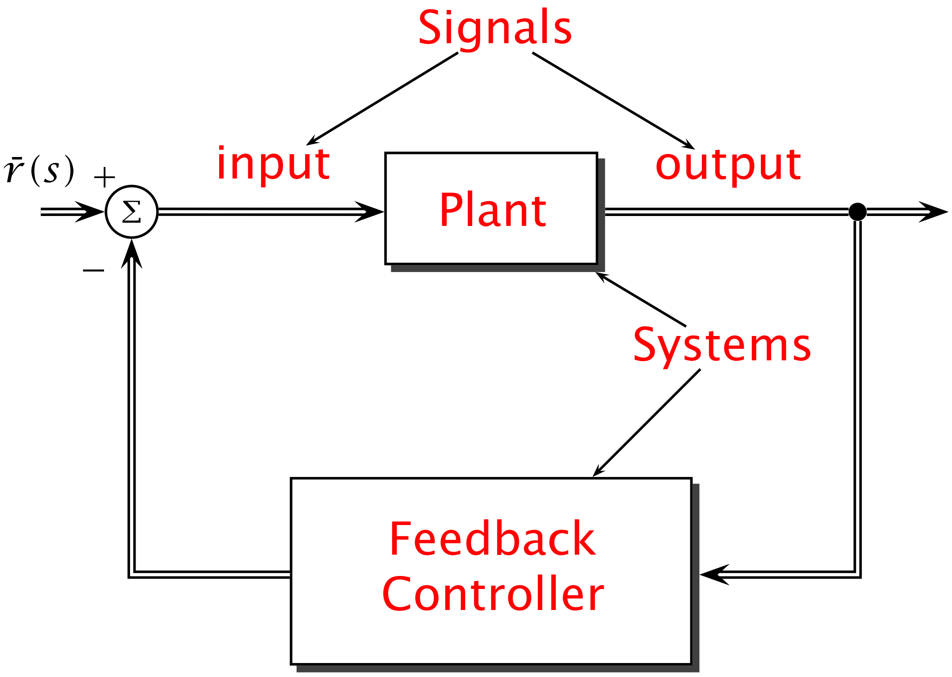
Part IB Paper 6: Information Engineering

LINEAR SYSTEMS AND CONTROL

Dr Glenn Vinnicombe

HANDOUT 1

"Signals, systems and feedback"



The **Aims** of the course are to:

- Introduce and motivate the use of feedback control systems.
- Introduce analysis techniques for linear systems which are used in control, signal processing, communications, and other branches of engineering.
- Introduce the specification, analysis and design of feedback control systems.
- Extend the ideas and techniques learnt in the IA Linear Systems and Vibrations course.

By the end of the course students should:

- Be able to develop and interpret block diagrams and transfer functions for simple systems.
- Be able to relate the time response of a system to its transfer function and/or its poles.
- Understand the term 'stability', its definition, and its relation to the poles of a system.
- Understand the term 'frequency response' (or 'harmonic response'), and its relation to the transfer function of a system.
- Be able to interpret Bode and Nyquist diagrams, and to sketch them for simple systems.
- Understand the purpose of, and operation of, feedback systems.
- Understand the purpose of proportional, integral, and derivative controller elements, and of velocity feedback.
- Possess a basic knowledge of how controller elements may be implemented using operational amplifiers, software, or mechanical devices.
- Be able to apply Nyquist's stability theorem, to predict closed-loop stability from open-loop Nyquist or Bode diagrams.
- Be able to assess the quality of a given feedback system, as regards stability margins and attenuation of uncertainty, using open-loop Bode, Nyquist and/or root-locus diagrams.

SYLLABUS

Course material	Section numbers	
	book 1	book 2
Examples of feedback control systems. Use of block diagrams. Differential equation models. Meaning of 'Linear System'.	1.1- 1.11 2.2-2.3	1.1-1.3 2.1- 2.6.1
Review of Laplace transforms. Transfer functions. Poles (characteristic roots) and zeros. Impulse and step responses. Convolution integral. Block diagrams of complex systems.	2.4-2.6	3.1-3.2
Definition of stability. Pole locations and stability. Pole locations and transient characteristics.	5.6 6.1	3.3-3.4 4.4.1
Frequency response (harmonic response). Nyquist (polar) and Bode diagrams.	8.1-8.3	6.1
Terminology of feedback systems. Use of feedback to reduce sensitivity. Disturbances and steady-state errors in feedback systems. Final value theorem.	4.1-4.2 4.4-4.5	4.1 3.1.6
Proportional, integral, and derivative control. Velocity (rate) feedback. Implementation of controllers in various technologies.	7.7 12.6	4.2
Nyquist's stability theorem. Predicting closed-loop stability from open-loop Nyquist and Bode plots.	9.1-9.3	6.3
Performance of feedback systems: Stability margins, Speed of response, Sensitivity reduction. Interpretation of root locus diagrams.	6.3,8.5 9.4,9.6 12.5	6.4,6.6 6.9

References

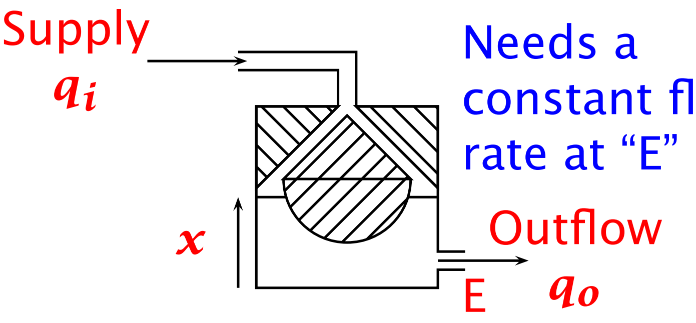
1. Dorf,R.C, and Bishop,R.H, Modern Control Systems, 8th ed., (Addison-Wesley), 1998.

2. Franklin,G.F, Powell,J.D, and Emami-Naeini,A, Feedback Control of Dynamic Systems, 3rd ed., (Addison-Wesley), 1994.

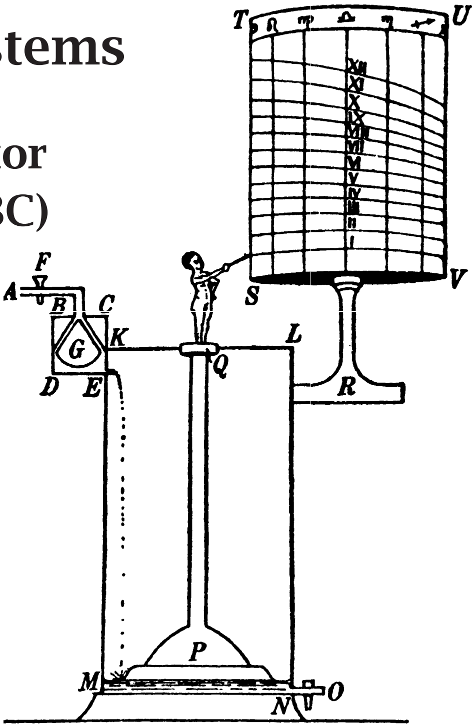
Contents

1.1 Examples of feedback systems

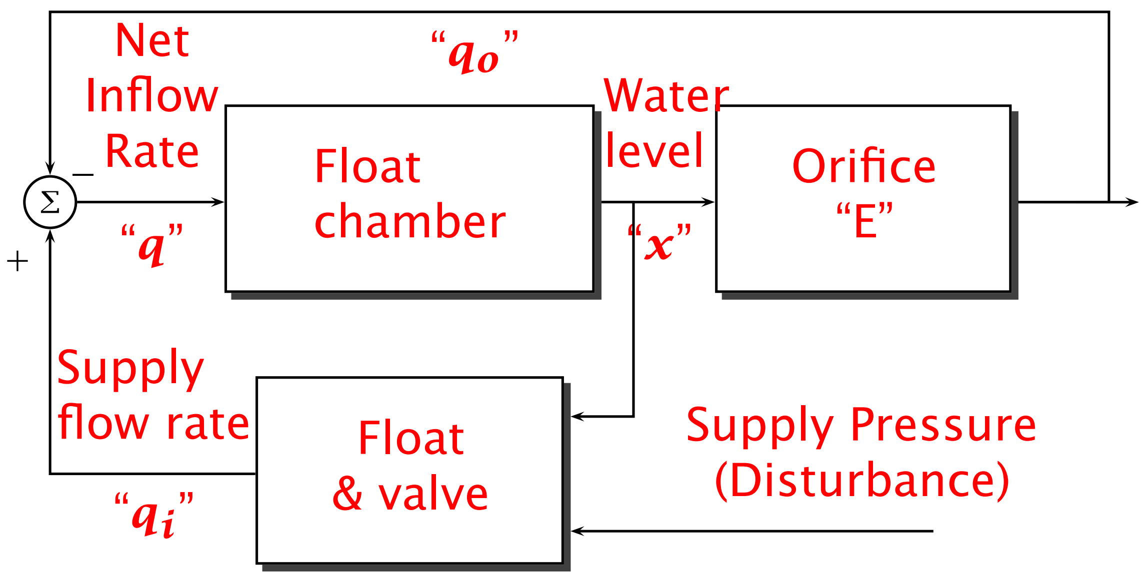
1.1.1 Ktesibios' Float Valve regulator (Water-clock, Alexandria 250BC)



Needs a constant flow rate at "E"



is a feedback control system.
Block Diagram:



Signals have units (usually), are functions of time, and are represented by the *connections*:

e.g. Net inflow "$q(t)$" is measured in m^3/s
Water level "$x(t)$" is measured in m

Systems have equations, and are represented by the *blocks*:
e.g. the Float chamber is described by

$$x(t) = \frac{1}{A}\int_0^t q(\tau)\,d\tau$$

cross-sectional area

1.1.2 Watt's Governor

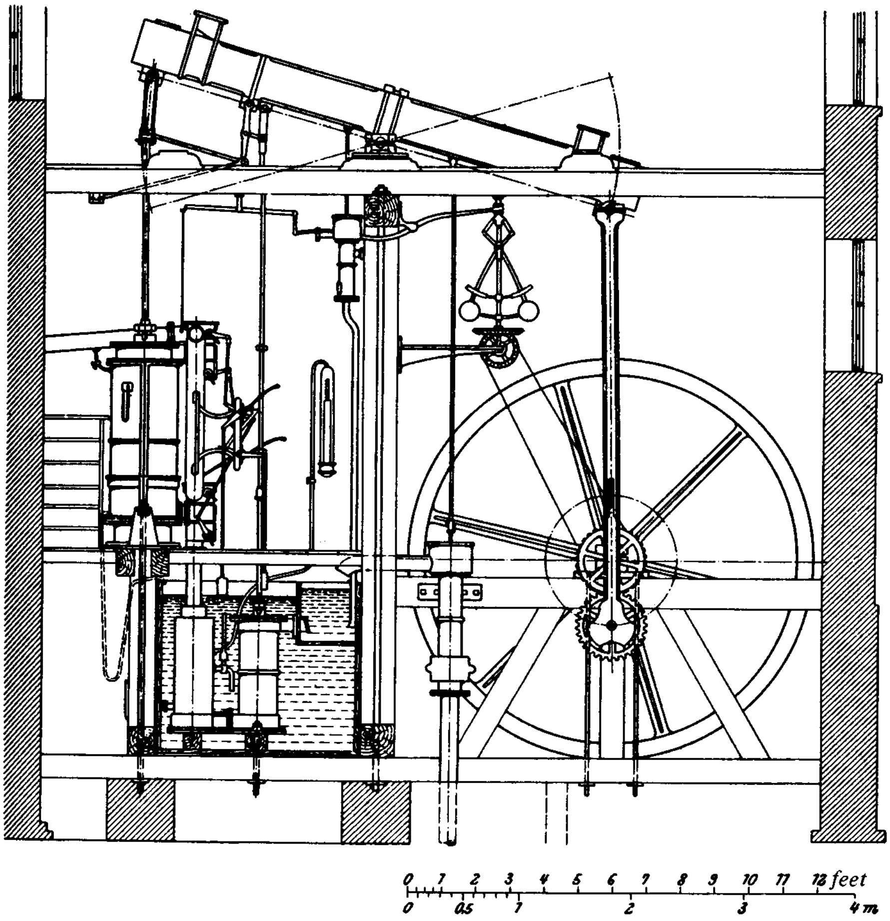
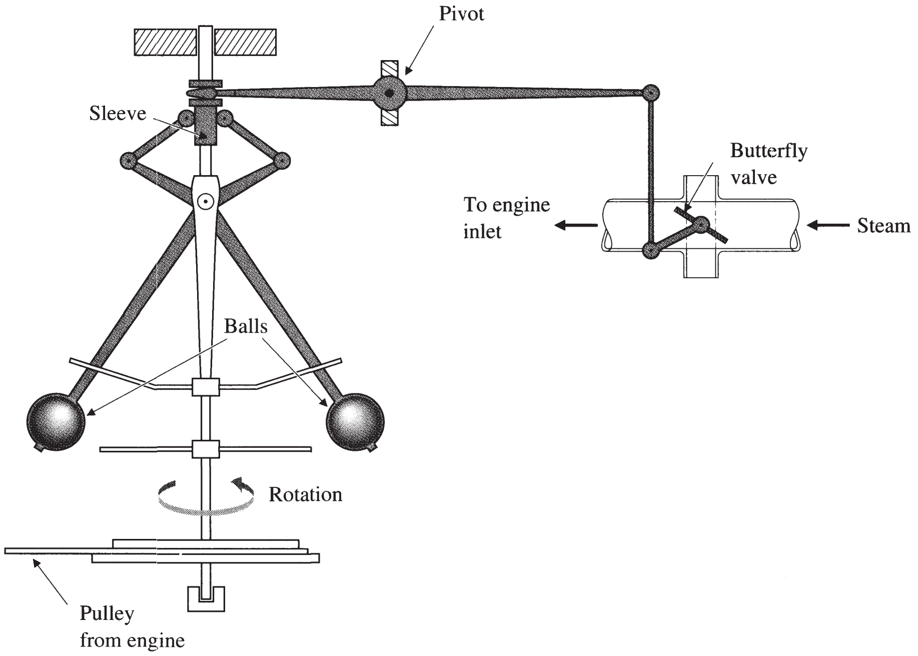


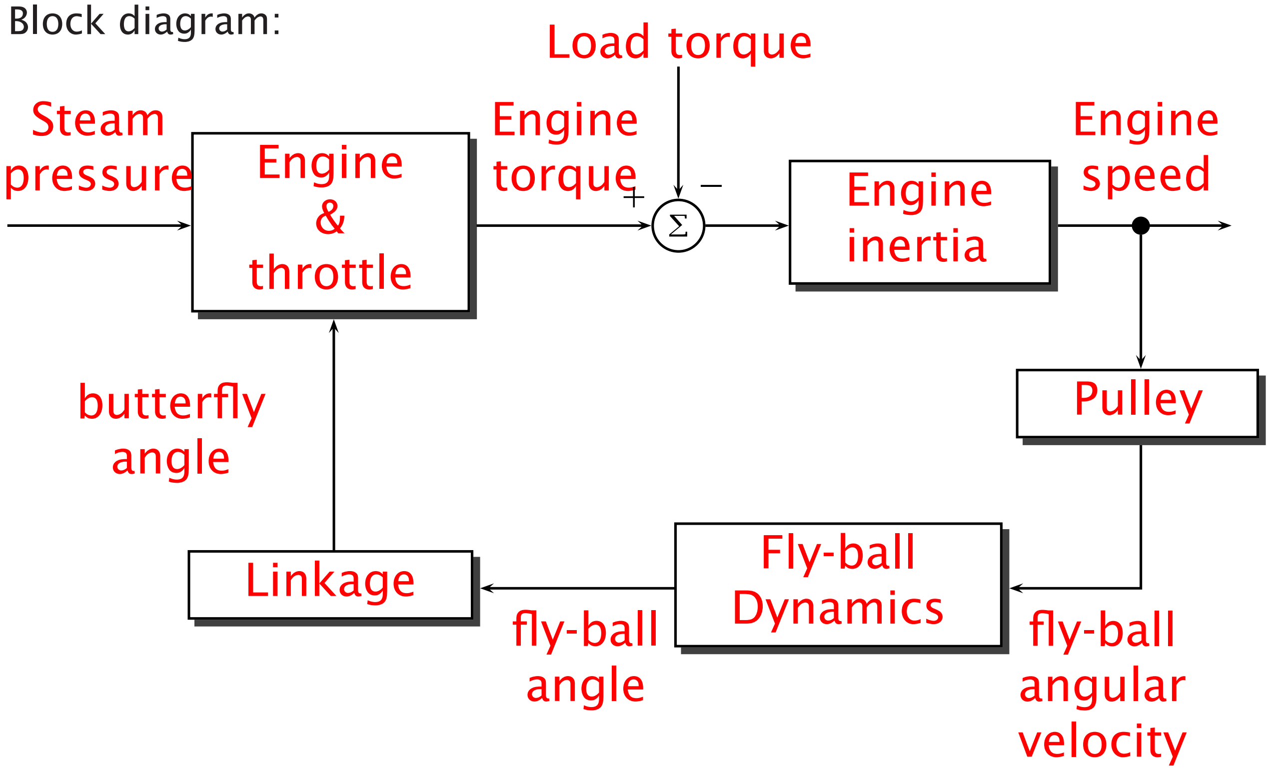
Figure 1 Watt steam engine (1789-1800) with centrifugal governor.

If the load upon the running engine is suddenly increased, its speed will decrease. The flyweights will swing back, and the sleeve will slide upward, causing the steam valve to open. The increase in the flow rate of steam, and hence in torque, will accelerate the engine. The centrifugal weights will fly outward again, reducing the aperture of the valve. Ultimately, the engine will reach an equilibrium at a new speed that lies somewhat below the equilibrium speed prior to the load increase. This *offset* due to lasting disturbances or changes in the command signal is a characteristic of all *proportional* control systems. The increased load requires an increased

Watt's Governor

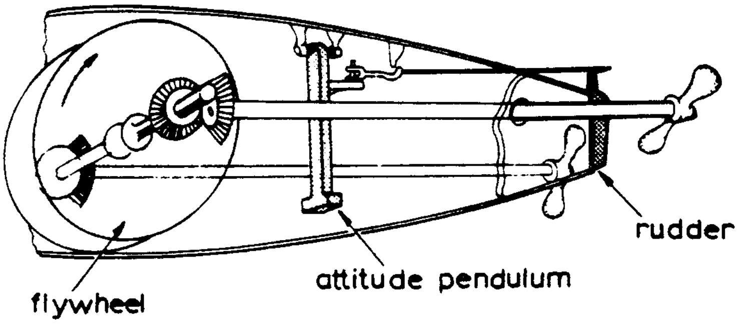


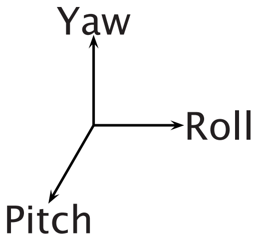
Is a feedback control system.
Block diagram:



Note: it would be wrong to label the input to the feedback system as simply "steam" rather than "steam pressure". Steam in itself is not a quantity (although its pressure, temperature or flow rate is).

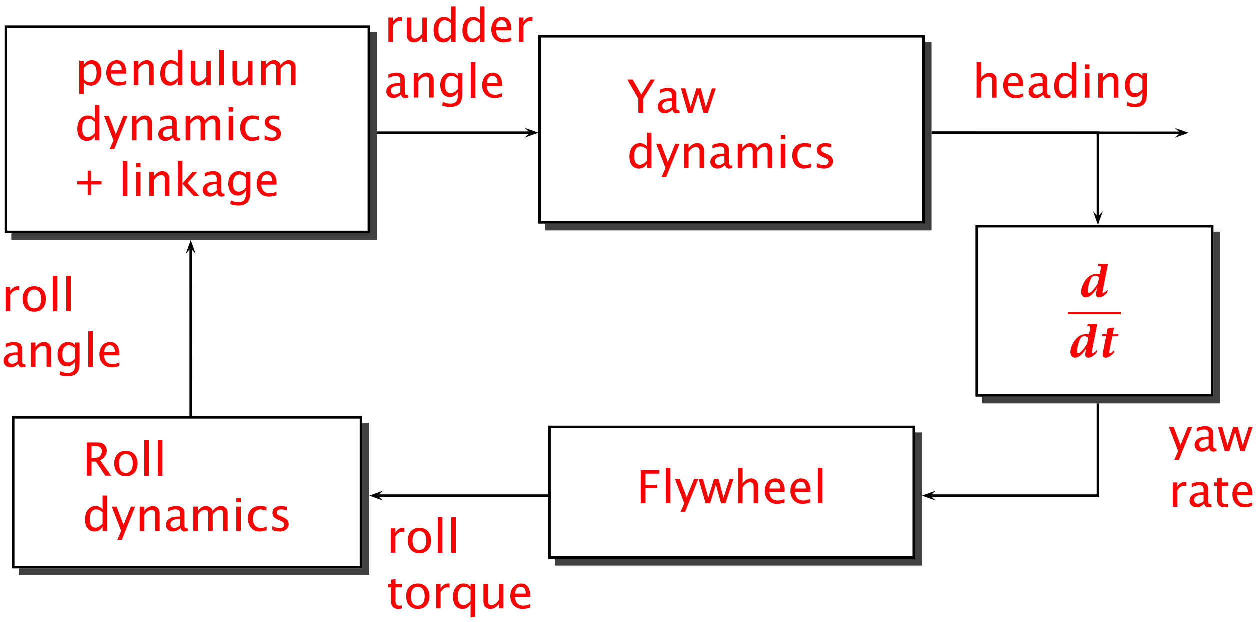
1.1.3 Howell's Torpedo



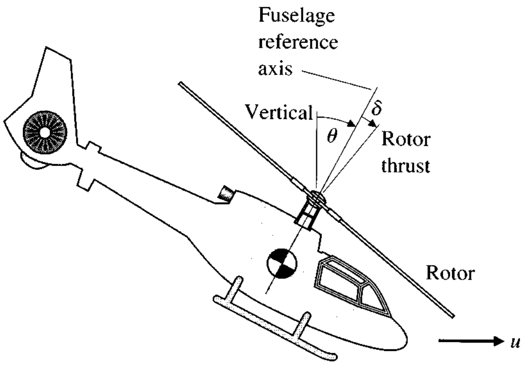


Is a feedback control system.

Block diagram:

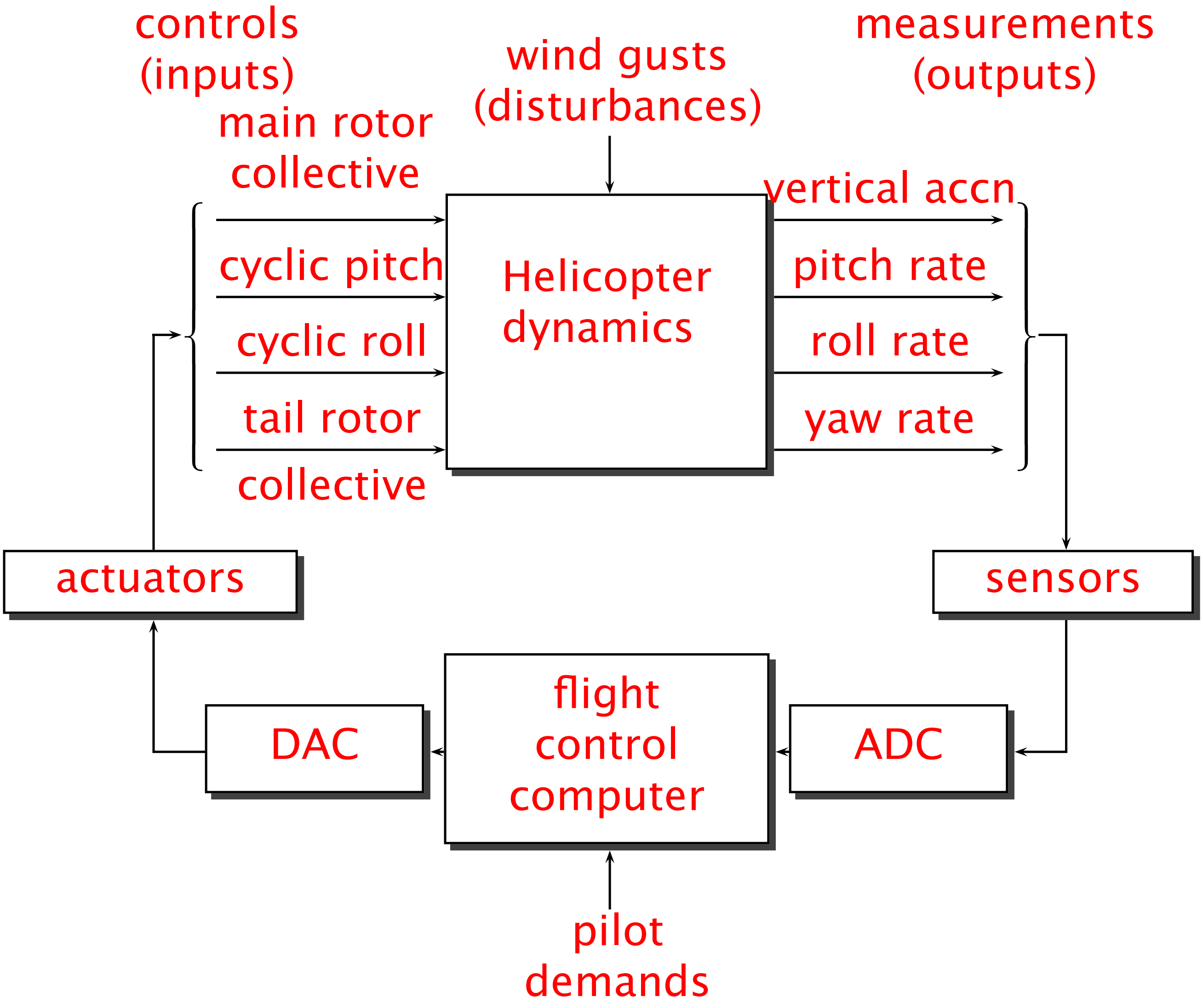


1.1.4 A Helicopter Flight Control System



Is a feedback control system

Block Diagram:

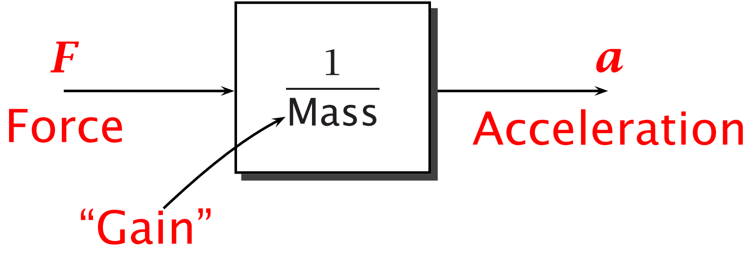


1.2 Block Diagrams

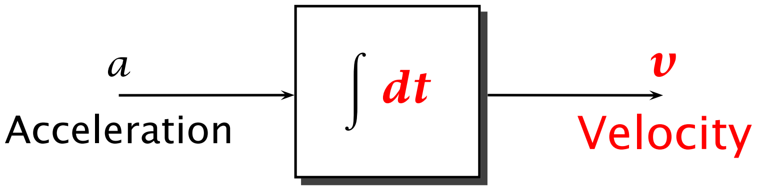
1.2.1 What goes in the blocks?

Some of them act like “amplifiers” or “attenuators”

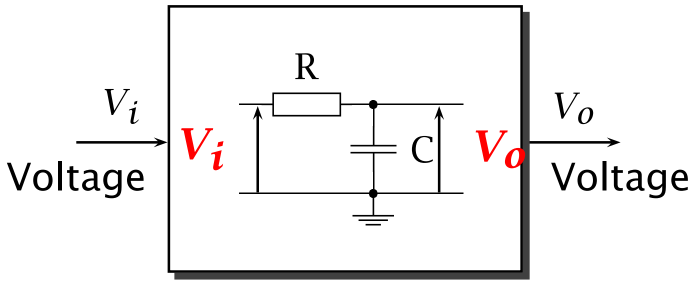
$$a = \frac{1}{Mass} \times F$$



But many are dynamic processes described by Ordinary Differential Equations (ODEs).



$$\dot{v} = a$$



We shall describe these by *transfer functions*.

Note: By drawing this circuit as a block, we are implicitly assuming that any current it draws has negligible effect on the preceding block and that the following block draws insignificant current from it.

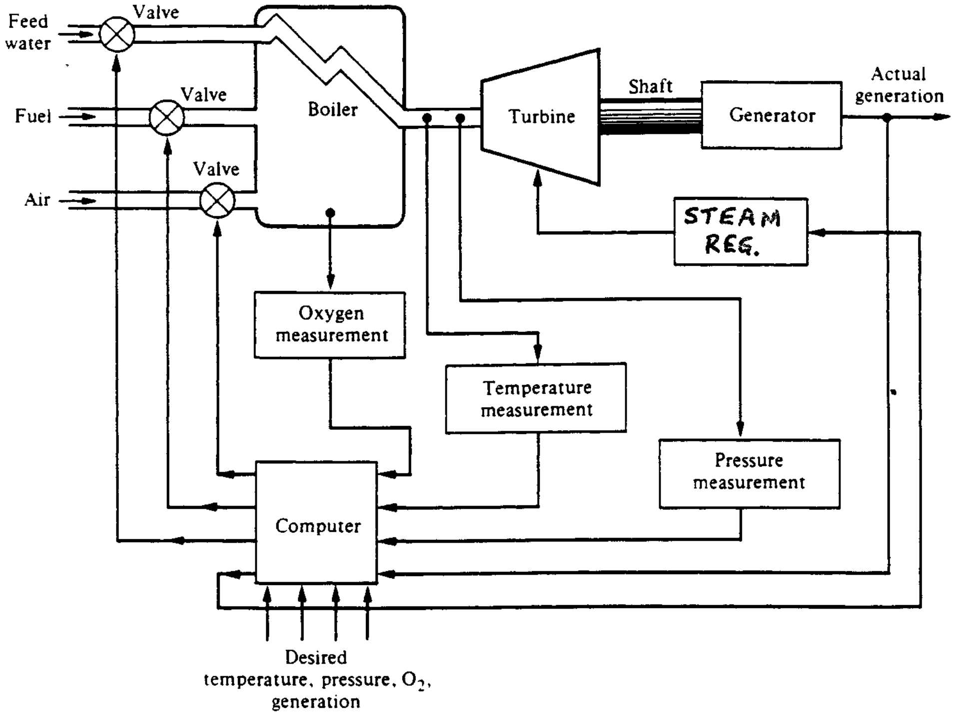
1.2.2 Signals and systems

Blocks represent “systems ”, whose inputs and outputs are “signals ”.

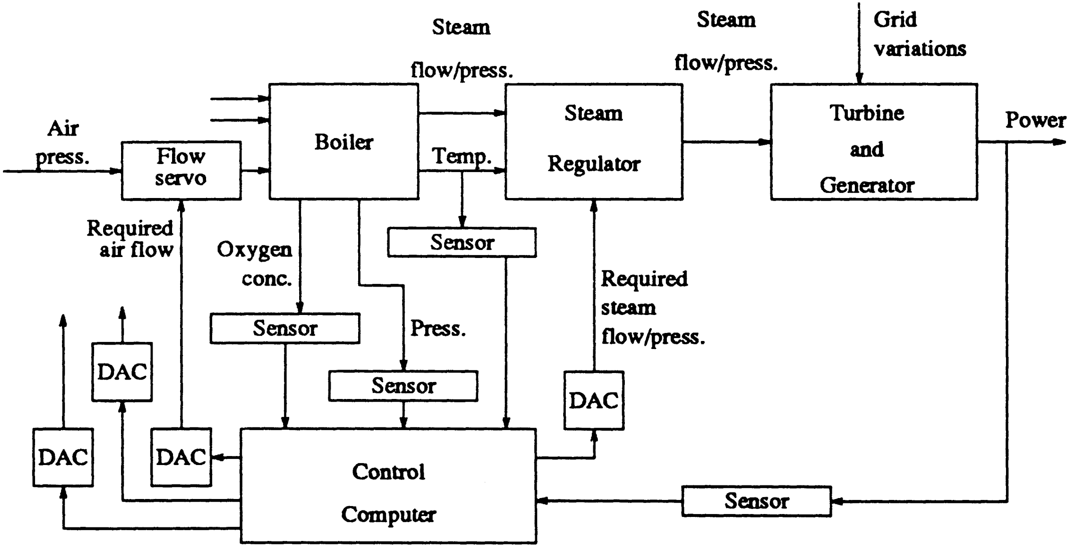
sets of equations/relationships which map inputs into outputs

which take a numeric value as a function of time

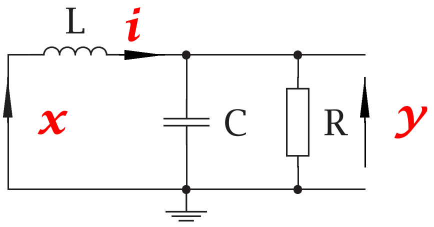
This is **NOT** a block diagram
(in our sense)



This **IS** a block diagram



1.2.3 ODE models – A circuits example



$$x - y = L\frac{di}{dt}$$

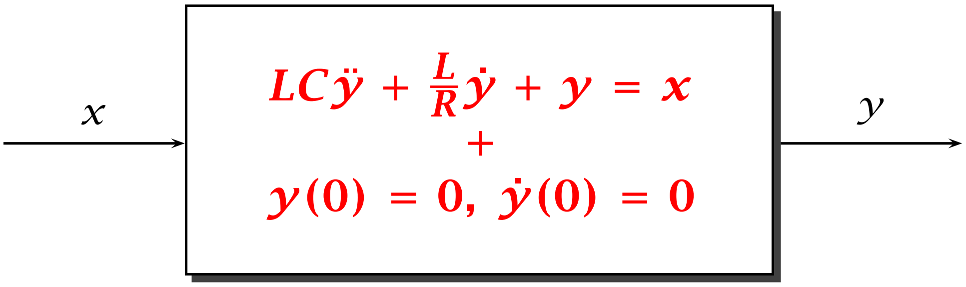
$$i = C\dot{y} + \frac{y}{R}$$

$$\Rightarrow \qquad x - y = L\left(C\ddot{y} + \frac{\dot{y}}{R}\right)$$

which gives a 2nd-order *linear* Ordinary Differential Equation:

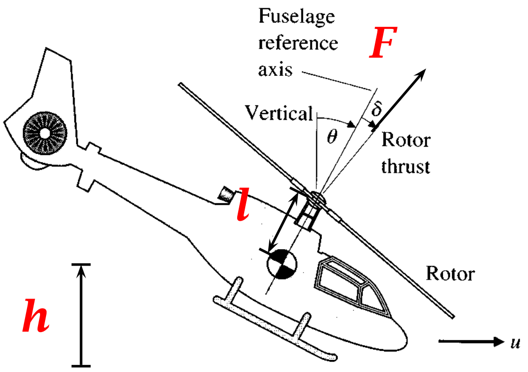
$$\Rightarrow \qquad \boxed{LC\ddot{y} + \frac{L}{R}\dot{y} + y = x}$$

Note: in order to find $y(t)$ (for $t > 0$) given $x(t)$ (for $t > 0$) we would also need to know initial conditions for y, $\dot{y}$. For example, $y(0) = 0$, $\dot{y}(0) = 0$.



$$LC\ddot{y} + \tfrac{L}{R}\dot{y} + y = x$$
$$+$$
$$y(0) = 0,\ \dot{y}(0) = 0$$

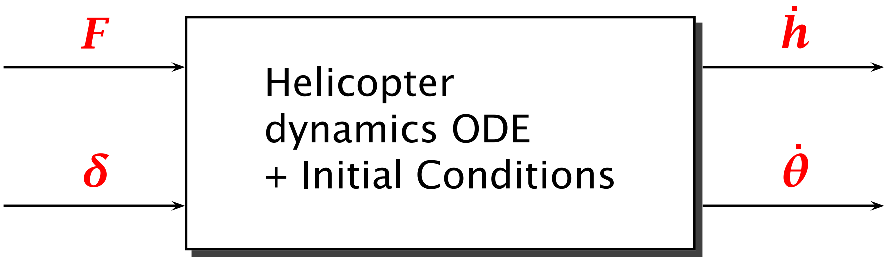
1.2.4 ODE models – A helicopter example



$$I\ddot{\theta} = Fl\sin\delta$$

$$M\ddot{h} = F\cos(\theta + \delta) - Mg$$

This is a non-linear system, with inputs F, δ and outputs $\dot{h}$, $\dot{\theta}$ (for example)



1.2.5 Block diagrams and the control engineer

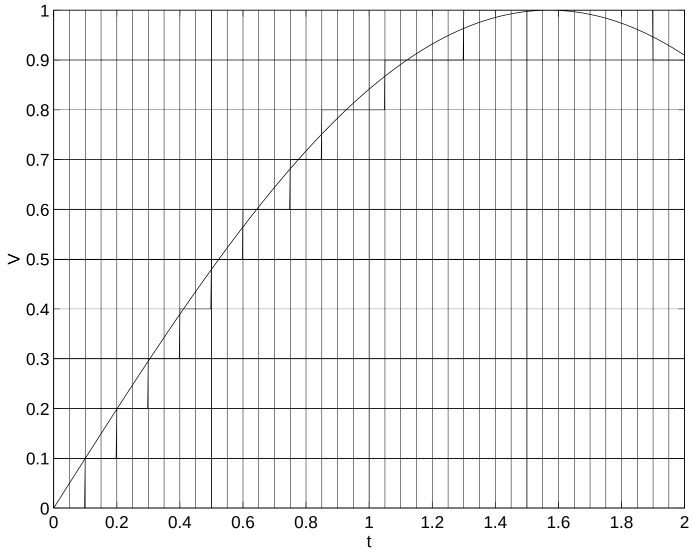
For the control engineer:
Some blocks are given (fixed)
eg

1. Aircraft Dynamics
2. Actuator Dynamics

while other blocks are to be designed
eg

1. Geometry of fly-ball mechanism in Watt governor.
2. The program in an aircraft's flight control computer.

Strictly speaking, the action of control computer cannot be described exactly by ordinary differential equations. However, if the sampling rates are fast enough, and the resolution of the Analogue to Digital and Digital to Analogue converters great enough, then the computer can be approximated in this way.



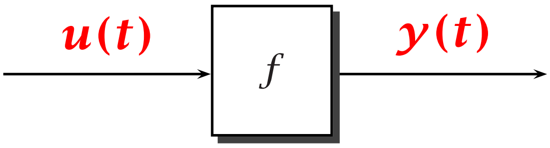
Quantization of a sine curve to nearest $0.1V$ and $0.05s$ (ie, sampling rate is $20Hz$ and resolution is $0.1V$).

(more on this in Communications, Lent term)

1.3 Linear Systems

1.3.1 What is a “linear system”

Consider a “system” f mapping inputs u into outputs y



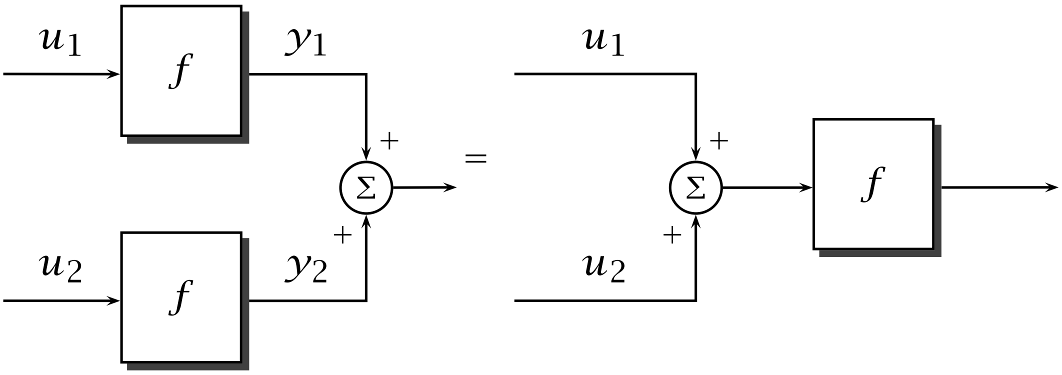
$$y = f\left(u\right)$$

the “system” f is *linear* if superposition holds, that is, if

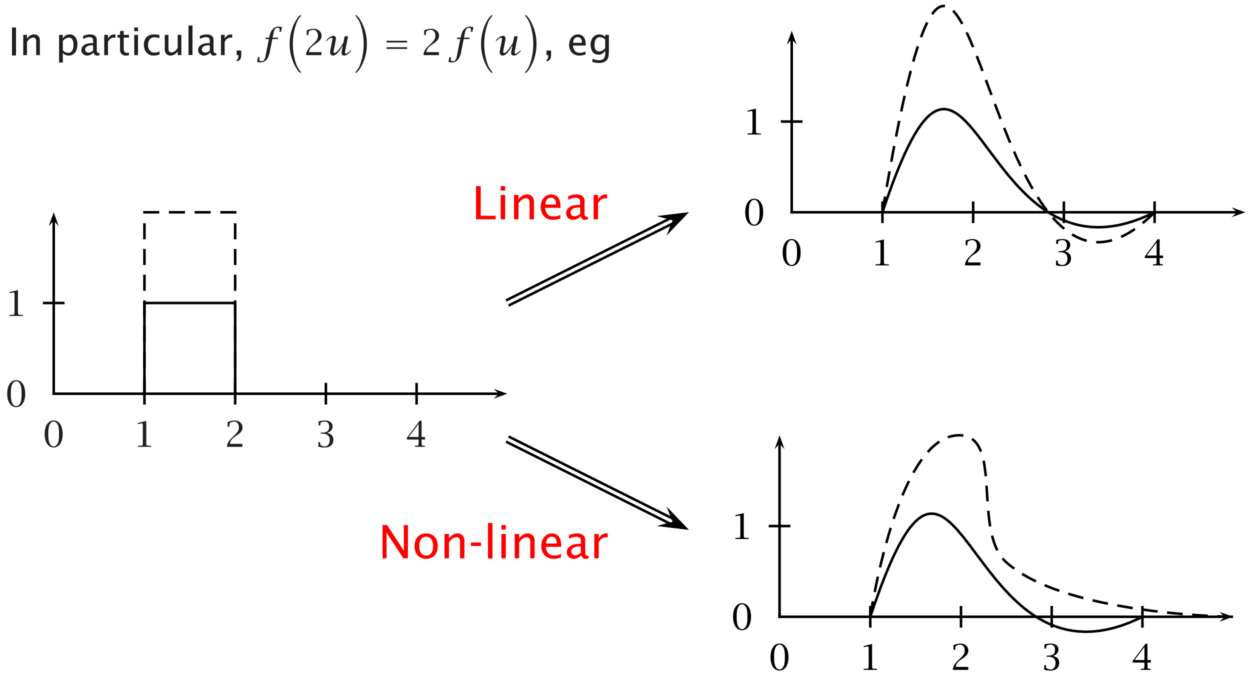
$$\underbrace{f\left(u_1\right)}_{y_1} + \underbrace{f\left(u_2\right)}_{y_2} = f\left(u_1 + u_2\right)$$

for any u_1 and u_2.

In terms of block diagrams. If f is a linear system,

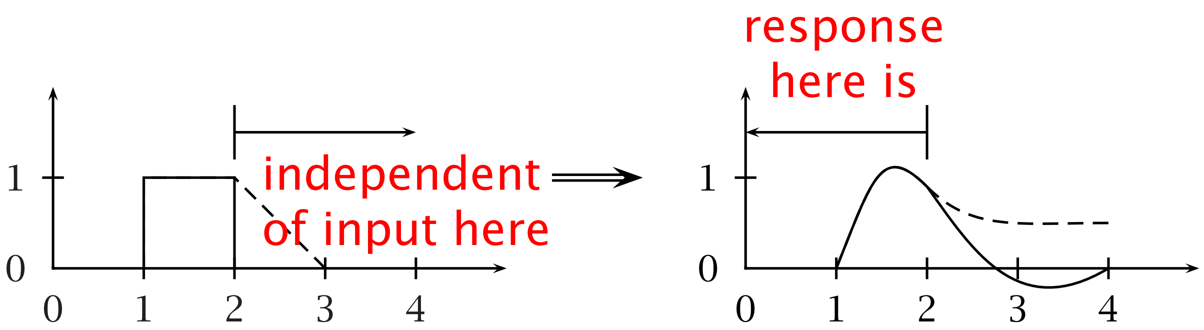


In particular, $f(2u) = 2f(u)$, eg



In addition, we shall also assume that all systems are:

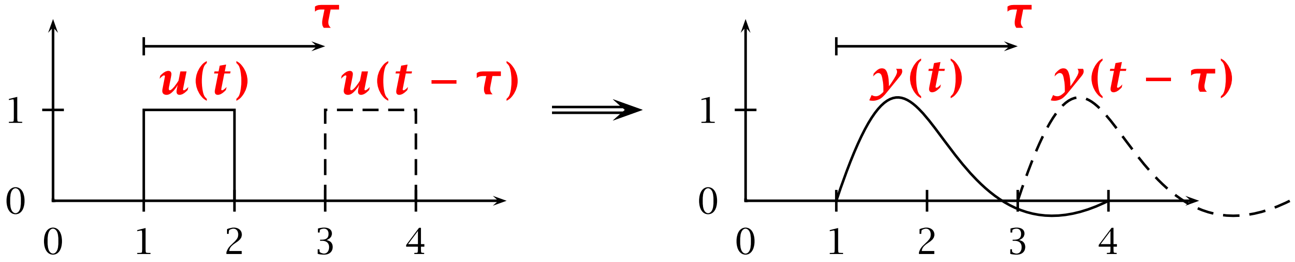
- *causal* – the output at time T, $y(T)$, depends only on the input up to time t (ie $y(t)$, $t \leq T$ is independent of $u(t)$, $t \geq T$).



- *time-invariant* – the response of the system to a particular input doesn't depend on when that input is applied, ie if

$$u(t) \to y(t), \text{ then } u(t-\tau) \to y(t-\tau)$$

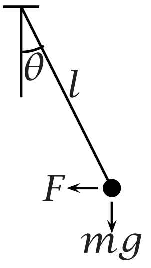
for any τ.



1.3.2 Linearization

Most real systems are actually nonlinear, but many of these behave approximately linearly for small perturbations from equilibrium.

1. Pendulum:



$$Fl\cos\theta + mlg\sin\theta = -ml^2\ddot{\theta}$$

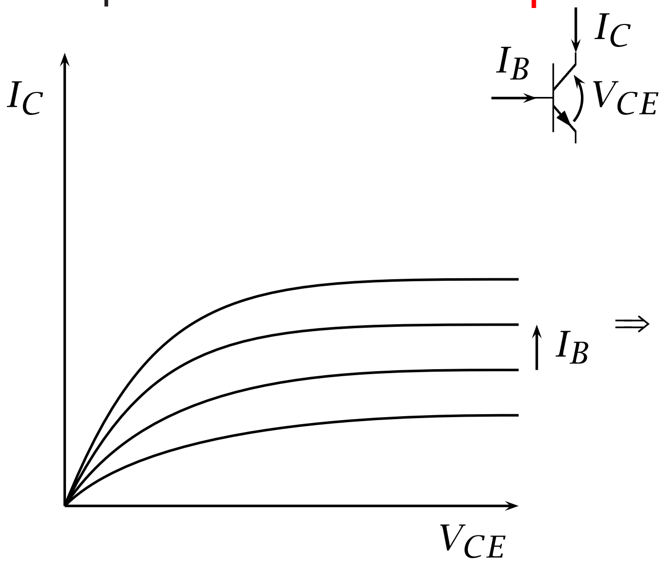
But, for *small* θ

$$\boldsymbol{Fl + mlg\theta \approx -ml^2\ddot{\theta}}$$

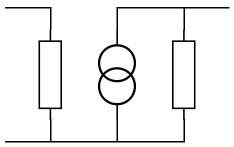
or

$$\boxed{\boldsymbol{l\ddot{\theta} + g\theta \approx -F/m}}$$ which is a linear ODE

2. Bipolar Transistor. Paper 5



⇒



Non-linear large signal characteristics ⇒ linear small-signal model

3. General case

Suppose a system is described by an ODE of the form

$$\dot{x} = f(x,u)$$

where f is a smooth function. Assume that this system has an *equilibrium* at (x_0, u_0), by which we mean that

$$f(x_0,u_0) = 0.$$

where x_0 and u_0 are constants.

Let $x = x_0 + \delta x$, $u = u_0 + \delta u$,
and use a Taylor series expansion to obtain:

$$\cancelto{0}{\dot{x}_0} + \dot{\delta x} = f(x_0 + \delta x, u_0 + \delta u)$$

$$= \cancelto{0}{f(x_0,u_0)} + \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x_0,u_0}}_{a} \delta x + \underbrace{\left.\frac{\partial f}{\partial u}\right|_{x_0,u_0}}_{b} \delta u + \overset{\text{neglect}}{\cancel{\text{higher order terms}}}$$

which results in the linear ODE

$$\boxed{\dot{\delta x} = a\delta x + b\delta u}$$

Systems designed to behave linearly:

- Hi-Fi audio system (non-linearities are called distortion).
- Aircraft fly-by-wire system (to make the aircraft respond predictably)

Systems designed to behave nonlinearly:

- Switch (because it is either on or off).
- Clutch (because it depends on friction).

1.4 Laplace Transforms

Laplace transforms are an essential tool for the analysis of linear, time-invariant, causal systems. We shall now briefly review some pertinent facts that you learnt at Part IA and introduce some new ideas.

DEFINITION:

$$\bar{y}(s) \;=\; \int_{0^-}^{\infty} y(t)e^{-st}\,dt$$

(provided the integral converges for sufficienty large and positive values of s.)

Note, a Laplace transform

- is *NOT* a function of t
- *IS* a function of s.

Various notations:

$$\mathcal{L}\{y(t)\} = \mathcal{L}y = \bar{y}(s) = \int_{0^-}^{\infty} y(t)e^{-st}\,dt$$

Notation for the *inverse transform*:

$$y(t) \;=\; \mathcal{L}^{-1}\bar{y}(s)$$

EXAMPLES

Find $\bar{y}(s)$ if $y(t) = C$ (a constant)

$$\bar{y}(s) \;=\; \int_0^{\infty} Ce^{-st}dt = C\left[\frac{-e^{-st}}{s}\right]_0^{\infty} = \frac{C}{s} \quad \text{(taking Real}(s) > 0\text{).}$$

Find $\bar{y}(s)$ if $y(t) = e^{-at}$

$$\bar{y}(s) = \int_0^{\infty} e^{-(s+a)t}dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^{\infty} = \frac{1}{s+a} \quad \text{(taking Real}(s) > -a\text{).}$$

Addition or Superposition property

If

$$y(t) = Ay_1(t) + By_2(t)$$

then

$$\bar{y}(s) = A\bar{y}_1(s) + B\bar{y}_2(s)$$

(A, B constants)

Proof:

$$\begin{aligned} \bar{y} &= \int_0^\infty (Ay_1 + By_2)e^{-st}dt \\ &= A\int_0^\infty y_1 e^{-st}dt + B\int_0^\infty y_2 e^{-st}dt \\ &= A\bar{y}_1 + B\bar{y}_2 \end{aligned}$$

The operation of taking a Laplace transform is linear.

Transforms of derivatives

$$
\begin{aligned}
\mathcal{L}\dot{y}(t) &= \int_0^\infty \frac{dy}{dt} e^{-st} dt \\
&= \left[y(t) e^{-st} \right]_0^\infty + s \int_0^\infty y(t) e^{-st} dt \\
&= s\bar{y} - y(0)
\end{aligned}
$$

……

$$
\begin{aligned}
\mathcal{L}\ddot{y} &= \int_0^\infty \frac{d^2y}{dt^2} e^{-st} dt \\
&= \left[\frac{dy}{dt} e^{-st} \right]_0^\infty + s \int_0^\infty \frac{dy}{dt} e^{-st} dt \\
&= -\dot{y}(0) + s(s\bar{y} - y(0)) \\
&= s^2\bar{y} - sy(0) - \dot{y}(0)
\end{aligned}
$$

Obvious pattern:

$$
\boxed{
\begin{aligned}
\mathcal{L}y &= \bar{y} \\
\mathcal{L}\dot{y} &= s\bar{y} - y(0) \\
\mathcal{L}\ddot{y} &= s^2\bar{y} - sy(0) - \dot{y}(0) \\
\vdots \;&\vdots\; \vdots \\
\mathcal{L}\frac{d^ny}{dt^n} &= s^n\bar{y} - s^{n-1}y(0) - s^{n-2}\dot{y}(0) - \\
&\quad - \ldots - \left(\frac{d^{n-1}y}{dt^{n-1}} \right)(0)
\end{aligned}
}
$$

In particular, if $y(0) = \dot{y}(0) = \ddot{y}(0) = \cdots = 0$, then

$$\begin{aligned}
\mathcal{L}y &= \bar{y} \\
\mathcal{L}\dot{y} &= s\bar{y} \\
\mathcal{L}\ddot{y} &= s^2\bar{y} \\
\vdots \quad & \vdots \quad \vdots \\
\mathcal{L}\frac{d^n y}{dt^n} &= s^n\bar{y}
\end{aligned}$$

i.e. in the absense of initial conditions, differentiation (in the time domain) corresponds to multiplication by s (in the s domain)

Laplace Transform of t^n

Define $\bar{y}_n(s) \;=\; \mathcal{L}\,\frac{t^n}{n!}$.

$$\begin{aligned}
\bar{y}_n &= \int_0^\infty \frac{t^n}{n!} e^{-st} dt \\
&= \left[-\frac{1}{s}\frac{t^n}{n!} e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty \frac{n t^{n-1}}{n!} e^{-st} dt \\
&= \frac{1}{s}\int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-st} dt \\
&= \frac{1}{s}\bar{y}_{n-1},
\end{aligned}$$

(since for Real$(s) > 0$, and as $t \to \infty$, then $\left|e^{-st}\right| \to 0$ faster than $t^n \to \infty$).

Thus we have

$$\begin{aligned}
\bar{y}_0 &= \mathcal{L}\,1 = \frac{1}{s} \\
\bar{y}_1 &= \mathcal{L}\,t = \frac{1}{s^2} \\
\bar{y}_2 &= \mathcal{L}\,\frac{t^2}{2} = \frac{1}{s^3} \\
\bar{y}_3 &= \mathcal{L}\,\frac{t^3}{3 \times 2} = \frac{1}{s^4} \\
\text{Similarly} \quad \bar{y}_n &= \mathcal{L}\,\frac{t^n}{n!} = \frac{1}{s^{n+1}}
\end{aligned}$$

i.e. integration (in the time domain) corresponds to division by s (in the s domain)

Poles and Zeros

Suppose $G(s)$ is a *rational* function of s, by which we mean

$$G(s) = \frac{n(s)}{d(s)}$$

where $n(s)$ and $d(s)$ are polynomials in s.

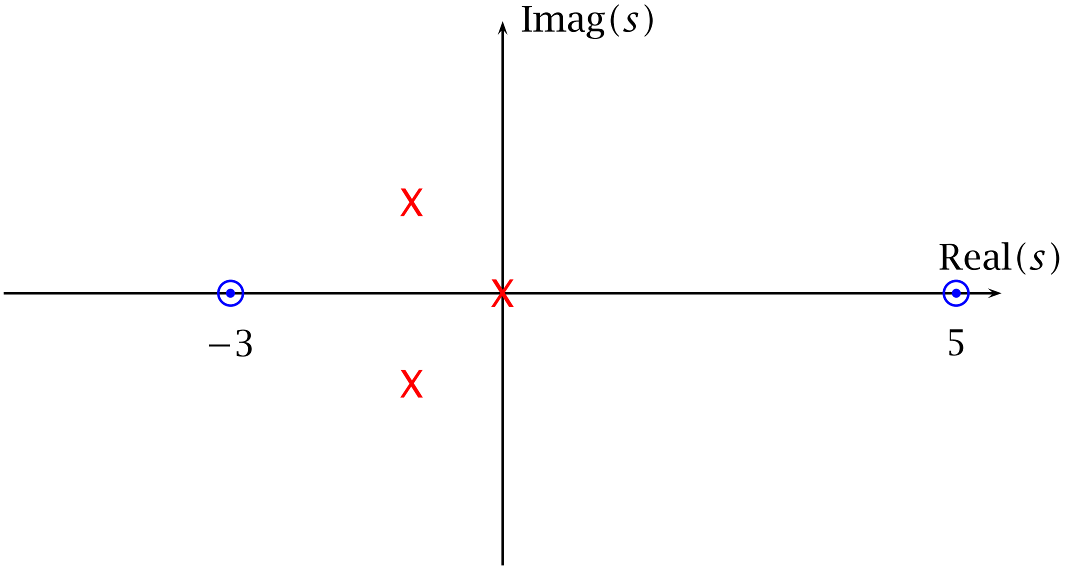
Then the roots of $n(s)$ are called the *zeros* of $G(s)$
and the roots of $d(s)$ are called the *poles* of $G(s)$

Example:

$$\begin{aligned} G(s) &= \frac{4s^2 - 8s - 60}{s^3 + 2s^2 + 2s} \\ &= \frac{\mathbf{4(s+3)(s-5)}}{\mathbf{s(s+1+j)(s+1-j)}} \end{aligned}$$

Zeros of $G(s)$ are **$-3, +5$.**

Poles of $G(s)$ are **$-1 - j, -1 + j, 0$**



X - denote poles
⊙ - denote zeros

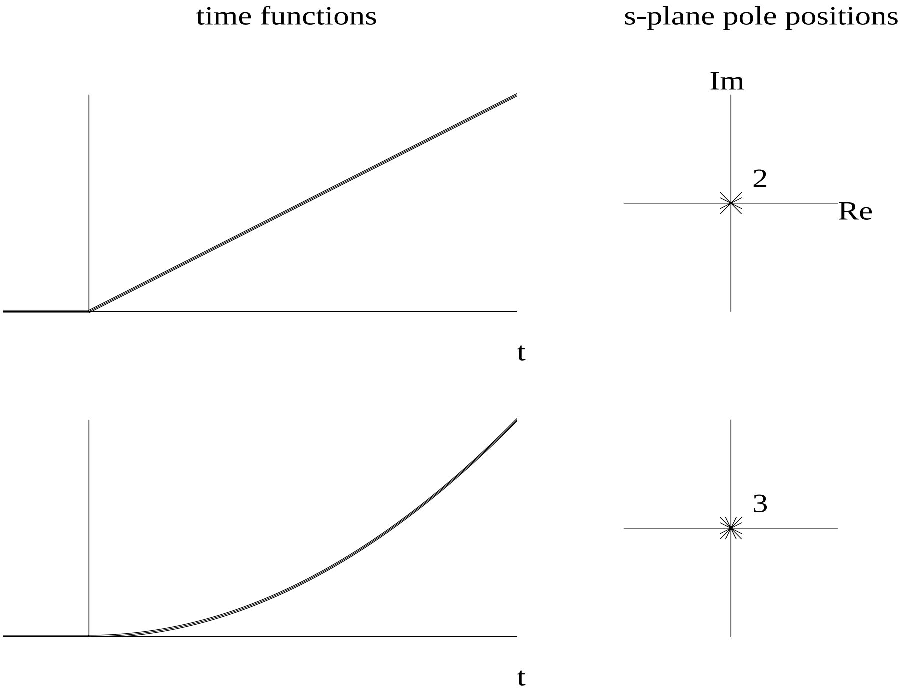


Figure 1.1: Time functions and pole positions for $y(t) = t$ and $y(t) = t^2$

Shift in s theorem

$$\boxed{\begin{aligned} \text{If } \mathcal{L}y(t) &= \bar{y}(s) \\ \text{then } \mathcal{L}e^{at}y(t) &= \bar{y}(s-a). \end{aligned}}$$

Proof:

$$\begin{aligned} \mathcal{L}e^{at}y(t) &= \int_0^\infty e^{-(s-a)t}y(t)\,dt \\ &= \bar{y}(s-a), \end{aligned}$$

Example of use:

$$\begin{aligned} \mathcal{L}^{-1}\frac{20}{s^2+2s+101} &= \mathcal{L}^{-1}\frac{20}{(s+1)^2+100} \\ &= 2e^{-t}\sin 10t \end{aligned}$$

because $\mathcal{L}^{-1}\dfrac{10}{s^2+100} = \sin 10t$

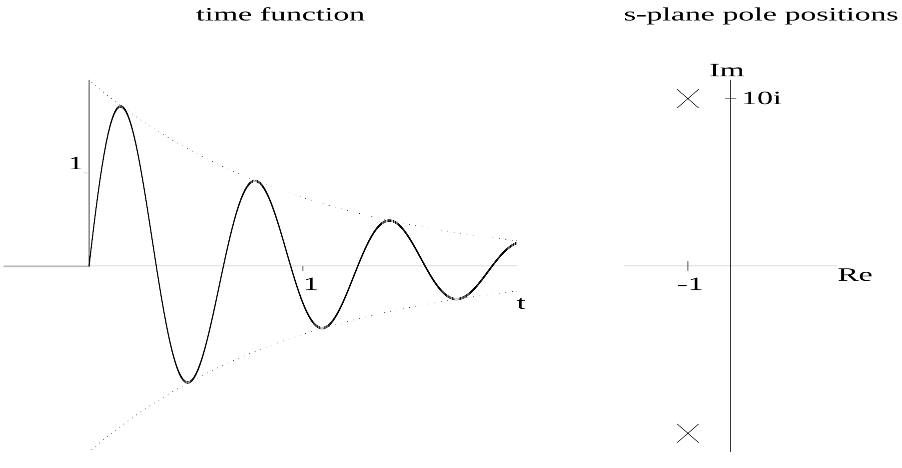


Figure 1.2: Time functions and pole positions for $y(t) = 2e^{-t}\sin 10t$

Initial and Final Value Theorems

If $\bar{y}(s) = \mathcal{L}\, y(t)$ then *whenever the indicated limits exist* we have

Final Value Theorem:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\bar{y}(s)$$

Initial Value Theorem:

$$\lim_{t\to 0^+} y(t) = \lim_{s\to\infty} s\bar{y}(s)$$

Proofs omitted (as it’s a little tricky to prove these properly.)

However, for rational functions of s it is easy to demonstrate that these relationships hold:
Let a partial fraction of $\bar{y}(s)$ be given as:

$$\bar{y}(s) = \frac{b_0}{s} + \sum_{i=1}^{n} \frac{b_i}{s + a_i}$$

and so

$$y(t) = b_0 + \sum_{i=1}^{n} b_i e^{-a_i t}.$$

Hence

$$\underline{y(0) = b_0 + \sum_{i=1}^{n} b_i} \quad \text{and, provided } a_i > 0, \quad \underline{y(\infty) = b_0.}$$

On the other hand,

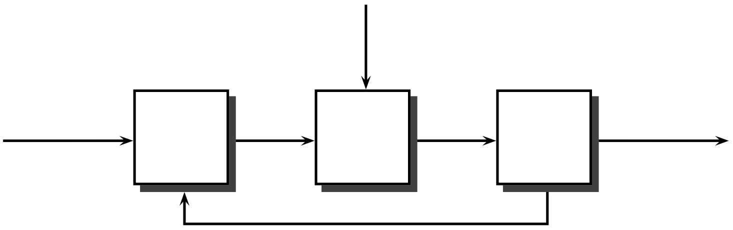
$$s\bar{y}(s) = b_0 + \sum_{i=1}^{n} \frac{s b_i}{s + a_i}$$

hence

$$\underline{s\bar{y}(s)|_{s=\infty} = b_0 + \sum_{i=1}^{n} b_i} \quad \text{and, provided } a_i \neq 0, \quad \underline{s\bar{y}(s)|_{s=0} = b_0}$$

which are the same expressions as above.

1.5 Key points



- Feedback is used to reduce sensitivity.
- We use block diagrams to represent feedback interconnections.
- Each block represents a “system”.
- Each connection carries a “signal”.
- We shall assume that systems are described by *linear*, *time-invariant* and *causal* ODE’s.
- We distinguish between *causes* (the input signals) and *effects* (the output signals).
- Large and complex systems can be constructed by connecting together simpler sub-systems.
- Laplace transforms are central to the study of linear, time-invariant systems.