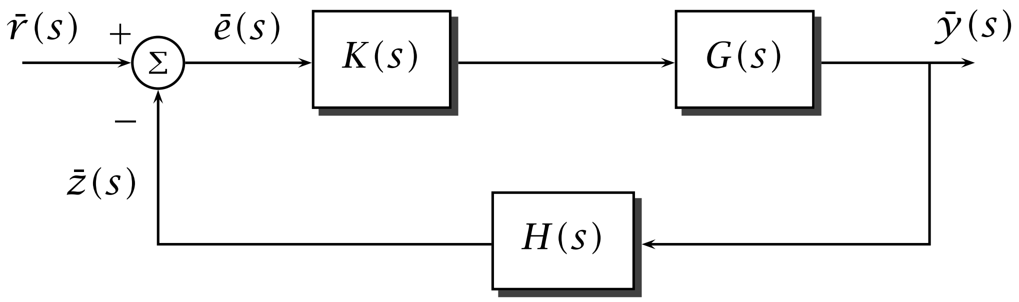
Part IB Paper 6: Information Engineering

LINEAR SYSTEMS AND CONTROL

Glenn Vinnicombe

HANDOUT 5

"An Introduction to Feedback Control Systems"



$$\bar{z}(s) = \underbrace{H(s)G(s)K(s)}_{L(s)}\,\bar{e}(s)$$

Return ratio

$$\bar{e}(s) = \underbrace{\frac{1}{1+L(s)}}\,\bar{r}(s)$$

Closed-loop transfer function
relating $\bar{e}(s)$ and $\bar{r}(s)$

$$\bar{y}(s) = G(s)K(s)\bar{e}(s) = \underbrace{\frac{G(s)K(s)}{1+L(s)}}\,\bar{r}(s)$$

Closed-loop transfer function
relating $\bar{y}(s)$ and $\bar{r}(s)$

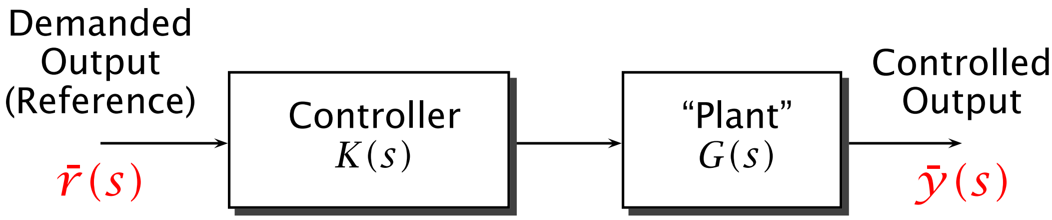
Key Points

- The *Closed-Loop Transfer Functions* are the *actual* transfer functions which determine the behaviour of a feedback system. They relate signals around the loop (such as the plant input and output) to external signals injected into the loop (such as reference signals, disturbances and noise signals).
- It is possible to infer much about the behaviour of the feedback system from consideration of the *Return Ratio* alone.
- The aim of using feedback is for the plant output $y(t)$ to follow the reference signal $r(t)$ in the presence of uncertainty. A persistent difference between the reference signal and the plant output is called a steady state error. Steady-state errors can be evaluated using the final value theorem.

- Many simple control problems can be solved using combinations of proportional, derivative and integral action:
  - Proportional action is the basic type of feedback control, but it can be difficult to achieve good damping and small errors simultaneously.
  - Derivative action can often be used to improve damping of the closed-loop system.
  - Integral action can often be used to reduce steady-state errors.

Contents

5.1 Open-Loop Control



In principle, we could could choose a “desired” transfer function $F(s)$ and use $K(s) = F(s)/G(s)$ to obtain

$$\bar{y}(s) = G(s)\frac{F(s)}{G(s)}\bar{r}(s) = F(s)\bar{r}(s)$$

In practice, **this will not work**

– because it requires an exact model of the plant and that there be no disturbances (i.e. no uncertainty).

Feedback is used to combat the effects of uncertainty

For example:

- Unknown parameters
- Unknown equations
- Unknown disturbances

5.2 Closed-Loop Control (Feedback Control)

<u>For Example:</u>

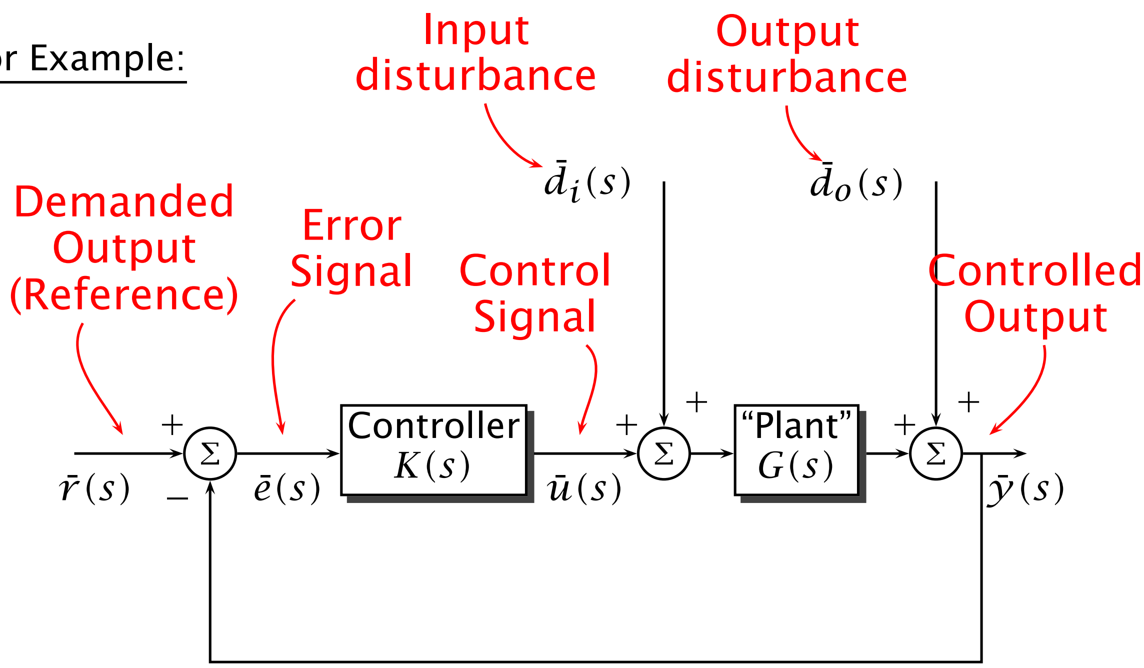


Figure 5.1

5.2.1 Derivation of the closed-loop transfer functions:

$$\bar{y}(s) = \bar{d}_o(s) + G(s)[\bar{d}_i(s) + K(s)\bar{e}(s)]$$

$$\bar{e}(s) = \bar{r}(s) - \bar{y}(s)$$

$$\Longrightarrow \quad \bar{y}(s) = \bar{d}_o(s) + G(s)[\bar{d}_i(s) + K(s)(\bar{r}(s) - \bar{y}(s))]$$

$$\Longrightarrow \quad \Big(1 + G(s)K(s)\Big)\bar{y}(s) = \bar{d}_o(s) + G(s)\bar{d}_i(s) + G(s)K(s)\bar{r}(s)$$

$$\Longrightarrow \quad \bar{y}(s) = \frac{1}{1 + G(s)K(s)}\bar{d}_o(s) + \frac{G(s)}{1 + G(s)K(s)}\bar{d}_i(s) + \frac{G(s)K(s)}{1 + G(s)K(s)}\bar{r}(s)$$

Also:

$$\begin{aligned}\bar{e}(s) &= \bar{r}(s) - \bar{y}(s)\\ &= -\frac{1}{1+G(s)K(s)}\bar{d}_o(s) - \frac{G(s)}{1+G(s)K(s)}\bar{d}_i(s)\\ &\quad + \underbrace{\left(1 - \frac{G(s)K(s)}{1+G(s)K(s)}\right)}_{\dfrac{\mathbf{1}}{\mathbf{1}+G(s)K(s)}}\bar{r}(s)\end{aligned}$$

5.2.2 The Closed-Loop Characteristic Equation and the Closed-Loop Poles

Note: All the *C*losed-Loop Transfer Functions of the previous section have the same denominator:

$$\mathbf{1} + G(s)K(s)$$

The *Closed-Loop Poles* (*ie* the poles of the closed-loop system, or feedback system) are the zeros of this denominator.

For the feedback system of Figure 5.1, the Closed-Loop Poles are the roots of

$$\boxed{1 + G(s)K(s) = 0}$$

Closed-Loop Characteristic Equation
(for Fig 5.1)

The closed-loop poles determine:

- The stability of the closed-loop system.
- Characteristics of the closed-loop system's transient response.(e.g. speed of response, presence of any resonances etc)

5.2.3 What if there are more than two blocks?

For Example:

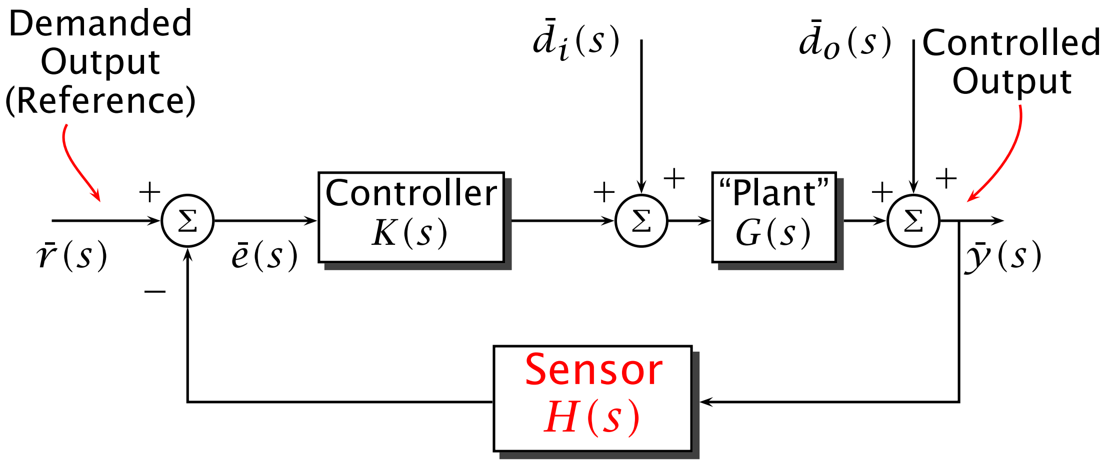


Figure 5.2

We now have

$$\bar{y}(s) = \frac{G(s)K(s)}{1 + H(s)G(s)K(s)}\bar{r}(s) + \frac{1}{1 + H(s)G(s)K(s)}\bar{d}_o(s) + \frac{G(s)}{1 + H(s)G(s)K(s)}\bar{d}_i(s)$$

This time $\mathbf{1} + H(s)G(s)K(s)$ appears as the denominator of all the closed-loop transfer functions.

Let,

$$L(s) = H(s)G(s)K(s)$$

i.e. the product of all the terms around loop, not including the -1 at the summing junction. $L(s)$ is called the *Return Ratio* of the loop (and is also known as the *Loop Transfer Function*).

The *Closed-Loop Characteristic Equation* is then

$$\mathbf{1} + L(s) = \mathbf{0}$$

and the *Closed-Loop Poles* are the roots of this equation.

5.2.4 A note on the Return Ratio

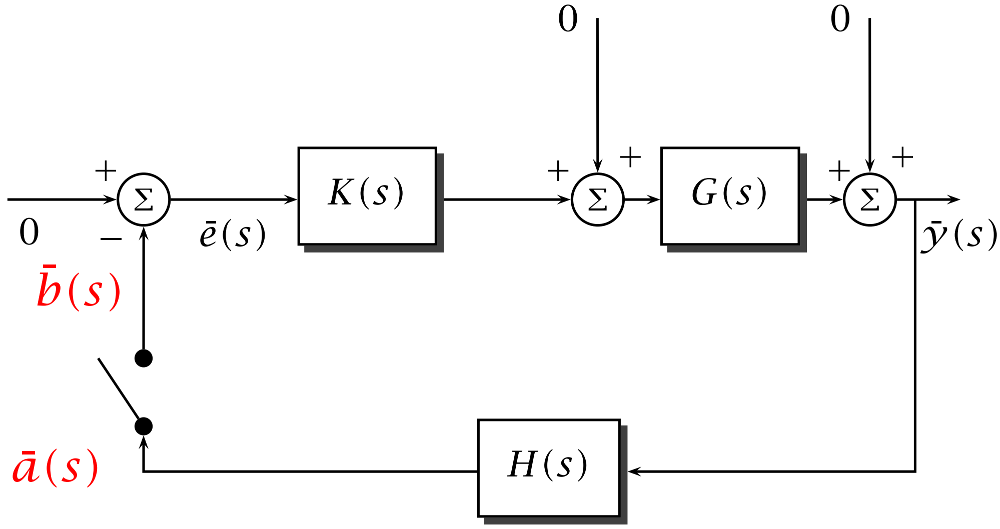


Figure 5.3

With the switch in the position shown (i.e. open), the loop is *open*. We then have

$$\bar{a}(s) = H(s)G(s)K(s) \times -\bar{b}(s) = -H(s)G(s)K(s)\bar{b}(s)$$

Formally, the *Return Ratio* of a loop is defined as -1 times the product of all the terms around the loop. In this case

$$L(s) = -1 \times -H(s)G(s)K(s) = H(s)G(s)K(s)$$

Feedback control systems are often tested in this configuration as a final check before “closing the loop” (i.e. flicking the switch to the closed position).

Note: In general, the block denoted here as $H(s)$ could include filters and other elements of the controller in addition to the sensor dynamics. Furthermore, the block labelled $K(s)$ could include actuator dynamics in addition to the remainder of the designed dynamics of the controller.

5.2.5 Sensitivity and Complementary Sensitivity

The *Sensitivity* and *Complementary Sensitivity* are two particularly important closed-loop transfer functions. The following figure depicts just *one* configuration in which they appear.

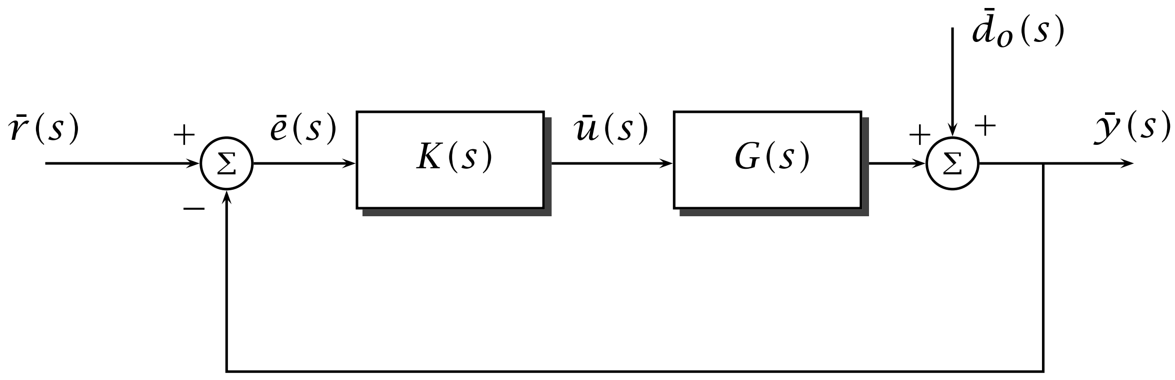


Figure 5.4

$$\Big(L(s) = G(s)K(s) \Big)$$

$$\begin{aligned}
\bar{y}(s) &= \frac{G(s)K(s)}{1+G(s)K(s)}\bar{r}(s) + \frac{1}{1+G(s)K(s)}\bar{d}_o(s) \\
&= \underbrace{\frac{L(s)}{1+L(s)}}_{\substack{\text{Complementary}\\ \text{Sensitivity}\\ T(s)}}\bar{r}(s) + \underbrace{\frac{1}{1+L(s)}}_{\substack{\text{Sensitivity}\\ S(s)}}\bar{d}_o(s)
\end{aligned}$$

Note:

$$S(s) + T(s) = \frac{\mathbf{1}}{\mathbf{1}+L(s)} + \frac{L(s)}{\mathbf{1}+L(s)} = \mathbf{1}$$

5.3 Summary of notation

- The system being controlled is often called the *“plant”*.
- The control law is often called the *“controller”*; sometimes it is called the *“compensator”* or *“phase compensator”*.
- The *“demand”* signal is often called the *“reference”* signal or *“command”*, or (in the process industries) the *“set-point”*.
- The *“Return Ratio”*, the *“Loop transfer function”* always refer to the transfer function of the opened loop, that is the product of all the transfer functions appearing in a standard negative feedback loop (our $L(s)$). Figure 5.1 has $L(s) = G(s)K(s)$, Figure 5.2 has $L(s) = H(s)G(s)K(s)$.

- The *“Sensitivity function”* is the transfer function $S(s) = \dfrac{1}{1 + L(s)}$. It characterizes the sensitivity of a control system to disturbances appearing at the output of the plant.
- The transfer function $T(s) = \dfrac{L(s)}{1 + L(s)}$ is called the *“Complementary Sensitivity”*. The name comes from the fact that $S(s) + T(s) = 1$. When this appears as the transfer function from the demand to the controlled output, as in Fig 5.4 it is often called simply *the* “Closed-loop transfer function” (though this is ambiguous, as there are many closed-loop transfer functions).

5.4 The Final Value Theorem (revisited)

Consider an asymptotically stable system with impulse response $g(t)$ and transfer function $G(s)$, i.e.

$$\underbrace{g(t)}_{\text{Impulse response}} \quad \rightleftharpoons \quad \underbrace{G(s)}_{\substack{\text{Transfer Function} \\ \text{(assumed asymptotically stable)}}}$$

Let $y(t) = \int_0^t g(\tau)\,d\tau$ denote the step response of this system and note that $\bar{y}(s) = \dfrac{G(s)}{s}$.

We now calculate the final value of this step response:

$$\begin{aligned}\lim_{t\to\infty} y(t) &= \int_0^\infty g(\tau)\,d\tau \\ &= \int_0^\infty \underbrace{\exp(-0\tau)}_{\mathbf{1}}\, g(\tau)\,d\tau = \mathcal{L}(g(t))\,\big|_{s=\mathbf{0}} = G(\mathbf{0})\end{aligned}$$

Hence,

$$\boxed{\text{Final Value of Step Response} \equiv \begin{array}{c}\text{Transfer Function} \\ \text{evaluated at } \; s = \mathbf{0}\end{array}}$$

"Steady-State Gain" or "DC gain"

Note that the same result can be obtained by using the Final Value Theorem:

$$\begin{aligned}\lim_{t\to\infty} y(t) &= \lim_{s\to 0} s\bar{y}(s) \quad \left(\begin{array}{l}\textit{for any } y \textit{ for which} \\ \textit{both limits exist.}\end{array}\right) \\ &= \lim_{s\to\mathbf{0}} \; s \cdot \frac{G(s)}{s} = G(\mathbf{0})\end{aligned}$$

5.4.1 The "steady state" response – summary

The term "steady-state response" means two different things, depending on the input.

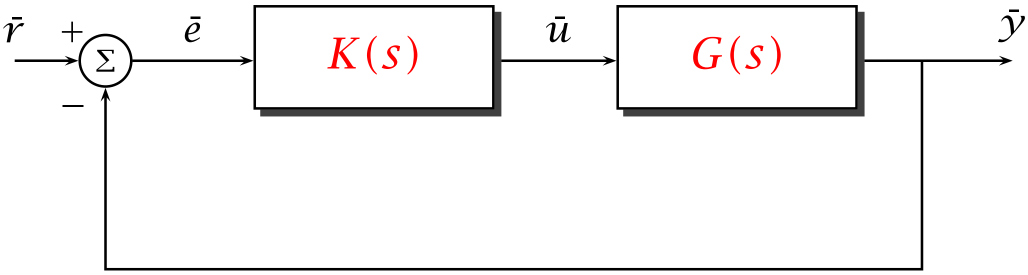
Given an asymptotically stable system with transfer function $G(s)$:

- The *steady-state* response of the system to a *constant* input U is a *constant*, $G(0)U$.
- The *steady-state* response of the system to a *sinusoidal* input $\cos(\omega t)$ is the *sinusoid* $|G(j\omega)|\cos(\omega t + \arg G(j\omega))$.

These two statements are not entirely unrelated, of course: The *steady-state gain* of a system, $G(0)$ is the same as the frequency response evaluated at $\omega = 0$ (i.e. the DC gain).

5.5 Some simple controller structures

5.5.1 Introduction – steady-state errors



Return Ratio: $L(s) = G(s)K(s)$.

CLTFs: $\bar{y}(s) = \dfrac{L(s)}{1 + L(s)}\bar{r}(s)$ and $\bar{e}(s) = \dfrac{1}{1 + L(s)}\bar{r}(s)$

Steady-state error:(for a step demand) If $r(t) = H(t)$, then $\bar{y}(s) = \frac{L(s)}{1+L(s)} \times \frac{1}{s}$ and so

$$\lim_{t\to\infty} y(t) = s \times \frac{L(s)}{1 + L(s)} \times \frac{1}{s}\bigg|_{s = 0} = \frac{L(\mathbf{0})}{\mathbf{1} + L(\mathbf{0})}$$

and

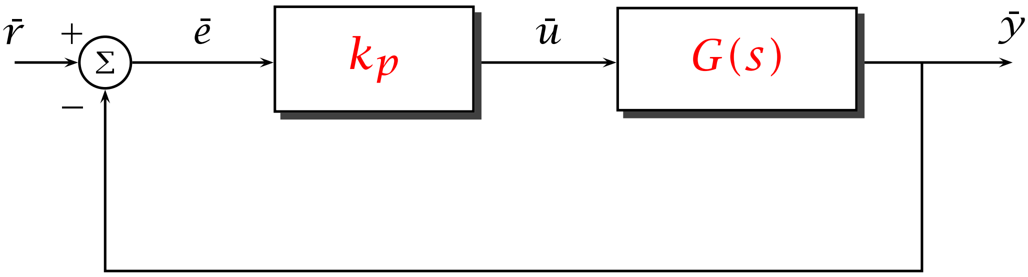
$$\lim_{t\to\infty} e(t) = s \times \frac{1}{1 + L(s)} \times \frac{1}{s}\bigg|_{s = 0} = \underbrace{\frac{\mathbf{1}}{\mathbf{1} + L(\mathbf{0})}}_{\text{Steady-state error}}$$

(using the final-value theorem.)

Note: These particular formulae only hold for this simple configuration – where there is a unit step demand signal and no constant disturbances (although the final value theorem can always be used).

5.5.2 Proportional Control

$$K(s) = k_p$$



Typical result of increasing the gain k_p, (for control systems where $G(s)$ is itself stable):

- Increased accuracy of control. "good"
- Increased control action.
- Reduced damping. "bad"
- Possible loss of closed-loop stability for large k_p.

Example:

$$G(s) = \frac{1}{(s+1)^2}$$

(A critically damped 2nd order system)

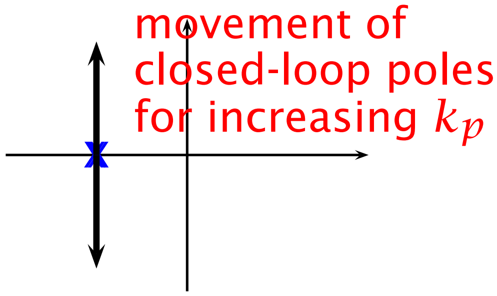
$$\bar{y}(s) = \frac{k_p G(s)}{1 + k_p G(s)} \bar{r}(s) = \frac{k_p \frac{1}{(s+1)^2}}{1 + k_p \frac{1}{(s+1)^2}} \bar{r}(s)$$

$$= \frac{k_p}{s^2 + 2s + 1 + k_p} \bar{r}(s)$$

So, $\omega_n^2 = \mathbf{1} + k_p, \quad 2\zeta\omega_n = \mathbf{2}$

$$\Longrightarrow \omega_n = \sqrt{\mathbf{1} + k_p}, \quad \zeta = \frac{\mathbf{1}}{\sqrt{\mathbf{1} + k_p}}$$

Closed-loop poles at $s = -\mathbf{1} \pm j\sqrt{k_p}$



"root locus diagram"

Steady-state errors using the final value theorem:

$$\bar{y}(s) = \frac{k_p}{s^2 + 2s + 1 + k_p}\bar{r}(s)$$

and

$$\bar{e}(s) = \frac{1}{1 + k_p G(s)} = \frac{(s+1)^2}{s^2 + 2s + 1 + k_p}\bar{r}(s).$$

So, if $r(t) = H(t)$,

final value of step response

$$\lim_{t\to\infty} y(t) = \left.\frac{k_p}{s^2 + 2s + 1 + k_p}\right|_{\mathbf{s\,=\,0}} = \frac{k_p}{\mathbf{1} + k_p}$$

and

$$\lim_{t\to\infty} e(t) = \left.\frac{(s+1)^2}{s^2 + 2s + 1 + k_p}\right|_{\mathbf{s\,=\,0}} = \underbrace{\frac{\mathbf{1}}{\mathbf{1} + k_p}}$$

Steady-state error

(Note: $L(s) = k_p \dfrac{1}{(s+1)^2}$ $\quad\Longrightarrow\quad L(\mathbf{0}) = k_p \times \mathbf{1} = k_p$)

Hence, in this example, *increasing* k_p gives *smaller* steady-state errors, but a larger and *more oscillatory* transient response .

- However, by using more complex controllers it *is* usually possible to remove steady state errors *and* increase damping at the same time:

To increase damping –
can often use derivative action (or velocity feedback).

To remove steady-state errors – can often use integral action.

For reference, the step response: (i.e. response to $\bar{r}(s) = \frac{1}{s}$) is given by

$$\bar{y}(s) = -\frac{\frac{k_p}{1+k_p}(2+s)}{s^2+2s+1+k_p} + \frac{\frac{k_p}{1+k_p}}{s}$$

so

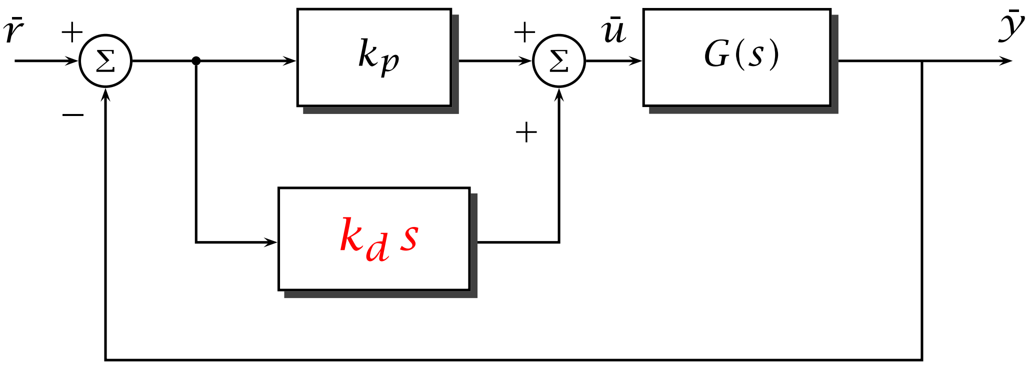
$$\begin{aligned} y(t) &= -\frac{k_p}{1+k_p}\exp(-t)\left(\cos(\sqrt{k_p}t) + \frac{1}{\sqrt{k_p}}\sin(\sqrt{k_p}t)\right) + \frac{k_p}{1+k_p} \\ &= \underbrace{-\sqrt{\tfrac{k_p}{1+k_p}}\exp(-t)\left(\cos(\sqrt{k_p}t - \phi)\right)}_{\text{Transient Response}} \qquad + \qquad \underbrace{\frac{k_p}{1+k_p}}_{\text{Steady-state response}} \end{aligned}$$

where $\phi = \arctan \frac{1}{\sqrt{kp}}$

But you don't need to calculate this to draw the conclusions we have made.

5.5.3 Proportional + Derivative (PD) Control

$$\boxed{K(s) = k_p + k_d\, s}$$



Typical result of increasing the gain k_d, (when $G(s)$ is itself stable):

- Increased Damping.
- Greater sensitivity to noise.

(It is usually better to measure the rate of change of the error directly if possible – i.e. use velocity feedback)

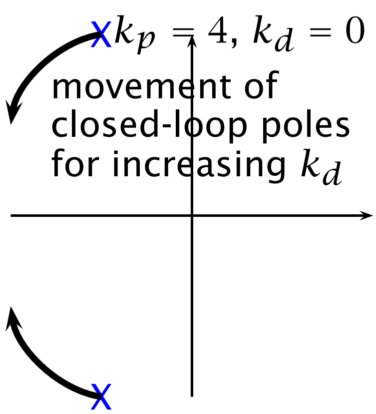
Example: $\qquad G(s) = \dfrac{1}{(s+1)^2}, \quad K(s) = k_p + k_d\, s$

$$\bar{y}(s) = \frac{K(s)G(s)}{1 + K(s)G(s)}\,\bar{r}(s) = \frac{(k_p + k_d\, s)\frac{1}{(s+1)^2}}{1 + (k_p + k_d\, s)\frac{1}{(s+1)^2}}\,\bar{r}(s)$$

$$= \frac{k_p + k_d s}{s^2 + (2 + k_d)s + 1 + k_p}\,\bar{r}(s)$$

So, $\omega_n^2 = 1 + k_p, \quad 2c\omega_n = 2 + k_d \implies$

$$\omega_n = \sqrt{\mathbf{1} + k_p}\ , \quad c = \frac{\mathbf{2} + k_d}{\mathbf{2}\sqrt{\mathbf{1} + k_p}}$$



5.5.4 Proportional + Integral (PI) Control

In the absence of disturbances, and for our simple configuration,

$$\bar{e}(s) = \frac{1}{1 + G(s)K(s)}\bar{r}(s)$$

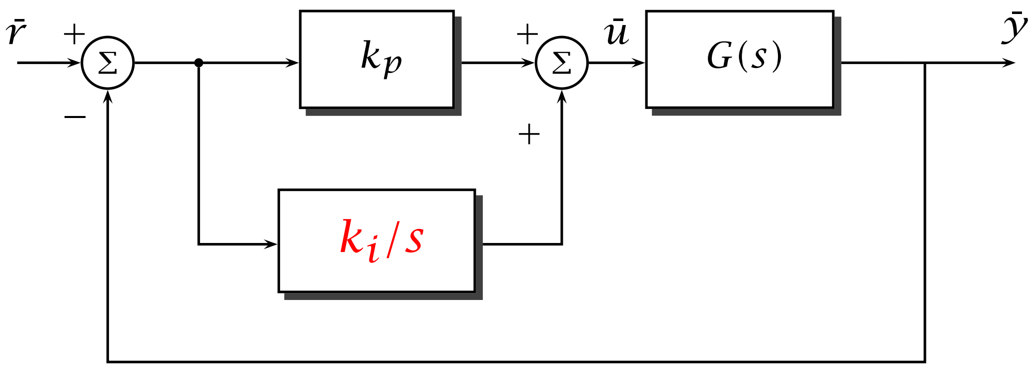
Hence,

$$\begin{array}{l}\text{steady-state error,}\\ \text{(for step demand)}\end{array} = \left.\frac{1}{1 + G(s)K(s)}\right|_{\mathbf{s=0}} = \frac{\mathbf{1}}{\mathbf{1 + G(0)K(0)}}$$

To remove the steady-state error, we need to make $K(\mathbf{0}) = \infty$ (assuming $G(0) \neq 0$).

e.g

$$\boxed{K(s) = k_p + \frac{k_i}{s}}$$



Example:

$$G(s) = \frac{1}{(s+1)^2}, \quad K(s) = k_p + k_i/s$$

$$
\begin{aligned}
\bar{y}(s) &= \frac{K(s)G(s)}{1+K(s)G(s)}\,\bar{r}(s) = \frac{(k_p + k_i/s)\frac{1}{(s+1)^2}}{1+(k_p+k_i/s)\frac{1}{(s+1)^2}}\,\bar{r}(s) \\
&= \frac{k_p\, s + k_i}{s(s+1)^2 + k_p\, s + k_i}\,\bar{r}(s)
\end{aligned}
$$

$$
\begin{aligned}
\bar{e}(s) &= \frac{1}{1+K(s)G(s)}\,\bar{r}(s) = \frac{1}{1+(k_p+k_i/s)\frac{1}{(s+1)^2}}\,\bar{r}(s) \\
&= \frac{s(s+1)^2}{s(s+1)^2 + k_p\, s + k_i}\,\bar{r}(s)
\end{aligned}
$$

Hence, for $r(t) = H(t)$,

$$
\lim_{t\to\infty} y(t) = \left.\frac{k_p\, s + k_i}{s(s+1)^2 + k_p\, s + k_i}\right|_{s=0} = 1
$$

and

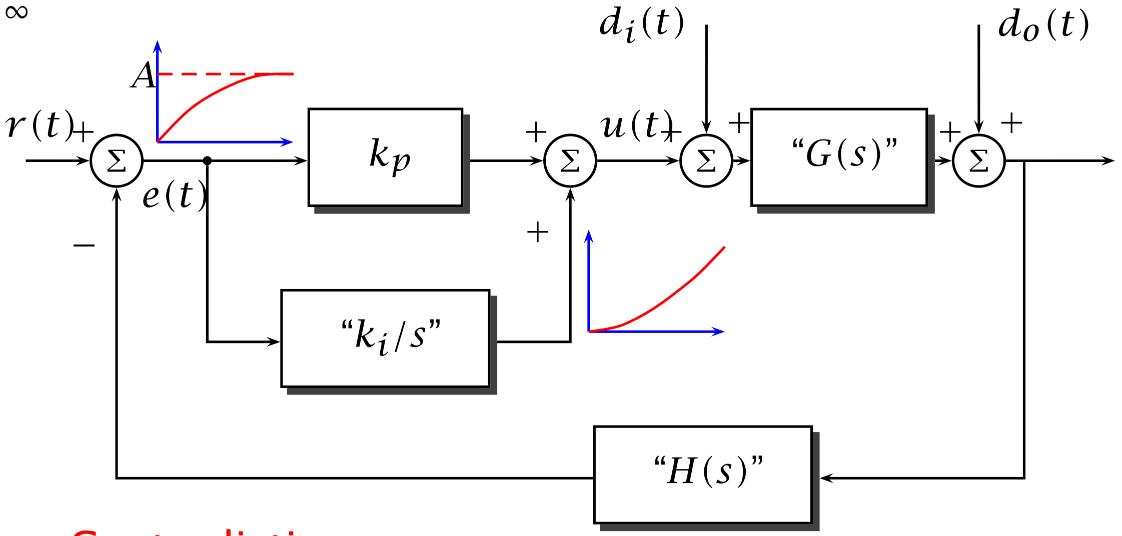
$$
\lim_{t\to\infty} e(t) = \left.\frac{s(s+1)^2}{s(s+1)^2 + k_p\, s + k_i}\right|_{s=0} = 0
$$

$\Longrightarrow$ no steady-state error

PI control – General Case

In fact, integral action (if stabilizing) always results in zero steady-state error, in the presence of constant disturbances and demands, as we shall now show.

Assume that the following system settles down to an equilibrium with $\lim_{t\to\infty} e(t) = A \neq 0$, then:



$\Longrightarrow$ Contradiction

(as system is not in equilibrium)

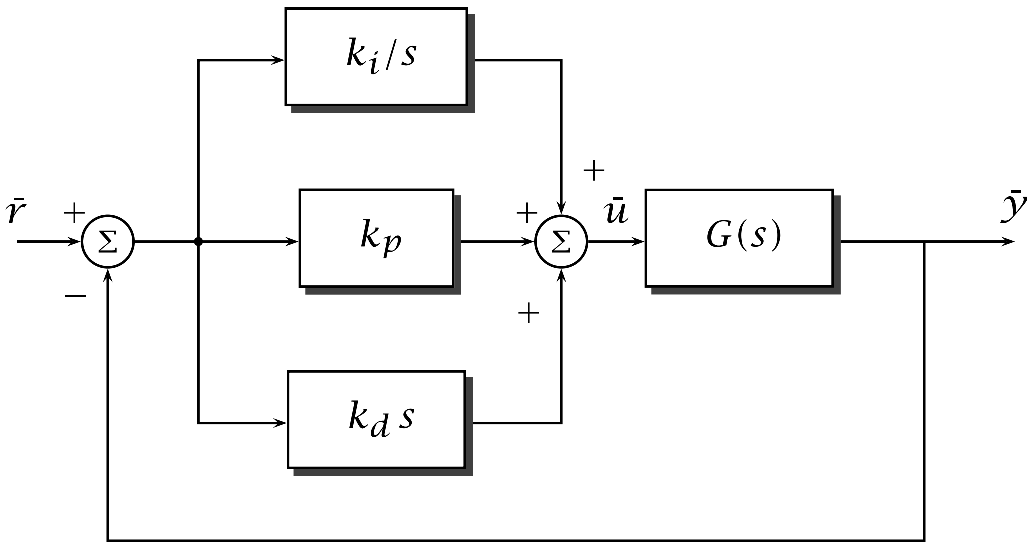
Hence, with PI control the only equilibrium possible has

$$\lim_{t\to\infty} e(t) = \mathbf{0}.$$

That is, $\lim_{t\to\infty} e(t) = 0$ provided the closed-loop system is asymptotically stable.

5.5.5 Proportional + Integral + Derivative (PID) Control

$$\boxed{K(s) = k_p + \frac{k_i}{s} + k_d\, s}$$



Characteristic equation:

$$\mathbf{1} + G(s)(k_p + k_d\, s + k_i/s) = \mathbf{0}$$

• can potentially combine the advantages of both derivative and integral action:

but can be difficult to “tune”.

There are many empirical rules for tuning PID controllers (Ziegler-Nichols for example) but to get any further we really need some more theory . . .