_0\tilde{K}^* + \tilde{K}G_0G_0^*\tilde{K}^* = \tilde{K}\tilde{K}^* = I,\ \tilde{G}_0\tilde{K}^*\tilde{K}\tilde{G}_0^* + \tilde{G}_0KK^*\tilde{G}_0^* = \tilde{G}_0\tilde{G}_0^* = I.$$

So

$$\underline{\sigma}^2(\tilde{K}G_0)(j\omega) = 1 - \bar{\sigma}^2(\tilde{K}\tilde{G}_0^*)(j\omega) = \underline{\sigma}^2(\tilde{G}_0K)(j\omega). \tag{21}$$

□

The proof of the following theorem is, in essence, identical to the proof that δ_ν obeys the triangle inequality.

Theorem III.5. *For any* P_0, P_1 *and* C*,*

$$\arcsin b_{P_1,C} \geq \arcsin b_{P_0,C} - \arcsin \delta_\nu(P_0, P_1).$$

Proof: This is trivially true if $\delta_\nu(P_0, P_1) \geq b_{P_0,C}$; so we can assume $\delta_\nu(P_0, P_1) < b_{P_0,C}$ which implies that $\delta_\nu(P_0, P_1) < 1$ and that $[P_0, C]$ is stable. Note that

$$\tilde{K}G_1 = \tilde{K}G_0G_0^*G_1 + \tilde{K}\tilde{G}_0^*\tilde{G}_0G_1. \tag{22}$$

Now, $\delta_\nu(P_0, P_1) < b_{P_0,C}$

$$\implies \qquad \bar{\sigma}(\tilde{G}_0G_1)(j\omega) < \underline{\sigma}(\tilde{K}G_0)(j\omega) \quad \forall\omega \tag{22}$$

(by Proposition III.4(a))

$$\iff \qquad \frac{\bar{\sigma}(\tilde{G}_0 G_1)(j\omega)}{\underline{\sigma}(G_0^* G_1)(j\omega)} < \frac{\underline{\sigma}(\tilde{K}G_0)(j\omega)}{\bar{\sigma}(\tilde{K}\tilde{G}_0^*)(j\omega)} \quad \forall\omega \tag{23}$$

(using (15) and (21))

$$\implies \qquad \bar{\sigma}(\tilde{K}\tilde{G}_0^*\tilde{G}_0 G_1)(j\omega) < \underline{\sigma}(\tilde{K}G_0 G_0^* G_1)(j\omega) \quad \forall\omega. \tag{24}$$

Hence,

$$\operatorname{wno}\det(\tilde{K}G_1) = \operatorname{wno}\det(\tilde{K}G_0) + \operatorname{wno}\det(G_0^* G_1) = 0.$$

So, $[P_1, C]$ is stable and $b_{P_1,C} = \inf_\omega \underline{\sigma}(\tilde{K}G_1)$, which can be bounded from (22);

$$\begin{aligned}\underline{\sigma}(\tilde{K}G_1)(j\omega) &\geq \underline{\sigma}(\tilde{K}G_0)(j\omega)\underline{\sigma}(G_0^* G_1)(j\omega) - \bar{\sigma}(\tilde{K}\tilde{G}_0^*)(j\omega)\bar{\sigma}(\tilde{G}_0 G_1)(j\omega)\ \forall\omega \\ &= \sin\left(\arcsin\underline{\sigma}(\tilde{K}G_0)(j\omega) - \arcsin\bar{\sigma}(\tilde{G}_0 G_1)(j\omega)\right) \end{aligned} \tag{26}$$

and consequently

$$\arcsin b_{P_1,C} \geq \arcsin b_{P_0,C} - \arcsin \delta_\nu(P_0, P),$$

as $\sin(\cdot)$ is monotonically increasing in the range $[-\pi/2, \pi/2]$. This completes the proof $\square$

COROLLARY III.6. *Given a nominal plant $P_0 \in \mathcal{P}^{p\times q}$, and a nominal compensator $C_0 \in \mathcal{P}^{q\times p}$ such that $[P_0, C_0]$ is stable, then*

$$\arcsin b_{P,C} \geq \arcsin b_{P_0,C_0} - \arcsin\delta_\nu(P_0, P) - \arcsin\delta_\nu(C_0, C)$$

for all $P \in \mathcal{P}^{p\times q}, C \in \mathcal{P}^{q\times p}$.

PROOF: Theorem III.5 states that,

$$\arcsin b_{P,C_0} \geq \arcsin b_{P_0,C_0} - \arcsin\delta_\nu(P_0, P).$$

Clearly it is also true that

$$\arcsin b_{C,P} \geq \arcsin b_{C_0,P} - \arcsin\delta_\nu(C_0, C),$$

which is equivalent to

$$\arcsin b_{P,C} \geq \arcsin b_{P,C_0} - \arcsin\delta_\nu(C_0, C),$$

by Proposition III.4. $\square$

The preceding theorems give sufficient conditions for robust stability. The following theorem shows that the bound given in Theorem III.5 is the best possible if either P_0 and C are given and P_1 is known only to satisfy $\delta_\nu(P_0, P_1) \leq \alpha$ for some α, or P_0 and P_1 are given and C is known only to satisfy $b_{P_0,C} > \alpha$ for some α.

THEOREM III.7.

i) *Given* P_0, C *and numbers* $\beta \leq \alpha < b_{opt}(P_0)$ *then:*
$\arcsin b_{P_1,C} > \arcsin \alpha - \arcsin \beta$ *for all* P_1 *satisfying* $\delta_\nu(P_0, P_1) \leq \beta$ *if, and only if,* $b_{P_0,C} > \alpha$.

ii) *Given* P_0, P_1 *and numbers* $\beta \leq \alpha < b_{opt}(P_0)$ *then:*
$\arcsin b_{P_1,C} > \arcsin \alpha - \arcsin \beta$ *for all* C *satisfying* $b_{P_0,C} > \alpha$ *if, and only if,* $\delta_\nu(P_0, P_1) \leq \beta$.

Remark III.8. This theorem includes the claim of the introduction, when $\alpha = \beta$. Note that part i) of the theorem also holds for the gap and the pointwise gap metrics [QD92a], though part ii) does not.

The following lemma will aid us in our constructions.

Lemma III.9.

i) *If* $\delta_\nu(P_0, P_1) < 1$, *then*

$$\|\tilde{G}_0 G_1 (G_0^* G_1)^{-1}\|_\infty = \frac{\delta_\nu(P_0, P_1)}{\sqrt{1 - \delta_\nu^2(P_0, P_1)}}.$$

Furthermore, given any $\omega_0 \in \mathbb{R} \cup \infty$, $P_0 \in \mathcal{P}^{p \times q}$, $T \in \mathbb{C}^{p \times q}$ *satisfying* $\bar{\sigma}(T) \leq \frac{b_{opt}(P_0)}{\sqrt{1 - b_{opt}^2(P_0)}}$, *there exists a* P_1 *satisfying:*

$$(\tilde{G}_0 G_1)(G_0^* G_1)^{-1}(j\omega_0) = T, \quad \frac{\delta_\nu(P_0, P_1)}{\sqrt{1 - \delta_\nu^2(P_0, P_1)}} = \bar{\sigma}(T).$$

ii) *If* $[P_0, C]$ *is stable, then*

$$\|(\tilde{K} G_0)^{-1} \tilde{K} \tilde{G}_0^*\|_\infty = \frac{\sqrt{1 - b_{P_0,C}^2}}{b_{P_0,C}}.$$

Furthermore, given any $\omega_0 \in \mathbb{R} \cup \infty$, $P_0 \in \mathcal{P}^{p \times q}$, $T \in \mathbb{C}^{q \times p}$ *and* $\gamma > \max\left\{\bar{\sigma}(T), \frac{\sqrt{1 - b_{opt}^2(P_0)}}{b_{opt}^2(P_0)}\right\}$, *there exists a* C *satisfying:*

$$(\tilde{K} G_0)^{-1}(\tilde{K} \tilde{G}_0^*)(j\omega_0) = T, \quad \frac{\sqrt{1 - b_{P_0,C}^2}}{b_{P_0,C}} < \gamma.$$

Proof: The initial statement of part i) follows from the fact that

$$\left(\begin{bmatrix} \tilde{G}_0 \\ G_0^* \end{bmatrix} G_1\right)^* \left(\begin{bmatrix} \tilde{G}_0 \\ G_0^* \end{bmatrix} G_1\right) = I,$$

and similar reasoning establishes the initial statement of ii). We shall now demonstrate the existence of a suitable P_1 and C by construction.

For part i), put $X = (G_0 + \tilde{G}_0^* \Omega \Delta)$, where Ω is an unitary function satisfying $\tilde{G}_0^* \Omega \in \mathcal{H}_\infty$ and $\Delta \in \mathcal{RH}_\infty$ satisfies

$$\Delta(j\omega_0) = \Omega^*(j\omega_0) T, \ \bar{\sigma}(\Delta(s)) < \bar{\sigma}(T) \ \forall s : \Re(s) > 0.$$

We can always construct such a Δ. We shall now show that X is left invertible (*ie* a legitimate graph symbol). Let C satisfy $b_{P_0,C} = b_{opt}(P_0)$, then

$$\underline{\sigma}(X) \geq \underline{\sigma}(\tilde{K}X)/\bar{\sigma}(\tilde{K}) \geq \underline{\sigma}(\tilde{K}X) \; \forall s : \Re(s) > 0$$

(as $\tilde{K}$ is co-inner)

$$\begin{aligned} &\geq \underline{\sigma}(\tilde{K}G_0)\underline{\sigma}(I - (\tilde{K}G_0)^{-1}(\tilde{K}\tilde{G}_0^*)\Omega\Delta) \\ &> b_{opt}(P_0)\left(1 - \frac{\sqrt{1 - b_{opt}^2(P_0)}}{b_{opt}(P_0)}\bar{\sigma}(T)\right)/\|Q\|_\infty \geq 0 \forall s : \Re(s) > 0. \end{aligned}$$

Furthermore,

$$\underline{\sigma}(X) \geq \underline{\sigma}(G_0^* X)/\bar{\sigma}(G_0^*) = 1 \; \forall s = j\omega.$$

Let P_1 be the plant with graph symbol X in which case,

$$\|\tilde{G}_0 G_1 (G_0^* G_1)^{-1}\|_\infty = \|\tilde{G}_0 X (G_0^* X)^{-1}\|_\infty = \|\Omega\Delta\|_\infty$$

and

$$\operatorname{wno}\det(G_0^* G_1) = \operatorname{wno}\det(G_0^* X) = \operatorname{wno}\det(I) = 0$$

as required.

For part ii), substitute the Youla parameterization $\tilde{K} =$ inner factor of $(\tilde{X}_0 + \Delta\tilde{G}_0)$, where $\Delta \in \mathcal{H}_\infty$ and $\tilde{X}_0 \in \mathcal{H}_\infty$ is such that $\tilde{X}_0 G_0 = I$. Then, $(\tilde{K}G_0)^{-1}\tilde{K}\tilde{G}_0^* = \tilde{X}_0\tilde{G}_0^* + \Delta$. Note that

$$\|\mathbf{H}_{\tilde{G}_0 \tilde{X}_0^*}\| = \inf_{\Delta \in \mathcal{H}_\infty} \|\tilde{X}_0\tilde{G}_0^* + \Delta\|_\infty = \sqrt{1 - b_{opt}(P_0)^2}/b_{opt}(P_0),$$

(because as Δ ranges over $\mathcal{H}_\infty$, $\tilde{K}$ ranges over the graph symbols of all stabilizing compensators). The Hankel operator $\mathbf{H}_\phi$ is defined as $\mathbf{H}_\phi : \mathcal{H}_2^\perp \to \mathcal{H}_2, \; u \mapsto \mathbf{\Pi}_+\phi u$.

So, in order to construct the required C, it suffices to find a $\Delta \in \mathcal{H}_\infty$ satisfying $\|\tilde{X}_0\tilde{G}_0^* + \Delta\|_\infty < \gamma$ with the additional boundary constraint that $(\tilde{X}_0\tilde{G}_0^* + \Delta)(j\omega_0) = T$. A standard result from interpolation theory is that this is always possible, as $\bar{\sigma}(T) < \gamma$ and $\gamma > \|\mathbf{H}_{\tilde{X}_0\tilde{G}_0^*}\|$ by assumption. (In fact, given $R \in \mathcal{RH}_\infty$, Glover [Glo89](Theorem 3.2) parameterizes all solutions $\Delta \in \mathcal{RH}_\infty$ to $\|R^* - \Delta\|_\infty < \sigma$ for $\sigma > \|\mathbf{H}_R\|$ in the form $\Delta - R^* = \mathcal{F}(J, \Phi) : \Phi \in \mathcal{RH}_\infty, \|\Phi\|_\infty < 1$ where, frequency by frequency, $Y = \mathcal{F}(J(j\omega), X)$ defines an invertible mapping between the sets of all constant matrices $\{Y \in \mathbb{C}^{p\times q} : \bar{\sigma}(Y) < \sigma\}$ and $\{X \in \mathbb{C}^{p\times q} : \bar{\sigma}(X) < 1\}$. The problem is thus reduced to the simple one of finding a stable contraction that interpolates a single point on the boundary.) This completes the proof. □

Proof: of Theorem III.7 Sufficiency for both parts of the theorem is equivalent to Theorem III.5. Necessity for each part requires a slightly different construction. The common part of the proofs is given first.

Suppose that, at some frequency ω_0,

$$\tilde{G}_0 G_1 (G_0^* G_1)^{-1}(j\omega_0) = U \operatorname{diag}\left\{ \frac{\hat{\beta}}{\sqrt{1-\hat{\beta}^2}}, \dots \right\} V^* \tag{27}$$

and

$$(\tilde{K}G_0)^{-1}\tilde{K}\tilde{G}_0^*(j\omega_0) = -V \operatorname{diag}\left\{ \frac{\sqrt{1-\hat{\alpha}^2}}{\hat{\alpha}}, \dots \right\} U^* \tag{28}$$

are singular value decompositions, with the same U and V in each. (In the sequel, either P_0 and P_1 will have been given, fixing U and V, and C constructed to satisfy (28), for some $\hat{\alpha}$ and the same U and V; or P_0 and C will have been given, and P_1 constructed to satisfy (27) for some $\hat{\beta}$). Since $\left(\left[\begin{smallmatrix}\tilde{G}_0\\ G_0^*\end{smallmatrix}\right] G_1\right)^* \left(\left[\begin{smallmatrix}\tilde{G}_0\\ G_0^*\end{smallmatrix}\right] G_1\right) = I$, it would follow that

$$\begin{aligned} \tilde{G}_0 G_1(j\omega_0) &= U \ \operatorname{diag}\left\{\hat{\beta}, \dots\right\} W^* \\ G_0^* G_1(j\omega_0) &= V \ \operatorname{diag}\left\{\sqrt{1-\hat{\beta}^2}, \dots\right\} W^* \end{aligned}$$

for some unitary W. Similarly,

$$\begin{aligned} \tilde{K}\tilde{G}_0^*(j\omega_0) &= -X \ \operatorname{diag}\left\{\sqrt{1-\hat{\alpha}^2}, \dots\right\} U^* \\ \tilde{K}G_0(j\omega_0) &= X \ \operatorname{diag}\left\{\hat{\alpha}, \dots\right\} V^* \end{aligned}$$

for some unitary X. Substituting these expressions into (22) leads to

$$\tilde{K}G_1(j\omega_0) = X \ \operatorname{diag}\left\{\hat{\alpha}\sqrt{1-\hat{\beta}^2} - \hat{\beta}\sqrt{1-\hat{\alpha}^2}, \dots\right\} W^*,$$

from which it would follow that

$$b_{P_1,C} \le \left|\hat{\alpha}\sqrt{1-\hat{\beta}^2} - \hat{\beta}\sqrt{1-\hat{\alpha}^2}\right| = |\sin(\arcsin\hat{\alpha} - \arcsin\hat{\beta})|. \tag{29}$$

Part i) Necessity:

Given P_0, C and α satisfying $b_{P_0,C} \le \alpha < b_{opt}(P_0)$, we need to be able to construct a P_1 satisfying

$$\delta_\nu(P_0, P_1) \le \beta \tag{30}$$

$$b_{P_1,C} \le \sin(\arcsin\alpha - \arcsin\beta) \tag{31}$$

for any $\beta \le \alpha$.

Denote by ω_0 a frequency, which may be 0 or ∞, at which the norm of $(\tilde{K}G_0)^{-1}\tilde{K}\tilde{G}_0^*$ is achieved, and let (28) be a singular value decomposition. We can assume that $[P_0, C]$ is stable, as otherwise the construction $P_1 = P_0$ will do. So,

$$b_{P_0,C} = \inf_\omega \underline{\sigma}(\tilde{K}G_0)(j\omega) = \hat{\alpha} \le \alpha, \tag{32}$$

by assumption. We need now consider two possibilities:

a) $\hat{\alpha} \leq \beta$
In this case put $\hat{\beta} = \hat{\alpha}$, then we can always construct a P_1 satisfying $\delta_\nu(P_0, P_1) = \hat{\beta}$ and (27) for this $\hat{\beta}$ by Lemma III.9, as $\hat{\alpha} = b_{P_0,C} < b_{opt}(P_0)$ by assumption. So, $\delta_\nu(P_0, P_1) \leq \beta$ and $[P_1, C]$ is unstable by (29).

b) $\hat{\alpha} > \beta$ In this case put $\hat{\beta} = \beta$. Again, we can always construct a P_1 satisfying $\delta_\nu(P_0, P_1) = \hat{\beta}$ and (27) for this $\hat{\beta}$ by Lemma III.9, as $\beta < b_{opt}(P_0)$ by assumption. So, $\arcsin b_{P_1,C} \leq \arcsin b_{P_0,C} - \arcsin\beta \leq \arcsin\alpha - \arcsin\beta$ from (29) and (32).

Part ii) Necessity:

Given P_0, P_1 and β satisfying $\delta_\nu(P_0, P_1) > \beta$ and $\beta < b_{opt}(P_0)$, we need to be able to construct a C satisfying

$$b_{P_0,C} > \alpha \tag{33}$$
$$b_{P_1,C} \leq \sin(\arcsin\alpha - \arcsin\beta). \tag{34}$$

for any $\alpha :\ \beta \leq \alpha < b_{opt}(P_0)$.

Denote by ω_0 a frequency, which may be 0 or ∞, at which the norm of $\tilde{G}_0 G_1 (G_0^* G_1)^{-1}$ is achieved, and let (27) be a singular value decomposition. So $\delta_{\mathcal{L}_2}(P_0, P_1) = \hat{\beta}$. If $\hat{\beta} \leq \beta$, then it must follow that $\delta_\nu(P_0, P_1) = 1$, as $\delta_\nu(P_0, P_1) > \beta$ by assumption. In this case, any C satisfying (33) will actually destabilize P_1, as

$$b_{P_0,C} \geq \bar{\sigma}(\tilde{G}_0 G_1)(j\omega)\forall\omega \implies \quad \bar{\sigma}(\tilde{K}\tilde{G}_0^*\tilde{G}_0 G_1)(j\omega) \leq \underline{\sigma}(\tilde{K} G_0 G_0^* G_1)(j\omega) \quad \forall\omega$$

as in the proof of Theorem III.5. So, (22) implies that, either $\det(\tilde{K}G_1)(j\omega) = 0$ for some ω or

$$\operatorname{wno}\det(\tilde{K}G_1) = \operatorname{wno}\det(\tilde{K}G_0) + \operatorname{wno}\det(G_0^* G_1) \neq 0.$$

So, we can assume that $\hat{\beta} > \beta$. We need now consider two possibilities:

a) $\hat{\beta} > \alpha$.
In this case put $\hat{\alpha} = \hat{\beta}$, then we can always construct a C satisfying $b_{P_0,C} > \alpha$ and (28) for this $\hat{\alpha}$ by Lemma III.9, as $\hat{\alpha} = \hat{\beta} > \alpha$ and $\alpha < b_{opt}(P_0)$ by assumption. So, $[P_1, C]$ is unstable by (29).

b) $\hat{\beta} \leq \alpha$. In this case we can always construct a C satisfying $b_{P_0,C} > \alpha$ and (28) for any $\hat{\alpha} > \alpha$ by Lemma III.9, as $\alpha < b_{opt}(P_0)$ by assumption. So, there exists a C satisfying (33) and $\arcsin b_{P_1,C} \leq \arcsin\hat{\alpha} - \arcsin\hat{\beta}$ for any $\hat{\alpha} > \alpha$ by (29). Since $\hat{\beta} > \beta$ by assumption, we can always choose $\hat{\alpha}$ sufficiently close to α to obtain (34).

This completes the proof. □

Remark III.10. Theorem III.7 still holds both ways if all the systems are required to be real rational, as if the given systems are real rational, then the constructions can also be made real rational (by choosing the functions Δ in the proof of Lemma III.9 to be real rational).

3. Extending the metric

A common criticism of the gap metric, which also applies to δ_ν, is that there are many plants P at a distance $\delta_\nu(P_0, P) = 1$ which the metrics fail to differentiate between; although intuitively some will be "nearer" to P_0 than others. For example, consider the following three plants:

$$P_0 = .3\frac{s-1}{s^2+1} \quad P_1 = .3\frac{s-1}{s^2+2} \quad P_2 = .3\frac{(s-1)(s-2)}{(s^2+2)(s-\sqrt{2})}$$

Here, both $\delta_\nu(P_0, P_1) = 1$ and $\delta_\nu(P_0, P_2) = 1$, although intuitively P_2 is further away from P_0 than P_1 is. It is a straightforward exercise to find a compensator to stabilize both P_0 and P_1; however, there is *no* compensator that stabilizes both P_0 and P_2. The main result of this section uses the intuition gained from the frequency response interpretation of $\delta_\nu(\cdot,\cdot)$ to extend the equivalent metric $\theta(\cdot,\cdot)$ beyond $\pi/2$.

Definition III.2. Define a function $\theta_e(\cdot,\cdot)$, $\mathcal{P}^{p\times q}\times\mathcal{P}^{p\times q}\to[0,\pi]$ as follows:

$$\theta_e(P, P_0) = \begin{cases} \arcsin\|\tilde{G}_2 G_1\|_\infty & \text{if } \det(G_2^* G_1)(j\omega) \neq 0 \forall\omega\in(-\infty,\infty), \\ & \text{and } \operatorname{wno}\det(G_2^* G_1) = 0 \\ \pi - \arcsin\|\tilde{G}_2 G_1\|_\infty & \text{otherwise} \end{cases}$$

So, if $\delta_\nu(P, P_0) = 1$ (indicating that the encirclement condition is violated) then, the closer the two frequency responses are, the further away the two systems are adjudged to be. This new function is still a metric. To prove this we need the following lemma.

Lemma III.11. *Given* $A, A^{-1}, B \in \mathcal{L}_\infty \cap C_0$, *with either* $\det(A+B)(j\omega) = 0$ *for some* ω *or* $\operatorname{wno}\det(A+B) \neq \operatorname{wno}\det(A)$, *then at some frequency* ω_0

$$\underline{\sigma}(A+B)(j\omega_0) \leq \bar{\sigma}(B)(j\omega_0) - \underline{\sigma}(A)(j\omega_0).$$

Proof: Note that

$$\operatorname{wno}\det(A+B) - \operatorname{wno}\det(A) = \operatorname{wno}\det(I + A^{-1}B) = \sum_i \operatorname{wno}\lambda_i(I + A^{-1}B) \neq 0.$$

Therefore, there must be some ω_0 for which $(I+A^{-1}B)(j\omega_0)$ has an eigenvalue at zero or on the negative real axis *ie* there exist a constant vector $\hat{x} : \|\hat{x}\|_2 = 1$ such that $(I + A^{-1}B)(j\omega_0)\hat{x} = -\epsilon\hat{x}$ for some $\epsilon \geq 0$ (or $B(j\omega_0)\hat{x} = -(1+\epsilon)A(j\omega_0)\hat{x}$). Furthermore,

$$\begin{aligned}\underline{\sigma}(A+B)(j\omega_0) &\leq \|(A+B)(j\omega_0)\hat{x}\|_2 \\ &= \|B(j\omega_0)\hat{x}\|_2 - \|A(j\omega_0)\hat{x}\|_2\end{aligned}$$

(as $b(j\omega_0)\hat{x}$ and $A(j\omega_0)\hat{x}$ are aligned in opposite directions, with $\|B(j\omega_0)\hat{x}\|_2 \geq \|A(j\omega_0)\hat{x}\|_2$)

$$\leq \bar{\sigma}(B)(j\omega_0) - \underline{\sigma}(A)(j\omega_0)$$

as required. □

We can now prove the following theorem.

THEOREM III.12. *$\theta_e(\cdot,\cdot)$ is a metric on $\mathcal{P}^{p\times q}$. Furthermore, $\theta_e(P,P_0) < \pi \forall P, P_0 \in \mathcal{P}^{p\times q}$.*

PROOF: As $\theta(\cdot,\cdot)$ is itself a metric, it is immediate that we only have to prove the triangle inequality

$$\theta_e(P_1,P_2) \leq \theta_e(P_1,P_0) + \theta_e(P_0,P_2).$$

As in the proof of Theorem III.2 it is necessary to consider a number of cases.

(a) $\theta_e(P_1,P_2), \theta_e(P_1,P_0), \theta_e(P_0,P_2) < \pi/2$.

In this case $\theta_e = \theta$ for each combination, and the triangle inequality has already been proven for θ in Theorem III.2.

(b) $\theta_e(P_1,P_2) \geq \pi/2,\ \theta_e(P_1,P_0), \theta_e(P_0,P_2) < \pi/2$

In this case, as $\theta(P_1,P_2) = \pi/2$, in order to prove the triangle inequality for θ it was necessary to show that

$$\theta(P_1,P_0) + \theta(P_2,P_0) \geq \pi/2.$$

It is now necessary to prove the somewhat stronger result, that

$$\theta(P_1,P_0) + \theta(P_2,P_0) \geq \pi - \arcsin \|\tilde{G}_2 G_1\|_\infty.$$

Apply Lemma III.11 with $A = G_2^* G_0 G_0^* G_1$ and $B = G_2^* \tilde{G}_0^* \tilde{G}_0 G_1$ (and thus $A + B = G_2^* G_1$ and the assumptions of the lemma are satisfied) to show that there exists an ω_0 at which

$$\begin{aligned}
&\underline{\sigma}(G_2^* G_1)(j\omega_0) \\
&\quad \leq \bar{\sigma}(G_2^* \tilde{G}_0^* \tilde{G}_0 G_1)(j\omega_0) - \underline{\sigma}(G_2^* G_0 G_0^* G_1)(j\omega_0) \\
&\quad \leq \bar{\sigma}(\tilde{G}_0 G_2)(j\omega_0)\bar{\sigma}(\tilde{G}_0 G_1)(j\omega_0) - \underline{\sigma}(G_2^* G_0)(j\omega_0)\underline{\sigma}(G_0^* G_1)(j\omega_0) \\
&\quad = -\cos(\arcsin \bar{\sigma}(\tilde{G}_0 G_1)(j\omega_0) + \arcsin \bar{\sigma}(\tilde{G}_0 G_2)(j\omega_0)) \\
&\quad = \cos(\pi - \arcsin \bar{\sigma}(\tilde{G}_0 G_1)(j\omega_0) - \arcsin \bar{\sigma}(\tilde{G}_0 G_2)(j\omega_0))
\end{aligned}$$

thus

$$\arcsin \bar{\sigma}(\tilde{G}_1 G_2)(j\omega_0) \geq \pi - \arcsin \bar{\sigma}(\tilde{G}_0 G_1)(j\omega_0) - \arcsin \bar{\sigma}(\tilde{G}_0 G_2)(j\omega_0)$$

(as $\cos(\cdot)$ is monotonically decreasing in the range $[0,\pi)$). Hence,

$$\pi - \sup_\omega \arcsin \bar{\sigma}(\tilde{G}_1 G_2) \leq \sup_\omega \arcsin \bar{\sigma}(\tilde{G}_0 G_1) + \sup_\omega \arcsin \bar{\sigma}(\tilde{G}_0 G_2) \tag{35}$$

as required.

(c) $\theta_e(P_1,P_2), \theta_e(P_1,P_0) \geq \pi/2,\ \theta_e(P_0,P_2) < \pi/2$

Notice that

$$\sup_\omega \arcsin \bar{\sigma}(\tilde{G}_0 G_1) \leq \sup_\omega \arcsin \bar{\sigma}(\tilde{G}_0 G_2) + \sup_\omega \arcsin \bar{\sigma}(\tilde{G}_2 G_1),$$

as $\delta_{\mathcal{L}_2}(P_0,P_1) = \|\tilde{G}_0 G_1\|_\infty$ is itself a metric, so

$$\pi - \sup_\omega \arcsin \bar{\sigma}(\tilde{G}_2 G_1) \leq \pi - \sup_\omega \arcsin \bar{\sigma}(\tilde{G}_0 G_1) + \sup_\omega \arcsin \bar{\sigma}(\tilde{G}_0 G_2)$$

as required.

For all other combinations the triangle inequality either follows trivially or by symmetry, thus the proof of the first part of the theorem is complete. For the second part, note that for $\theta_e(P_1,P_2)$ to be equal to π it is necessary that $\tilde{N}_1 M_2 = \tilde{M}_1 N_2$; *ie* either $P_1 = P_2$, which implies $\theta_e(P_1,P_2) = 0$, or one of $\{\tilde{N}_1, M_1\}$, $\{N_2, M_2\}$ are not coprime, leading to a contradiction. □

If we now use this metric on the examples given above we find that $\theta_e(P_0,P_1) = \pi/2$, as $\bar{\sigma}(\tilde{G}_0 G_1)(j\omega)$ just reaches 1 on the imaginary axis, however $\theta_e(P_0,P_2)$ is much larger at 2.984, as P_0 and P_2 are quite close on the imaginary axis but there is a near *rhp* pole/zero cancellation in P_2. So, the extended metric correctly places P_2 significantly further away from P_0 than it does P_1.

4. The $\mathcal{L}_2$-gap and the graph topology

In developing the results of the last two sections we have only differentiated between those perturbed plants P_1 for which $\operatorname{wno}\det(G_0^* G_1) = 0$, and those for which $\operatorname{wno}\det(G_0^* G_1) \neq 0$. That is, we have partitioned the set $\mathcal{B}_{\mathcal{L}_2}(P_0, r) := \{P : \delta_{\mathcal{L}_2}(P,P_0) < r\}$, for $r \leq 1$ into two components — one comprising of the subset $\mathcal{B}_\nu(P_0,r) := \{P : \delta_\nu(P,P_0) < r\}$ and the other containing the rest. However, it is interesting to note how the winding number can be used to partition an $\mathcal{L}_2$ gap ball in a way that characterizes how it splits under the graph topology. Define

$$\mathcal{B}^i_{\mathcal{L}_2}(P_0,r) := \{P_1 : \delta_{\mathcal{L}_2}(P_1,P_0) < r, \operatorname{wno}\det(G_0^* G_1) = i\}.$$

The first theorem proves that the sets $\mathcal{B}^i_{\mathcal{L}_2}(P_0,r)$ and $\mathcal{B}^j_{\mathcal{L}_2}(P_0,r)$, $i \neq j$ are disconnected in the graph topology, and is a straightforward modification of Theorem III.12. We shall use P_2 for the centre of the ball to tie in with the notation of that theorem.

Theorem III.13. *Given P_2, an $r \leq 1$, and any $P_0 \in \mathcal{B}^i_{\mathcal{L}_2}(P_2,r)$, $P_1 \in \mathcal{B}^j_{\mathcal{L}_2}(P_2,r)$, $i \neq j$, then $\delta_\nu(P_0,P_1) > 0$.*

Proof: We shall actually prove that,

i) if $r \leq 1/\sqrt{2}$, then $\delta_\nu(P_0,P_1) = 1$;
ii) if $r > 1/\sqrt{2}$, then $\delta_\nu(P_0,P_1) > \sin(\pi - 2\arcsin r) \geq 0$.

Note that $\operatorname{wno}\det(G_1^* G_2) \neq \operatorname{wno}\det(G_0^* G_2)$. Furthermore, if $\delta_\nu(P_0,P_1) = 1$ then the theorem is trivially true, so we assume that $\operatorname{wno}\det(G_1^* G_0) = 0$. So

$$\begin{aligned}\arcsin \delta_\nu(P_0,P_1) &= \arcsin \delta_{\mathcal{L}_2}(P_0,P_1)\\ &\geq \pi - \arcsin \delta_{\mathcal{L}_2}(P_0,P_2) - \arcsin \delta_{\mathcal{L}_2}(P_1,P_2)\end{aligned}$$

(by the same reasoning as was used to establish (35) in Theorem III.12 Part b))

$$> \pi - 2 \arcsin r,$$

which is a contradiction to the assumption $\operatorname{wno}\det(G_1^* G_0) = 0$ if $r \leq 1/\sqrt{2}$, and the required bound otherwise. $\square$

The following theorem comes from a similar modification to Theorem III.5. We define

$$b_{\mathcal{L}_2}(P, C) := \left\| \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} [-C \;\; I] \right\|_{\mathcal{L}_\infty},$$

irrespective of the stability of $[P, C]$.

THEOREM III.14. *Given P_0, P_1, C satisfying $b_{\mathcal{L}_2}(P_0, C) > \delta_{\mathcal{L}_2}(P_0, P_1)$ then*

i) $\eta[P_1, C] = \eta[P_0, C] + \operatorname{wno}\det(G_0^* G_1)$

ii) $\arcsin b_{\mathcal{L}_2}(P_1, C) > \arcsin b_{\mathcal{L}_2}(P_0, C) - \arcsin \delta_{\mathcal{L}_2}(P_0, P)$

Part i) follows directly from Theorem III.7(22) and the subsequent arguments establishing that

$$\operatorname{wno}\det(\tilde{K} G_1) = \operatorname{wno}\det(\tilde{K} G_0) + \operatorname{wno}\det(G_0^* G_1) \neq 0.$$

These still hold for $b_{\mathcal{L}_2}(P_0, C) > \delta_{\mathcal{L}_2}(P_0, P_1)$, as do the subsequent arguments to establish (26) which translates to part ii) of the present theorem. $\square$

Thus, if $b_{P_0,C} > \delta_{\mathcal{L}_2}(P_0, P_1)$ and $[P_1, C]$ is stable, then $\delta_\nu(P_0, P_1) = \delta_{\mathcal{L}_2}(P_0, P_1)$ as noted in [GSS92].

CHAPTER IV

The New Metric and the Graph Topology

1. Operator theory, and the infinite dimensional case

The only special property of finite dimensional linear systems we have used up until now is the existence of normalized coprime factorizations that are continuous on the imaginary axis (in order that the winding number may be evaluated). Any infinite dimensional system has normalized coprime factorizations provided it is stabilizable ([Smi89]), but these may fail to be continuous [GS92]. As in [GS92], define C_0 as the class of functions continuous on $j\mathbb{R} \cup \{\infty\}$ and $A_0 := \mathcal{H}_\infty \cap C_0$; then all the theorems of the previous sections go through unchanged for the class of systems that admit normalized coprime factorizations over A_0. So, for a wide class of interesting systems — *eg* many strictly proper retarded delay systems — $\delta_\nu(\cdot,\cdot)$ is well defined and is easily evaluated in the frequency plane (remembering that it is not necessary to obtain normalized coprime factors to calculate $\delta_{\mathcal{L}_2}$). However, in order to give the results full generality we must generalize the idea of a winding number. A suitable tool for our purpose is the Index of a Fredholm operator. An operator $\mathbf{F}$ is Fredholm if it has closed range and dim Ker $\mathbf{F}$ and codim Ra $\mathbf{F}$ are finite, in which case its index is defined as follows (see [Nik80], p318):

$$\begin{aligned}\operatorname{Ind}\mathbf{F} &:= \dim\ \operatorname{Ker}\mathbf{F} - \dim\ \operatorname{Ker}\mathbf{F}^* \\ &= \dim\ \operatorname{Ker}\mathbf{F} - \operatorname{codim}\ \operatorname{Ra}\mathbf{F}\end{aligned}$$

where Ra denotes the range of an operator, and $\mathbf{F}^*$ is the operator adjoint to $\mathbf{F}$. In particular, if $\phi \in \mathcal{L}_\infty$ and $\mathbf{T}_\phi$ is Fredholm, then

$$\operatorname{Ind}\mathbf{T}_\phi = \dim\ \operatorname{Ker}\mathbf{T}_\phi - \dim\ \operatorname{Ker}\mathbf{T}_{\phi^*},$$

where $\mathbf{T}_\phi$ denotes the Toeplitz operator with symbol ϕ, *ie*:

$$\mathbf{T}_\phi : \mathcal{H}_2 \to \mathcal{H}_2 \quad f \mapsto \mathbf{\Pi}_+ \phi f,$$

and $\mathbf{\Pi}_+$ denotes projection onto $\mathcal{H}_2$. Now, for $\phi \in C_0$, $\mathbf{T}_\phi$ is Fredholm if, and only if, $\det(\phi)(j\omega) \neq 0$ for all $\omega \in \mathbb{R} \cup \infty$; in which case $\operatorname{Ind}\mathbf{T}_\phi = \operatorname{wno}\det(\phi)$, [GK60].

So the Index and the winding number can be used equivalently in the finite dimensional case. For the general case it can be shown that

$$[P_0, C] \text{ is stable } \iff \mathbf{T}_{\tilde{K}G_0} \text{ is Fredholm and } \operatorname{Ind}\mathbf{T}_{\tilde{K}G_0} = 0, \tag{36}$$

(this is noted in [Sef91] for rational systems), which suggests we define

$$\delta_\nu(P_1, P_2) < 1 \iff \mathbf{T}_{G_1^* G_2} \text{ is Fredholm and } \operatorname{Ind}\mathbf{T}_{G_1^* G_2} = 0.$$

It is unknown whether this gives a metric with the correct properties on the class of all stabilizable, linear time-invariant systems.

The definition of the gap metric comes from the concept of angles between closed manifolds of the Hilbert space $\mathcal{H}_2$. Recently, a number of works have appeared that use the gap metric to examine feedback stability from a geometric perspective [FGS90] [OS91] [QD92a] [Sef91]. We shall now examine δ_ν from a similar point of view, giving a geometric interpretation of $\operatorname{Ind}\mathbf{T}_{G_1^* G_2}$.

As we showed in Chapter II, any transfer function P can be associated with two distinct time domain operators. The first,

$$\mathbf{M}_P : \mathcal{H}_2 \to \mathcal{H}_2,\ u \mapsto Pu$$

is a possibly unbounded operator on $\mathcal{H}_2$ (which we identify with $\mathcal{L}_2[0, \infty)$ in the time domain) with domain $\mathcal{D}(\mathbf{M}_P) = \{u : Pu \in \mathcal{H}_2\}$. The second,

$$\mathbf{L}_P : \mathcal{L}_2 \to \mathcal{L}_2,\ u \mapsto Pu$$

is an operator on $\mathcal{L}_2$ that will be bounded provided $P \in \mathcal{L}_\infty$. If P is rational, then $\mathbf{M}_P$ is always causal, whereas $\mathbf{L}_P$ is only causal when $P \in \mathcal{H}_\infty$.

Recall that

$$\delta_g(P_1, P_2) := \operatorname{gap}(\mathcal{G}_1, \mathcal{G}_2)$$

where

$$\mathcal{G}_i := \left\{ \begin{bmatrix} P_i u \\ u \end{bmatrix} : u, P_i u \in \mathcal{H}_2 \right\} = G_i \mathcal{H}_2 = \operatorname{Ra}\mathbf{T}_{G_i} = \operatorname{Ker}\mathbf{T}_{\tilde{G}_i}$$

the graph of $\mathbf{M}_{P_i}$, and $\operatorname{gap}(\mathcal{G}_1, \mathcal{G}_2) := \|\mathbf{\Pi}_{\mathcal{G}_1} - \mathbf{\Pi}_{\mathcal{G}_2}\|$ where $\mathbf{\Pi}_{\mathcal{G}}$ denotes orthogonal projection onto $\mathcal{G}$. Now,

$$\operatorname{gap}(\mathcal{G}_1, \mathcal{G}_2) < 1 \iff \mathbf{\Pi}_{\mathcal{G}_1}|_{\mathcal{G}_2} \text{ is Fredholm, and } \mathcal{G}_1 \cap \mathcal{G}_2^\perp = \mathcal{G}_1^\perp \cap \mathcal{G}_2 = \{0\}, \tag{37}$$

(see [Nik80] p201) where

$$\mathcal{G}_i^\perp = \operatorname{Ra}\mathbf{T}_{\tilde{G}_i^*} = \operatorname{Ker}\mathbf{T}_{G_i^*}$$

the orthogonal complement of the graph of $\mathbf{M}_P$ in $\mathcal{H}_2$. Notice that

$$\operatorname{Ker}\mathbf{\Pi}_{\mathcal{G}_1}|_{\mathcal{G}_2} = \mathcal{G}_1^\perp \cap \mathcal{G}_2,\ \operatorname{Ker}(\mathbf{\Pi}_{\mathcal{G}_1}|_{\mathcal{G}_2})^* = \operatorname{Ker}\mathbf{\Pi}_{\mathcal{G}_2}|_{\mathcal{G}_1} = \mathcal{G}_2^\perp \cap \mathcal{G}_1.$$

So, provided $\mathbf{\Pi}_{\mathcal{G}_1}|_{\mathcal{G}_2}$ is Fredholm,

$$\operatorname{Ind}\mathbf{\Pi}_{\mathcal{G}_1}|_{\mathcal{G}_2} = \dim \mathcal{G}_1^\perp \cap \mathcal{G}_2 - \dim \mathcal{G}_1 \cap \mathcal{G}_2^\perp.$$

Furthermore, as $\mathbf{\Pi}_{\mathcal{G}_1} = \mathbf{T}_{G_1}\mathbf{T}_{G_1^*}$ and $\mathbf{T}_{G_1}$ is bounded from below, it follows that

$$\mathbf{T}_{G_1^*}\mathbf{T}_{G_2}\left(= \mathbf{T}_{G_1^*G_2}\right) \text{ is Fredholm } \iff \mathbf{\Pi}_{\mathcal{G}_1}|_{\mathcal{G}_2} \text{ is Fredholm;}$$

in which case

$$\operatorname{Ind} \mathbf{T}_{G_1^*G_2} = \operatorname{Ind} \mathbf{\Pi}_{\mathcal{G}_1}|_{\mathcal{G}_2}.$$

So, in contrast to (37),

$$\delta_\nu(P_1, P_2) < 1 \iff \mathbf{\Pi}_{\mathcal{G}_1}|_{\mathcal{G}_2} \text{ is Fredholm, and } \dim \mathcal{G}_1 \cap \mathcal{G}_2^\perp = \dim \mathcal{G}_1^\perp \cap \mathcal{G}_2. \tag{38}$$

Clearly then, $\delta_\nu(P_1, P_2) = 1 \implies \delta_g(P_1, P_2) = 1$. However, the converse is not true. It is possible that $\mathcal{G}_1 \cap \mathcal{G}_2^\perp \neq \{0\}$ *and* $\mathcal{G}_1^\perp \cap \mathcal{G}_2 \neq \{0\}$ with the intersections of equal dimensions. In such a case we would have $\delta_g(P_1, P_2) = 1$, but $\delta_\nu(P_1, P_2) < 1$, an example of this is given in the next section. Note, however, that this can only happen for plants with more than one input; as Coburn's lemma ([Nik80] p318) states that if $\phi \in \mathcal{L}_\infty^{1\times 1}, \phi \neq 0$, then either $\operatorname{Ker} \mathbf{T}_\phi = \{0\}$ or $\operatorname{Ker} \mathbf{T}_\phi^* = \{0\}$ (or both). Thus, if $P_1, P_2 \in \mathcal{P}^{p\times 1}$, then $\delta_\nu(P_1, P_2) < 1 \iff \delta_g(P_1, P_2) < 1$ (apply the lemma with $\phi = G_1^* G_2$). This result also appears in [GSS92] for single input, single output systems.

The condition that $\dim \mathcal{G}_1^\perp \cap \mathcal{G}_2 = \dim \mathcal{G}_1 \cap \mathcal{G}_2^\perp$ is necessary and sufficient for the existence of a direct rotation between the two graph spaces, as defined by Davis and Kahan in [DK70]. (In [Sef91] and [QD92a] this rotation is examined from a control theoretic viewpoint under the stronger assumption that $\operatorname{gap}(\mathcal{G}_1, \mathcal{G}_2) < 1$.)

It was shown in Chapter II that $\|\tilde{G}_0 G_1\|_\infty$ is the gap between the graph spaces of $\mathbf{L}_{P_0}$ and $\mathbf{L}_{P_1}$ in $\mathcal{L}_2$. Thus, δ_ν arises from defining a metric on shift invariant subspaces of $\mathcal{H}_2$ by lifting them onto $\mathcal{L}_2$ and taking the gap there (which is generally smaller), provided a direct rotation exists between the original $\mathcal{H}_2$ subspaces. The extended metric $\theta_e(\cdot,\cdot)$ of Section 3 has a somewhat more attractive interpretation. Here, we take the acute angle between the lifted spaces if a direct rotation exists between the original subspaces, and the obtuse angle otherwise.

2. Comparison with the gap metric

In this section we establish that the metric δ_ν and the $\mathcal{H}_2$-gap metric δ_g induce the same topology. We also give direct proofs of some the properties of this topology, known as the graph topology, using the metric δ_ν. The results of this section hold for systems with normalized coprime factorizations over A_0. We first show that the formula for $\delta_\nu(P_1, P_2)$ can be written as a minimization problem which resembles that given for the gap metric in (9).

Lemma IV.1.

$$\delta_\nu(P_1, P_2) = \inf_{\substack{Q, Q^{-1} \in \mathcal{L}_\infty \\ \operatorname{wno}\det(Q) = 0}} \|G_1 - G_2 Q\|_\infty \tag{39}$$

Proof: Recall that

$$\|G_1 - G_2Q\|_\infty = \left\| \begin{matrix} G_2^*G_1 - Q \\ \tilde{G}_2G_1 \end{matrix} \right\|_\infty$$

(pre-multiplying by $\begin{bmatrix} G_2 & \tilde{G}_2^* \end{bmatrix}^*$, which is unitary). So, if $\delta_\nu(P_1, P_2) < 1$ the claim follows trivially, as then $\operatorname{wno}\det(G_2^*G_1) = 0$, and $\delta_\nu(P_1, P_2) = \|\tilde{G}_2G_1\|_\infty$, with the infimum attained at $Q = G_2^*G_1$. Furthermore, for any Q satisfying

$$\|G_1 - G_2Q\|_\infty < 1, \tag{40}$$

it follows that

$$\|G_1^*(G_1 - G_2Q)\|_\infty = \|I - G_1^*G_2Q\|_\infty < 1$$

and consequently $\det(G_1^*G_2Q)(j\omega) \neq 0 \forall\omega$ and $\operatorname{wno}\det(G_1^*G_2Q) = 0$. So, if $\delta_\nu(P_1, P_2) = 1$ (*ie* $\det(G_2^*G_1)(j\omega) = 0$ for some ω or $\operatorname{wno}\det(G_2^*G_1) \neq 0$) there is no admissible Q satisfying (40).

A similar formula holds for the gap metric (a closely related theorem appears in [SO]).

Lemma IV.2.

$$\delta_g(P_1, P_2) = \inf_{Q, Q^{-1} \in \mathcal{H}_\infty} \|G_1 - G_2Q\|_\infty \tag{41}$$

Proof: If $\delta_g(P_1, P_2) < 1$, then $\delta_g(P_1, P_2) = \vec{\delta}_g(P_1, P_2)$ [GS90] and $\operatorname{wno}\det(G_1^*G_2) = 0$ (from (37) and (38)). In this case, restricting Q to be invertible in $\mathcal{H}_\infty$ does not affect the answer (by the same reasoning as used in the proof of Lemma IV.1). If $\delta_g(P_1, P_2) = 1$ but $\vec{\delta}_g(P_1, P_2) < 1$ then $\mathcal{G}_1 \cap \mathcal{G}_2^\perp = \{0\}$ (as $\vec{\delta}_g(P_1, P_2) = \|\mathbf{\Pi}_{\mathcal{G}_2^\perp}\mathbf{\Pi}_{\mathcal{G}_1}\|$, [GS90] — and see Chapter II) and so either $\mathcal{G}_1^\perp \cap \mathcal{G}_2 \neq \{0\}$ or $\mathbf{\Pi}_{\mathcal{G}_1}|_{\mathcal{G}_2}$ is not Fredholm. In either case $\delta_\nu(P_1, P_2) = 1$, so there is no $Q, Q^{-1} \in \mathcal{H}_\infty$ that makes $\|G_1 - G_2Q\|_\infty < 1$. □

In [GS90] it is shown that, in general, $\delta_g(P_1, P_2) \neq \delta_g(P_1^T, P_2^T)$. That the same is not true for the metric δ_ν can be inferred from Theorem III.7 and the fact that $b_{P^T, C^T} = b_{C,P} = b_{P,C}$. In the case of rational plants, the fact that $\delta_\nu(P_1, P_2) = \delta_\nu(P_1^T, P_2^T)$ can be proven from (5) and (6), which are both invariant to transposition. A more general proof of this result follows.

Theorem IV.3. $\delta_\nu(P_1, P_2) = \delta_\nu(P_1^T, P_2^T)$

Note that $\{\tilde{N}_i^T, \tilde{M}_i^T\}$ and $\{N_i^T, M_i^T\}$ are normalized right and left coprime factorizations respectively of P_i^T. So the result depends on proving that $\operatorname{wno}\det(G_2^*G_1) = \operatorname{wno}\det(\tilde{G}_1\tilde{G}_2^*)$. Note that $\underline{\sigma}(\tilde{G}_1\tilde{G}_2^*) = \underline{\sigma}(G_2^*G_1)$ (from (11)), and so $\mathbf{T}_{\tilde{G}_1\tilde{G}_2^*}$ is Fredholm if, and only if, $\mathbf{T}_{G_2^*G_1}$ is. That they share the same index follows from the fact that, for continuous symbols, $\operatorname{Ind}\mathbf{T}_{\tilde{G}_1\tilde{G}_2^*} = \operatorname{Ind}\mathbf{T}_{\tilde{G}_1}\mathbf{T}_{\tilde{G}_2^*}$ (see [Nik80] p350). So, as $\mathbf{T}_{\tilde{G}_2^*}$ is bounded from below,

$$\dim\ker\mathbf{T}_{\tilde{G}_1}\mathbf{T}_{\tilde{G}_2^*} = \dim(\ker\mathbf{T}_{\tilde{G}_1}\cap\operatorname{Ra}\mathbf{T}_{\tilde{G}_2^*}) = \dim(\mathcal{G}_1\cap\mathcal{G}_2^{\perp});$$

and similarly for the adjoint. $\square$

The next theorem establishes bounds for $\delta_\nu(P_1, P_2)$ in terms of the gap.

THEOREM IV.4. *The following inequality holds:*

$$\delta_g(P_1, P_2)b_{opt}(P_1) \;\le\; \delta_\nu(P_1, P_2) \;\le\; \delta_g(P_1, P_2)$$

REMARK IV.5. The second inequality shows that any set which is δ_ν-open is also δ_g-open as, for any P and any b, $\mathcal{B}_g(P, b) \subseteq \mathcal{B}_\nu(P, b)$, where $\mathcal{B}_g(P, b) = \{\hat{P} : \delta_g(P, \hat{P}) < b\}$ and $\mathcal{B}_\nu(P, b) = \{\hat{P} : \delta_\nu(P, \hat{P}) < b\}$. The first inequality shows that, for any P and any b, we can choose an ϵ such that $\mathcal{B}_\nu(P, \epsilon) \subseteq \mathcal{B}_g(P, b)$ (any ϵ less than $b \times b_{opt}(P)$ will do), so any set that is δ_g-open is also δ_ν-open. Hence, this result shows that the topology induced by δ_ν and that induced by the gap metric are identical.

PROOF: of Theorem IV.4

Note that the first inequality is trivially satisfied if $\delta_\nu(P_1, P_2) \ge b_{opt}(P_1)$ as $\delta_g(P_1, P_2)$ is always bounded by 1 anyway. So we need only consider the situation where $\delta_\nu(P_1, P_2) < b_{opt}(P_1)$, *ie* when P_2 is within the maximal stabilizable ball around P_1. In this case we can always choose a C that stabilizes both P_1 and P_2. Given such a C, put

$$\begin{aligned}\Delta &:= \begin{bmatrix}P_2\\ I\end{bmatrix}(I - CP_2)^{-1}\begin{bmatrix}-C & I\end{bmatrix} - \begin{bmatrix}P_1\\ I\end{bmatrix}(I - CP_1)^{-1}\begin{bmatrix}-C & I\end{bmatrix}\\ &= G_2(\tilde{K}G_2)^{-1}\tilde{K} - G_1(\tilde{K}G_1)^{-1}\tilde{K}. \end{aligned} \tag{42}$$

or, equivalently

$$\begin{aligned}\Delta &= \begin{bmatrix}I\\ C\end{bmatrix}(I - P_1C)^{-1}\begin{bmatrix}I & -P_1\end{bmatrix}\begin{bmatrix}P_2\\ I\end{bmatrix}(I - CP_2)^{-1}\begin{bmatrix}-C & I\end{bmatrix}\\ &= K(\tilde{G}_1K)^{-1}\tilde{G}_1G_2(\tilde{K}G_2)^{-1}\tilde{K}\end{aligned} \tag{43}$$

If we now define

$$Q := (\tilde{K}G_1)^{-1}\tilde{K}G_2, \tag{44}$$

(so Q, $Q^{-1} \in \mathcal{H}_\infty$), then it follows from (42) that

$$\Delta = \{G_2 - G_1Q\}\,(\tilde{K}G_2)^{-1}\tilde{K}. \tag{45}$$

Thus, comparing (43) and (45),

$$\begin{aligned}\|G_2 - G_1Q\|_\infty &\le \|(\tilde{G}_1K)^{-1}\|_\infty\|\tilde{G}_1G_2\|_\infty\\ &= \frac{\delta_\nu(P_1, P_2)}{b_{P_1,C}}\end{aligned}$$

The choice of C is free, so take the optimally robust compensator for P_1 — in which case

$$\delta_g(P_1,P_2) \le \|G_2 - G_1 Q\|_\infty \le \frac{\delta_\nu(P_1,P_2)}{b_{opt}(P_1)} < 1.$$

by Proposition IV.2. That $\delta_\nu(P_1,P_2) \le \delta_g(P_1,P_2)$, follows from the comparison of Proposition IV.1 and Proposition IV.2. □

The proof of the last theorem suggests the following bounds on the difference in the closed loops due to perturbations of the plant. Here,

$$H(P,C) := \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix},$$

Corollary IV.6. *For any P_1, P_2, C such that $[P_1,C]$ and $[P_2,C]$ are stable then,*

$$\delta_\nu(P_1,P_2) \le \delta_g(P_1,P_2) \le \|H(P_1,C) - H(P_2,C)\|_\infty \le$$
$$\frac{\delta_\nu(P_1,P_2)}{b_{P_1,C}\, b_{P_2,C}} \le \frac{\delta_\nu(P_1,P_2)}{b_{P_1,C} \sin\left(\arcsin b_{P_1,C} - \arcsin \delta_\nu(P_1,P_2)\right)}.$$

Proof: The upper bound follows from (43), and the lower bound from (45), noting that $\bar\sigma(\tilde K G_2)(j\omega) \le 1$ for all ω. □

Remark IV.7. The final upper bound in Corollary IV.6 is, in fact, the best possible, in the sense that, for $\beta < b_{P_1,C}$,

$$\sup_{P_2 : \delta_\nu(P_1,P_2) \le \beta} \|H(P_1,C) - H(P_2,C)\|_\infty = \frac{\beta}{b_{P_1,C} \sin(\arcsin b_{P_1,C} - \arcsin \beta)}$$

and, for $\alpha > \delta_\nu(P_1,P_2)$,

$$\sup_{C : b_{P_1,C} \ge \alpha} \|H(P_1,C) - H(P_2,C)\|_\infty = \frac{\delta_\nu(P_1,P_2)}{\alpha \sin\left(\arcsin \alpha - \arcsin \delta_\nu(P_1,P_2)\right)}.$$

In fact, in each case the suprema are attained by the same constructions as those employed in the proof of Theorem III.7 (to show this, it is also necessary to use the fact that $(\tilde K G_1)^{-1}(\tilde K \tilde G_1^*) = (G_1^* K)(\tilde G_1 K)^{-1})$.

These upper and lower bounds allow us to give simple and direct proofs of the following properties of the graph topology.

Corollary IV.8. [Vid84] *Given a P and an infinite sequence $\{P_i\}$ in $\mathcal{P}^{p\times q}$ the following are equivalent.*

i) $\delta_\nu(P,P_i) \to 0$
ii) *there exists a C such that $[P,C]$ is stable and $H(P_i,C) \to H(P,C)$ in $\mathcal{H}_\infty$.*
iii) *for any C, such that $[P,C]$ is stable, $H(P_i,C) \to H(P,C)$ in $\mathcal{H}_\infty$.*

PROOF: That i) $\implies$ iii) is immediate from the upper bound in Corollary IV.6. ii) $\implies$ i) is immediate from the lower bound, and iii) $\implies$ ii) is obvious. $\square$

For the next corollary, which appears in various forms in [ZES80] and [Vid84], we put $\mathcal{S}(C) := \{P \in \mathcal{P}^{p\times q} : [P, C] \text{ is stable}\}$

COROLLARY IV.9. *For any $C \in \mathcal{P}^{q\times p}$, the map $P \to H(P,C)$ is a continuous map from $\mathcal{P}^{p\times q}$ to $\mathcal{H}_\infty$ at all points $P \in \mathcal{S}(C)$ when $\mathcal{P}^{p\times q}$ is equipped with the graph topology (and $\mathcal{H}_\infty$ the norm topology). Furthermore, the graph topology is the weakest topology on $\mathcal{P}^{p\times q}$ for which this holds.*

PROOF: For the first part, we need to show that, for any $P \in \mathcal{S}(C)$ and for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\delta_\nu(P, P_i) < \delta \implies H(P_i, C) \in \mathcal{H}_\infty, \; \|H(P,C) - H(P_i, C)\|_\infty < \epsilon.$$

The existence of such a δ is guaranteed by the upper bound of Corollary IV.6.

For the second part, note that for any topology $\mathcal{T}$ to have this property it is necessary that for any $P \in \mathcal{S}(C)$ and for any $\epsilon > 0$ there exists an open neighbourhood of P, $\mathcal{N}(P,\epsilon)$, such that

$$H(P_i, C) \in \mathcal{H}_\infty, \; \|H(P,C) - H(P_i, C)\|_\infty < \epsilon \forall P \in \mathcal{N}(P,\epsilon).$$

But the lower bound of Corollary IV.6 then ensures that

$$\delta_\nu(P, P_i) < \epsilon \forall P \in \mathcal{N}(P,\epsilon).$$

So, for any $P \in \mathcal{S}(C)$ and for any $\epsilon > 0$ there exists a $\mathcal{T}_i \in \mathcal{T}$ such that $\mathcal{T}_i \subset \mathcal{B}_\nu(P,\epsilon)$, *ie* $\mathcal{T}$ contains the graph topology. $\square$

It is an open question whether the graph topology is also the weakest topology in which feedback stability alone is a robust property (*ie* preserved in small enough neighbourhoods). The weakest topology on $\mathcal{P}^{p\times q}$ with this property is that with sub-basis $\{\mathcal{S}(C_i) : C_i \in \mathcal{P}^{q\times p}\}$, *ie* the topology comprising of arbitrary unions and finite intersections of such sets (notice that if we were to allow infinite intersections we would end up with the discrete topology). By construction, this topology is the weakest on $\mathcal{P}^{p\times q}$ with the following properties.

i) For any $P \in \mathcal{P}^{p\times q}, C \in \mathcal{P}^{q\times p}$ then, if $[P, C]$ is stable there exists an open neighbourhood $\mathcal{N}(P)$ of P such that $[P_i, C]$ is stable for all P_i in $\mathcal{N}(P)$.
ii) $\{P_i\} \to P$ if, and only if, each C in $\mathcal{S}(P)$ also stabilizes all but a finite number of $\{P_i\}$

It is easy to see that this topology is weaker than the graph topology as, given any finite set $\{C_i\}$,

$$\mathcal{B}_\nu(P,\epsilon) \subset \cap_i \mathcal{S}(\{C_i\}) \text{ for any } \epsilon < \min_i b_{P,C_i}.$$

Now, for any open ball $\mathcal{B}_\nu(P,\epsilon)$, one can construct an infinite intersection

$$\cap_{C:b_{P,C}\geq\delta} \mathcal{S}(C) \subseteq \mathcal{B}_\nu(P,\epsilon) \text{ for } \; \delta \leq \epsilon.$$

(in fact these sets are equal if δ is equal to ϵ — by Theorem III.7). The question, which appears difficult to answer, is whether there is a finite intersection that does the same job.

3. Examples

In this section we look at two examples, taken from [GS90], which demonstrate both the quantitative and qualitative differences between the gap metric and the metric defined here.

EXAMPLE IV.1. ([GS90] Example 2) Put

$$P_1 = \frac{s-1}{s+1}, \qquad P_2 = \frac{2s-1}{s+1}. \tag{46}$$

It is shown in [QD92b] that $\delta_p(P_1, P_2) \geq 1/\sqrt{5}$, there being no way as yet to calculate the pointwise gap metric exactly. However, $\delta_g(P_1, P_2) = 1/3$ ([GS90]), with the implication that any compensator C which stabilizes P_1 and achieves $b_{P_1,C} > 1/3$ also stabilizes P_2. Now,

$$G_1^* G_2 = \frac{(3s-2)(s+1)}{(\sqrt{10}s+2)(s-1)},$$

with a winding number of zero, thus $\delta_\nu(P_1, P_2) < 1$; and

$$\tilde{G}_1 G_2 = \frac{-s}{\sqrt{10}s+2},$$

thus $\delta_\nu(P_1, P_2) = 1/\sqrt{10}$. Therefore, any compensator C that stabilizes P_1 and achieves only $b_{P_1,C} > 1/\sqrt{10}$ will actually stabilize P_2. Furthermore, there exists a compensator such that $b_{P_1,C} = 1/\sqrt{10}$ that *destabilizes* P_2. Such a compensator is $C = k$, with $k = 1/2$ (*ie* $\tilde{N}_c = 1/\sqrt{5}$, $\tilde{M}_c = 2/\sqrt{5}$), as then

$$\tilde{K} G_1 = \frac{s-1}{\sqrt{10}(s+1)}$$

and the characteristic polynomial of the feedback combination of P_2 and k is $(1-2k)s + (1+k) = 0$.

The next example is one where $\delta_\nu(P_1, P_2) < 1$ but $\delta_g(P_1, P_2) = 1$.

EXAMPLE IV.2. ([GS90] Example 5) Put

$$P_1 = \left[\frac{\epsilon}{s-1},\ 0\right], \qquad P_2 = \left[0,\ \frac{\epsilon}{s-1}\right]. \tag{47}$$

Now, $\delta_g(P_1, P_2) = 1$, as

$$G_1^* G_2 = \begin{bmatrix} \dfrac{s+1}{s-\gamma} & \dfrac{\epsilon^2}{-s^2+\gamma^2} \\ 0 & \dfrac{s-1}{s+\gamma} \end{bmatrix},$$

where $\gamma^2 = 1 + \epsilon^2$, and thus $\mathbf{T}_{G_1^* G_2}$ has a 1-dimensional Kernel,

$$\begin{bmatrix} \dfrac{s+1}{s-\gamma} & \dfrac{\epsilon^2}{-s^2+\gamma^2} \\ 0 & \dfrac{s-1}{s+\gamma} \end{bmatrix} \begin{bmatrix} \dfrac{1}{s+1} \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s-\gamma} \\ 0 \end{bmatrix} \in \mathcal{H}_2^\perp.$$

However, $\mathbf{T}_{G_2^* G_1}$ also has a 1-dimensional Kernel,

$$\begin{bmatrix} \dfrac{s-1}{s+\gamma} & 0 \\ \dfrac{\epsilon^2}{-s^2+\gamma^2} & \dfrac{s+1}{s-\gamma} \end{bmatrix} \begin{bmatrix} 0 \\ \dfrac{1}{s+1} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{1}{s-\gamma} \end{bmatrix} \in \mathcal{H}_2^\perp,$$

thus Ind $\mathbf{T}_{G_1^* G_2} = 0$ and so $\delta_\nu(P_1, P_2) = \delta_{\mathcal{L}_2}(P_1, P_2)$. Alternatively,

$$\det(G_1^* G_2) = \frac{(s-1)(s+1)}{(s-\gamma)(s+\gamma)},$$

which has a zero winding number, again showing that $\delta_\nu(P_1, P_2) = \delta_{\mathcal{L}_2}(P_1, P_2)$. In fact,

$$\delta_\nu(P_1, P_2) \quad (= \delta_\nu(P_1^T, P_2^T)) = \frac{\epsilon\sqrt{2+\epsilon^2}}{1+\epsilon^2} = \epsilon + O(\epsilon^2),$$

which is equal to $\delta_g(P_1^T, P_2^T)$, the gap between the transposed plants, as given in [GS90]. (Note that

$$b_{opt}(P_1) = b_{opt}(P_1) = \sqrt{1 - 1/\gamma}/\sqrt{2} = \epsilon/2 + O(\epsilon^2) > \delta_\nu(P_1, P_2),$$

so although the plants get closer as $\epsilon \to 0$, the distance between them is still larger than radius of the maximal stabilizable ball. However, both plants are simultaneously stabilizable with constant feedback.

CHAPTER V

Robust Design in the Graph Topology

We begin this chapter by giving further motivation for the use of $b_{P,C}$ as a performance measure. We shall investigate the loop shaping design procedure of McFarlane and Glover ([MG90] [MG92]) in some detail, giving, in the scalar case, sharp bounds on the attained sensitivity and complementary sensitivity functions in terms of $b_{p,c}$ and the loop shape $|p(j\omega)|$. We also look at the relationship between $|p(j\omega)|$ and $b_{opt}(p)$, the best attainable $b_{p,c}$.

We have developed the metric $\delta_\nu(\cdot,\cdot)$ as the natural dual to the stability/performance measure $b_{P,C}$. One of the strengths of this metric is that it readily admits of a frequency response interpretation. In this chapter we elaborate on this interpretation and generalize our main robust stability result, Theorem III.7 to the case where the metric is measured pointwise in the frequency domain. We also discuss how the metric makes sense, quantitatively, as a measure of the importance of the difference between two plants in a feedback context. In the scalar case, this involves viewing the projection of the Nyquist diagram of a plant onto the Riemann sphere.

We shall then go on to investigate the use of the loop shaping design procedure in conjunction with the pointwise metric. The idea is to use bounds on the uncertainty, obtained by measuring the size of the uncertainty frequency by frequency, to provide an *a priori* weight for an $\mathcal{H}_\infty$ optimization problem.

The treatment to this point has been restricted to the robust disturbance rejection problem. In the final section we look at robust tracking problems.

1. The loop shaping design procedure of McFarlane and Glover

Recall the definition of $b_{P,C}$ that was given in the introduction

$$b_{P,C} := \begin{cases} \left\| \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} [-C \;\; I] \right\|_\infty^{-1}, & \text{if } [P,C] \text{ is stable} \\ 0, & \text{otherwise.} \end{cases}$$

We motivated the use of this as a performance measure by noting that it bounds the gain of all eight[1] closed loop transfer functions between inputs and outputs at any point in the loop. However, tighter bounds on these gains are possible for the reason that these transfer functions are not independent. For example, given P then fixing the so-called control sensitivity function $C(I-PC)^{-1}$ is sufficient to determine C and thus all of the other closed loop transfer functions.

McFarlane and Glover ([MG90]) show that, given a nominal plant P_0, the problem of finding $\sup_C b_{P_0,C}$ has a useful loop shaping interpretation. If a particular compensator C achieves $b_{P_0,C} \geq b$ then the following bounds hold in frequency ranges where the minimum singular value of the plant is large or the maximum singular value of the plant is small respectively:

$$\underline{\sigma}(P_0) \gg \frac{\sqrt{1-b^2}}{b} \implies \underline{\sigma}(C) \gtrsim \frac{b}{\sqrt{1-b^2}} > b$$

and

$$\bar{\sigma}(P_0) \ll \frac{b}{\sqrt{1-b^2}} \implies \bar{\sigma}(C) \lesssim \frac{\sqrt{1-b^2}}{b} < \frac{1}{b}.$$

Here, $\gtrsim$ and $\lesssim$ mean asymptotically greater than and less than as $\underline{\sigma}(P_0) \to \infty$ and $\bar{\sigma}(P_0) \to 0$ respectively. Hence, if P_0 has a desirable loop shape, then $b_{P_0,C}$ gives a bound on the degradation of that loop shape by C at high and low gains. If P_0 does not itself have a desirable loop shape to start with then it is modified by multiplying by a weight W_s. The optimal compensator is then found for this shaped plant (*ie* C_{opt} is the compensator achieving $\sup_C b_{P_0W_s,C}$) and the shaping elements are then absorbed into this compensator to give a feedback compensator for the original plant, W_sC_{opt} — it being a simple matter to show that the stability of $[P_0W_s, C_{opt}]$ implies the stability of $[P_0, W_sC_{opt}]$ provided there are no unstable pole/zero cancellations between P_0 and W_s. More details of this approach may be found in [MG90] and [Hyd91], the latter giving a number of nontrivial examples, and guidelines for choosing the weights in the multivariable case.

The justification for this loop shaping approach is as a means of indirectly shaping the classical sensitivity function S and the complementary sensitivity function T, where

$$S := (I-P_0C)^{-1} \text{ and } T := -P_0(I-CP_0)^{-1}C.$$

S is the transfer function from output disturbances to the output of the plant (*ie* from w_2 to y in Figure I.1 on page 2). T is the transfer function from measurement noise to the output of the plant (*ie* from v_2 to y in the same figure). Because $S+T=I$, it is impossible to make them both S and T small at the same frequency, and the resultant trade off between the two sensitivities is a popular paradigm for robust control. One of the fundamental objectives of

[1] recall that $b_{P,C} = b_{C,P}$

feedback is to make $S(jw)$ small, whilst the size of $T(jw)$ determines the tolerance to multiplicative uncertainty at that frequency. In the sequel we shall think of $T(jw)$ purely as a performance measure, in terms of the effect of measurement noise, and use the metric δ_ν to deal with uncertainty.

If we have a compensator C with $b_{P_0,C} > 1/\beta$, then the foregoing bounds ensure that:
In frequency ranges where

$$\underline{\sigma}(P_0) \gg \frac{\sqrt{1-\beta^2}}{\beta} \text{ we are assured that } \underline{\sigma}(C_{opt}) \gtrsim \frac{\beta}{\sqrt{1-\beta^2}},$$

which in turn implies that

$$\bar{\sigma}((I - P_0 C_{opt})^{-1}) \lesssim 1/\left(\underline{\sigma}(P_0)\frac{\beta}{\sqrt{1-\beta^2}} - 1\right).$$

Similarly, in frequency ranges where

$$\bar{\sigma}(P_0) \ll \frac{\beta}{\sqrt{1-\beta^2}} \text{ we are assured that } \bar{\sigma}(C_{opt}) \lesssim \frac{\sqrt{1-\beta^2}}{\beta},$$

which in turn implies that

$$\bar{\sigma}(P_0(I - C_{opt}P_0)^{-1}C_{opt}) \lesssim 1/\left(\frac{1}{\bar{\sigma}(P_0)}\frac{\beta}{\sqrt{1-\beta^2}} - 1\right).$$

So W_s is chosen with the aim of ensuring that the shaped plant satisfies $\underline{\sigma}(P_0 W_s)(j\omega) \gg 1$ in frequency ranges where sensitivity reduction is required and $\underline{\sigma}(P_0 W_s)(j\omega) \ll 1$ in frequency ranges where the complementary sensitivity is required to be small. Typically, W_s is composed of a constant gain to give the desired crossover frequency, and possibly includes integrators in series to achieve asymptotic tracking and disturbance rejection and filters to modify the high frequency roll-off. Clearly, another constraint on W_s is that $b_{opt}(P_0 W_s)$ should be sufficiently large, otherwise the bounds are useless. The experience reported in [MG90] and [Hyd91], and motivated by classical loop shaping ideas, is that this is achieved by ensuring that the shaped plant has a small gain around the frequency of any *rhp* zeros, a large gain around the frequency of any *rhp* poles and includes compensation to obtain an attenuation rate of not much greater than 20dB/decade at any crossover frequencies. In the sequel we shall give theoretical justification for these claims as well as giving sharper upper bounds (and also lower bounds) on the classical closed loop objectives in terms of $b_{P,C}$ and the gain of the plant. We shall concentrate our attention on the scalar case; the results are then sharper and their interpretation more transparent. First, we define $\rho(P, C)$ as a pointwise version of $b_{P,C}$:

$$\begin{aligned}
&\rho(P(j\omega), C(j\omega)) \\
&\quad := \bar{\sigma}^{-1}\left(\begin{bmatrix} P(j\omega) \\ I \end{bmatrix}(I - C(j\omega)P(j\omega))^{-1}[-C(j\omega) \;\; I]\right), \\
&\quad := \bar{\sigma}^{-1}((I + C^*C)^{1/2}(I - PC)^{-1}(I + PP^*)^{1/2})(j\omega)
\end{aligned}$$

thus $b_{P,C} = \inf_\omega \rho(P(j\omega), C(j\omega))$ provided $[P, C]$ is stable.

Our first lemma characterizes the best possible bound for the singular values of the sensitivity function in terms of the singular values of the plant and a bound on $\rho(P(j\omega), C(j\omega))$.

Lemma V.1. *Given an invertible matrix $P \in \mathbb{C}^{p\times p}$, then $\rho(P, C) \geq 1/\gamma \iff$*

$$(I - PC)^{-1} = (I + PP^*)^{-1} + \left(I + (PP^*)^{-1}\right)^{-1/2} X(I + PP^*)^{-1/2} \quad (48)$$

for some $X : \bar{\sigma}(X) \leq \sqrt{\gamma^2 - 1}$. Furthermore, if

$$P = U\Sigma V^*$$

is a singular value decomposition, then

$$\begin{aligned}
&\max_{C:\rho(P,C)\geq 1/\gamma} \sigma_i(I - PC)^{-1} = \\
&\qquad \max_{X:\bar{\sigma}(X)\leq\sqrt{\gamma^2-1}} \sigma_i\left((I + \Sigma^2)^{-1} + (I + \Sigma^{-2})^{-1/2}X(I + \Sigma^2)^{-1/2}\right) \quad (49)
\end{aligned}$$

Proof: Put

$$S := (I - PC)^{-1} \quad (50)$$

and

$$H := (I + C^*C)^{1/2}(I - PC)^{-1}(I + PP^*)^{1/2}.$$

So $\rho(P, C) \geq 1/\gamma \iff \bar{\sigma}(H) \leq \gamma$. Note that

$$\begin{aligned}
S^*(I + C^*C)S &= S^*(I + C^*P^*(PP^*)^{-1}PC)S \\
&= S^*S + (S^* - I)(PP^*)^{-1}(S - I) \\
&= (S^* - (I + PP^*)^{-1})(I + (PP^*)^{-1})(S - (I + PP^*)^{-1}) + (I + PP^*)^{-1},
\end{aligned}$$

completing the square, and so

$$\begin{aligned}
&(H^*H - I) = \\
&\quad (I + PP^*)^{1/2}(S^* - (I + PP^*)^{-1})(I + (PP^*)^{-1})(S - (I + PP^*)^{-1})(I + PP^*)^{1/2}.
\end{aligned}$$

It follows that $\rho(P, C) \geq 1/\gamma$ if, and only if,

$$(I + (PP^*)^{-1})^{1/2}(S - (I + PP^*)^{-1})(I + PP^*)^{1/2} = X \quad (51)$$

for some $X : \bar{\sigma}(X) \leq \sqrt{\gamma^2 - 1}$. Solving this for S, and then substituting the singular value decomposition of P, yields (48) and (49). □

In the multivariable case, the bound expressed in (49) is not immediately useful. However, we shall show below that, if $\Sigma = \operatorname{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_p$, then, for α_i sufficiently large,

$$\max_{C : \rho(P,C) \geq 1/\gamma} \sigma_{p+1-i}(I - PC)^{-1} = (\sqrt{\gamma^2 - 1}\alpha_i + 1)/(\alpha_i^2 + 1).$$

We shall consider the scalar case first though, as things are much clearer there.

Theorem V.2. *Given $p, c \in \mathcal{P}_{1\times 1}$:*
If $\rho(p(j\omega), c(j\omega)) \geq 1/\gamma$, then

$$\frac{1}{|1 - pc|}(j\omega) \leq \frac{\sqrt{\gamma^2 - 1}|p(j\omega)| + 1}{|p(j\omega)|^2 + 1},$$

and

$$\frac{|pc|}{|1 - pc|}(j\omega) \leq \frac{\sqrt{\gamma^2 - 1}/|p(j\omega)| + 1}{|1/p(j\omega)|^2 + 1}.$$

Conversely, if

$$\frac{1}{|1 - pc|}(j\omega) \leq \frac{\sqrt{\gamma^2 - 1}|p(j\omega)| - 1}{|p(j\omega)|^2 + 1},$$

or

$$\frac{|pc|}{|1 - pc|}(j\omega) \leq \frac{\sqrt{\gamma^2 - 1}/|p(j\omega)| - 1}{|1/p(j\omega)|^2 + 1},$$

then $\rho(p(j\omega), c(j\omega)) \geq 1/\gamma$.

Furthermore, these are the best bounds possible (in terms of the information used).

Proof: In the scalar case (49) becomes $\rho(p(j\omega), c(j\omega)) \geq 1/\gamma$ if, and only if,

$$\frac{1}{1 - pc} = \frac{\sqrt{\gamma^2 - 1}|p(j\omega)|x + 1}{|p(j\omega)|^2 + 1},$$

for some $x : |x| \leq 1$. The bounds involving sensitivity follow immediately, those for complementary sensitivity follow after noticing that

$$\rho(p(j\omega), c(j\omega)) = \rho(1/p(j\omega), 1/c(j\omega)).$$

□

Remark V.3. It is easy to show that

$$\sup_{x \in \mathbb{R}^+} \frac{\sqrt{\gamma^2 - 1}x + 1}{x^2 + 1} = (\gamma + 1)/2$$

and

$$\sup_{x\in\mathbb{R}^+} \frac{\sqrt{\gamma^2-1}x-1}{x^2+1} = (\gamma-1)/2.$$

So, if $b_{p,c} \geq 1/\gamma$ then both the sensitivity and complementary sensitivity functions are bounded by $(\gamma+1)/2$ at all frequencies. Furthermore, the upper and lower bounds on the sensitivity and complementary sensitivity are close for $|p(j\omega)|$ sufficiently large and sufficiently small respectively. Our conclusion is that, if $b_{p,c}$ is large then we are guaranteed small sensitivity at frequencies where $|p(j\omega)|$ is large (with respect to 1), small complementary sensitivity at frequencies where $|p(j\omega)|$ is small and that both functions are well behaved around $|p(j\omega)| \approx 1$. Conversely, if $b_{p,c}$ is small then either the compensator is not achieving sufficient sensitivity reduction at some frequency where $|p(j\omega)|$ is large *or* complementary sensitivity is overly large at some frequency where $|p(j\omega)|$ is small *or* there is a large peak in one of these functions around frequencies where $|p(j\omega)| \approx 1$. Thus, within the leeway of the bounds we have presented, that $b_{p,c}$ should be large is both necessary and sufficient for good feedback properties.

We now give the related result for the multivariable case that we alluded to above.

THEOREM V.4. *Given $P \in \mathbb{C}^{n\times n}$, and $P = U\Sigma V^*$ a singular value decomposition of P with $\Sigma = \operatorname{diag}\{\sigma_1,\ldots,\sigma_n\}$, $\sigma_1 \geq \ldots \geq \sigma_n$ and $U = [u_1 \ \ldots \ u_n]$ then, provided $\sigma_i \geq \sqrt{\gamma^2-1}$,*

$$\max_{C:\rho(P,C)\geq 1/\gamma} \sigma_{n+1-i}(I-PC)^{-1} = \max_{C:\rho(P,C)\geq 1/\gamma} \|(I-PC)^{-1}u_i\|_2 = \frac{\sqrt{\gamma^2-1}\sigma_i+1}{1+\sigma_i^2}.$$

REMARK V.5. Unlike Theorem V.2, we do not give bounds in both directions here. Although such results are certainly possible, it is difficult to see how to state them in a way that does not appear unduly conservative — this being due to problems involving directionality.

We should also point out that it may be proved much more simply, using methods employed in [MG90], that if $\rho(P,C) \geq 1/\gamma$ then

$$\sigma_{n+1-i}(I-PC)^{-1} \leq \frac{\gamma}{\sqrt{1+\sigma_i^2(P)}},$$

irrespective of the ordering of σ_i and $\sqrt{\gamma^2-1}$. Although this bound is worse than that given here, it is only marginally so when $\sigma_i \geq \sqrt{\gamma^2-1}$. It is, furthermore, the best bound possible when $\sigma_i < \sqrt{\gamma^2-1}$. (To see this, put $P_\epsilon = \operatorname{diag}\{1/\epsilon,\alpha\}$ and $X = \begin{bmatrix} \alpha & \sqrt{(\gamma^2-1)-\alpha^2} \\ -\sqrt{(\gamma^2-1)-\alpha^2} & \alpha \end{bmatrix}$ with $\alpha < \sqrt{\gamma^2-1}$. Now, define C_ϵ via (51) and (50). Then, as $\epsilon \to 0$, $\rho(P_\epsilon, C_\epsilon) \to 1/\gamma$ and $\bar{\sigma}(I-P_\epsilon C_\epsilon)^{-1} \to \gamma/\sqrt{1+\alpha^2}$.)

PROOF: of Theorem V.4 It is clear from Lemma V.1 that equality is achieved in each case with $X = \sqrt{\gamma^2 - 1}I$ in (49), so it is only necessary to prove that the maxima are bounded by the given expressions. We shall first show that

$$\sigma_{n+1-i}(I - PC)^{-1} \leq \frac{\sqrt{\gamma^2-1}\sigma_i + 1}{1+\sigma_i^2} \;\forall C : \rho(P,C) \geq 1/\gamma,\; i : \sigma_i \geq \sqrt{\gamma^2 - 1}.$$

From the minimax characterization of singular values,

$$\begin{aligned}\sigma_{n+1-i}(I - PC)^{-1} &= \min_{\mathcal{S}:\dim\mathcal{S}=i} \max_{\substack{x\in\mathcal{S}\\ \|x\|_2\leq 1}} \|(I-PC)^{-1}x\|_2 \\ &\leq \max_{\substack{x\in\mathcal{X}^i\\ \|x\|_2\leq 1}} \|(I-PC)^{-1}x\|_2,\end{aligned}$$

where $\mathcal{X}^i$ is defined as the subspace of $\mathbb{C}^n$ with zeros in the last $n-i$ positions. Furthermore,

$$\max_{\substack{x\in\mathcal{X}^i\\ \|x\|_2\leq 1}} \|X(I+\Sigma^2)^{-1/2}x\|_2 \leq \bar{\sigma}(X)\frac{1}{\sqrt{1+\sigma_i^2}}$$

So, from (49),

$$\max_{C:\rho(P,C)\geq 1/\gamma} \sigma_{n+1-i}(I-PC)^{-1} \leq \max_{\substack{x\in\mathcal{X}^i\\ \|x\|_2\leq 1\\ \|y\|_2\leq 1}} \left\|(I+\Sigma^2)^{-1}x + \frac{g}{\sqrt{1+\sigma_i^2}}(I+\Sigma^{-2})^{-1/2}y\right\|_2$$

where $g := \sqrt{\gamma^2-1}$. Clearly, this maximum will be obtained when the elements of x and y are real and positive. So we put

$$x := \left[x_1, x_2, \ldots, x_{i-1}, \sqrt{1 - \sum_{j<i} x_j^2}, 0, \ldots, 0\right]'$$

$$y := \left[y_1, y_2, \ldots, y_{i-1}, \sqrt{1 - \sum_{j\neq i} y_j^2}, y_{j+1}, \ldots, y_n\right]',$$

and abbreviate $\sum_{j<i} x_j^2$ to $\sum x^2$ and $\sum_{j\neq i} y_j^2$ to $\sum y^2$, to obtain

$$\begin{aligned}&\left\|(I+\Sigma^2)^{-1}x + \frac{g}{\sqrt{1+\sigma_i^2}}(I+\Sigma^{-2})^{-1/2}y\right\|_2 \\ &\quad= \left(\frac{\sqrt{1-\sum x^2} + \sqrt{1-\sum y^2}g\sigma_i}{1+\sigma_i^2}\right)^2 + \sum_{j<i}\left(\frac{x_j}{1+\sigma_j^2} + \frac{y_j\sigma_j g}{\sqrt{1+\sigma_j^2}\sqrt{1+\sigma_i^2}}\right)^2 \\ &\qquad+ \sum_{j>i}\left(\frac{y_j\sigma_j g}{\sqrt{1+\sigma_j^2}\sqrt{1+\sigma_i^2}}\right)^2.\end{aligned}$$

Note that, fixing x, Σ and $\sum_{j\neq i} y_j^2$, the maximum over the remaining elements of y will always have $y_j = 0\,\forall j > i$.

We shall show that

$$
\begin{aligned}
w(x,y,\Sigma) := \left(\frac{1+g\sigma_i}{1+\sigma_i^2}\right)^2 &- \left(\frac{\sqrt{1-\sum x^2}+\sqrt{1-\sum y^2}g\sigma_i}{1+\sigma_i^2}\right)^2 \\
&- \sum_{j<i}\left(\frac{x_j}{1+\sigma_j^2}+\frac{y_j\sigma_j g}{\sqrt{1+\sigma_j^2}\sqrt{1+\sigma_i^2}}\right)^2 \\
&\geq 0 \quad \forall x : \sum x^2 < 1, y : \sum y^2 < 1,
\end{aligned}
$$

under the assumptions that $\sigma_j \geq \sigma_i \, \forall j < i$ and $\sigma_i \geq g$.

Expanding the above, making the substitution $z_j := \dfrac{1+s_i^2}{1+s_j^2}$ (so $0 \leq z_j \leq 1$) we get

$$
\begin{aligned}
\frac{(1+\sigma_i^2)}{2g\sigma_i} w(x,y,\Sigma) = 1 - \sqrt{1-\sum x^2}\sqrt{1-\sum y^2} &- \sum_{j<i} \frac{g}{2\sigma_i}(1-z_j)y_j^2 \\
&+ \frac{z_j\sqrt{1+\sigma_i^2-z_j}}{\sigma_i} y_j x_j - \frac{1-z_j^2}{2\sigma_i g} x_j^2
\end{aligned}
$$

Now note that

$$
1-\sqrt{1-\sum x^2}\sqrt{1-\sum y^2} \geq 1/2\left(\sum x^2 + \sum y^2\right) \iff \left(\sum x^2 - \sum y^2\right)^2 \geq 0,
$$

which is always. To complete the proof, we shall show that the left hand inequality above is indeed satisfied, *ie*

$$
\frac{g}{2\sigma_i}(1-z_j)y_j^2 + \frac{z_j\sqrt{1+\sigma_i^2-z_j}}{\sigma_i} y_j x_j - \frac{1-z_j^2}{2\sigma_i g} x_j^2 \leq 1/2(x_j^2+y_j^2). \tag{52}
$$

Now, $ax^2/2 + bxy + cy^2/2 \leq x^2/2 + y^2/2$ for $a, b \leq 1$ and all x, y if, and only if, $(1-a)(1-c) \geq b^2$ (completing the square). In the present case we require

$$
\left(1+\frac{1-z_j^2}{\sigma_i g}\right)\left(1-\frac{g}{\sigma_i}(1-z_j)\right) \geq \frac{z_j^2(1+\sigma_i^2-z_j)}{\sigma_i^2}
$$

$$
\Updownarrow
$$

$$
-z_j^2\left(\frac{1}{\sigma_i g}+1\right) + z_j\left(\frac{g}{\sigma_i}+\frac{1}{\sigma_i^2}\right) + \left(\frac{1}{\sigma_i g}+1-\frac{g}{\sigma_i}-\frac{1}{\sigma_i^2}\right) \geq 0.
$$

As the second derivative with respect to z_j is negative we only need to check this at $z_j = 1$, when the expression is zero, and $z_j = 0$, when we require $(\sigma_i - g)(\sigma_i g + 1)/\sigma_i^2 g \geq 0$, which is true by assumption.

This completes the proof of the first part of the theorem. The proof of the second part, regarding the gain in the left basis directions of P, is a restriction of the above with $x_j = 0 : j \neq i$. $\square$

It is also possible to relate $b_{p,c}$ to classical gain and phase margins. For a stable $[p,c]$ define gain and phase margins as follows

If $pc(j\omega) \in \mathbb{R}^+$ for some ω, then

$$\mathrm{GM}(p,c) := \min\{k \in \mathbb{R} : k \geq 1, kpc(j\omega) = 1 \text{ or } pc(j\omega)/k = 1 \text{ for some } \omega\},$$

otherwise $\mathrm{GM}(p,c) := \infty$.

If $|pc(j\omega)| = 1$ for some ω, then

$$\mathrm{PM}(p,c) := \min\{\theta \in \mathbb{R} : \theta \geq 0, e^{j\theta}pc(j\omega) = 1 \text{ or } e^{-j\theta}pc(j\omega)/k = 1 \text{ for some } \omega\},$$

otherwise $\mathrm{PM}(p,c) := \infty$.

THEOREM V.6. *If $[p,c]$ is stable, then $\mathrm{GM}(p,c) \geq (1+b_{p,c})/(1-b_{p,c})$ and $\mathrm{PM}(p,c) \geq 2\arcsin(b_{p,c})$.*

PROOF:

$$|1-pc|/\sqrt{1+|p|^2}\sqrt{1+|c|^2} \geq b_{p,c} \ \forall\omega.$$

So, at frequencies where $pc(j\omega) = k \in \mathbb{R}^+$,

$$(1-k)^2 \geq b^2(p,c)(1+|p|^2)(1+k^2/|p|^2) \geq b^2(p,c)(1+k)^2,$$

from which the gain margin result follows. Similarly, at frequencies where $pc(j\omega) = e^{j\theta}$,

$$|1-e^{j\theta}|^2 = 4\sin^2(\theta/2) \geq b^2(p,c)(1+|p|^2)(1+1/|p|^2) \geq 4b^2(p,c),$$

which is the claimed result for the phase margin. □

REMARK V.7. Unlike the previous theorems, it is impossible to bound the gain and phase margins from both directions. This merely reflects the inadequacy of these indicators of performance and robustness. It is possible to construct examples where the gain and phase margins are both arbitrarily good with $b_{P,C}$, and both sensitivities, arbitrarily bad. The theorem is useful, though, in relating numerical values of $b_{P,C}$ to numbers that all designers will be familiar with — for example, a $b_{P,C}$ of $1/2$ guarantees a gain margin of 3 and a phase margin of $60°$, whereas a $b_{P,C}$ of $1/4$ only guarantees a gain margin of $4/3$ and a phase margin of $29°$.

The remaining question we have to answer is, given a plant p, for what weights w can we guarantee the existence of a compensator c that makes $b_{pw,c/w}$ sufficiently large to get the benefit of the bounds discussed above. Equivalently, how does $b_{opt}(p)$ depend on p? Fortunately, we have an explicit formula for $b_{opt}(P)$ from [GM89].

PROPOSITION V.8. [GM89] *Given a $P_0 \in \mathcal{P}^{p\times q}$, the following are equivalent*

i) *$[P_0, C]$ is stable and $b_{P_0,C} > \beta$.*

ii) $C = UV^{-1}$ *for some* $U, V \in \mathcal{H}_\infty$ *satisfying.*

$$\left\| \tilde{G}_0^* - \begin{bmatrix} V \\ U \end{bmatrix} \right\|_\infty < \sqrt{1-\beta^2}$$

(where, as usual, $\tilde{G}_0 = \begin{bmatrix} -\tilde{M}_0 & \tilde{N}_0 \end{bmatrix}$ *and* $\{\tilde{N}_0, \tilde{M}_0\}$ *is a normalized* lcf *of* P_0*).*

An immediate consequence of this result, by Nehari's theorem, is that

$$\sqrt{1 - b_{opt}(P_0)^2} = \|\mathbf{H}_{\tilde{G}_0^*}\|$$

Furthermore, an optimally robust compensator, *ie* one achieving $b_{P_0,C} = b_{opt}(P_0)$, can be constructed from the optimal $\mathcal{H}_\infty$ approximation to $\tilde{G}_0^*$.

Here is an alternative proof of this proposition in the spirit of the development of this monograph.

Proof: of Proposition V.8. This follows by showing that both *i)* and *ii)* are equivalent to

iii) $$\inf_{Q \in \mathcal{H}_\infty} \left\| \tilde{G}_0^* - KQ \right\|_\infty < \sqrt{1-\beta^2}.$$

(where, $K = \begin{bmatrix} M_c \\ N_c \end{bmatrix}$ and $\{N_c, M_c\}$ is a normalized *rcf* of C.) Note that

$$\begin{aligned} \inf_{Q \in \mathcal{H}_\infty} \left\| \tilde{G}_0^* - KQ \right\|_\infty &\geq \inf_{Q \in \mathcal{L}_\infty} \left\| \tilde{G}_0^* - KQ \right\|_\infty \\ &= \inf_{Q \in \mathcal{L}_\infty} \left\| \begin{bmatrix} K^* \tilde{G}_0^* - Q \\ \tilde{K} \tilde{G}_0^* \end{bmatrix} \right\|_\infty \\ &= \sqrt{1 - b_{P,C}^2}. \end{aligned}$$

by Proposition III.4. If $[P, C]$ is stable then a choice of $\hat{Q} = (\tilde{G}_0 K)^{-1} b_{P,C}^2 \in \mathcal{H}_\infty$ achieves this lower bound, as

$$\begin{aligned} \left\| \tilde{G}_0^* - K\hat{Q} \right\|_\infty &= \left\| \begin{bmatrix} I - \tilde{G}_0 K \hat{Q} \\ G_0^* K \hat{Q} \end{bmatrix} \right\|_\infty \\ &= \sqrt{(1 - b_{P,C}^2)^2 + \frac{1 - b_{P,C}^2}{b_{P,C}^2} b_{P,C}^4} \\ &= \sqrt{1 - b_{P,C}^2}. \end{aligned}$$

Thus, $i) \implies iii)$. To show that $iii) \implies i)$ note that

$$\left\| \tilde{G}_0^* - KQ \right\|_\infty < 1 \implies \left\| \tilde{G}_0 (\tilde{G}_0^* - KQ) \right\|_\infty < 1$$

and so

$$\operatorname{wno}\det(\tilde{G}_0 K) = -\operatorname{wno}\det(Q),$$

(since $\tilde{G}_0 KQ = I - (I - \tilde{G}_0 KQ)$). But Q is in $\mathcal{H}_\infty$, so $\operatorname{wno}\det(Q) \geq 0$; and $\operatorname{wno}\det(\tilde{G}_0 K) \geq 0$ for the same reason. Hence, $\operatorname{wno}\det(\tilde{G}_0 K) = 0$ and $[P, C]$ is stable. Clearly $iii) \implies ii)$, to show that $ii) \implies iii)$ put $Q = M_c^{-1} V$. As $\{N_c, M_c\}$ are right coprime there exist $X, Y \in \mathcal{H}_\infty$ such that $XM_c + YN_c = I$, thus

$$Q = (XM_c + YN_c)M_c^{-1}V = XV + YU \in \mathcal{H}_\infty.$$

$\square$

Remark V.9. Although not necessary for the previous theorem, it is easy to show using these methods that any $U, V \in \mathcal{H}_\infty$ satisfying

$$\left\| \tilde{G}_0^* - \begin{bmatrix} V \\ U \end{bmatrix} \right\|_\infty < 1 \tag{53}$$

is, in fact, a *coprime* factorization of a stabilizing compensator. Given (53) we have shown that there exists a $Q \in \mathcal{H}_\infty$ such that

$$\begin{bmatrix} V \\ U \end{bmatrix} = KQ.$$

Now, $[U, V]$ are coprime if $Q^{-1} \in \mathcal{H}_\infty$. As in the previous theorem, (53) implies that

$$\operatorname{wno}\det(\tilde{G}_0 K) = -\operatorname{wno}\det(Q),$$

but $\operatorname{wno}\det(\tilde{G}_0 K) \geq 0$ as $\tilde{G}_0 K \in \mathcal{H}_\infty$, and similarly $\operatorname{wno}\det(Q) \geq 0$, thus

$$\operatorname{wno}\det(\tilde{G}_0 K) = \operatorname{wno}\det(Q) = 0,$$

guaranteeing that $[U, V]$ are coprime.

So, the explicit formula for b_{opt} is

$$\sqrt{1 - b_{opt}(P_0)^2} = \|\mathbf{H}_{\tilde{G}_0^*}\|.$$

Unfortunately though, although this means that $b_{opt}(P_0)$ is easy to calculate, the formula gives us little insight into the problem. An upper bound on $b_{opt}(P_0)$ is given in [GS90].

Proposition V.10. [GS90]

$$b_{opt}(P_0) \leq \tau(G_0) := \inf_{s:\Re(s)>0} \underline{\sigma}(G_0(s))$$

Proof: There exists a compensator satisfying $\underline{\sigma}(\tilde{K}G_0)(s) \geq b_{opt}(P_0)$ $\forall s : \Re(s) > 0$, and thus $\bar{\sigma}(\tilde{K}(s))\underline{\sigma}(G_0(s)) \geq b_{opt}(P_0)$ for all s. But, $\bar{\sigma}(\tilde{K}(s)) \leq 1$, which gives the result. $\square$

An open question concerns how good this bound is. Experience shows that it is normally close to b_{opt}, provided b_{opt} is not too small, and numerical experiments in the case of scalar systems have failed to provide a counterexample to the (rather wild!) conjecture that $\tau(G_0) \leq b_{opt}^{1/2}(P_0)$. (We give below a fairly extreme example where $\tau(G_0)$ is larger than $b_{opt}(P_0)$ by a factor of 3, for $b_{opt}(P_0) = 1/15$.) In any case, $\tau(G_0)$ being small is a sufficient condition for $b_{opt}(P_0)$ being poor; the open question is whether $b_{opt}(P_0)$ can be poor for reasons not revealed by $\tau(G_0)$ (although it appears likely that it cannot).[2]

Some frequency response interpretations of b_{opt} have recently been reported in [CF92], we now give some related bounds.

In the scalar case Proposition V.10 simplifies to

$$b_{opt}^2(p_1) \leq |n_1(s)|^2 + |m_1(s)|^2 \ \forall s : \Re(s) > 0$$

where $\{n_1, m_1\}$ is a normalized *rcf* of p_1. We can factor a rational scalar transfer function p_1 as

$$p_1 = p_0 \Omega_z \Omega_p^{-1}$$

where p_0 has no open *rhp* poles or zeros, and $\Omega_z, \Omega_p \in \mathcal{RH}_\infty$ are unitary functions whose zeros correspond to any open *rhp* zeros and poles of p_1 respectively. Thus, if $\{n_0, m_0\}$ is a normalized *rcf* of p_0, then $\{n_0\Omega_z, m_0\Omega_p\}$ is a normalized *rcf* of p_1 and

$$b_{opt}^2(p_1) \leq |n_0(s)|^2|\Omega_z(s)|^2 + |m_0(s)|^2|\Omega_p(s)|^2 \ \forall s : \Re(s) > 0. \tag{54}$$

Now, $\ln(n_0(s))$ and $\ln(m_0(s))$ are both analytic in $\Re(s) > 0$ and so can be determined from their boundary values on $\Re(s) = 0$ via Poisson integrals. Furthermore, $n_0 = (1 + 1/(p(s)p^*(s)))^{-1/2}$ and $m_0 = (1 + p(s)p^*(s))^{-1/2}$ is a normalized *rcf* of p_0 (taking the outer spectral factors here). So,

$$\ln|n_0(re^{j\theta})| = \frac{1}{\pi}\int_{-\infty}^{\infty} -\frac{1}{2}\ln(1 + 1/|p(j\omega)|^2) P_\theta(\omega/r) d(\ln\omega) \tag{55}$$

and

$$\ln|m_0(re^{j\theta})| = \frac{1}{\pi}\int_{-\infty}^{\infty} -\frac{1}{2}\ln(1 + |p(j\omega)|^2) P_\theta(\omega/r) d(\ln\omega), \tag{56}$$

assuming that p is *real* rational, where the Poisson kernel is defined as

$$P_\theta(k) = \frac{2k(1+k^2)\sin(\theta)}{(1-k^2)^2 + 4\sin^2(\theta)k^2}.$$

(This formula follows from transforming the standard result on the disk to the half plane, and then making the change of variables $\phi = \ln\omega$. Note that the Poisson formula can normally only be applied to functions analytic in the *closed* disk), however for the log of a rational function is it only necessary to be analytic in

[2]This question has, in fact, been studied in some detail by operator theorists (as is noted in [GS90]), and is known as the problem of finding the best constant in the Corona problem, (see [Nik80]p295). It is known that $b_{opt} \geq 1/(1/\tau + 1/\tau^2(7\sqrt{\log 1/\tau} + 20\log 1/\tau))$. Although it is useful to know that a bound in this direction does exist, this one is rather too far away from τ to be useful in giving strength to our argument. It appears that the worst known example of τ being far from b_{opt} (which is due to Tolokonnikov) has $b_{opt} = \tau^2/\sqrt{1-\tau^2} > \tau^2$.

the *open* disk.) The Kernel $P_\theta(\omega/r)$ is plotted in Figure V.7 against logarithmic frequency, for $r = 1$ and values of θ between 0 and $\pi/2$. Note that it is a maximum at $\omega = r$ and is symmetric in $\log \omega$ about this point, with the height of the peak increasing with θ (and, in fact, approaching an impulse at $\omega = r$ as θ approaches $\pi/2$). Thus $|f(re^{j\theta})|$ is determined by a weighted integral, on log/log axis, of $|f(jw)|$ with the weighting greatest at $w = r$ and the contribution from nearby ω greatest for small θ. Even when $\theta = 0$, 87% of the area of the weight is concentrated in a 2 decade range and 96% in a 3 decade range.

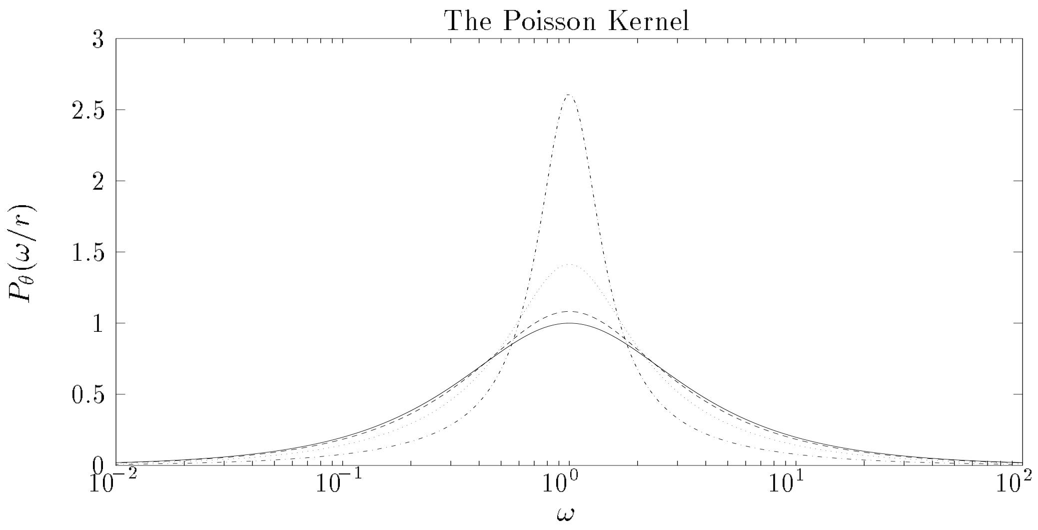


FIGURE V.7. The Poisson Kernel $P_\theta(\omega/r)$ against ω, for $r = 1$
Key: —— is $\theta = 0$, ----- is $\theta = \pi/8$, ... is $\theta = \pi/4$, -·-·- is $\theta = 3\pi/8$

So, (55) implies that $n_0(s)$ is small if, and only if $|p(j\omega)|$ is small at frequencies ω near $|s|$ — any nearby frequencies where $|p(j\omega)|$ is large making no contribution to the integral. Similarly, (56) implies that $m_0(s)$ is small if, and only if $|p(j\omega)|$ is large at frequencies ω near $|s|$.

If we now return to the bound (54) we see that there are four ways in which the bound may be small (guaranteeing that $b_{opt}(p)$ is poor).

A: $|\Omega_z(s)|$ and $|\Omega_p(s)|$ both small for some s.
This corresponds to p having a zero near a *rhp* pole.

B: $|\Omega_z(s)|$ and $|m_0(s)|$ both small for some s.
This corresponds to $|p(j\omega)|$ being large at frequencies around the modulus of a *rhp* zero of p.

C: $|\Omega_p(s)|$ and $|n_0(s)|$ both small for some s.
This corresponds to $|p(j\omega)|$ being small at frequencies around the modulus of a *rhp* pole of p.

D: $|m_0(s)|$ and $|n_0(s)|$ both small for some s.
The only way in which these can both be small is if $|p(j\omega)|$ is both small *and* large at frequencies near the absolute value of s, *ie* $|p(j\omega)|$ is approximately unity around this frequency, with a nonzero average gradient against frequency. This effect will usually be worse for real values of s, as the weight is then more spread out. The greater the average rate of change of $\log|p(j\omega)|$ against $\log(w)$ around such frequencies then the smaller the bound will be.

We now compare these results to those reported in [CF92]. The authors note, by a different reasoning, that $b_{opt}(p)$ is bounded by $|n(s)|$ at *rhp* zeros of p and $|m(s)|$ at *rhp* poles of p, corresponding to our cases B and C, and give a similar interpretation in terms of Poisson integrals. They also give a bound that has a similar interpretation to case D above, although the bound is weaker than the one given here (their bound is in fact equivalent to $b_{opt}(p) \leq |n(s)+xm(s)|$, where $x = 1$, if $\Re(n(s)/m(s)) >= 0$, or $x = -1$, if $\Re(n(s)/m(s)) < 0$. This bound is larger than $\sqrt{|n(s)|^2 + |m(s)|^2}$, with equality achieved only when $p(s) = n(s)/m(s)$ is purely imaginary).

We now give an example where the upper bound is somewhat larger than b_{opt}. This was actually the worst example of $b_{opt}(p)$ found under the constraint that the upper bound should be greater than 1/5. (The reasoning here being that, if the upper bound is less than 1/5, then this is enough in itself to signal that the loop shape of p should be changed). The example does, however, illustrate most of the above points. The plant is

$$p = \frac{(s - 1/v)(s - 1/\bar{v})(s + |v|)^2}{s^3(s - v)(s - \bar{v})(s + 1/|v|^2)}$$

for $v = 2\exp(0.865j\pi/2)$. The Bode plot of $|p(jw)|$ is given in Figure V.8 from which it is seen that the average attenuation rate around the unity gain crossover frequency of $\omega = 1$ is approximately third order. Furthermore, the pair of complex *rhp* zeros are just inside the bandwidth at $\omega = 1/2$ and the pair of complex *rhp* poles are just outside the bandwidth at $\omega = 2$.

In Figure V.9 we show $1/\sqrt{|n(s)|^2 + |m(s)|^2}$ plotted against s, for the real part of s logarithmically spaced in the range 0.1 to $\sqrt{10}$ and the imaginary part of s uniformly spaced in the range -3 to 3. The viewpoint is from above the third quadrant of the complex plane (or, rather, would be if the real axis were not logarithmically spaced!), and the axes are at a height corresponding to 1.0. The rounded hump in the middle corresponds to $s = 1$ and is due to the relatively high attenuation rate, the sharp ridges either side of it peak near (though not actually at) the two *rhp* poles and the other two peaks are due to the *rhp* zeros. The height of the rounded peak is 4.66 and those of the other four are 5.05. So we know that $b_{opt}(p) \leq 1/5.05$. In fact $b_{opt}(p) = 1/14.9$. The discrepancy is probably due to the fact that the *rhp* poles, zeros and the attenuation rate at crossover are all contributing together to the poor b_{opt}; yet their effects on

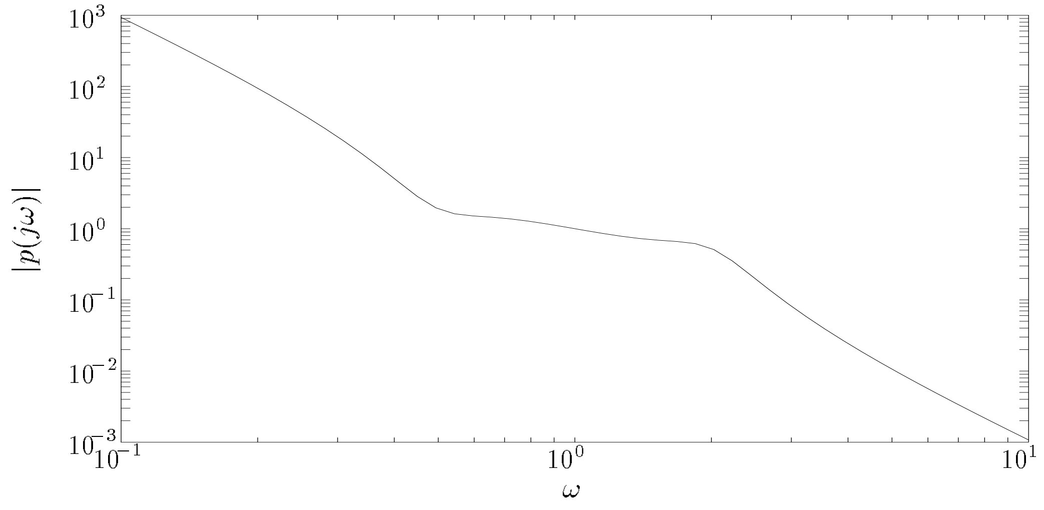


FIGURE V.8. $|p(j\omega)|$ against frequency

the bound over the *rhp* are isolated. Examples where one of our cases A-D is dominant in determining the upper bound tend to give closer ratios of the bound to the true b_{opt}.

Extra insight into the bound of Proposition V.10 is given by the following argument. For any non-zero vector $v \in \mathcal{H}_2^{\perp}$,

$$\|\mathbf{H}_{G_0}\| \geq \frac{\|\mathbf{\Pi}_+ G_0 v\|_2}{\|v\|_2} = \sqrt{1 - \left(\frac{\|\mathbf{\Pi}_- G_0 v\|_2}{\|v\|_2}\right)^2},$$

as G_0 is inner. So,

$$b_{P_0,C} = \sqrt{1 - \|\mathbf{H}_{G_0}\|^2} \leq \frac{\|\mathbf{\Pi}_- G_0 v\|_2}{\|v\|_2}$$

for all $v \in \mathcal{H}_2^{\perp}$, equality being achieved when v is parallel to the right singular vector of $\mathbf{H}_{G_0}$ corresponding to its maximum singular value. In contrast, $\tau(G_0)$ is equal to the infimum of $\|\mathbf{\Pi}_- G_0 v\|_2 / \|v\|_2$ over all vectors of order 1 as the following argument shows. Put

$$v_1 = \frac{w}{s - \alpha},$$

then

$$\mathbf{\Pi}_- G_0 v_1 = \frac{G(\alpha) w}{s - \alpha}$$

(by the cover up rule) and

$$\inf_{w,\alpha:\Re(\alpha)>0} \frac{\|\mathbf{\Pi}_- G_0 v_1\|_2}{\|v_1\|_2} = \inf_{w,\alpha:\Re(\alpha)>0} \frac{\|G(\alpha) w\|_2}{\|w\|_2} = \inf_{\alpha:\Re(\alpha)>0} \underline{\sigma}(G(\alpha)) = \tau(G_0).$$

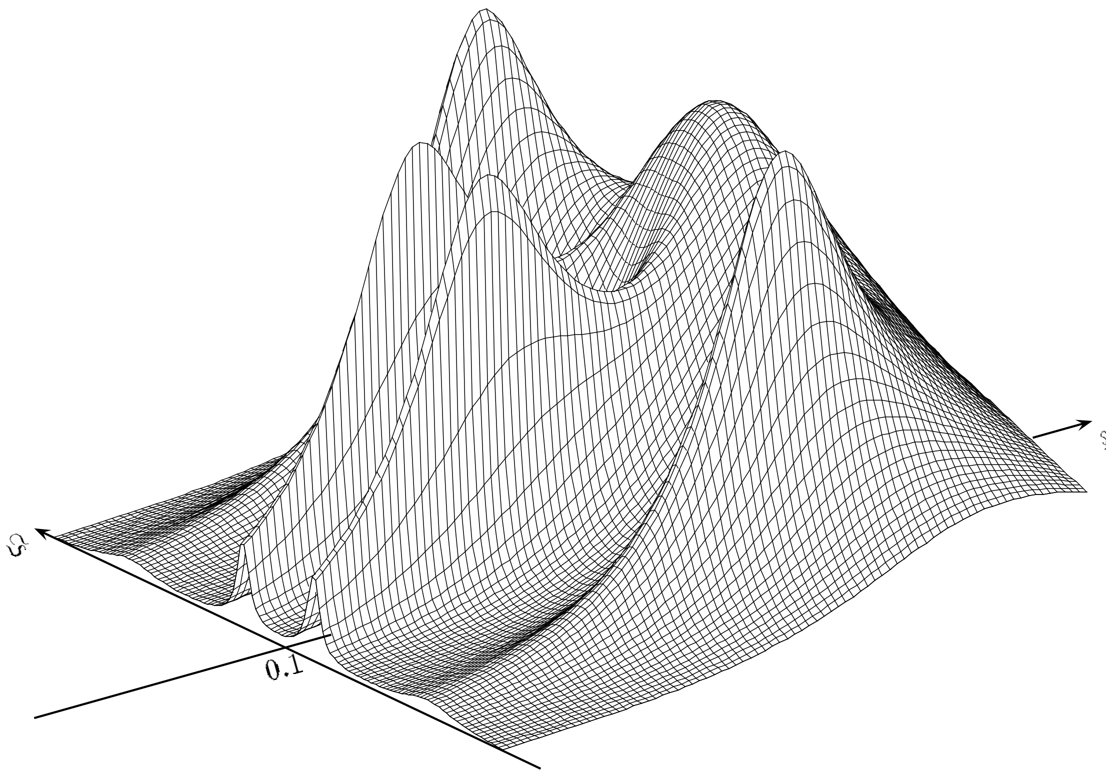


FIGURE V.9. Reciprocal of the upper bound for b_{opt} against s

So, the vector of order 1 that minimizes $\|\mathbf{\Pi}_{-}G_0 v\|_2/\|v\|_2$ puts its pole at the largest peak of $1/\underline{\sigma}(G_0)(s)$. The maximizing right singular vector for $\mathbf{H}_{G_0}$ is easily calculated from the state space formulae given in [Glo89], and is actually of order equal to the McMillan degree of G_0 — with its poles corresponding those of G_0^*. So, if G_0 has McMillan degree one, the bound of Proposition V.10 is always tight. With higher order systems, the minimizing vector for $\|\mathbf{\Pi}_{-}G_0 v\|_2/\|v\|_2$ tends to have its poles around the peaks of $1/\underline{\sigma}(G_0)(s)$. In the above example, the poles of G_0^* are at

$$s = \left\{1,\, \exp(\pm 0.47 j\pi/2),\, 2.05\exp(\pm 0.86 j\pi/2),\, \frac{1}{2.05}\exp(\pm 0.86 j\pi/2)\right\}.$$

Plots of $1/\underline{\sigma}(G_0)(s)$, and evaluation of the maximizing right singular vector for $\mathbf{H}_{G_0}$, form an important part of the extension to the design procedure of McFarlane and Glover that we envisage, as an aid to choosing a suitable loop shaping weight.

2. Frequency response interpretation of δ_ν

In the introduction, it was claimed that an advantage of the metric δ_ν is that it can be regarded as the supremum over frequency of a pointwise distance. We define this pointwise distance as

$$\kappa(P_1(j\omega), P_2(j\omega)) := \bar{\sigma}\left((I + P_1 P_1^*)^{-1/2}(P_2 - P_1)(I + P_2^* P_2)^{-1/2}\right)(j\omega)$$

Thus, the metric can be written as

$$\delta_\nu(P_1, P_2) = \begin{cases} \sup_\omega \kappa(P_1(j\omega), P_2(j\omega)) & \text{if } \operatorname{wno}\det(G_1^* G_2) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

This is important, as it means that the distance between two plants can be obtained directly from frequency response measurements, provided the winding number condition is satisfied. Equally, the distance between a measured frequency response and a state-space realization, or between an infinite dimensional model and a finite dimensional approximation, are easily calculated. Also, whereas the gap metric between two plants is just a single number, the metric δ_ν is seen to be the maximum over frequency of a pointwise distance. The value of this function at other frequencies gives even more information about the difference between two plants.

In contrast to the above, the pointwise gap metric (δ_p) of Qiu and Davison [QD92b] (which is the multivariable generalization of the chordal metric introduced into control theory by El-Sakkary in [ES89]) is the supremum over all s in the *rhp* of the pointwise distance on the right hand side of (2), provided we interpret P^* as $\{P(s)\}^*$ rather than $P^*(s)$. In general, this supremum occurs in the interior of the right half plane rather than on the boundary $s = j\omega$, which makes the metric difficult to calculate and explains why it is conservative compared with the metric δ_ν.

If we restrict our attention to the scalar case, then the pointwise distance κ takes on a particularly sharp form, as then

$$\kappa(p_1(j\omega), p_2(j\omega)) = \frac{|p_1(j\omega) - p_2(j\omega)|}{(1 + |p_1(j\omega)|^2)^{1/2}(1 + |p_2(j\omega)|^2)^{1/2}}.$$

This expression is precisely that of the chordal distance on the Riemann sphere. Hence, provided $\delta_\nu(p_1, p_2)$ is less than one, it is given by the supremum over ω of the distance between the Nyquist diagrams of p_1 and p_2 when projected onto the surface of the Riemann sphere. Similarly,

$$\begin{aligned} \rho(p_0(j\omega), c(j\omega)) &= \frac{|p_0(j\omega) - 1/c(j\omega)|}{(1 + |p_0(j\omega)|^2)^{1/2}(1 + |1/c(j\omega)|^2)^{1/2}} \\ &= \kappa(p_0(j\omega), 1/c(j\omega)) \end{aligned}$$

Hence, $b_{p_0,c}$ is simply the infimum over ω of the chordal distance between the Nyquist diagrams of p_0 and the reciprocal of c on the surface of the Riemann sphere. Figure V.10 illustrates projection onto the Riemann sphere. The Riemann sphere is depicted as a sphere of unit diameter sitting on the origin of the complex plane. To project a point onto the sphere, draw a straight line between the point and the "North pole" of the sphere, the projection of the point is where this line intersects the surface of the sphere. Thus, the "North pole" represents the point at infinity and the unit circle is projected onto the "equator". The chordal metric between two points is the straight line (chordal) distance between their projections.

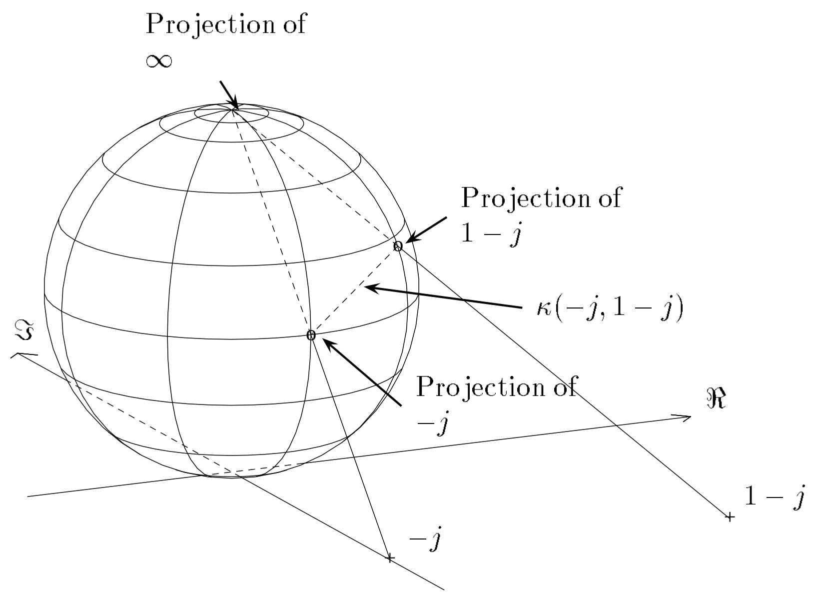
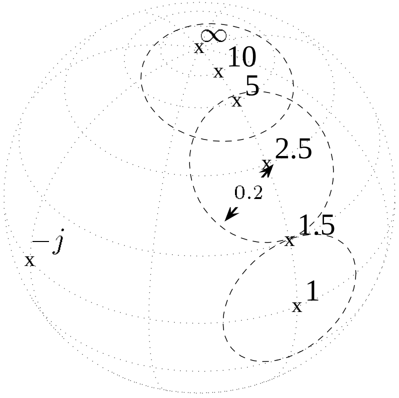


FIGURE V.10. Projection onto the Riemann sphere

Figure V.11 demonstrates how this metric makes sense quantitatively in a control context. Circles of chordal radius 0.2 are drawn around the projected points 0.1, 0.4, 1, 2.5 and 10. At unity gain, an uncertainty of 0.2 represents a small relative uncertainty, as, if $|p_0(j\omega_0)| = 1$ for some ω_0 and $\delta_\nu(p_0, p) \leq 0.2$, then this ensures $0.67 \leq |p(j\omega_0)| \leq 1.5$ and the phase difference between p_0 and p is no more than about 25° at ω_0. However, at frequencies where $|p_0(j\omega)|$ is much greater than one or much less than one, the relative error allowed by $\delta_\nu(p_0, p) = 0.2$ is large. If $|p_0(j\omega)| = 10$ at some frequency then $|p(j\omega)|$ can be as small as 3.25, and is unbounded from above, at that frequency. The phase difference is also unbounded, as the point -10 is included in the circle around the point at 10. Similarly, if $|p_0(j\omega)| = 1/10$ at some frequency then $|p(j\omega)|$ can be as large as 1/3.25, or as small as zero, at that frequency. Again the phase difference is unbounded.



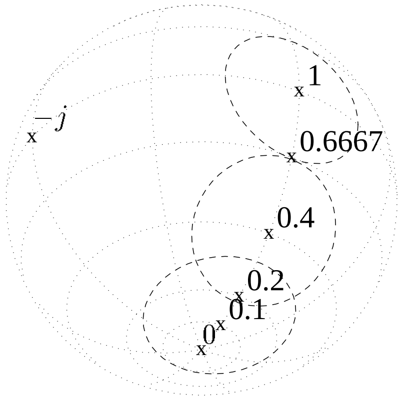


FIGURE V.11. Uncertainty on the Riemann sphere

All this ties in very well with a classical view of control when $\delta_\nu(\cdot,\cdot)$ is applied to a plant that is to be stabilized by a feedback compensator of near to unity gain at all frequencies. At frequencies where the open loop gain is 'large' then the closed loop sensitivity and complementary sensitivity transfer functions will be little changed by a perturbation provided the perturbed open loop gain is still reasonably large. Similarly, at frequencies where the open loop gain is 'small' the closed loop is insensitive to perturbations that leave the open loop gain reasonably small. However, the closed loop *is* very sensitive to phase around the crossover, when the modulus of the open loop gain is near one. Thus, $\delta_\nu(P_1, P_2)$ is a good measure of the *importance* of the difference between P_1 and P_2, were both to be controlled by the same, near unity gain, feedback compensator.

3. Robustness, frequency by frequency

We shall now show how the frequency response interpretation of the metric δ_ν allows it to be linked to the variation in physical parameters of the systems considered. An important point is that, when evaluating the distance $\delta_\nu(P_0, P)$ the winding number condition will always be satisfied provided the system P can be obtained by continuously perturbing parameters of the system P_0 without κ becoming 1 on the path at some frequency. The following theorem states this more formally.

THEOREM V.11. *Given a metric space Λ, a mapping*

$$\Lambda \to \mathcal{P}^{p\times q} \quad \lambda \mapsto P_\lambda,$$

continuous in the graph topology, and a pathwise connected closed subset $\bar{\mathcal{U}} \subset \Lambda$, with $\lambda_0 \in \bar{\mathcal{U}} \mapsto P_0$, then

$$\sup_{\lambda\in\bar{\mathcal{U}}} \delta_\nu(P_0, P_\lambda) = \sup_{\omega} \sup_{\lambda\in\bar{\mathcal{U}}} \kappa(P_0(j\omega), P_\lambda(j\omega)),$$

PROOF: This follows from continuity arguments, since if there exists a $\lambda \in \bar{\mathcal{U}}$ such that $\operatorname{wno}\det(G_\lambda^* G_0) \neq 0$ there will also be a $\lambda \in \bar{\mathcal{U}}$ such that $\det(G_\lambda^* G_0) = 0$ at some frequency. $\square$

REMARK V.12. The condition that the mapping $\lambda \mapsto P_\lambda$ needs to be continuous is not a serious constraint in practice. For example, if $\Lambda = \mathbb{R}^n$, *ie* the plant depends on a number of real parameters, this continuity is implied if the mapping is continuous in the topology induced by the Euclidean norm on the state-space matrices of P_λ (see [Vid85] p252), which is the case for the real μ set up of Doyle (see [Doy82]). A result of this argument is that the calculation of $\sup_{\lambda \in \bar{\mathcal{U}}} \delta_\nu(P_0, P_\lambda)$, when the mapping is polynomial onto the elements of the state-space representation of P_λ, is equivalent to calculating μ with several real blocks and one nontrivial complex block. This is the case considered in [FTD91], where it is shown that the calculation of good bounds, at least, is reasonably tractable.

The previous theorem shows that, even in the case of complicated uncertainty sets, we can measure the "size" of an uncertainty set frequency by frequency - given mild continuity assumptions. Let us now consider how we may use this "frequency by frequency" idea in the synthesis of robust controllers. The first step is to give a more general version of the our main robust stability result so far, Theorem III.7. This theorem effectively requires that

$$\inf_\omega \underline{\sigma}(\tilde{K} G_0) > \sup_\omega \bar{\sigma}(\tilde{G}_0 G_1)).$$

However, a glance at the proof of Theorem III.5 reveals that the condition

$$\underline{\sigma}(\tilde{K} G_0) > \bar{\sigma}(\tilde{G}_0 G_1) \quad \forall \omega$$

is sufficient to guarantee the stability of $[P_1, C]$. So we can recast the theorem as follows, to take advantage of this frequency by frequency interpretation. For convenience, given a P_0 and an $f(\omega)$ satisfying $0 \leq f(\omega) < 1$ for all ω, we define a set of plants

$$\bar{\mathcal{B}}(P_0; f) := \{P : \delta_\nu(P_0, P) < 1, \kappa(P_0(j\omega), P(j\omega)) \leq f(\omega) \forall \omega\}.$$

THEOREM V.13. *Provided there exists a $g(s)$ satisfying $g, g^{-1} \in \mathcal{H}_\infty$ and $|g(j\omega)| = f(\omega)$:*

i) *Given a plant P_0, and a compensator C, then:* $b_{P_1,C} > \alpha$ *for all* $P_1 \in \bar{\mathcal{B}}(P; f)$ *if, and only if,* $[P_0, C]$ *is stable and*

$$\arcsin \rho(P_0, C) > \arcsin f(\omega) + \arcsin \alpha \text{ for all } \omega.$$

ii) *Given plants P_0 and P_1 then, provided there exists a C such that $[P_0, C]$ is stable and which satisfies $\rho(P_0, C) > \arcsin f(\omega) + \arcsin \alpha$ for all ω:* $b_{P_1,C} > \alpha$ *for all C such that $[P_0, C]$ is stable, and which satisfy*

$$\arcsin \rho(P_0, C) > \arcsin f(\omega) + \arcsin \alpha \text{ for all } \omega$$

if, and only if, $P_1 \in \bar{\mathcal{B}}(P_0; f)$.

PROOF: Sufficiency for both parts is immediate from the proof of Theorem III.5. Necessity requires constructions that are, in principle, no more involved than those of Theorem III.7, and so are omitted. (In fact, it only really necessary to extend Lemma III.9 part i) to allow for the possibility of constructing a P_1 satisfying

$$\kappa(P_0, P_1)(j\omega)/\sqrt{1-\kappa^2(P_0, P_1)(j\omega)} \leq f(\omega)\,\forall\omega \text{ and } (\tilde{G}_0 G_1)(G_0^* G_1)^{-1}(j\omega_0) = T$$

for some T satisfying $\bar{\sigma}(T) \leq f(j\omega_0)$, and extend part ii) similarly.) □

This theorem is the basis of a design methodology to be employed in the next section.

4. An extended loop shaping design procedure

In Section 1 we gave motivation for the maximization of $b_{P,C}$ as a performance measure, even in the absence of plant uncertainty. If, instead of a single known plant, we have a nominal plant P_0 and an uncertain set of plants $\mathfrak{P}$ (i.e. $P_0 \in \mathfrak{P}$ and all we know about the 'real' plant P is that $P \in \mathfrak{P}$), then a sensible approach is to try to find

$$\sup_{C} \inf_{P \in \mathfrak{P}} b_{P,C}.$$

Depending on the nature of the set $\mathfrak{P}$, this can be a very difficult problem. The first stage of our design procedure is to embed the set $\mathfrak{P}$ inside a set with nicer properties, as follows. Obtain $f(w)$ as the maximum distance from the nominal plant P_0 to the boundary of the set, frequency by frequency. So,

$$f(\omega) := \sup_{P \in \mathfrak{P}} \kappa(P_0(j\omega), P(j\omega))$$

This could be obtained from frequency response measurements of a finite number of perturbed plants, or from a μ-analysis as in Remark V.12 for example, depending on the nature of the set $\mathfrak{P}$. As before, define a set of plants

$$\bar{\mathcal{B}}(P_0; f) := \{P : \delta_\nu(P_0, P) < 1, \kappa(P_0(j\omega), P(j\omega)) \leq f(\omega)\forall\omega\}.$$

Then, provided $\mathfrak{P}$ is pathwise connected in the graph topology — a mild assumption — we have that $\mathfrak{P} \subset \bar{\mathcal{B}}(P; f)$. It is then easy to find

$$\sup_{C} \inf_{P \in \bar{\mathcal{B}}(P_0; f)} b_{P,C}, \tag{57}$$

or the compensator that uniformly maximizes the residual stability/performance margin over the enclosing set. According to Theorem V.13, all we need to find is the largest α such that there exists a C satisfying

$$[P_0, C] \text{ is stable and } \arcsin\rho(P_0, C) > \arcsin f(\omega) + \arcsin\alpha \text{ for all } \omega.$$

Or, equivalently, the largest α such that

$$\inf_{C stabilizing} \sup_{\omega} w(\omega)/\rho(P_0, C) < 1. \tag{58}$$

where $w(\omega) = \sin(\arcsin f(\omega) + \arcsin \alpha)$. For fixed α, the infimum can be found explicitly, as it is equivalent to a "one-block" $\mathcal{H}_\infty$ optimization problem. To see this, recall that $\bar{\sigma}\{(\tilde{K}G_0)^{-1}\tilde{K}\tilde{G}_0^*\}(j\omega) = \sqrt{1 - \rho^2(P_0, C)(j\omega)}/\rho(P_0, C)(j\omega)$, and substitute the Youla parameterization $\tilde{K} =$ inner factor of $(\tilde{X}_0 + Q\tilde{G}_0)$, where $Q \in \mathcal{H}_\infty$ and $\tilde{X}_0 \in \mathcal{H}_\infty$ is such that $\tilde{X}_0 G_0 = I$. Then $(\tilde{K}G_0)^{-1}\tilde{K}\tilde{G}_0^* = \tilde{X}_0\tilde{G}_0^* + Q$ and (58) holds if, and only if,

$$\inf_{Q \in \mathcal{H}_\infty} \left\| w_1(\tilde{X}_0\tilde{G}_0^* + Q) \right\|_\infty < 1, \tag{59}$$

where $w_1(s)$ is the outer spectral factor of $|w|^2/(1-|w|^2)$. Solving this problem in the frequency domain via Nevanlinna-Pick interpolation means that it is not even necessary to fit the weight w_1 with a rational transfer function. It *is* necessary to determine $w_1(j\omega)$, and $w_1(s)$ at the *rhp* poles of $\tilde{G}_0^*$, but this can be done via a Hilbert transform, and Poisson integrals, respectively. This is a considerably more straightforward task provided $f(\omega)$ has been determined on a sensible grid of frequencies to begin with. Note that the infimum in (59) varies monotonically with α; and so the solution to (57) is easily obtained by iterating over α.

The first stage of the design will have given us a compensator C such that

$$\inf_{P \in \bar{\mathcal{B}}(P; f)} \rho(C(j\omega), P(j\omega)) = \alpha \forall \omega,$$

whereas

$$\inf_{P \in \mathfrak{P}} \rho(C(j\omega), P(j\omega)) = g(\omega) \geq \alpha \forall \omega$$

At frequencies where $g(\omega) > \alpha$, indicating an excess stability margin, the radius function $f(\omega)$ can be relaxed, by putting

$$f_1(\omega) = \sin\left(\arcsin f(\omega) - \epsilon(\arcsin g(\omega) - \alpha)\right),$$

for some small ϵ. The first stage is now repeated with this f and the procedure iterated until $g(\omega)$ is flat at all frequencies. There seems no reason why this process should converge for general $\mathfrak{P}$. However, it does seem to work well in practice, as demonstrated by the benchmark design example covered next.

5. A benchmark design example

The plant consists of two unit masses constrained to slide rectilinearly on a frictionless table and coupled by a spring of uncertain stiffness k, as shown in Figure V.12. Alternatively, it could consist of two flywheels rotating on frictionless bearings and connected by a shaft with torsional stiffness k. The input is the force on the first mass and the measurement is of the position of the second mass. This example has been considered by a number of authors (see, for example, [WB92] and the references therein). Two sets of performance specifications are given, resulting in two distinct designs.

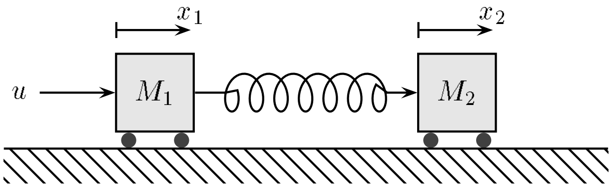


FIGURE V.12. The benchmark design problem

The plant is given by

$$P(s) = \frac{k}{s^2(s^2+2k)}$$

with $0.5 \leq k \leq 2$ and the nominal plant at $k = 1$, *ie*

$$P_0(s) = \frac{1}{s^2(s^2+2)}.$$

Design 1. The problem is to design a compensator that gives a settling time of around 15 seconds to an impulsive disturbance at the plant input for the whole range of k. We shall design a compensator achieving the minimum open loop bandwidth consistent with this. The loop shaping weight is chosen as follows. A constant gain w_1 is chosen to give a gain crossover frequency for the nominal plant of $\omega_c = 2.5$, just above the maximum frequency of the resonant mode. The rationale here is that, if the undamped resonance is to be controlled, then it must be within the control bandwidth. Some dynamic weighting is required. The nominal plant itself provides double integral action, however, it is necessary to roll off the loop gain faster above the crossover to reduce the control effort in response to an impulsive disturbance at the output. A second order roll off above $\omega_c \times 2.5$ was used. In addition, the attenuation rate of the nominal plant at crossover is $80dB$ per decade, and so this needs to be reduced. Thus the chosen loop shaping weight is

$$w(s) = w_1 w_2(s) = 30.8125 \frac{(2.5s+2.5)^4}{(s+6.25)^6}.$$

Note that this weight is fixing the performance objectives in the frequency domain — the choice of crossover frequency determining the control bandwidth, and the order and severity of the lead network determining the trade off between loop shape degradation and stability margins at crossover as discussed in Section 1. (The original specifications were, of course, given in the time domain — here we have roughly approximated them in the frequency domain, and it will be necessary to check after the design that the original specifications are satisfied.)

Let us now consider the problem of providing robust performance. Denote $P|_{k=.5}$ by P_l and $P|_{k=2}$ by P_u, then, in this case $\delta_\nu(P_0 w, P_l w) = .5624$ and $\delta_\nu(P_0 w, P_u w) = .5036$. The upper curve in Figure V.13 is the maximum of $\kappa(P_0 w, P_l w)$ and $\kappa(P_0 w, P_u w)$ against frequency (note that $\delta_\nu(P_0 w, P_l w)$ and $\delta_\nu(P_0 w, P_u w)$ are the two peaks of this curve) and clearly shows that it is not necessary to make $\rho(P_0 w, C) > .5624$ at all frequencies, but only at frequencies around the chosen crossover frequency.

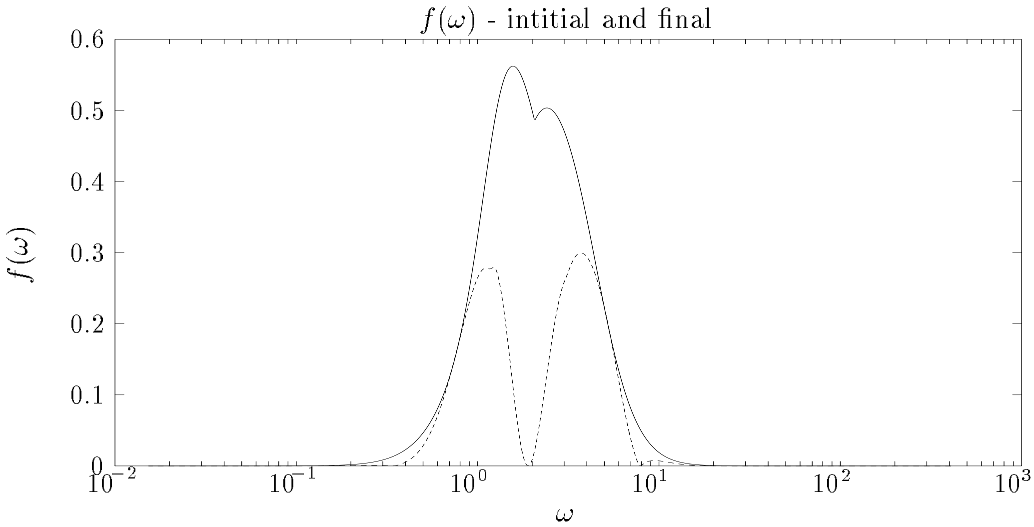


FIGURE V.13. $f(\omega)$ *Key:* —— initial, ----- final

We now proceed to design a robust compensator using the above procedure. The weight $f(\omega)$ converged rapidly to the lower curve in Figure V.13, with $\alpha = 0.229$. Thus, the final compensator is guaranteed to achieve $b > .229$, with the above weights, for all perturbations. This compensator was then approximated with a seventh order rational transfer function with negligible error. This approximated compensator has a DC gain of 0.1839 with poles and zeros as follows;

poles	zeros
-8.9609	-0.1878
$-0.7407 \pm 1.1639i$	$-0.9277 \pm 1.5099i$
$-12.4311 \pm 12.2987i$	$0.0219 \pm 0.7638i$
$-2.0679 \pm 7.8204i$	

Figure V.14 shows the Bode magnitude plot of the nominal plant, scaled up to give the desired crossover, and the final open loop gain given by the product of the nominal plant and this compensator. The nominal gain and phase margins are $11.7dB$ and $34.2°$. Figures V.15 and V.16 show the time responses for the three plants (x_2 is the position of the second mass), with $k = 1$ (solid), $k = 0.5$ (dashed) and $k = 2$ (dotted), for an impulsive disturbance on the first mass. The design is clearly very robust. Insensitivity to sensor noise is guaranteed by the bounds on the complementary sensitivity.

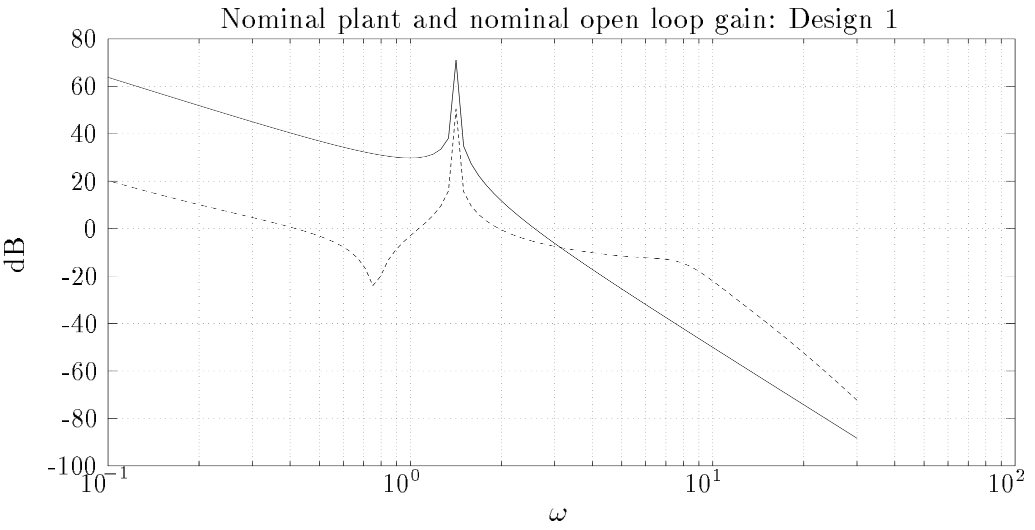


FIGURE V.14. Nominal plant and nominal open loop gain: Design 1

Key: —— is $|30.8125 \times P_0(j\omega)|$, ----- is $|P_0(j\omega)C(j\omega)|$

Design 2. Here, the objective was to track a step reference, with as short a rise time as possible, subject to $|u| < 1$. Similar weights to those for the first design were used, although with ω_c pushed up to $5rad/s$. The final compensator has a DC gain of 3.1846 with poles and zeros as follows;

poles	zeros
-39.1275	$-1.8263 \pm 15.7319i$
$-19.3232 \pm 20.1139i$	$-0.6328 \pm 5.4039i$
$-4.6833 \pm 16.1871i$	$-2.8091 \pm 2.6104i$
$-1.5237 \pm 15.7114i$	$-0.4353 \pm 1.0216i$
$-0.6395 \pm 5.3790i$	-0.5488
$-1.7123 \pm 2.2164i$	

This compensator was used in the feedback path, with a simple precompensator of $1/(s + 1)$ conditioning the reference. Figures V.17 and V.18 show the time responses for the three plants, with $k = 1, 0.5$ and 2.

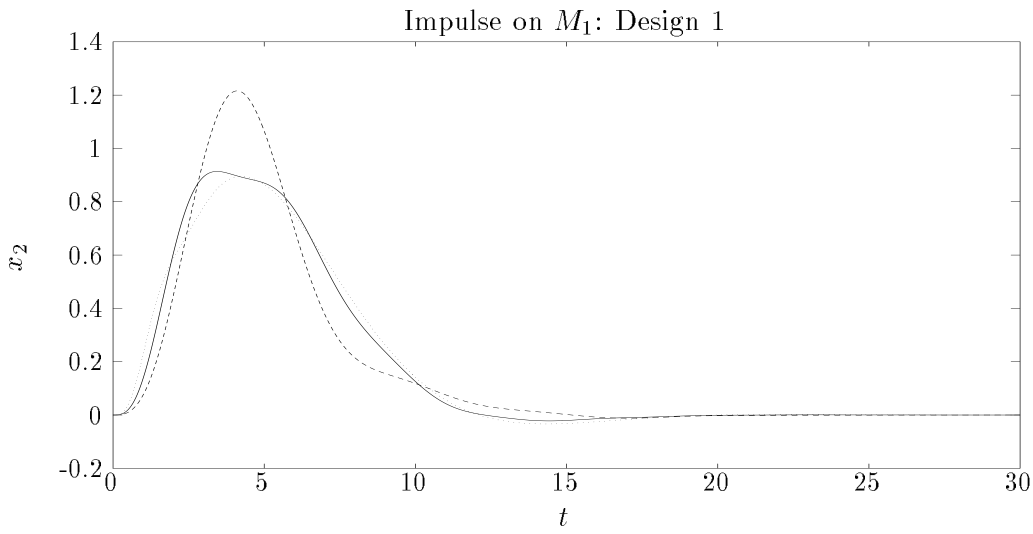


FIGURE V.15. Response of x_2 to impulse at plant input
Key: —— is with $k = 1$, ----- is with $k = 0.5$, ········ is with $k = 2$

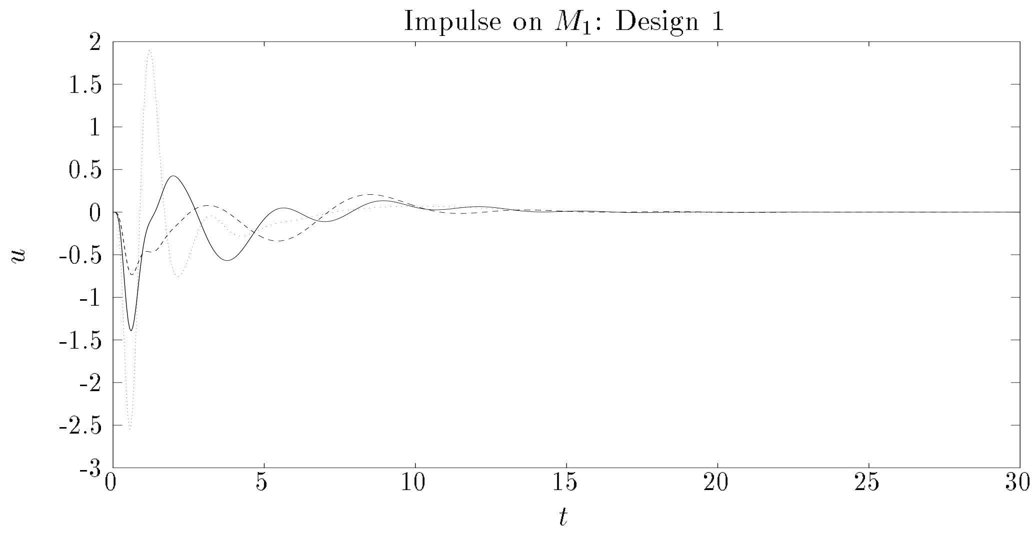


FIGURE V.16. Response of u to impulse at plant input
Key: —— is with $k = 1$, ----- is with $k = 0.5$, ········ is with $k = 2$

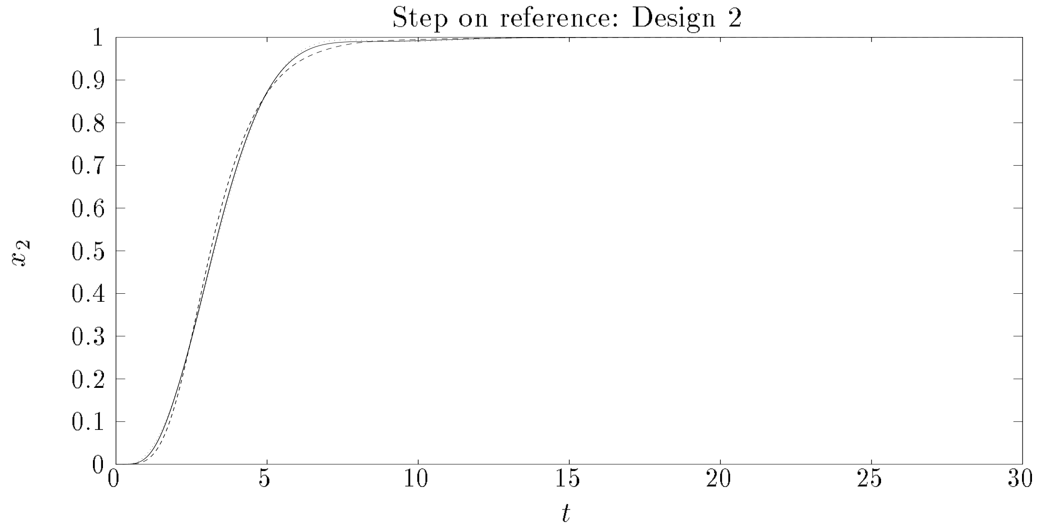


FIGURE V.17. Response of x_2 to reference
Key: —— is with $k = 1$, ----- is with $k = 0.5$, ······· is with $k = 2$

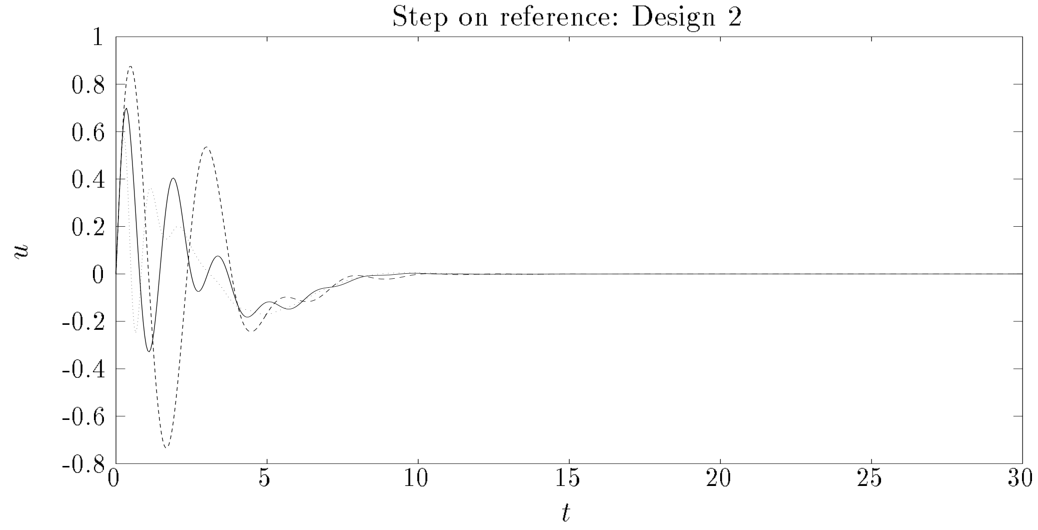


FIGURE V.18. Response of u to reference
Key: —— is with $k = 1$, ----- is with $k = 0.5$, ······· is with $k = 2$

6. Robust tracking

Apart from part 2 of the preceding design example we have only, so far, looked at the so-called regulator, or disturbance rejection problem — where the control objective is to maintain the output of the plant near zero (or, equivalently, near some fixed setpoint) in the face of disturbances acting on the plant and uncertainties in the plant description. In this section we shall look at the tracking problem, where the objective is to force the output of the plant to follow a reference trajectory. The robust version of this problem is to maintain this behaviour in the face of disturbances and uncertainty. There are a number of versions of the robust tracking problem, which depend essentially on how much is known about the reference trajectory when the control scheme is designed. If the reference trajectory is known in advance then the problem is equivalent to the regulator problem, but with a time-varying setpoint. More realistically, the reference trajectory may be known to lie within a class of signals, for example steps, ramps, higher order polynomials, sinusoids or combinations of these. The natural class of references to consider in our framework is those of bounded energy. This class may be criticized as being both too small, including none of the above list, and too large, including trajectories that vary arbitrarily quickly and are unbounded pointwise as functions of time. These are valid criticisms, although it should be pointed out that steps, ramps etc do have bounded energy on any finite interval and that we can restrict the frequency content of $\mathcal{L}_2$ signals using suitable weights. The advantage of this approach, as we shall see, is that if we ask the right question we get an explicit solution — this can give substantial insight into the general problem.

The nominal tracking problem, in the absence of plant uncertainty and disturbances, is essentially an open loop problem. We shall start by looking at the set of achievable trajectories. Let $\{N, M\}$ be a normalized *rcf* of P, then the graph of P is given by

$$\mathcal{G}(\mathbf{M}_P) = \begin{bmatrix} N \\ M \end{bmatrix} \mathcal{H}_2.$$

Thus, if the signals at the plant input are to be bounded (in $\mathcal{H}_2$), then the only possible bounded outputs that the plant can produce are of the form Nq for some $q \in \mathcal{H}_2$. So, the set of all possible operators from a reference to the plant output, given any combination of pre- post- or feedback compensators, is at most $N\mathbf{Q}$ for some bounded operator $\mathbf{Q}$. The set of linear responses is $NQ \ : \ Q \in \mathcal{H}_\infty$. Note that N contains all the right half plane zeros of the plant, which thus provide a fundamental limitation, in fact the only limitation, on the set of achievable responses. It should be emphasised that these limitations are not a particular consequence of interpreting boundedness in terms of energy, the arguments continue to hold if we characterize boundedness pointwise in the time domain. The design of a suitable Q is relatively straightforward, and would normally be done in the time domain. The full set of achievable responses can be achieved in an open loop configuration, using a precompensator MQ as in Figure V.19, even if the plant itself is unstable.

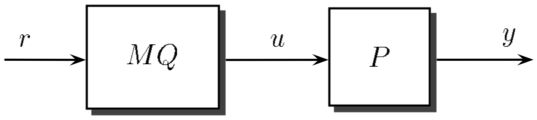


FIGURE V.19. Open Loop Scheme Achieving $y = NQr$

The open loop scheme can, of course, be extremely sensitive to uncertainty in the plant description, particularly so if the plant is unstable, and does nothing to lessen the effect of disturbances acting on the plant. The standard way to reduce these effects is to use feedback. Assume, for the moment, that we have designed a suitable feedback compensator C to achieve these aims. If we wish to add this compensator without affecting the nominal tracking response then the closed loop should be implemented as in Figure V.20, where $C = \tilde{V}^{-1}\tilde{U}$ is an *lcf* (not necessarily normalized) of C satisfying the Bezout identity $\tilde{V}M - \tilde{U}N = I$. That such a factorization exists is equivalent to C being a stabilizing compensator for P. This scheme includes Figure V.19 as a special case, with $\tilde{V} = M^{-1}$, $\tilde{U} = 0$, which is an *lcf* of the zero compensator provided P is stable. So, in Figure V.20 we have

$$\begin{aligned} y &= P(I - CP)^{-1}\tilde{V}^{-1}Qr \\ &= N(\tilde{V}M - \tilde{U}N)^{-1}Qr = NQr \end{aligned}$$

as required.

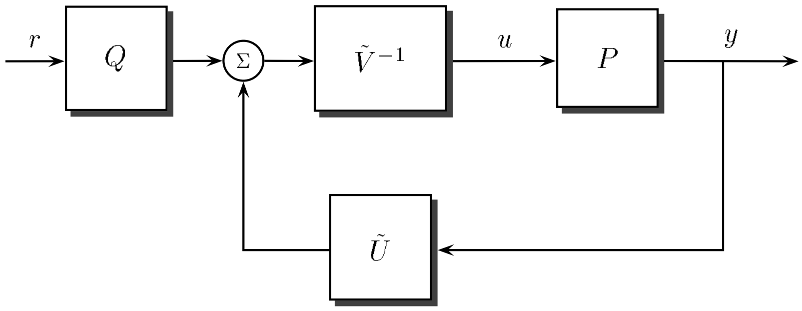


FIGURE V.20. Closed Loop Scheme achieving $y = NQr$

This illustrates the well known result that the disturbance rejection problem is entirely decoupled from the nominal tracking problem (see [Vid85] and the references therein). If the feedback part of the compensator is implemented in any other way then the set of achievable tracking responses may be compromised. In particular, if the feedback compensator is not minimum phase and is implemented entirely in the forward path (the position occupied by $\tilde{V}^{-1}$) then its *rhp* zeros must appear in the transfer function from r to y, for any Q. Similarly, if the feedback compensator is unstable and is implemented entirely in the feedback path (the position occupied by $\tilde{U}$), then these unstable poles will always appear as *rhp* zeros of the tracking response.

We shall now consider the problem of designing a feedback compensator to minimize the effect of plant uncertainty on the tracking response. We first point out that, given any reasonable Q, we can choose a weight W so that $N_W = NQ$ where $\{N_W, M_W\}$ is a normalized *rcf* of PW. In fact, given any outer $Q \in \mathcal{RH}_\infty$ such that $\|NQ\|_\infty \leq 1$ and $\det(I - (NQ)^*NQ)$ is not identically zero[3], choose W such that W, W^{-1} are analytic in the open *rhp* and

$$WW^* = MQ(I - (NQ)^*NQ)^{-1}(MQ)^*. \tag{60}$$

Then, $N_W = NQ, M_W = W^{-1}MQ$ is a normalized *rcf* of PW as required. (It would normally be regarded as undesirable to introduce extra *rhp* zeros in Q, or for the gain of the nominal tracking response to be greater than one at any frequency - this corresponding to overshoot in the frequency domain). So, in the sequel we shall assume that $Q = I$, and that the plant has been suitably weighted so that the numerator of its normalized coprime fraction gives the desired tracking response. It is actually straightforward to find a suitable W directly (rather than first choosing a Q) using classical loop shaping ideas. Note that

$$\sigma_i(N_W(j\omega)) = \frac{\sigma_i(PW)}{\sqrt{1 + \sigma_i^2(PW)}}(j\omega). \tag{61}$$

So, we need to choose W such that $\underline{\sigma}(PW) \gg 1$ in frequency ranges where the tracking response is required to be near to unity, *ie* in the bandwidth, and $\bar{\sigma}(PW) \ll 1$ outside the bandwidth. As $\bar{\sigma}(N_W) \leq 1$ everywhere, the response is guaranteed to be well behaved in between.

We now evaluate the tracking error resulting from perturbations to the plant, and then show that the compensator which minimizes this error over a ball of uncertainty of any radius in the metric $\delta_\nu(\cdot,\cdot)$ is precisely the one which achieves $\sup_C b_{P,C}$, *ie* the compensator which achieves optimal robust stability in the face of the same class of perturbations. The transfer function from r to u and y in Figure V.20, with $Q = I$ and $C := \tilde{V}^{-1}\tilde{U}$, where $\tilde{V}M - \tilde{U}N = I$, is given by

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} P_0 \\ I \end{bmatrix} (I - CP_0)^{-1}\tilde{V}^{-1}r = \begin{bmatrix} N_0 \\ M_0 \end{bmatrix} r.$$

[3] It would normally be regarded as undesirable to introduce extra *rhp* zeros in Q, or for the gain of the nominal tracking response to be greater than one at any frequency - this corresponding to overshoot in the frequency domain.

Define E, the error due to replacing P_0 with P_1, as

$$E(P_0, P_1, C) := \begin{bmatrix} N_0 \\ M_0 \end{bmatrix} - \begin{bmatrix} P_1 \\ I \end{bmatrix} (I - CP_1)^{-1}\tilde{V}^{-1}. \tag{62}$$

There are two main reasons for including the error at the plant input in this criterion. Firstly, in choosing the nominal tracking response it is presumed that the designer has paid attention to the resultant signals at the plant input, in order to ensure that saturation, rate limit and energy constraints are not violated on the expected reference trajectories. It would then be desirable for the performance at the plant input not to degrade substantially in the face of plant uncertainty. Secondly, as we shall see, this formulation leads to a cleaner solution.

THEOREM V.14. *Define the tracking error $E(P_0, P_1, C)$, due to P_0 being perturbed to P_1, as in (62) where $\{\tilde{U}, \tilde{V}\}$ is an* lcf *of C satisfying $\tilde{V}M_0 - \tilde{U}N_0 = I$. Then, for any $\alpha < b_{P_0,C}$,*

$$\sup_{P_1:\delta_\nu(P_0,P_1)\leq\alpha} \|E(P_0, P_1, C)\|_\infty = \frac{\alpha}{\sin(\arcsin b_{P_0,C} - \arcsin \alpha)}.$$

PROOF: We can rewrite E as follows:

$$E = K(\tilde{G}_1 K)^{-1}\tilde{G}_1 G_0(\tilde{K}G_0)^{-1}\tilde{M}_c\tilde{V}^{-1}$$

from (42) and (43), where $\tilde{K}$ and K are *normalized* left and right inverse graph symbols for C. Note that $\begin{bmatrix}\tilde{U} & \tilde{V}\end{bmatrix} = Q\tilde{K}$ for some $Q, Q^{-1} \in \mathcal{H}_\infty$, so

$$(\tilde{K}G_0)^{-1}\tilde{M}_c\tilde{V}^{-1} = (Q\tilde{K}G_0)^{-1}Q\tilde{M}_c\tilde{V}^{-1} = I.$$

Thus,

$$\begin{aligned}\|E\|_\infty &= \|(\tilde{G}_1 K)^{-1}\tilde{G}_1 G_0\|_\infty \\ &\leq \frac{\delta_\nu(P, P_1)}{b_{P_1,C}} \leq \frac{\delta_\nu(P, P_1)}{\sin(\arcsin b_{P,C} - \arcsin \delta_\nu(P, P_1))}.\end{aligned} \tag{63}$$

We shall now show that this bound is tight. Denote by ω_0 the frequency at which the norm of $G_0^* K$ is achieved, and let

$$\tilde{G}_0 K(j\omega_0) = U \operatorname{diag}\{b_{P_0,C}, \ldots\}W^*$$

and

$$G_0^* K(j\omega_0) = V \operatorname{diag}\{\sqrt{1 - b_{P_0,C}^2}, \ldots\}W^*$$

be singular value decompositions (noting that $\left(\begin{bmatrix} G_0^* \\ \tilde{G}_0 \end{bmatrix} K\right)^* \left(\begin{bmatrix} G_0^* \\ \tilde{G}_0 \end{bmatrix} K\right) = I$). Now construct an Ω satisfying $\Omega G_0^* \in \mathcal{H}_\infty$, $U^*\Omega(j\omega_0)V = -\alpha/\sqrt{1-\alpha^2}I$ and $\|\Omega\|_\infty = \alpha/\sqrt{1-\alpha^2}$ and put

$$\tilde{G}_1 = \text{ inner part of } (\tilde{G}_0 + \Omega G_0^*).$$

Then $\tilde{G}_1$ is a normalized left graph symbol of a P_1 satisfying $\delta_\nu(P_0, P_1) \le \alpha$ as in a dual version of Lemma III.9, since $\alpha < b_{opt}(P_0)$.

For this P_1, direct substitution in (63) yields

$$\|E(P_0, P_1, C)\|_\infty \ge \alpha / \left(\sqrt{1-\alpha^2} b_{P_0,C} - \alpha\sqrt{1-b_{P_0,C}^2}\right),$$

as required. □

So, to minimize the norm of the tracking error $\|E\|_\infty$ over a ball in the modified $\mathcal{L}_2$-gap metric, δ_ν, it is merely necessary to choose a feedback compensator that is optimally robust to those perturbations, *ie* a compensator achieving $\sup_C b_{P,C}$!. Notice that (61), and the bounds on the loop shape degradation, imply that this compensator attempts to match the open loop crossover frequency to the closed loop $3dB$ bandwidth — this is regarded as good practice.

Note that the scheme of Figure V.20 does not inflate the McMillan degree of the compensator above the McMillan degree of the coprime factors $\tilde{V}, \tilde{U}$, as it may be implemented with internal feedback as

$$u = \begin{bmatrix} \tilde{U} & I - \tilde{V} \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} + Qr. \tag{64}$$

However, the degree of $[\tilde{V}, \tilde{U}]$ may in general be higher than the degrees of PW and C. An important case where the degree of $[\tilde{V}, \tilde{U}]$ is precisely equal to that of C (and, in fact, that of PW) is when C is the so-called central (or maximum entropy) controller amongst the set satisfying $b_{P,C} > \beta$ for any $\beta < b_{opt}(P)$. In this case (64) is actually a state space observer form for the controller, for which we give formulae in the appendix.

We finish with a few words on decoupling in the multivariable case, for a system with at least as many inputs as outputs. The above method automatically achieves good decoupling in frequency regions where all the $\sigma_i(PW)(j\omega) \gg 1$, as then $\sigma_i(N_W)(j\omega) \approx 1$ and so N_W is close to being unitarily equivalent to the identity. In particular, if $\lim_{w\to 0} 1/\sigma_i(PW(j\omega)) = 0$, *ie* the weighted plant has integrators in all channels, then we can always choose a real unitary matrix Q such that $N(0)Q = I$. To get exact decoupling for square plants, and maintain it through crossover, choose an extra compensator of $Q = N_{adj}$ (the matrix of cofactors of N). The transfer function from r to y will then be $\det(N)I$. Since $\|N_{adj}\| \le 1$, the bounds on the tracking error are still applicable and the scheme is certainly robust to uncertainty in the metric δ_ν. The result is to equalize the bandwidth in each direction to that of the worst case direction.

CHAPTER VI

Optimal Compensators and Nearest Approximations

It is noted in [Glo86] that the largest stabilizable $\mathcal{H}_\infty$ additive uncertainty ball around a nominal plant has a nonstabilizable plant on the boundary. That is, if we define an $\mathcal{H}_\infty$ additive uncertainty ball as

$$\mathcal{B}_{\mathcal{A}}(P_0, b) := \{P : \|P - P_0\|_\infty < b,\ \eta(P) = \eta(P_0)\},$$

and define $b_{max}(P_0)$ as the largest b for which there exists a single C that stabilizes every $P \in \mathcal{B}_{\mathcal{A}}(P_0, b)$, then there are plants in $\mathcal{B}_{\mathcal{A}}(P_0, b_{max})$ that are arbitrarily close to having an unstable pole/zero cancellation. This implies that a time-varying or nonlinear compensator can do no better [KGP87]. It is further noted in [MG90] that this is not necessarily the case for normalized coprime factor uncertainty, (and hence uncertainty in the gap metric as well).

The main result of the first section of this chapter (Theorem VI.5) shows that if we use the metric $\delta_\nu(\cdot,\cdot)$, rather than the $\mathcal{H}_2$-gap metric, then we *do* find a nonstabilizable plant on the boundary. Therefore, the radius of the largest stabilizable ball in the metric $\delta_\nu(\cdot,\cdot)$ is precisely the distance in this metric to the nearest nonstabilizable plant; and a nonlinear, time-varying controller can do no better. Furthermore, it will be shown that this value is also the distance to the nearest reduced degree approximation of the conjugate transpose of the nominal plant (as measured in the same metric). In fact the nearest nonstabilizable plant is constructed from the conjugate transpose of this approximation by adding to it an arbitrarily close *rhp* pole/zero cancellation.

In the second section we go on to look at the problem of finding approximations of lower degree in the metric δ_ν. In the main result (Theorem VI.6) we give upper and lower bounds on the achievable error, together with two approximation schemes based on Hankel norm approximation, one of which is guaranteed to achieve the upper bound. The other, in general, gives approximation errors significantly nearer to the lower bound; however, we have not been able to prove that it always achieves the upper bound (although this is almost certainly true). These results, though seemingly of little relation to those of the first section, are placed here because all the techniques needed for their derivation are developed in that section.

The results of this chapter hold for rational systems only. Indeed, most of them only make sense for such systems. The exceptions are the technical Lemma VI.1 and Theorem VI.5, which characterizes the largest stabilizable ball around P_0 in terms of the distance to the nearest nonstabilizable plant. This theorem still holds, with only small modifications to the proof, provided P_0 can be approximated by rationals (*ie* provided there exists a sequence of rational plants $\{P_i\}$ with the property $\delta_g(P_0, P_i) \to 0$.

1. The nearest nonstabilizable plant

Before proving the main results of this chapter, Theorems VI.5 and VI.6, it is necessary to develop a number of lemmas. The first gives a test for the coprimeness of a normalized symbol in terms of the largest singular value of its associated Hankel operator.

LEMMA VI.1. *Given an inner matrix* $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right] \in \mathcal{H}_\infty$*, then* $\{X, Y\}$ *are right coprime if, and only if,*

$$\bar{\sigma}(\mathbf{H}_{\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]}) < 1.$$

PROOF: McFarlane and Glover ([MG90] Lemma 4.5) show that the coprimeness of $\{X, Y\}$ implies that $\sigma_1 < 1$. The converse follows from the following argument

$$\bar{\sigma}(\mathbf{H}_{\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]}) < 1. \iff \left\|\mathbf{\Pi}_- \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right] v\right\| > 0 \forall v \in \mathcal{H}_2^\perp,$$

(as $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ is inner,)

$$\implies \underline{\sigma}\left(\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]\right) > 0 \forall s : \Re(s) > 0$$

which implies that $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ is left invertible. □

The following lemma is proved for the scalar case, where the extension is unique, by Georgiou and Smith [GS91]. We use here a modification of their proof, with the construction of a particular optimal Nehari extension, in the multivariable case. Sefton [Sef91] proves a similar result for the optimal antistable approximation; although the singular value bounds are only obtained for square plants. The methods used in [Sef91] do not appear to generalize naturally to reduced degree stable approximations.

LEMMA VI.2. *Given a plant* $P_0 \in \mathcal{P}^{p\times q}$ *and* $G_0, \tilde{G}_0$ *normalized right and left graph symbols for* P_0*, denote by* σ_n *the minimum singular value of the Hankel operator* $\mathbf{H}_{G_0}$*, and let* r *be the multiplicity of this singular value. Then, provided* $p \geq q$*, there exists a* P_1 *of degree* $\deg(P_1) = \deg(P_0) - r$*, and* G_1 *a normalized right graph symbol of* P_1 *with the properties:*

$$\left\|G_0 - \sqrt{1-\sigma_n^2}G_1\right\|_\infty = \sigma_n, \qquad \sigma_i\left(\mathbf{H}_{G_1}\right) \leq \sigma_i\left(\mathbf{H}_{G_0}\right).$$

If $p \le q$ there a exists a P_1 of degree $\deg(P_1) = \deg(P_0) - r$*, and $\tilde{G}_1$ a normalized left graph symbol of P_1 with the properties:*

$$\left\| \tilde{G}_0 - \sqrt{1-\sigma_n^2}\tilde{G}_1 \right\|_\infty = \sigma_n, \qquad \sigma_i\left(\mathbf{H}_{\tilde{G}_1}\right) \le \sigma_i\left(\mathbf{H}_{\tilde{G}_0}\right).$$

Proof: see Appendix

The next two lemmas establish some connections between coprime factorizations of P and P^*, and are necessary for the proof of the main result of this section, Theorem VI.5, which follows.

Proposition VI.3. *Given $\{N_0, M_0\}$, a normalized* rcf *of P_0, there exists an Ω_0 with the following properties:*

i) $\Omega_0 \in \mathcal{H}_\infty$
ii) $\operatorname{wno}\det \Omega_0 = \deg(P_0)$
iii) $\Omega_0^*\Omega_0 = I$
iv) $\{\Omega_0 N_0^*, \Omega_0 M_0^*\}$ *is a normalized* lcf *of P_0^*.*

Proof: Let $\{\tilde{N}_1, \tilde{M}_1\}$ be a normalized *lcf* of $P_1 := P_0^*$ and put $\Omega_0 := \tilde{M}_1 M_0^{*-1}$, then it is readily verified that Ω_0 has the required properties. Property iv) follows from direct substitution, and iii) from

$$I = \tilde{M}_1\tilde{M}_1^* + \tilde{N}_1\tilde{N}_1^* = \Omega_0(M_0^*M_0 + N_0^*N_0)\Omega_0^*.$$

To show i), note that

$$\tilde{M}_1 M_0 + \tilde{N}_1 N_0 = \Omega_0(M_0^*M_0 + N_0^*N_0) = \Omega_0.$$

The number of unstable poles of the feedback combination of P_0 and P_0^*, $(\eta[P_0, -P_0^*])$ is equal to the McMillan degree of P_0, as $(I + P_0P_0^*)$ cannot be singular on the imaginary axis. Alternatively, $\eta[P_0, -P_0^*] = \operatorname{wno}\det(\tilde{M}_1 M_0 + \tilde{N}_1 N_0)$, which verifies ii). □

Lemma VI.4. *Given a plant $P_0 \in \mathcal{P}^{p\times q}$ of McMillan degree n, with $\tilde{G}_0$ a normalized left graph symbol for P_0 and $\tilde{G}_1$ a normalized left graph symbol of $P_1 := P_0^*$, then*

$$\sigma_i^2\left(\mathbf{H}_{\tilde{G}_0}\right) = 1 - \sigma_{n-i+1}^2\left(\mathbf{H}_{\tilde{G}_1}\right),$$

where σ_1 denotes the maximum singular value of the Hankel operator, and σ_n the minimum.

Proof: see Appendix

The following theorem, which is the main result of this section, relates the largest stabilizable ball to the distance to the nearest nonstabilizable plant. This plant is obtained by first finding the adjoint of the nearest reduced degree approximation to P^*, and then adding to it a near right half plane pole/zero cancellation.

THEOREM VI.5. *Given a plant* $P_0 \in \mathcal{P}^{p\times q}$, *then there exists a sequence of plants,* $\{P_\epsilon\}$, *with the properties that* $\lim_{\epsilon\to 0}\delta_\nu(P_0,P_\epsilon)=b_{opt}(P_0)$, *and, given any (possibly nonlinear and time-varying) compensator* C, *there exists an* ϵ_0 *such that* C *destabilizes* $P_\epsilon \forall \epsilon < \epsilon_0$.

PROOF: From Lemma VI.2, it follows that, provided $q \geq p$, there exists a P_1 of degree less than or equal to $\deg(P_0)-1$ such that

$$\|\Omega_0 G_0^* - \sqrt{1-\sigma_n^2}\Omega_1 G_1^*\|_\infty = \sigma_n = b_{opt}(P_0)$$

where $\Omega_0 G_0^*$ and $\Omega_1 G_1^*$ are normalized left graph symbols for P_0^* and P_1^* respectively and σ_n refers to the minimum singular value of $\mathbf{H}_{\Omega_0 G_0^*}$, which is equal to $b_{opt}(P_0)$ by Lemma VI.4 and Proposition V.8.

The rest of this proof is restricted to the case $q \geq p$. The proof for $q < p$ is similar, or can be inferred by working with P^T rather than P.

It follows that

$$\left\|G_0 - \sqrt{1-\sigma_n^2}G_1\Omega_1^*\Omega_0\right\|_\infty = b_{opt}(P).$$

Note that $\operatorname{wno}\det(\Omega_1^*\Omega_0)\geq 1$. Now choose a

$$\theta\in\mathcal{RH}_\infty : \theta^{-1}\in\mathcal{L}_\infty, \eta(\theta^{-1}) = \operatorname{wno}\det(\Omega_1^*\Omega_0)$$

and the *rhp* zeros $\det(\theta)$ are disjoint from those of N_1 and M_1, and put $Q = \theta^{-1}\Omega_1^*\Omega_0$, thus $\operatorname{wno}\det(Q)=0$. Next, put $\phi = \theta+\epsilon I$ for $\epsilon > 0$, and $P_\epsilon = N_1\phi\theta^{-1}M_1^{-1}$. As $\theta^{-1}\notin\mathcal{H}_\infty$, and $\phi\theta^{-1} = I+\epsilon\theta^{-1}$, it is clear that by choosing ϵ small enough, $\phi\theta^{-1}$ can be made arbitrarily close to the identity on the imaginary axis. Hence, since $\theta^{-1}\notin\mathcal{H}_\infty$, as $\epsilon\to 0$, $P_\epsilon\to$ a nonstabilizable plant. For a formal proof of something equivalent to the precise statement in the theorem refer to {[KGP87]: proof of Theorem 4.1}, noting that C stabilizes P_ϵ if, and only if, $(I+M_1^{-1}CN_1)^{-1}M_1^{-1}CN_1$ stabilizes $I+\epsilon\theta^{-1}$.

Now, for ϵ sufficiently small (so that $\{N_1\phi, M_1\theta\}$ are coprime)

$$\delta_\nu(P_\epsilon,P)\leq\left\|\begin{bmatrix}N_0\\M_0\end{bmatrix} - \sqrt{1-\sigma_n^2}\begin{bmatrix}N_1\phi\\M_1\theta\end{bmatrix}Q\right\|_\infty$$

as $\operatorname{wno}\det(Q)=0$

$$\begin{aligned}&\leq\left\|G_0-\sqrt{1-\sigma_n^2}G_1\Omega_1^*\Omega_0\right\|_\infty+\epsilon\|N_1\theta^{-1}\|_\infty\\&\leq b_{opt}(P)+\epsilon\|\theta^{-1}\|_\infty.\end{aligned}\tag{65}$$

Thus, as $\epsilon\to 0$, $\delta_\nu(P_\epsilon,P_0)\to b_{opt}(P_0)$. This completes the proof. □

2. Upper and lower bounds for approximation

Theorem VI.5 has another interesting interpretation. In the proof, we constructed a P_1 of degree less than P_0 and satisfying $\delta_\nu(P_0^*, P_1^*) = b_{opt}(P_0)$. Notice that, if there were a P_1 of degree less that P_0 and satisfying $\delta_\nu(P_0^*, P_1^*) < b_{opt}(P_0)$, then we could construct a sequence of plants $\{P_\epsilon\}$, in the same manner as the proof of Theorem VI.5, with the property that

$$\text{as } \ \epsilon \to 0, \ \delta_\nu(P, P_\epsilon) \to b < b_{opt}(P).$$

This would be a contradiction, as there exists a compensator that stabilizes the tail of this sequence. So, as a side-effect, we have shown that the optimal degree $n-1$ approximation to a plant in the metric $\delta_\nu(\cdot,\cdot)$ can be obtained via an optimal Hankel norm approximation of the inner symbol of the graph. This is encouraging, giving further weight to the idea that useful lower order approximations can also be obtained by this method (as studied in [GS91]). We now study this idea in more detail, giving both upper and lower bounds for approximation.

Theorem VI.6. *Let $\tilde{G}_0$ be a normalized left graph symbol for $P_0 \in \mathcal{P}^{p\times q}$, and let σ_i denote the singular values of the Hankel operator $\mathbf{H}_{\tilde{G}_0}$, ordered such that $\sigma_1 \geq \ldots \geq \sigma_{n-r} > \sigma_{n-r+1} = \ldots = \sigma_n$. Now put*

$$s_k := \sum_{i=k+1}^{n-r+1} \arcsin \sigma_i$$

and

$$u_k := \begin{cases} \sin(s_k) & \text{if } s_k < \pi/2 \\ 1 & \text{otherwise,} \end{cases}$$

then the following bounds hold for approximation in the metric $\delta_\nu(\cdot,\cdot)$:

$$\sigma_{k+1} \leq \inf_{P_1 : \deg(P_1) \leq k} \delta_\nu(P_0, P_1) \leq u_k$$

Note that the bounds coincide for $k = n - r$.

Remark VI.7. These bounds have a similar form to those obtained by Glover for approximation in the norm [Glo84], and are probably the best possible bounds given in terms of the Hankel singular values of the normalized graph symbol. The bounds given in Theorem VI.6 also hold for approximation in the gap metric. The lower bound follows immediately, as $\delta_\nu(P_0, P_1) \leq \delta_g(P_0, P_1)$, and the upper bound can be proved in exactly the same way as we shall do for δ_ν. The upper bound given in [GS91]) for the gap metric is slightly weaker than that obtained here, whilst the lower bound given in [GS91]) is considerably weaker.

PROOF: of Theorem VI.6

Lower bound

A well known theorem of Adamjan, Arov and Krein [AAK71] (see [Glo84]) implies that

$$\inf_{X \in \mathcal{H}_{\infty(-),k}} \|G_0 - X\|_\infty = \sigma_{k+1}(\mathbf{H}_{G_0}),$$

where $\mathcal{H}_{\infty(-),k}$ denotes the space of functions analytic in the open *left* half plane except for at most k stable poles. Equivalently $\theta \in \mathcal{H}_{\infty(-),k}$ if, and only if, $rank\,(\mathbf{H}_\theta) \leq k$. Assume that there exists a P_1, $\deg(P_1) \leq k$ such that $\delta_\nu(P_0, P_1) = \epsilon$ and put

$$X = G_1(G_0^* G_1)^{-1}(1 - \epsilon^2). \tag{66}$$

Then,

$$X \in \mathcal{H}_{\infty(-),k} \tag{67}$$

as $X^*/(1-\epsilon^2) = (G_1^* G_0)^{-1} G_1^* = (\Omega_1 G_1^* G_0)^{-1} \Omega_1 G_1^*$, where Ω_1 is defined as in Proposition VI.3. So, $\Omega_1 G_1^*$ and $\Omega_1 G_1^* G_0$ are $\mathcal{RH}_\infty$ functions and $\operatorname{wno}\det(\Omega_1 G_1^* G_0) = \operatorname{wno}\det(\Omega_1) \leq k$. It follows that $(\Omega_1 G_1^* G_0)^{-1}$ has no more than k unstable poles.

Furthermore,

$$\|G_0 - X\|_\infty = \epsilon, \tag{68}$$

as is easily verified by pre-multiplying by $\begin{bmatrix} G_0^* \\ \tilde{G}_0 \end{bmatrix}$ and using Lemma III.9.

The combination of (67) and (68) yields a contradiction if $\epsilon < \sigma_{k+1}(\mathbf{H}_{G_0})$.

Upper bound

This follows immediately from Lemma VI.2. Given a P_0, of degree n, first use Lemma VI.2 to obtain a P_1 of degree equal to $n-r$ such that $\delta_\nu(P_0, P_1) \leq \sigma_n$. It is then guaranteed that $\sigma_i(\mathbf{H}_{G_1}) \leq \sigma_i(\mathbf{H}_{G_0})$ and so, from P_1, we can obtain a P_2 of degree no greater than $n-r-1$ such that $\delta_\nu(P_1, P_2) \leq \sigma_{n-r}$. It follows from the metric property that $\arcsin \delta_\nu(P, P_2) \leq \arcsin \delta_\nu(P, P_1) + \arcsin \delta_\nu(P_1, P_2)$. Repeat this step the required number of times to obtain the bound. □

REMARK VI.8. The proof of the lower bound settles a question as to the efficacy of Hankel norm approximation for approximation in the gap metric and related problems. The worry is that, in simply finding an optimal Hankel norm approximation of rank k to G_0, we would appear to have sacrificed some of the degrees of freedom in the Q of (39) or (41). The proof shows that there is *no* extra freedom to be gained from this Q, as for any approximation P_1, there is a Q which satisfies $\|G_0 - G_1 Q\|_\infty = \delta_\nu(P_0, P_1)$ *without* inflating the rank of $\mathbf{H}_{G_1 Q}$.

REMARK VI.9. Although the upper bound has been proved using one step at a time approximation, it probably also holds if the approximation is done in one step, following the procedure of Glover [Glo84] for the additive case, although a proof for this appears difficult. In fact, for all examples tried, the one step procedure has given better results, giving an approximation error reasonably near to the lower bound. The one step at a time procedure, on the other hand, tends to give errors near the upper bound. A slightly weaker upper bound has been proved by Georgiou and Smith [GS91] for this one step procedure. We present below a modified one step procedure, which is suggested by the particular properties of δ_ν. Again, no upper bound proofs for this procedure have been found; however, the structure of the procedure appears more amenable to a proof. Neither of the one step methods gives consistently better results than the other. We now summarize both one-step procedures for this application, and indicate their important differences and where the difficulties in the proof of the tighter upper bound lie.

Firstly, both procedures use [Glo84] Corollary 7.3 to obtain an $X \in \mathcal{H}_{\infty(-),k}$ satisfying

$$\|G_0 - X\|_\infty = \sigma_{k+1}\left(\mathbf{H}_{G_0}\right).$$

There is a choice of a matrix U at this stage (as in Lemma VI.2). A choice of $U = C_2 B_2^{T^\dagger}$ (notation as in the proof of Lemma VI.2) has been found to give favourable approximation errors near to the lower bound, as it does in the additive case. The next stage is to remove the unstable poles of X. The difference between the two procedures lies in how this is achieved.

The Glover procedure decomposes X into its stable and unstable parts as follows

$$X = H + F^*,\ \ H, F \in \mathcal{H}_\infty, \deg H \le k.$$

It can be shown that

$$\sigma_i\left(\mathbf{H}_F\right) \le \sigma_{k+1+i}\left(\mathbf{H}_{G_0}\right)$$

and that for any $G \in \mathcal{H}_\infty$ there exists a constant matrix D satisfying $\|G - D\|_\infty \le \sum_i \sigma_i$. Construct such a D for F. It then follows that

$$\|G_0 - H - D\|_\infty \le \sum_{i>k} \sigma_i.$$

Now, provided $H + D$ is left invertible in $\mathcal{H}_\infty$, we let P_1 be the plant with right graph symbol $H + D$ (if not, then we can always perturb D slightly — note that $H + D$ is always left invertible provided the achieved norm is less than $b_{opt}(P)$ [GS90].) Then,

$$\delta_\nu(P_0, P_1) \le \sum_{i>k} \sigma_i.$$

However, in order to get the same upper bound as the one step at a time procedure it is necessary to turn the $\sum \sigma_i$ into $\sin \sum \arcsin \sigma_i$ somehow. Note that these points are of purely academic interest (as the difference between the two upper bounds is small unless the approximation errors are large) *unless* it suggests that a different choice of D, or U, might be more appropriate for this rather special problem.

The modified one step procedure removes the unstable poles in an entirely different way, which makes more use of the structure of this problem. Firstly, let P_1 be the plant with $\mathcal{L}_2$ graph symbol X. This plant will have McMillan degree

$n - r$ and the $\mathcal{L}_2$-gap between it and P_0 will be equal to the lower bound, σ_{k+1}. It can be shown that the presence of $n - k - r$ unstable poles in X means that, if $\delta_\nu(P_0, P_1)$ were calculated, the winding number would come out as $n-k-r$. This implies the presence of the same number of near *rhp* pole/zero cancellations in P_1, provided $\delta_{\mathcal{L}_2}(P_0, P_1)$ is not too large, and that the last $n-k-r$ singular values of the normalized graph symbol of P_1^* will be small. We then apply the same procedure to P_1^*, and iterate until the required degree is reached. As mentioned above, this procedure works well in practice — giving approximation errors near to the lower bound.

CHAPTER VII

Admissible Perturbations in $\mathcal{H}_\infty$ Control

One of the central themes of $\mathcal{H}_\infty$ control has been its ability to deal in a systematic way with uncertain plants. Any feedback compensator designed using $\mathcal{H}_\infty$ methods can, *a priori*, be guaranteed to stabilize a set of plants in the neighbourhood of some nominal model, characterized in terms of a stable, norm bounded perturbation. However, it is known that, in some specific cases (in particular multiplicative and additive uncertainty descriptions on an unstable plant) there are classes of unstable $\mathcal{L}_\infty$ norm bounded perturbations which are admissible, [Glo86][VK86] in the sense that any $\mathcal{H}_\infty$ controller designed to tolerate the stable perturbations will also tolerate this class of unstable perturbations. In a more general setting, an open question however has been "what is the largest set of linear, time-invariant plants that can, *a priori*, be guaranteed to be stabilized by any compensator which stabilizes a nominal plant model and satisfies a specified $\mathcal{H}_\infty$ norm bound on some closed loop transfer function". In this chapter we provide an answer to this question, and also look at the case where the perturbations are structured (the "μ-analysis" problem). We then go on to reproduce the standard results for multiplicative and additive uncertainty, and finally show how the metric $\delta_\nu(\cdot,\cdot)$ arises in a natural manner when the closed loop transfer function concerned is

$$\begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix}.$$

1. The general case

First, we shall make the question posed in the introduction more precise. As usual, we define upper and lower linear fractional maps with respect to the interconnection structure G, partitioned conformally with a Δ and a C as follows:

$$G = \left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array} \right].$$

Then,

$$\mathcal{F}_u(G,\Delta) := G_{22} + G_{21}\Delta(I - G_{11}\Delta)^{-1}G_{12}$$
$$\mathcal{F}_l(G,C) := G_{11} + G_{12}C(I - G_{22}C)^{-1}G_{21}$$

whenever the inverses exist. Now, define a set of $\mathcal{H}_\infty$ suboptimal compensators as:

$$\mathfrak{C}(G,\gamma) := \{C : [G_{22}, C] \text{ is stable, } \|\mathcal{F}_l(G,C)\|_\infty < \gamma\}$$

provided $\gamma > \inf_{C stabilizing} \|\mathcal{F}_l(G,C)\|_\infty$. We can now define the largest set of plants guaranteed, *a priori*, to be stabilized by a stabilizing compensator satisfying $\|\mathcal{F}_l(G,C)\|_\infty < \gamma$ as

$$\mathfrak{S}(G,\gamma) := \{P : [P, C] \text{ is stable } \forall C \in \mathfrak{C}(G,\gamma)\}.$$

Our aim is to characterize this set in a more computable manner.

The following proposition is well known, and has provided much of the motivation for solving the general $\mathcal{H}_\infty$ control problem. It states that

$$\{\mathcal{F}_u(G,\Delta) : \Delta \in \mathcal{H}_\infty, \|\Delta\| \leq 1/\gamma\} \subseteq \mathfrak{S}(G,\gamma).$$

PROPOSITION VII.1. *Given a compensator C such that $[G_{22}, C]$ is stable then the following are equivalent:*

i) $\mathcal{F}_l(G,C) \in \mathcal{H}_\infty$, $\|\mathcal{F}_l(G,C)\|_\infty < \gamma$
ii) $[\mathcal{F}_u(G,\Delta), C]$ *is stable* $\forall \Delta \in \mathcal{H}_\infty \|\Delta\|_\infty \leq 1/\gamma$

PROOF: That $i) \implies ii)$ follows from the small gain theorem applied to the loop $\mathcal{F}_l(G,C)\Delta$ noting that internal stability of this loop requires the stability of Δ. The reverse direction can be proved by the construction of a Δ such that $\det(I - \mathcal{F}_l(G,C)\Delta) = 0$ at some frequency if the norm bound in part i) is violated. □

The set described in part ii) of the proposition is, in general, not the largest set that can be guaranteed to be stabilized given part i). This is because the requirement that the pair $[\mathcal{F}_l(G,C), \Delta]$ be internally stable is overly strong. In general we are not interested in whether the compensator internally stabilizes the feedback combination of the operators $\mathbf{M}_G$ and $\mathbf{M}_\Delta$, which may be impossible when there are unstable pole/zero cancellations between the G and the Δ, but whether the compensator stabilizes a perturbed plant with the *transfer function*

$$\mathcal{F}_u(G(s), \Delta(s)).$$

The following theorem characterizes a larger class of admissible perturbations, and we shall see later that this is, in fact, the largest class of such perturbations.

THEOREM VII.2. *Given*

i) G, C *with* $[G_{22}, C]$ *stable,* $\mathcal{F}_l(G,C) \in \mathcal{H}_\infty$ $\|\mathcal{F}_l(G,C)\|_\infty < \gamma$, *and*
ii) $\Delta \in \mathcal{L}_\infty$, $\|\Delta\|_\infty \leq 1/\gamma$ *such that* $\det(I - G_{11}(s)\Delta(s)) \neq 0$ *almost everywhere,*

and defining

$$P_0(s) := \mathcal{F}_u(G(s), 0) = G_{22}(s)$$
$$P_1(s) := \mathcal{F}_u(G(s), \Delta(s)),$$

then

$$C \text{ stabilizes } P_1 \iff \eta(P_1) = \eta(P_0) + \operatorname{wno}\det(I - G_{11}\Delta) \tag{69}$$

where the winding number is evaluated on the standard Nyquist D-contour indented to the right around any imaginary axis poles of P_0 and P_1 (with the indentation sufficiently small that the winding number is constant for any smaller indentation).

PROOF: $\det(I - G_{11}(j\omega)\Delta(j\omega)) \neq 0$ almost everywhere by assumption and $\det(I - G_{22}(j\omega)C(j\omega)) \neq 0 \forall\omega$ as $[G_{22}, C]$ is stable. Hence we can expand

$$\det\left(I - G\begin{bmatrix}\Delta & 0\\ 0 & C\end{bmatrix}\right)$$

in terms of Schur complements to obtain

$$\det(I - G_{11}\Delta)\det(I - P_1 C) = \det(I - P_0 C)\det(I - \mathcal{F}_l(G, C)\Delta). \tag{70}$$

We shall now evaluate the winding number of these determinants on a contour indented around any imaginary axis poles of P_0, P_1 and C. Firstly,

$$\operatorname{wno}\det(I - P_0 C) = -\eta(P_0) - \eta(C),$$

by the Nyquist stability criterion, as $[P_0, C]$ is stable, and

$$\operatorname{wno}\det(I - \mathcal{F}_l(G, C)\Delta) = 0 \text{ as } \|\mathcal{F}_l(G, C)\Delta\|_\infty < 1. \tag{71}$$

The assumption that $\mathcal{F}_l(G, C) \in \mathcal{H}_\infty$ ensures that G_{11} cannot have any imaginary axis poles which do not appear in P_0, as then $\det(I - P_1 C)$ would need to be zero at any such poles for all $\Delta = \epsilon I$, $\epsilon \leq 1/\gamma$ to maintain consistency of (70), contradicting the fact that $[P_1, C]$ is stable for any such Δ by Proposition VII.1. Thus, both terms on the LHS of (70) are bounded and nonzero on the given contour, and

$$\operatorname{wno}\det(I - P_1 C) = \operatorname{wno}\det(I - G_{11}\Delta) - \eta(P_0) - \eta(C). \tag{72}$$

But, $[P_1, C]$ is stable if, and only if

$$\operatorname{wno}\det(I - P_1 C) = -\eta(P_1) - \eta(C), \tag{73}$$

with comparison of (72) and (73) giving the result, noting that we can drop the requirement to indent the contour around any imaginary axis poles of C as this is irrelevant to $\det(I - G_{11}\Delta)$. □

Theorem VII.2 is readily extended to the case of block-structured perturbations [Doy82]. We define

$$\boldsymbol{\Delta} := \{\Delta : \Delta = \operatorname{diag}(\Delta_1, \Delta_2, \ldots, \Delta_n), \Delta_i \in \mathcal{L}_\infty\},$$

and, for $M \in \mathcal{H}_\infty$

$$\mu_{\mathbf{\Delta}}(M) := 1/\inf\{\|\Delta\|_\infty : \Delta \in \mathbf{\Delta}, \det(I - M(j\omega)\Delta(j\omega)) = 0 \text{ for some } \omega\}.$$

As an analogue of Proposition VII.1, we then have the following well known result:

PROPOSITION VII.3. *Given a compensator C such that $[G_{22}, C]$ is stable then the following are equivalent:*

i) $\mathcal{F}_l(G,C) \in \mathcal{H}_\infty$, $\mu(\mathcal{F}_l(G,C)) < \gamma$
ii) $[\mathcal{F}_u(G,\Delta), C]$ *is stable* $\forall \Delta \in \mathcal{H}_\infty \cap \mathbf{\Delta}$, $\|\Delta\|_\infty \leq 1/\gamma$

PROOF: Directly analogous to that of Proposition VII.1

In our framework, Theorem VII.1 extends to:

THEOREM VII.4. *Given*

i) G, C *with* $[G_{22}, C]$ *stable,* $\mathcal{F}_l(G,C) \in \mathcal{H}_\infty$ $\quad \mu_{\mathbf{\Delta}}(\mathcal{F}_l(G,C)) < \gamma$, *and*
ii) $\Delta \in \mathbf{\Delta}$, $\|\Delta\|_\infty \leq 1/\gamma$ *such that* $\det(I - G_{11}(j\omega)\Delta(j\omega)) \neq 0$ *almost everywhere,*

and defining

$$P_0(j\omega) := G_{22}(j\omega)$$
$$P_1(j\omega) := \mathcal{F}_u(G(j\omega), \Delta(j\omega)),$$

then

$$C \text{ stabilizes } P_1 \iff \eta(P_1) = \eta(P_0) + \operatorname{wno}\det(I - G_{11}\Delta) \tag{74}$$

where the winding number is evaluated on the standard Nyquist D-contour indented around any imaginary axis poles of P_0 and P_1.

PROOF: The only difference between the proof of this corollary and Theorem VII.2 is at line (71), which becomes

$$\mu_{\mathbf{\Delta}}(\mathcal{F}_l(G,C)) < \gamma$$

$$\Downarrow$$

$$\det(I - \lambda\mathcal{F}_l(G,C)\Delta) \neq 0 \quad \forall\omega, \Delta \in \mathbf{\Delta}, \|\Delta\|_\infty \leq 1/\gamma, \lambda \in [0,1]$$

(by the definition of μ)

$$\Downarrow$$

$$\operatorname{wno}\det(I - \mathcal{F}_l(G,C)\Delta) = 0.$$

□

So, we have shown that

$$\{\mathcal{F}_u(G,\Delta) : \Delta \in \mathcal{H}_\infty, \|\Delta\| \leq 1/\gamma\} \subseteq$$
$$\{\mathcal{F}_u(G,\Delta) : \Delta \in \mathcal{L}_\infty, \|\Delta\| \leq 1/\gamma, \eta(\mathcal{F}_u(G,\Delta)) = \eta(G_{22}) + \operatorname{wno}\det(I - G_{11}\Delta)\}$$
$$\subseteq \mathfrak{S}(G,\gamma).$$

The last inclusion is actually an equality. To show this we must deal with the problem that, unless G_{12} and G_{21} are right and left invertible respectively, the mapping $\Delta \mapsto \mathcal{F}_u(G,\Delta)$ is many to one. So, if $P_1 = \mathcal{F}_u(G,\Delta)$ for some Δ of norm greater than $1/\gamma$ we have to show that *either* there is a C satisfying $\mathcal{F}_l(G,C) < \gamma$ which destabilizes P_1 *or* there is another perturbation, $\hat{\Delta}$, of norm no greater than $1/\gamma$ such that $\mathcal{F}_u(G,\hat{\Delta})$ is also equal to P_1. The following constant matrix lemma is crucial to our proof.

LEMMA VII.5. *Given $P \in \mathbb{C}^{n,p}$ and*

$$G = \left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array}\right] \in \mathbb{C}^{n+m,p+q}$$

with G_{12} full column rank and G_{21} full row rank then, for $\gamma > \inf_C \bar{\sigma}(\mathcal{F}_l(G,C))$, the following are equivalent.

i) *$P = \mathcal{F}_u(G,\Delta)$ for some Δ satisfying $\bar{\sigma}(\Delta) \le 1/\gamma$.*
ii) *$\det(I - PC) \ne 0$ for all C satisfying $\bar{\sigma}(\mathcal{F}_l(G,C)) < \gamma$.*

PROOF: We can assume, without loss of generality, that G has the form

$$G = \left[\begin{array}{c|c} X & \begin{bmatrix} I \\ 0 \end{bmatrix} \\ \hline \begin{bmatrix} I & 0 \end{bmatrix} & 0 \end{array}\right].$$

(As otherwise we can employ the change of variables

$$\hat{P} = R_2^{-1}(P - G_{22})R_1^{-1}$$

$$\hat{C} = R_1 C(I - G_{22}C)R_2$$

$$\hat{\Delta} = Q_2^* \Delta Q_1$$

$$\hat{G} = \left[\begin{array}{c|c} Q_1^* G_{11} Q_2 & \begin{bmatrix} I \\ 0 \end{bmatrix} \\ \hline \begin{bmatrix} I & 0 \end{bmatrix} & 0 \end{array}\right],$$

where

$$G_{12} = Q_1 \begin{bmatrix} R_1 \\ 0 \end{bmatrix},\ G_{21} = \begin{bmatrix} R_2 & 0 \end{bmatrix} Q_2^*$$

and Q_1, Q_2 are unitary. All the conditions in parts i) and ii) hold for the original variables if, and only if, they hold for these transformed variables.)

That i) implies ii) follows immediately from the fact that, if $P = \mathcal{F}_u(G,\Delta)$, then

$$\det(I - X\Delta)\det(I - PC) = \det(I - \mathcal{F}_l(G,C)\Delta),$$

(from (70). To show the converse, we need to construct a suitable Δ. First note that

$$\mathcal{F}_l(G,C) = X + \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix},$$

and so

$$\bar{\sigma}(\mathcal{F}_l(G,C)) < \gamma \iff C = L^{-1}\Phi R^{-1} - Q$$

for some $\Phi : \bar{\sigma}(\Phi) < \gamma$, by Parrot's theorem, where

$$L := (I + X_{12}(\gamma^2 I - X_{12}^* X_{12} - X_{22}^* X_{22})^{-1} X_{12}^*)^{1/2}$$
$$R := (I + X_{21}^*(\gamma^2 I - X_{21} X_{21}^* - X_{22} X_{22}^*)^{-1} X_{21})^{1/2}$$

and

$$Q := X_{11} + X_{12}(\gamma^2 I - X_{22}^* X_{22})^{-1} X_{22}^* X_{21}.$$

(Note that $\inf_C \bar{\sigma}(\mathcal{F}_l(G, C)) < \gamma$ ensures that $\gamma^2 I - X_{12}^* X_{12} - X_{22}^* X_{22}$ and $\gamma^2 I - X_{21} X_{21}^* - X_{22} X_{22}^*$ are both positive.) Thus, $\det(I - PC) \neq 0$ for all such C if, and only if,

$$\bar{\sigma}(\gamma R^{-1} P (I - QP)^{-1} L^{-1}) \leq 1, \tag{75}$$

(as $\det(I - PC) = \det(I - PQ)\det(I - R^{-1}P(I - QP)^{-1}L^{-1}\Phi))$. So, ii) is equivalent to (75). Furthermore, (75) is equivalent to

$$P^*(\gamma^2 (RR^*)^{-1} - Q^* L^* L Q) P - L^* L Q P - P^* Q^* L^* L - L^* L \leq 0 \tag{76}$$

(as $\bar{\sigma}(AB^{-1}) \leq 1 \iff A^*A \leq B^*B$).

Now, $\mathcal{F}_u(G, \Delta) = P \iff \Delta = P_a(I - XP_a)^{-1}$, where P_a satisfies $[I\ \ 0]\, P_a \begin{bmatrix} I \\ 0 \end{bmatrix} = P$. It is easily shown that $\bar{\sigma}(P_a(I - XP_a)^{-1}) \leq 1/\gamma$ if, and only if,

$$(P_a^* - X(\gamma^2 I - X^*X)^{-1})(\gamma^2 I - X^*X)(P_a - (\gamma^2 I - X^*X)^{-1}X^*) \leq \gamma^2(\gamma^2 I - XX^*)^{-1} \tag{77}$$

(use $\bar{\sigma}(AB^{-1}) \leq 1 \iff A^*A \leq B^*B$ again). We shall now show that (76) guarantees the existence of a P_a satisfying (77) and $[I\ \ 0]\, P_a \begin{bmatrix} I \\ 0 \end{bmatrix} = P$ (and hence all the conditions of part i) of the theorem statement). We can factorize

$$\gamma^2 I - X^*X = \begin{bmatrix} I & T^* \\ 0 & I \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ T & I \end{bmatrix}$$

where

$$D_2 := \gamma^2 I - X_{12}^* X_{12} - X_{22}^* X_{22} > 0$$
$$T := -D_2^{-1}(X_{12}^* X_{11} + X_{22}^* X_{21})$$

and

$$D_1 := \gamma^2 I - X_{11}^* X_{11} - X_{21}^* X_{21} - T^* D_2 T.$$

Similarly,

$$\gamma^2 I - X^*X = \begin{bmatrix} I & S \\ 0 & I \end{bmatrix} \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ S^* & I \end{bmatrix}$$

where

$$E_2 := \gamma^2 I - X_{21} X_{21}^* - X_{22} X_{22}^* > 0$$
$$S := -(X_{11} X_{21}^* + X_{12} X_{22}^*) E_2^{-1}$$

and

$$E_1 := \gamma^2 I - X_{11} X_{11}^* - X_{12} X_{12}^* - S E_2 S^*.$$

With these substitutions (77) is equivalent to

$$\hat{P}_a^* \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \hat{P}_a \leq \gamma^2 \begin{bmatrix} E_1^{-1} & 0 \\ 0 & E_2^{-1} \end{bmatrix}$$

where

$$\hat{P}_a := \begin{bmatrix} I & 0 \\ T & I \end{bmatrix} (P_a - (\gamma^2 I - X^* X)^{-1} X^*) \begin{bmatrix} I & S \\ 0 & I \end{bmatrix}.$$

In terms of this new variable

$$[I \ \ 0]\, P_a \begin{bmatrix} I \\ 0 \end{bmatrix} = [I \ \ 0]\, \hat{P}_a \begin{bmatrix} I \\ 0 \end{bmatrix} + D_1^{-1}(X_{11}^* - T^* X_{12}^*),$$

and so choosing

$$\hat{P}_a = \begin{bmatrix} P - D_1^{-1}(X_{11}^* - T^* X_{12}^*) & 0 \\ 0 & 0 \end{bmatrix}$$

ensures that $\mathcal{F}_u(G,\Delta) = P$; the norm condition $\bar{\sigma}(\Delta) \leq 1/\gamma$ is then satisfied if

$$\left(P - D_1^{-1}(X_{11}^* - T^* X_{12}^*)\right)^* D_1 \left(P - D_1^{-1}(X_{11}^* - T^* X_{12}^*)\right) \leq \gamma^2 E_1^{-1}$$

(as $E_2 > 0$) or, equivalently, if

$$\begin{aligned} P^* D_1 P - (X_{11} - X_{12}T)P - P^*(X_{11} - X_{12}T)^* + \\ (X_{11} - X_{12}T) D_1^{-1} (X_{11} - X_{12}T)^* - \gamma^2 E_1^{-1} \leq 0. \end{aligned}$$

Some tedious algebraic manipulation shows that the LHS of this last inequality is precisely equal to the LHS of (76) (*ie* $X_{11} - X_{12}T = L^* L Q$ *etc*). Thus ii) implies that i) is satisfied for the Δ implied by this $\hat{P}_a$. This completes the proof. □

The following theorem, which states that

$$\mathfrak{S}(G,\gamma) = \\ \{\mathcal{F}_u(G,\Delta) : \Delta \in \mathcal{L}_\infty, \|\Delta\| \leq 1/\gamma, \eta(\mathcal{F}_u(G,\Delta)) = \eta(G_{22}) + \operatorname{wno}\det(I - G_{11}\Delta)\}$$

is the main result of this chapter.

THEOREM VII.6. *Given* $P \in \mathcal{P}^{n\times p}$ *and*

$$G = \left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array}\right] \in \mathcal{P}^{n+m \times p+q}$$

with $G_{12}(s)$ *full column rank and* $G_{21}(s)$ *full row rank for almost all* s *then, for* $\gamma > \inf_{C \, stabilizing} \|\mathcal{F}_l(G,C)\|_\infty$, *the following are equivalent.*

i) *There exists a* $\Delta \in \mathcal{L}_\infty$ *satisfying* $\|\Delta\|_\infty \leq 1/\gamma$, $\mathcal{F}_u(G,\Delta) = P$ *and* $\eta(G_{22}) + \operatorname{wno}\det(I - G_{11}\Delta) = \eta(P)$.

ii) $[P, C]$ *is stable for all* C *such that* $[G_{22}, C]$ *is stable and* $\|\mathcal{F}_l(G,C)\|_\infty < \gamma$.

PROOF: That i) implies ii) is the result of Theorem VII.2. The converse is a consequence of Lemma VII.5, for if ii) is true then it must also be true that $|I - P(j\omega)X| \neq 0$ for all X satisfying $\bar{\sigma}(\mathcal{F}_l(G(j\omega), X) < \gamma$ at each frequency ω (this is because all C satisfying $\|\mathcal{F}_l(G, C)\|_\infty < \gamma$ may be parameterized in the form $C = \mathcal{F}_l(J, Q) : Q \in \mathcal{H}_\infty \|Q\|_\infty < 1$ where J is an invertible LFT, *ie*

$$C = \mathcal{F}_l(J, Q) \iff Q = \mathcal{F}_u(J^{-1}, C),$$

and at each frequency

$$\bar{\sigma}(\mathcal{F}_l(G(j\omega), \mathcal{F}_l(J(j\omega), Y)) < \gamma \iff \bar{\sigma}(Y) < 1$$

(see [DGKF89]). So, if there exists a sufficiently small X that makes $|I - P(j\omega)X| = 0$ at some frequency, then we can always find a destabilizing C by interpolating via the Q.) We can then use Lemma VII.5 to find a Δ satisfying the norm bound in part i), and this Δ can be made be continuous, as all the steps in the construction of Lemma VII.5 can be carried out with rational matrices. A continuity argument may then be used to establish that $[P, C]$ is stable, and consequently that the winding number condition is satisfied. □

One of the important points about Theorem VII.4 is that the condition for robust stability is independent of the compensator, provided it stabilizes the nominal plant and satisfies the closed loop bound on μ. This gives us a way of checking whether P_1 is admissibly perturbed without having to calculate winding numbers, stated in the following corollary.

COROLLARY VII.7. *Define the sets*

$$\mathfrak{S}_s(G, b) := \{P : P(s) = \mathcal{F}_u(G(s), \Delta(s)),\ \Delta \in \mathbf{\Delta} \cap \mathcal{H}_\infty,\ \|\Delta\|_\infty \leq b\}$$

and

$$\mathfrak{S}_u(G, b) := \{P : P(s) = \mathcal{F}_u(G(s), \Delta(s)),\ \Delta \in \mathbf{\Delta} \cap \mathcal{L}_\infty,\ \|\Delta\|_\infty \leq b\}.$$

Then, given an arbitrary C that stabilizes all $P \in \mathfrak{S}_s$, and an arbitrary $P_1 \in \mathfrak{S}_u$:

i) *If $[P_1, C]$ is stable, then P_1 is stabilized by* any *C that stabilizes the set $\mathfrak{S}_s$.*
ii) *If $[P_1, C]$ is not stable, then there is* no *C that stabilizes the set $\mathfrak{S}_s \cup P_1$.*

PROOF: This is immediate from Proposition VII.3 and Theorem VII.4, noting that condition (74) is independent of which C is chosen to satisfy assumption ii) in Theorem VII.4. □

Thus, at the design stage there is no need to check whether a particular P_1 is admissibly perturbed. If we design a compensator to stabilize the set $\mathfrak{S}_s$ then, in general, there will be some plants in $\mathfrak{S}_u$ that we are guaranteed to stabilize "for free". If P_1 is not in this subset then there is nothing we can do about it (provided, that is, we genuinely need to stabilize the set $\mathfrak{S}_s$, which is a question we address in the following remark).

REMARK VII.8. It is interesting to remark that any compensator C which stabilizes the set $\mathfrak{S}_s(G,b)$ can be used to partition the set $\mathfrak{S}_u(G,b)$ into disjoint subsets, which are themselves connected in the graph topology, each component being identified with a particular value of $\eta[P,C]$. In fact, any compensator satisfying $\|\mathcal{F}_l(G,C)\|_{\mathcal{L}_\infty} < 1/b$, irrespective of whether or not it is stabilizing, will partition $\mathfrak{S}_u(G,b)$. The idea is a generalization of Theorem III.13, where we partitioned an $\mathcal{L}_2$-gap ball of any radius using $\operatorname{wno}\det(G_1^*G_0)$, which is related to the number of unstable closed loop poles when the feedback compensator is $-P_0^*$. Here, the result only holds for radii less than $1/\inf_C \|\mathcal{F}_l(G,C)\|_{\mathcal{L}_\infty}$ — as for larger radii a P_1 may be connected to more than one of the smaller partitions. An application of this result occurs if we wish to stabilize an alternative partition to that occupied by the "centre", $\mathcal{F}_u(G,0)$ (this situation may arise, for example, when the nominal plant has been estimated frequency by frequency from identification experiments and contains extraneous near pole/zero pairs in the right half plane). If we find a C satisfying $\|\mathcal{F}_l(G,C)\|_{\mathcal{L}_\infty} < 1/b$, and this C gives k fewer unstable closed loop poles with a $P_1 \in \mathfrak{S}_u(G,b)$ than it does with $\mathcal{F}_u(G,0)$, then any compensator satisfying $\|\mathcal{F}_l(G,C)\|_{\mathcal{L}_\infty} < 1/b$ and $\eta[\mathcal{F}_u(G,0),C] \leq k$ will stabilize the whole partition of $\mathfrak{S}_u(G,b)$ containing P_1. The proof of this is a simple modification of Theorem VII.4. In the unstructured case, suitable compensators may be found by the methods of [GLH92].

2. Additive and normalized coprime factor uncertainty

Let us now consider some particular uncertainty sets. Firstly, we shall consider additive and multiplicative perturbations. In both these cases $G_{11} = 0$ in the standard plant, hence (69) requires that $\eta(P_1) = \eta(P_0)$ for a perturbation to be admissible, which corresponds to the well-known result for these types of perturbation, which we gave on page 26. Things get more interesting when we look at perturbations to the normalized right coprime factors of the plant, *ie*

$$P_1 = (N_0 + \Delta_N)(M_0 + \Delta_M)^{-1},\ \Delta := \begin{bmatrix}\Delta_N \\ \Delta_M\end{bmatrix},\ \|\Delta\|_\infty \leq b, \tag{78}$$

where $\{N_0, M_0\}$ is a normalized *rcf* of P_0. The standard plant for this case is

$$G = \left[\begin{array}{cc|c} 0 & -M_0^{-1} & M_0^{-1} \\ \hline I & -P_0 & P_0 \end{array}\right]$$

thus

$$\det(I - G_{11}\Delta) = \det(I + M_0^{-1}\Delta_M) = \det(M_0^{-1})\det(M_0 + \Delta_M).$$

Note also that $\eta(P_0) = \operatorname{wno}\det(M_0)$ (on a contour indented as in Theorem VII.2), as is true for any coprime fraction [Vid85] and hence (69) requires that

$$\eta(P_1) = \operatorname{wno}\det(M_0 + \Delta_M) \tag{79}$$

for the perturbation to be admissible. A sufficient condition for this is that $\Delta \in \mathcal{H}_\infty$ and $[(N_0 + \Delta_N), (M_0 + \Delta_M)]$ are right coprime. Let $\{N_1, M_1\}$ be a normalized *rcf* of P_1 and put $Q = M_1^{-1}(M_0 + \Delta_M)$ then

$$\begin{bmatrix} N_0 \\ M_0 \end{bmatrix} + \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} = \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} Q$$

and it can be seen that $[(N_0+\Delta_N), (M_0+\Delta_M)]$ are right coprime if, and only if, $Q : Q, Q^{-1} \in \mathcal{H}_\infty$. So, if we are given the perturbed plant explicitly and wish to know if it can be written in the form of (78) for some $\Delta_N, \Delta_M \in \mathcal{H}_\infty$ such that $[(N_0 + \Delta_N), (M_0 + \Delta_M)]$ are right coprime, this is equivalent to asking if

$$\inf_{Q, Q^{-1} \in \mathcal{H}_\infty} \left\| \begin{bmatrix} N_0 \\ M_0 \end{bmatrix} - \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} Q \right\|_\infty \leq b$$

Or, equivalently, if $\delta_g(P_0, P_1) \leq b$, as the above is the formula given for the gap in Chapter 4. However, it is rather artificial to require that the particular representation be right coprime (or equivalently internally stable), as we are really interested in whether the plant with the transfer function $(N_0(s) + \Delta_N(s))(M_0(s) + \Delta_M(s))^{-1}$ is stabilized. So we shall now look at condition (79) a little more closely. Note that $\eta(P_1) = \mathrm{wno}\det(M_1)$ and $\mathrm{wno}\det(M_0 + \Delta_M) = \mathrm{wno}\det(M_1) + \mathrm{wno}\det(Q)$. So (79) is satisfied if, and only if, $\mathrm{wno}\det(Q) = 0$; and P_1 can be written in the form of (78) for some admissible Δ_N, Δ_M if, and only if,

$$\inf_{Q : \mathrm{wno}\det(Q) = 0} \left\| \begin{bmatrix} N_0 \\ M_0 \end{bmatrix} - \begin{bmatrix} N_1 \\ M_1 \end{bmatrix} Q \right\|_\infty \leq b$$

Or, equivalently, if, and only if, $\delta_\nu(P_0, P_1) \leq b$, as this is the formula established for δ_ν in Chapter 4. Hence, we have derived a formula for the new metric in a natural manner and, hopefully, given yet another perspective on why it should, in general, be less than the gap between two plants.

Concluding remarks

A new metric on the space of multivariable linear time-invariant systems has been introduced as the natural dual of the $\mathcal{H}_\infty$ stability/performance measure

$$b_{P,C} := \left\| \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} [-C \ \ I] \right\|_\infty^{-1}$$

This metric has manifest advantages over known others that induce the graph topology.

1) The metric gives the tightest possible robust stability/performance bounds with a compensator known only to satisfy a bound on $b_{P,C}$.
2) The ability to measure the metric frequency by frequency has been shown to be of great benefit in the synthesis of robust controllers for uncertain plants. The simplicity of the metric in the scalar case, and the intuition gained from viewing it as the distance between the frequency response loci of plants when plotted on the Riemann sphere, should make these results readily accessible to a wide variety of practising control engineers and genuinely usable in a design context.
3) Given finite dimensional systems, the metric can be substantially easier to work with than, for example, the gap metric; the approximation results of Chapter VI being particularly sharp.
4) It has been shown that the metric is the natural way of assessing the importance of any uncertainty if the control objective is to keep $b_{P,C}$ large over an uncertain set of plants. The same has been shown for a particular robust tracking problem. It has also been shown that the metric precisely captures much of the classical treatment of uncertainty in one measure. In frequency regions where the objective is to gain the benefits of feedback via a high loop gain, then acceptable perturbations are those that leave the magnitude of the plant large. In regions where it is impossible to bound the magnitude of the plant from below it is necessary to use small loop gain to make the uncertainty look small. In

regions of frequency in between it is necessary to accurately know both the gain and the phase of the plant.

We have left a number of questions unanswered. The metric was defined for finite dimensional, linear time invariant systems in order to do a particular job, but it has turned out to have many nice properties (the fact that it is a metric at all was not to be expected). An open question is whether the metric is a particular manifestation of a more general object. We have made some attempt to understand the metric in operator theoretic terms, with regard to subspaces of $\mathcal{L}_2$ and $\mathcal{H}_2$, but have failed to generalize the results fully to the infinite dimensional case. At least there we have a possible definition of the metric, in terms of whether a particular projection operator is Fredholm with zero index; in the time varying case it is not at all clear what the metric should look like, if it has a generalization at all. We have relied heavily on being able to lift shift invariant subspaces of $\mathcal{H}_2$ onto $\mathcal{L}_2$. This lift no longer makes sense for more general subspaces; presumably an alternative interpretation is required.

This work has been motivated by the conviction that the maximization of weighted versions of $b_{P,C}$ is the simplest possible paradigm of robust control to capture the essential features of the problem. The development of this metric, together with the improved understanding of the relationship between P and the closed loop properties implied by a compensator maximizing $b_{P,C}$, mark a substantial contribution to the understanding of this problem. However, there are still a number of obstacles that stand in the way of a clean and satisfying textbook treatment. We need a tight *lower* bound on $b_{opt}(P)$, in terms of the loop shape $P(j\omega)$ and the *rhp* pole/zero structure of P, to supplement the upper bound presented, and a better understanding of the upper bound in the multivariable case. Further work needs to be done on directionality with multivariable systems. We have only made use of the maximum singular values of

$$(I+P_0P_0^*)^{-1/2}(P_0-P_1)(I+P_1^*P_1)^{-1/2} \text{ and } (I+P_0^*P_0)^{1/2}(I-CP_0)^{-1}(I+CC^*)^{1/2}.$$

Less conservative results, both in terms of robustness and the relationship between open and closed loop properties, would take account of the gains in other directions.

APPENDIX A

State-Space Formulae and Proofs

We collect here state-space formulae needed for the evaluation of $\delta_\nu(\cdot,\cdot)$ and the implementation of the two degree of freedom compensator of Section 6, as well state-space proofs of two of the theorems from the main text. We use the notation

$$P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \iff P(s) = C(sI - A)^{-1}B + D$$

for $P \in \mathcal{P}^{p\times q}$ with McMillan degree $\leq n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times q}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times q}$. The quadruple (A, B, C, D) will be referred to as a realization of P. Such a realization is called minimal if $n = \deg(P)$.

1. Calculation of δ_ν

We shall need the following two Riccati equations associated with (A, B, C, D), a minimal realization of P.

The *Generalized Control Algebraic Riccati Equation* (GCARE) is defined as

$$(A - BS^{-1}D^TC)^TX + X(A - BS^{-1}D^TC) - XBS^{-1}B^TX + C^TR^{-1}C = 0,$$

where $R := I + DD^T$ and $S := I + D^TD$. Similarly, the *Generalized Filtering Algebraic Riccati Equation* (GFARE) is defined as

$$(A - BS^{-1}D^TC)Z + Z(A - BS^{-1}D^TC)^T - ZC^TR^{-1}CZ + BS^{-1}B^T = 0.$$

The minimality of (A, B, C, D) is sufficient to ensure that there exist unique solutions $X = X^T > 0$ and $Z = Z^T > 0$ to these equations which are stabilizing in the following sense. Define a generalized control gain, F, and a generalized filter gain, H, as

$$F := -S^{-1}(D^TC + B^TX) \quad \text{and} \quad H := -(BD^T + ZC^T)R^{-1},$$

then X and Z are stabilizing solutions provided the eigenvalues of $(A+BF)$ and $(A+HC)$ have strictly negative real parts. The following formula for all normalized right coprime factorizations of P was established by Vidyasagar ([Vid88]).

If

$$G = \begin{bmatrix} N \\ M \end{bmatrix} = \left[\begin{array}{c|c} A+BF & BS^{-1/2}U \\ \hline C+DF & DS^{-1/2}U \\ F & S^{-1/2}U \end{array}\right]$$

for some unitary matrix $U \in \mathbb{R}^{q\times q}$, then $\{N, M\}$ is a normalized *rcf* of P (and G a normalized right graph symbol). As U ranges over all such unitary matrices then the above realization ranges over all normalized *rcf*s. In the sequel a choice of $U = I$ will be used exclusively. The minimality of (A, B, C, D) is sufficient to ensure that the above realization is also minimal ([OM89]). A similar formula holds for all normalized *lcf*s of P. If

$$\tilde{G} = \begin{bmatrix} -\tilde{M} & \tilde{N} \end{bmatrix} = \left[\begin{array}{c|cc} A+HC & -H & B+HD \\ \hline UR^{-1/2}C & -UR^{-1/2} & UR^{-1/2}D \end{array}\right]$$

for some unitary matrix $U \in \mathbb{R}^{p\times p}$, then $\{\tilde{N}, \tilde{M}\}$ is a normalized *lcf* of P (and $\tilde{G}$ a normalized left graph symbol).

There are two obvious ways of calculating $\delta_\nu(\cdot,\cdot)$. Both involve calculating the $\mathcal{H}_\infty$ or $\mathcal{L}_\infty$ norm of a transfer matrix. Several authors have proposed procedures for calculating the $\mathcal{H}_\infty$ norm of a stable transfer matrix that improve over direct search over ω for $\sup_\omega \bar{\sigma}(P(j\omega)$. All of these methods use the following result, which may be found in [Wil71] and [And67]. For (A, B, C, D), a minimal realization of a stable P, define a Hamiltonian matrix H_γ as

$$H_\gamma = \begin{bmatrix} A + BR_\gamma^{-1}D^TC & \gamma BR_\gamma^{-1}B^T \\ -\gamma C^TS_\gamma^{-1}C & -(A+BR_\gamma^{-1}D^TC)^T \end{bmatrix},$$

where $R_\gamma := (\gamma^2 I - D^TD)$ and $S_\gamma := (\gamma^2 I - DD^T)$. Then, for $\gamma > \bar{\sigma}(D)$,

$$\|P\|_\infty < \gamma \iff H_\gamma \text{ has no } j\omega \text{ axis eigenvalues.}$$

Thus, calculating the $\mathcal{H}_\infty$ norm of a transfer matrix involves searching for the smallest γ for which H_γ has no imaginary axis eigenvalues. A bisection algorithm is suitable for this search. Boyd *et al* give a faster, quadratically convergent algorithm ([BB90]).

Provided $\delta_\nu(P_1, P_2) < 1$, it can be calculated, without first calculating normalized factors, via the equation

$$\sqrt{1-\delta_\nu(P_1,P_2)^2} = \left\| \begin{bmatrix} P_1 \\ I \end{bmatrix} (I + P_2^*P_1)^{-1} [P_2^* \;\; I] \right\|_\infty^{-1}.$$

It is also necessary to calculate $\operatorname{wno}\det(G_1^*G_2)$ as $\delta_\nu(P_1, P_2) < 1$ if, and only if, this winding number is zero. A realization of $G_1^*G_2$ is easily calculated from the realizations of the normalized factors given above, cascading the relevant representations. In fact, given (A_1, B_1, C_1, D_1), a minimal realization of P_1, and

(A_2, B_2, C_2, D_2), a minimal realization of P_2, with F_1, H_1 the generalized control and filter gains for P_1 and F_2, H_2 those for P_2,

$$G_1^* G_2 = \left[\begin{array}{cc|c} -(A_1+B_1F_1)^T & (C_1+D_1F_1)^T(C_2+D_2F_2)+F_1^TF_2 & \\ 0 & A_2+B_2F_2 & \\ \hline -S^{-1/2}B^T & S^{-1/2}F_2+S^{-1/2}D^T(C_2+D_2F_2) & \end{array}\right. \cdots$$

$$\cdots \left.\begin{array}{|c} F_1^T S_2^{-1/2}+(C_1+D_1F_1)^T D_2 S_2^{-1/2} \\ B_2 S_2^{-1/2} \\ \hline S^{-1/2}(I+D^TD_2)S_2^{-1/2} \end{array}\right]. \quad (80)$$

This realization may not be minimal (try, for example, $P_2 = (s-1)/(s+1)$, $P_1 = (s-1)(s+2)/(s+1)(s+3)$); this is not a problem though, as any hidden pole/zero cancellations in the representation of $G_1^*G_2$ will not affect its winding number. It is easily calculated that the "A" matrix of $(G_1^*G_2)^{-1}$, is given by

$$A_{(G_1^*G_2)^{-1}} = \begin{bmatrix} -(A_1 - B_1WD_2^TC_1)^T & C_1^TYC_2 \\ B_2W^TB_1^T & A_2 - B_2D_1^TYC_2 \end{bmatrix},$$

where $W := (I + D_2^TD_1)^{-1}$ and $Y := (I + D_2D_1^T)^{-1}$, (notice that this realization does not depend on the Riccati equation solutions). It follows from (80) that $\eta(G_1^*G_2) = \deg(P_1)$, thus

$$\operatorname{wno}\det(G_1^*G_2) = 0$$

$$\Updownarrow$$

$$A_{(G_1^*G_2)^{-1}} \text{ has precisely } \deg(P_1) \text{ eigenvalues with a positive real part}$$

2. Observer based two degree of freedom compensators

In Section 6 we showed that, given a feedback compensator designed to maximize $b_{P,C}$, injecting any reference signals in such a way that $y = Nr$ (where N is the numerator of a normalized right coprime factorization of P) gives a well behaved tracking response, which is robust to small perturbations of P as measured by the metric δ_ν, and which has a bandwidth approximately equal to the open loop bandwidth of P. Furthermore, we saw that such a structure could be implemented in terms of a particular coprime factorization of C. In this section, we show that, provided C is the so-called central compensator satisfying $b_{P,C} > 1/\gamma$, this scheme may be implemented in an observer form of McMillan degree equal to that of the plant. The structure we shall be using is shown in Figure A.21, for $\{A, B, C, 0\}$ a realization of a strictly proper P. This is an intuitively appealing way of injecting references into an observer based controller and is used, for example, in [Hyd91]).

Provided $A+HC$ is stable, the resulting transfer function from r to y is clearly equivalent to that obtained when applying a state feedback matrix F around P alone. The tracking response is otherwise independent of H. Thus $y = Nr$,

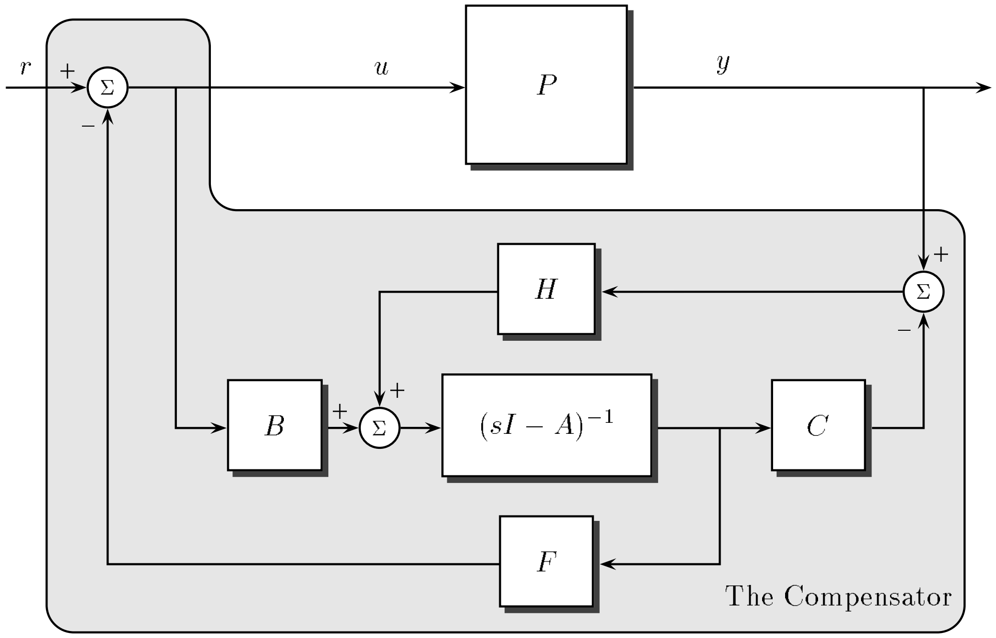


FIGURE A.21. An observer based two degree of freedom compensator

where N is the numerator of the (not necessarily normalized) right coprime factorization determined by state feedback gain F. In [MG90] all possible feedback compensators satisfying $b_{P,C} > 1/\gamma$, for $1/\gamma < b_{opt}(P)$ are parameterized as a linear fractional transformation on the unit ball of $\mathcal{H}_\infty$. A state space realization of the so-called central solution, corresponding to the origin, is given as

$$C = \left[\begin{array}{c|c} A + BF + HC & -H \\ \hline F & 0 \end{array}\right]$$

where

$$H = \gamma^2((1-\gamma^2) + ZX)^{-1}ZC^T, \quad F = -B^T X,$$

and X, Z solve GCARE and GFARE. This is precisely the observer form of Figure A.21 (without the reference). So, implementing this compensator with the reference injected as in Figure A.21 yields the tracking response $y = Nr$ where N is the numerator of a normalized right coprime factorization of P. It remains to be shown that H is a stabilizing output injection. This can be inferred from the dual result in [SG90], where an observer form of the same compensator is constructed using the output injection directly from the GFARE solution and the state feedback from an associated $\mathcal{H}_\infty$ control problem. Here we are using the state feedback directly from the GCARE solution, and the observer gain H actually solves an associated $\mathcal{H}_\infty$ filtering problem.

3. Proof of Lemmas from Chapter 6

PROOF: of Lemma VI.2 We prove this theorem for the case where $p \geq q$, the result for the other case is then easily inferred from this after noticing that $\{\tilde{N}^T, \tilde{M}^{-T}\}$ is a normalized *rcf* of P^T which satisfies the assumptions of the first case.

From results on optimal Hankel norm approximation (see [Glo84]) it follows that there exist extensions of the required degree which satisfy the norm bound, we need to construct a particular extension which also has the other properties. Suppose the Hankel operator $\mathbf{H}_{G_0}$ has singular values

$$\sigma_1 \geq \ldots \geq \sigma_{n-r} > \sigma_{n-r+1} = \ldots = \sigma_n.$$

Let $\{A, B, C, D\}$ be a minimal, balanced realization of G_0, satisfying

$$AR + RA^T + BB^T = 0, \quad \text{and } A^T S + SA + C^T C = 0 \tag{81}$$

where $R = S = \operatorname{diag}(\Sigma, \sigma_n I_r)$ and $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_{n-r})$. Note also that

$$D^T C + B^T S = 0, \quad \& \quad D^T D = I \tag{82}$$

as G_0 is inner. Now partition A, B, C conformally with R as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, C = [C_1 \ C_2]$$

and define

$$\begin{aligned}
\hat{A} &:= \Gamma^{-1}(\sigma_n^2 A_{11}^T + \Sigma A_{11} \Sigma - \sigma_n C_1^T U B_1^T) \\
\hat{B} &:= (1 - \sigma_n^2)^{-\frac{1}{2}} \Gamma^{-1}(\Sigma B_1 + \sigma_n C_1^T U) \\
\hat{C} &:= C_1 \Sigma + \sigma_n U B_1^T \\
\hat{D} &:= (1 - \sigma_n^2)^{-\frac{1}{2}}(D - \sigma_n U)
\end{aligned}$$

where

$$\Gamma := \Sigma^2 - \sigma_n^2 I$$

and U satisfies $U^T U \leq I$ and

$$B_2 = -C_2^T U. \tag{83}$$

By [Glo84] Corollary 7.3

$$G_1 = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$$

satisfies the norm bound of the theorem statement. We shall first show that if we choose U to satisfy the further constraints

$$U^T U = I \tag{84}$$

$$D^T U = \sigma_n I \tag{85}$$

then G_1 has the additional properties claimed. We shall then construct such a U. Define

$$\hat{S} := \Sigma\Gamma,$$

then

$$\hat{A}^T\hat{S} + \hat{S}\hat{A} + \hat{C}^T\hat{C} = 0,$$
$$\hat{D}^T\hat{D} = I$$
$$\hat{D}^T\hat{C} + \hat{B}^T\hat{S} = 0$$

as is easily verified, using the partitioned forms of (81) and (82). Hence, by [Glo84] Lemma 2.10, G_1 is inner. We shall now prove the bound on the singular values given in part ii) of the theorem statement. As in [GS91] we define

$$\hat{R} := \Gamma^{-1}\Sigma$$

from which it is easily verified that

$$\hat{A}\hat{R} + \hat{R}\hat{A}^T + \hat{B}\hat{B}^T = \sigma_n^2\Gamma^{-1}C_1^T(\hat{D}\hat{D}^T + UU^T - I)C_1\Gamma^{-1}.$$

Now put

$$X := \begin{bmatrix}\hat{D} & U\end{bmatrix} \in \mathbb{R}^{(p+q)\times(2q)}$$

and notice that $X^TX = I$, thus $XX^T \leq I$ and hence

$$\hat{A}\hat{R} + \hat{R}\hat{A}^T + \hat{B}\hat{B}^T \leq 0.$$

Because of the monotonicity of Lyaponov equation solutions, and the stability of $\hat{A}$, this means that

$$\sigma_i\left(\mathbf{H}_{G_1}\right) \leq \lambda_i^{1/2}(\hat{R}\hat{S}) = \lambda_i^{1/2}\Sigma = \sigma_i\left(\mathbf{H}_{G_0}\right).$$

Notice that if $p = q$, or the original system is square, then X is also square and equality is achieved for all singular values. The fact that G is left invertible follows from the bound on the maximum singular value of this operator and Lemma VI.1. It remains to show that we can always construct a U satisfying (83), (84) and (85). Note that $B_2 \in \mathbb{R}^{r\times q}$, and put $s := rank(B_2)$ Now, $B_2B_2^T = C_2^TC_2$ from (81), which implies that B_2 and C_2 have singular value decompositions

$$B_2 = U_1\begin{bmatrix}\Lambda_{s\times s} & 0_{s\times q-s}\\ 0_{r-s\times s} & 0_{r-s\times q-s}\end{bmatrix}V_1^T, \qquad -C_2^T = U_1\begin{bmatrix}\Lambda_{s\times s} & 0_{s\times p+q-s}\\ 0_{r-s\times s} & 0_{r-s\times p+q-s}\end{bmatrix}V_2^T,$$

with $\Lambda > 0$. Also, $D^TC_2 = -\sigma_nB_2^T$ from (82), which implies that D has the form

$$D = V_2\begin{bmatrix}\sigma_nI_s & 0_{s\times q-s}\\ \multicolumn{2}{c}{\tilde{D}_{p+q-s\times q}}\end{bmatrix}V_1^T.$$

Also, $D^TD = I_q$ implies that

$$\tilde{D}^T\tilde{D} = \begin{bmatrix}(1-\sigma_n^2)I_s & 0\\ 0 & I_{q-s}\end{bmatrix}$$

and thus $\tilde{D}$ has a singular value decomposition

$$\tilde{D} = U_2 \begin{bmatrix} \sqrt{1-\sigma_n^2} I_s & 0 \\ 0 & I_{q-s} \\ 0_{p-s \times s} & 0_{p-s \times q-s} \end{bmatrix}.$$

Now put

$$U := V_2 \begin{bmatrix} I-s & 0 \\ 0 & \tilde{U} \end{bmatrix} V_1^T, \quad \text{with } \tilde{U} := U_2 \begin{bmatrix} 0_{s \times q-s} \\ \sigma_n I_{q-s} \\ \sqrt{1-\sigma_n^2} I_{q-s} \\ 0_{p-q \times q-s} \end{bmatrix}.$$

This U then satisfies (83), (84) and (85). Note that if $s = q$, $s = r$ or $p = q$ then some of the submatrices in the construction may be empty; this, though, does not effect the argument. In particular, if $s = q$ then virtually all of the submatrices are empty and U is given more simply by $U = -C_2 B_2^{T\dagger}$ as in [Glo84]. If $p = q$ the choice of U is unique, otherwise there are alternative ways of distributing the identity between the lower two blocks of $\tilde{U}$. □

PROOF: of Lemma VI.4 Given $\{\tilde{N}, \tilde{M}\}$, a normalized *lcf* of P, then by [Glo84] Theorem 5.2 there exists a transfer function $\begin{bmatrix} \tilde{X} & \tilde{Y} \end{bmatrix}$ of McMillan degree n such that

$$Z := \begin{bmatrix} -\tilde{N} & \tilde{M} \\ \tilde{X} & \tilde{Y} \end{bmatrix}$$

is unitary and $\deg(Z) = n$, where $n = \deg(P)$. Now,

$$Z^* Z = I \implies \tilde{X}^* \tilde{X} = I - \tilde{N}^* \tilde{N} \geq 0,$$

equality holding only at imaginary axis zeros of $\tilde{M}$ (as $\tilde{N}\tilde{N}^* + \tilde{M}\tilde{M}^* = I$). Thus $\tilde{X}$ is invertible almost everywhere on the imaginary axis. Also,

$$ZZ^* = I \implies -\tilde{X}\tilde{N}^* + \tilde{Y}\tilde{M}^* = 0$$

thus $\tilde{X}^{-1}\tilde{Y} = \tilde{N}^*\tilde{M}^{-*} = P^*$. Furthermore, $\{\tilde{Y}, \tilde{X}\}$ is a normalized *lcf* of P^*, as $\deg(\begin{bmatrix} \tilde{Y} & \tilde{X} \end{bmatrix}) = n$ and

$$ZZ^* = I \implies \tilde{Y}\tilde{Y}^* + \tilde{X}\tilde{X}^* = I.$$

As Z is unitary, then, by [Glo84] Theorem 5.1, all the singular values of the Hankel operator $\mathbf{H}_Z$ are equal to 1, and Z has a balanced realization

$$Z = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

such that

$$AP + PA^T + BB^T = 0 \text{ and } A^T Q + QA + C^T C = 0 \tag{86}$$

for $P = Q = I$. Partition C as $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$, and D as $\begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$, such that

$$\begin{bmatrix} -\tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C_1 & D_1 \end{array}\right] \text{ and } \begin{bmatrix} \tilde{X} & \tilde{Y} \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C_2 & D_2 \end{array}\right].$$

Thus, $\sigma_i^2 \mathbf{H}_{\begin{bmatrix} -\tilde{N} & \tilde{M} \end{bmatrix}}$ are given by the eigenvalues of PQ_1, where $Q_1 = Q_1^T \geq 0$ solves

$$A^T Q_1 + Q_1 A + C_1^T C_1 = 0. \tag{87}$$

Similarly, $\sigma_i^2 \mathbf{H}_{\begin{bmatrix} \tilde{X} & \tilde{Y} \end{bmatrix}}$ are given by the eigenvalues of PQ_2, where $Q_2 = Q_2^T \geq 0$ solves

$$A^T Q_2 + Q_2 A + C_2^T C_2 = 0. \tag{88}$$

Comparison of (86),(87) and (88), noting that $C^T C = C_1^T C_1 + C_2^T C_2$, implies $Q_1 + Q_2 = Q = I$. Hence,

$$\lambda_i(Q_1) = 1 - \lambda_{n-i+1}(Q_2) \ \forall i$$

and

$$\sigma_i^2 \left(\mathbf{H}_{\begin{bmatrix} -\tilde{N} & \tilde{M} \end{bmatrix}} \right) = 1 - \sigma_{n-i+1}^2 \left(\mathbf{H}_{\begin{bmatrix} \tilde{X} & \tilde{Y} \end{bmatrix}} \right) \ \forall i.$$

To complete the proof, note that

$$\sigma_i \left(\mathbf{H}_{\begin{bmatrix} -\tilde{Y} & \tilde{X} \end{bmatrix}} \right) = \sigma_i \left(\mathbf{H}_{\begin{bmatrix} \tilde{X} & \tilde{Y} \end{bmatrix}} \right),$$

as the only difference between their two realizations is a reordering of the columns of the B matrix, which does not change the Lyaponov equation for P. $\square$

List of Symbols

$\mathbb{Z}, \mathbb{R}, \mathbb{C}$		integers, real and complex numbers
$\mathcal{L}_2$, $\mathcal{H}_2$, $\mathcal{H}_2^\perp$, $\|\cdot\|_2$	10	
$\mathcal{L}_\infty$, $\mathcal{RL}_\infty$, $\|\cdot\|_\infty$	11	
$\mathcal{H}_\infty$, $\mathcal{RH}_\infty$, $\|\cdot\|_\infty$	11	
C_0	39	functions continuous on $j\mathbb{R} \cup \infty$
A_0	39	$= \mathcal{H}_\infty \cap C_0$
$\det M$, M^T, M^*		determinant, transpose, conjugate transpose of a matrix M
$\bar{\sigma}(\cdot)$, $\underline{\sigma}(\cdot)$, $\sigma_i(\cdot)$		maximum, minimum and i'th singular value of a matrix or operator
$\lambda_i(\cdot)$		i'th eigenvalue of a matrix or operator
$\mathcal{P}^{p\times q}$	12	the set of rational transfer function matrices with p rows and q columns
$\mathcal{P}_s^{p\times q}$		$= \mathcal{P}^{p\times q} \cap \mathcal{RH}_\infty$
$P^*(s)$		$= \overline{P(-\bar{s})}^T$, thus $P^*(j\omega) = \{P(j\omega)\}^*$. If $P(s)$ is real rational, $P^*(s) = P(-s)^T$.
$\mathbf{M}_P$	12	multiplication operator on $\mathcal{H}_2$.
$\mathbf{L}_P$	12	multiplication operator on $\mathcal{L}_2$.
$\mathcal{D}(\mathbf{A})$, $\mathcal{G}(\mathbf{A})$	12	domain, and graph, of an operator $\mathbf{A}$
$\mathrm{Ra}(\mathbf{A})$, $\mathrm{Ker}(\mathbf{A})$	15	range, and kernel, of an operator $\mathbf{A}$
$\mathcal{G}_i$		abbreviation for $\mathcal{G}(\mathbf{M}_{P_i})$.
$\{N_i, M_i\}$, $\{\tilde{N}_i, \tilde{M}_i\}$	14	normalized right, and left, coprime factorizations of P_i.
$\{N_c, M_c\}$, $\{\tilde{N}_c, \tilde{M}_c\}$	14	normalized right, and left, coprime factorizations of C
G_i, $\tilde{G}_i$	15	normalized right, and left, graph symbols for P_i

K, $\tilde{K}$	15	normalized right, and left, inverse graph symbols for C
$\mathbf{H}_P$, $\mathbf{T}_P$, $\mathbf{\Pi}_+$, $\mathbf{\Pi}_-$ ..	13	
$\delta_g(P_1, P_2)$, $\vec{\delta}_g(P_1, P_2)$	19	(and see 20) $\mathcal{H}_2$-gap metric, directed $\mathcal{H}_2$-gap
$\delta_{\mathcal{L}_2}(P_1, P_2)$	21	$\mathcal{L}_2$-gap metric
$\delta_p(P_1, P_2)$	25	pointwise gap metric
$\delta_\nu(P_1, P_2)$	25	modified $\mathcal{L}_2$-gap metric
$\theta(P_1, P_2)$	35	$= \arcsin \delta_\nu(P_1, P_2)$
$\kappa(P_1(j\omega), P_2(j\omega))$ ...	64	pointwise $\mathcal{L}_2$-gap metric
$b_{P,C}$	3	generalized stability margin
$b_{\mathcal{L}_2}(P, C)$	38	
$\rho(P(j\omega), C(j\omega))$	51	pointwise stability margin
$[P, C]$	3	standard feedback connection of P and C
$\mathrm{wno}(g)$	25	winding number of $g(j\omega)$
$\eta(g)$		number of open right half plane poles of $g(s)$
rcf, lcf		right (left) coprime factorization
rhp		right half plane *ie* $\{s : \Re(s) > 0\}$
LHS, RHS		left, and right, hand side (of equation)
$\exists$, $\forall$		there exists, for all
$\{A : B\}$		set of As satisfying B

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