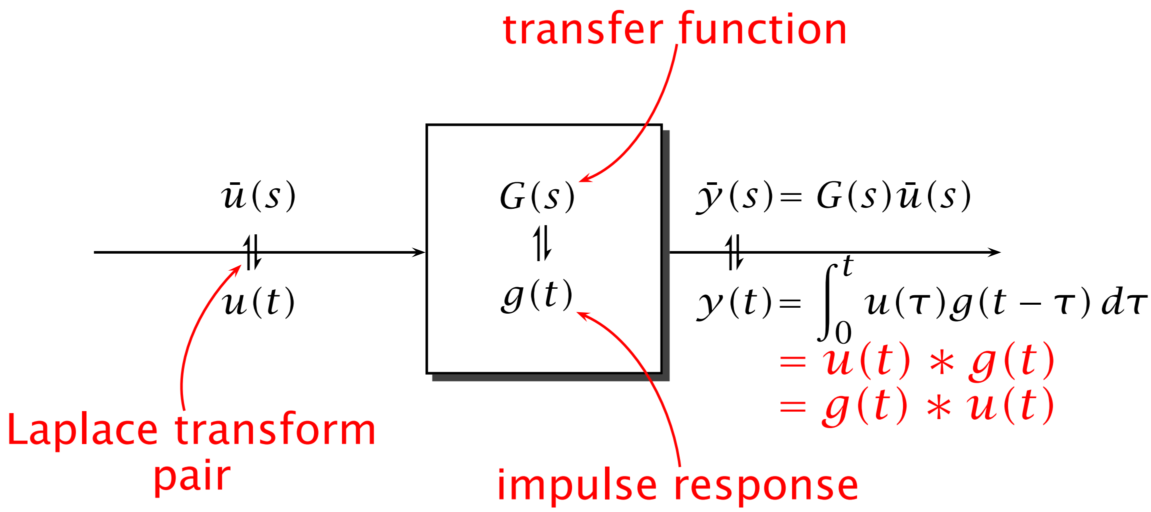
Part IB Paper 6: Information Engineering

LINEAR SYSTEMS AND CONTROL

Glenn Vinnicombe

HANDOUT 2

"Impulse responses, step responses and transfer functions."



Summary

The

- impulse response,
- step response and
- transfer function

of a Linear, Time Invariant and causal (LTI) system *each* completely characterize the input-output properties of that system.

Given the input to an LTI system, the output can be deterermined:

- In the time domain: as the *convolution* of the impulse response and the input.
- In the Laplace domain: as the *multiplication* of the transfer function and the Laplace transform of the input.

They are related as follows:

- The step response is the integral of the impulse response.
- The transfer function is the Laplace transform of the impulse response.

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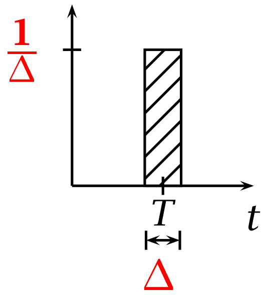
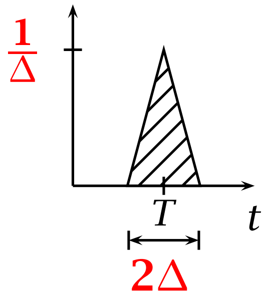
2.1 Preliminaries

2.1.1 Definition of the impulse "function"

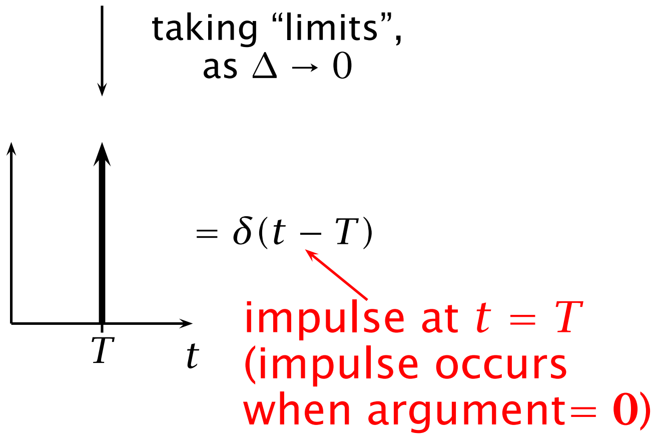
The impulse can be defined in many different ways, for example:

a) 

or b) 

as $\Delta \to 0$.

(area$= 1$ in each case.)

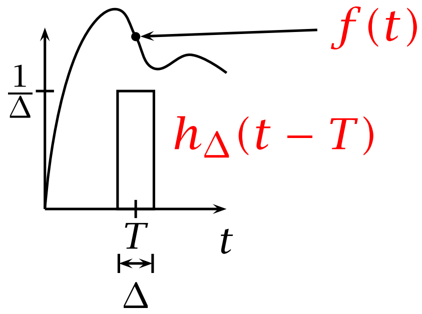


Note: The "convergence", as $\Delta \to 0$, occurs in the sense that both functions a) and b) have the same properties in the limit:

2.1.2 Properties of the impulse "function"

Consider a continuous function $f(t)$, and let $h_\Delta(t)$ denote the pulse approximation to the impulse ((a) on previous page):

$$h_\Delta(t) = \begin{cases} 1/\Delta & \text{if} - \Delta/2 < t < \Delta/2 \\ 0 & \text{otherwise} \end{cases}$$

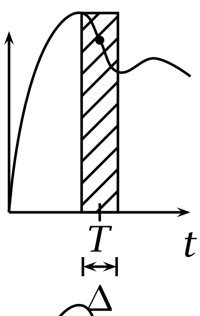


Let

$$I_\Delta = \int_{-\infty}^{\infty} f(t) h_\Delta(t-T)\, dt = \int_{T-\frac{\Delta}{2}}^{T+\frac{\Delta}{2}} f(t) \frac{1}{\Delta}\, dt \quad = \frac{1}{\Delta} \int_{T-\frac{\Delta}{2}}^{T+\frac{\Delta}{2}} f(t)\, dt$$

Now,

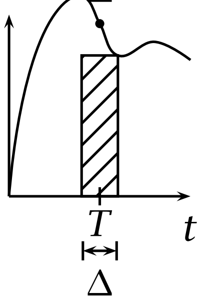
$$I_\Delta \leq \frac{1}{\Delta} \times$$



$$= \frac{1}{\cancel{\Delta}} \times \cancel{\Delta} \times \max_{T-\Delta/2 \leq t \leq T+\Delta/2} f(t)$$

and also

$$I_\Delta \geq \frac{1}{\Delta} \times$$



$$= \frac{1}{\cancel{\Delta}} \times \cancel{\Delta} \times \min_{T-\Delta/2 \leq t \leq T+\Delta/2} f(t)$$

So, it must be the case that $\lim_{\Delta \to 0} I_\Delta = f(T)$, that is

$$\lim_{\Delta \to 0} \int_{-\infty}^{\infty} f(t) h_\Delta(t-T)\, dt = f(T)$$

A similar argument leads to the same result for the triangular approximation to the impulse (defined above as (b)).

Formally, the *unit impulse* is defined as **any** "function" $\delta(t)$ which has the property

$$\int_{-\infty}^{\infty} f(t)\delta(t-T)\,dt = f(T)$$

(*Strictly speaking,* $\delta(t)$ *is a* distribution *rather than a function.*)

Laplace transform of $\delta(t)$:

$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)e^{-st}\,dt = e^{-s\times\mathbf{0}} = \mathbf{1}$$

2.1.3 The Impulse and Step Responses

Definition: The *impulse response* of a system is the output of the system when the input is an impulse, $\delta(t)$, and all initial conditions are zero.

Definition: The *step response* of a system is the output of the system when the input is a step, $H(t)$, and all initial conditions are zero.

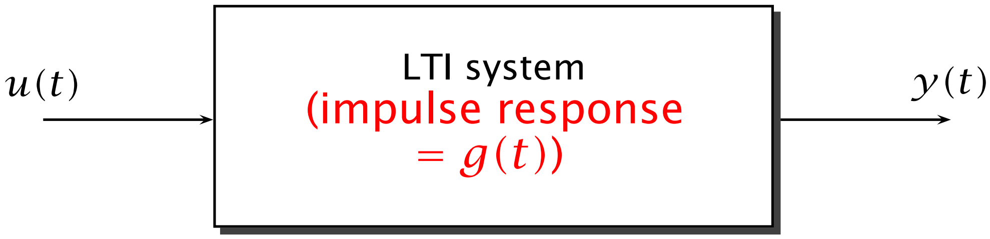
where $H(t)$ is the unit step function

$$H(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

If you know the impulse response of a system, then the response of that system to *any* input can be determined using convolution, as we shall now show:

2.2 The convolution integral

2.2.1 Direct derivation of the convolution integral



INPUT	OUTPUT
	impulse response
$\delta(t)$	$g(t)$
$\delta(t-\tau)$	$g(t-\tau)$
$u(\tau)\delta(t-\tau)$	$u(\tau)g(t-\tau)$
$\underbrace{\int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)\,d\tau}_{u(t)}$	$\int_{-\infty}^{\infty} u(\tau)g(t-\tau)\,d\tau$

Hence, the response to the input $u(t)$ is given by:

$$\boxed{y(t) = \int_{-\infty}^{\infty} u(\tau)g(t-\tau)d\tau}$$

2.2.2 Alternative statements of the convolution integral

The convolution integral

$$y(t) = \int_{-\infty}^{\infty} u(\tau)g(t-\tau)d\tau$$

is abbreviated as

$$y(t) = u(t) * g(t)$$

Let $T = t - \tau$, so $\tau = t - T$ and $d\tau = -dT$

It follows that

$$\boxed{y(t) = \int_{-\infty}^{\infty} u(t-T)g(T)\, dT}$$

$$= g(t) * u(t)$$

NOTE: in either form of the convolution integral,

the arguments of $u(\cdot)$ and $g(\cdot)$ add up to t

If, in addition,

- $g(t) = 0$ for $t < 0$ (CAUSALITY)
- $u(t) = 0$ for $t < 0$ (standing assumption), then

as $g(t-\tau) = 0$ for $t - \tau < 0$ (i.e. $\tau > t$)

$$y(t) = \int_0^t u(\tau)g(t-\tau)\, d\tau$$

as $u(\tau) = 0$ for $\tau < 0$

and also

as $u(t-\tau) = 0$ for $t - \tau < 0$ (i.e. $\tau > t$)

$$y(t) = \int_0^t u(t-\tau)g(\tau)\, d\tau$$

as $g(\tau) = 0$ for $\tau < 0$

2.3 The Transfer Function (for ODE systems)

As an alternative to convolution, the response of a linear system to arbitrary inputs can be determined using Laplace transforms. This is clear when the system is described as a linear ODE:

For example, if a linear system has input u and output y satisfying the ODE

$$\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \beta y = a\frac{du}{dt} + bu$$

and if *all initial conditions are zero*, i.e. $\left.\frac{dy}{dt}\right|_{t=0} = y(0) = u(0) = 0$, then taking Laplace transforms gives

$$s^2\bar{y}(s) + \alpha s\bar{y}(s) + \beta\bar{y}(s) = a\,s\bar{u}(s) + b\bar{u}(s)$$

or

$$\left(s^2 + \alpha s + \beta\right)\bar{y}(s) = \left(a\,s + b\right)\bar{u}(s)$$

and so

$$\bar{y}(s) = \underbrace{\frac{a\,s + b}{s^2 + \alpha s + \beta}}_{\text{Transfer function}}\bar{u}(s)$$

The function $\frac{a\,s+b}{s^2+\alpha s+\beta}$ is called the *transfer function* from $\bar{u}(s)$ (the input) to $\bar{y}(s)$ (the output).

Clearly the same technique will work for higher order linear ordinary differential equations (with constant coefficients). For such systems, the transfer function can be regarded as a placeholder for the coefficients of the differential equation.

2.3.1 Laplace transform of the convolution integral

We have seen that both convolution with the impulse reponse (in the time domain) and multiplication by the transfer function (in the Laplace domain) can both be used to determine the output of a linear system. What is the relationship between these techniques?

Assume that $g(t) = u(t) = 0$ for $t < 0$,

$$
\begin{aligned}
\boxed{\mathcal{L}\big(g(t) * u(t)\big)} &= \mathcal{L}\left(\int_{-\infty}^{\infty} g(\tau)u(t-\tau)\,d\tau\right) \\
&= \int_0^{\infty} e^{-st} \int_{-\infty}^{\infty} g(\tau)u(t-\tau)\,d\tau\,dt \\
&= \int_{-\infty}^{\infty} \int_0^{\infty} e^{-st} g(\tau)u(t-\tau)\,dt\,d\tau
\end{aligned}
$$

Note that τ is constant for the inner integration, and let $T = t - \tau$, which implies $t = T + \tau$ and $dt = dT$, giving

(so $e^{-st} = e^{-sT}e^{-s\tau}$)

$$
\begin{aligned}
\mathcal{L}\big(g(t) * u(t)\big) &= \int_{\cancel{-\infty}\ 0}^{\infty} \int_{\cancel{-\tau}\ 0}^{\infty} e^{-s\tau}e^{-sT} g(\tau)u(T)\,dT\,d\tau \\
&= \int_0^{\infty} e^{-s\tau} g(\tau)d\tau \times \int_0^{\infty} e^{-sT} u(T)dT \\
&= \boxed{\mathcal{L}\big(g(t)\big) \times \mathcal{L}\big(u(t)\big)} = \bar{g}(s)\,\bar{u}(s)
\end{aligned}
$$

In words, a Laplace transform (easy when you get used to them) turns a convolution (always hard!) into a multiplication (very easy).

2.4 The transfer function for *any* linear system

It follows from the result of the previous section that, if

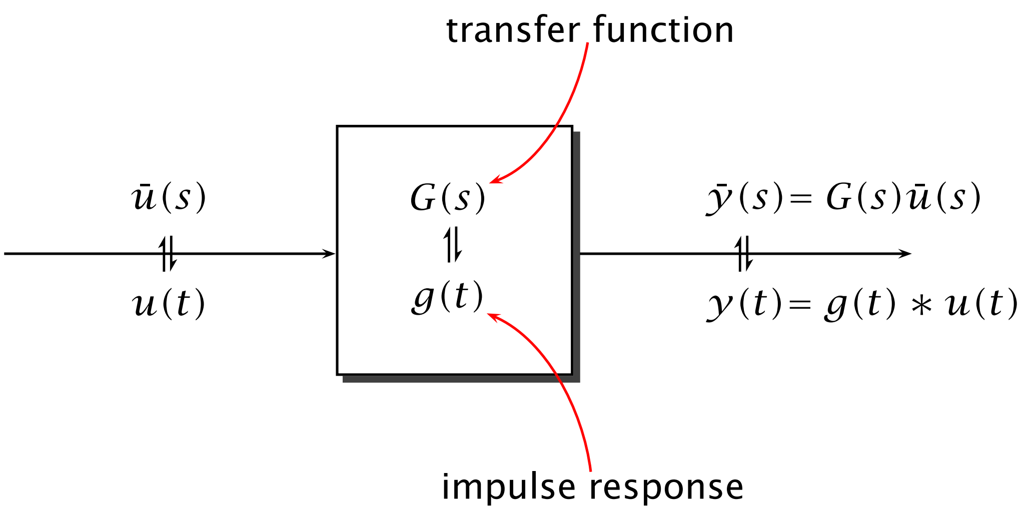
$$y(t) = g(t) * u(t)$$

is the response of an LTI system with impulse response $g(t)$ to the input $u(t)$, then we can also write

$$\bar{y}(s) = G(s)\,\bar{u}(s)$$

where $G(s) = \mathcal{L}g(t)$ is called the *transfer function* of the system.

It follows that *all* LTI systems have transfer functions (given by the Laplace transform of their impulse response).



We shall use the notation $\bar{x}(s)$ to represent the Laplace transform of a signal $x(t)$, and uppercase characters to represent transfer functions (e.g. $G(s)$).

In general a system may have more than one input. In this case, the transfer function from a particular input to a particular output is defined as the Laplace transform of that output when an impulse is applied to the given input, all other inputs are zero and all initial conditions are zero.

This is most easily seen in the Laplace domain: If an LTI system has an input u and an output y then we can always write

$$\bar{y}(s) = G(s)\bar{u}(s) + \text{other terms independent of } u.$$

$G(s)$ is then called the transfer function from $\bar{u}(s)$ to $\bar{y}(s)$. (Or, the transfer function relating $\bar{y}(s)$ and $\bar{u}(s)$)

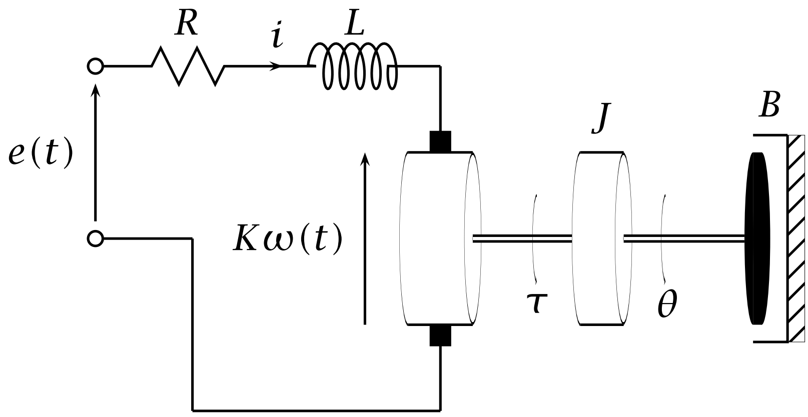
Here the "other terms" could be a result of non-zero initial conditions or of other non-zero inputs (disturbances, for example).

NOTE: Remember that although the transfer function is defined in terms of the impulse response, it is usually most easily calculated directly from the system's differential equations.

2.5 Example: DC motor

The following example will illustrate the impulse and step responses and the transfer function. We take the input to the motor to be the applied voltage $e(t)$, and the output to be the shaft *angular velocity*

$$\omega := \dot{\theta}(t).$$



1) $\tau(t) = Ki(t)$ Linearized motor equation

2) $\tau(t) - B\omega(t) = J\dot{\omega}(t)$ Newton

3) $e(t) = Ri(t) + L\dfrac{di(t)}{dt} + K\omega(t)$ Kirchoff

Find the effect of $e(t)$ on $\omega(t)$:

Take Laplace Transforms, assuming that $\omega(0) = i(0) = 0$:

1) $\bar{\tau}(s) = K\bar{i}(s)$

2) $\bar{\tau}(s) - B\bar{\omega}(s) = J\left(s\bar{\omega}(s) - \omega(\mathbf{0})\right)$

3) $\bar{e}(s) = R\bar{i}(s) + L\left(s\bar{i}(s) - i(\mathbf{0})\right) + K\bar{\omega}(s)$

We can now eliminate $\bar{i}(s)$ and $\bar{\tau}(s)$ to leave one equation relating $\bar{e}(s)$ and $\bar{\omega}(s)$:

First 1) and 2) give:

$$K\bar{i}(s) = (Js + B)\bar{\omega}(s)$$

and rearranging 3) gives:

$$\bar{e}(s) = (Ls + R)\bar{i}(s) + K\bar{\omega}(s)$$

putting these together leads to

$$\bar{e}(s) = \left((Ls + R)\frac{(Js + B)}{K} + K\right)\bar{\omega}(s)$$

or, in the standard form (outputs on the left)

$$\begin{aligned}\bar{\omega}(s) &= \frac{K}{(Ls + R)(Js + B) + K^2}\bar{e}(s) \\ &= \frac{k}{(sT_1 + 1)(sT_2 + 1)}\bar{e}(s)\end{aligned}$$

for suitable definitions of k, T_1 and T_2 (assuming real roots).

That is,

$$\overset{\text{output}}{\bar{\omega}(s)} = \underbrace{\frac{k}{(T_1 s + 1)(T_2 s + 1)}}_{\text{Transfer Function}} \overset{\text{input}}{\bar{e}(s)}$$

We call $G(s) = \frac{k}{(sT_1+1)(sT_2+1)}$ the transfer function *from* $\bar{e}(s)$ *to* $\bar{\omega}(s)$ (or, relating $\bar{\omega}(s)$ and $\bar{e}(s)$).

The diagram

$$\bar{e}(s) \longrightarrow \boxed{G(s)} \longrightarrow \bar{\omega}(s)$$

represents this relationship between the input and the output:

> By using Laplace transforms, all LTI blocks can be treated as multiplication by a transfer function.

For the lego NXT motor we have $k \approx 2.2$*,* $T_1 \approx 54$*ms and* $T_2 \approx 1$*ms, as demonstrated in lectures.*

2.5.1 Impulse Response of the DC motor

To find the impulse response, we let

$$e(t) = \delta(t) \leftarrow \text{unit impulse}$$

which implies

$$\bar{e}(s) = \mathbf{1}$$

and put

$$\omega(0) = i(0) = 0.$$

So,

$$\bar{\omega}(s) = \frac{k}{(T_2 s + 1)(T_1 s + 1)} \times \mathbf{1}$$

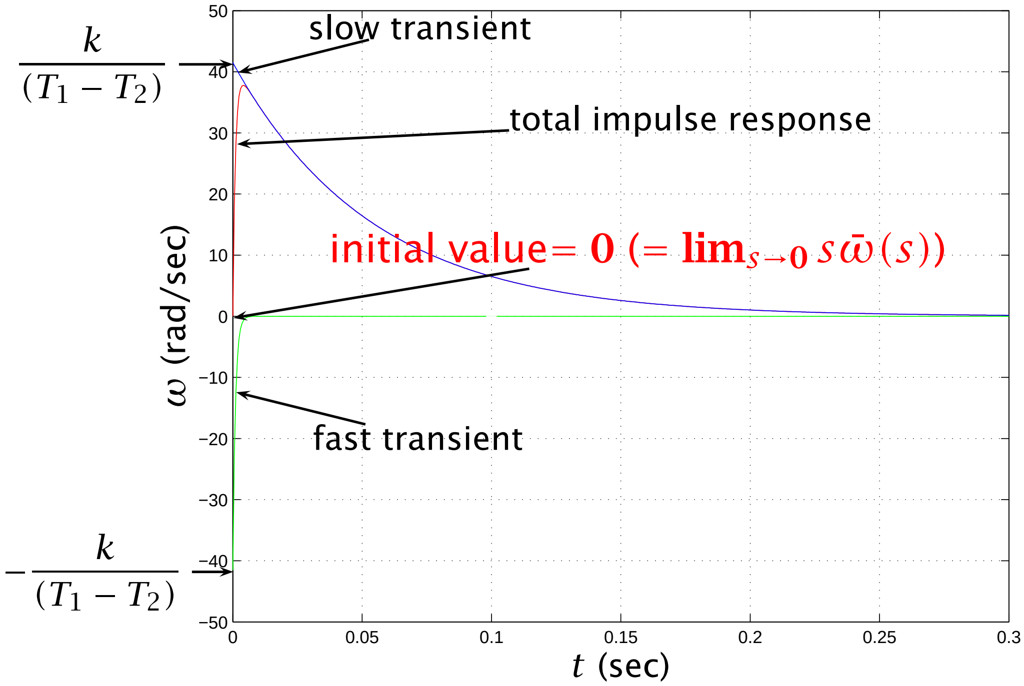
Split into partial fractions:

$$\begin{aligned} \bar{\omega}(s) &= k\left[\frac{1}{1 - \frac{T_1}{T_2}} \times \frac{1}{sT_2 + 1} + \frac{1}{1 - \frac{T_2}{T_1}} \times \frac{1}{sT_1 + 1}\right] \\ &= \frac{k}{T_1 - T_2}\left[-\frac{1}{s + 1/T_2} + \frac{1}{s + 1/T_1}\right] \end{aligned}$$

Hence,

$$\omega(t) = \frac{k}{T_1 - T_2}\Bigg[\underbrace{-e^{-t/T_2}}_{\text{Fast Transient}} + \underbrace{e^{-t/T_1}}_{\text{Slow Transient}}\Bigg]$$

$$\text{e.g.} \quad T_2 \approx 1\text{ms} \qquad T_1 \approx 54 \text{ ms}$$



2.5.2 Step Response of the DC motor

To find the step response, we let

$$e(t) = H(t)$$

which implies that

$$\bar{e}(s) = \mathbf{\frac{1}{s}}$$

and put

$$\omega(0) = i(0) = 0$$

(as for the impulse response.) So,

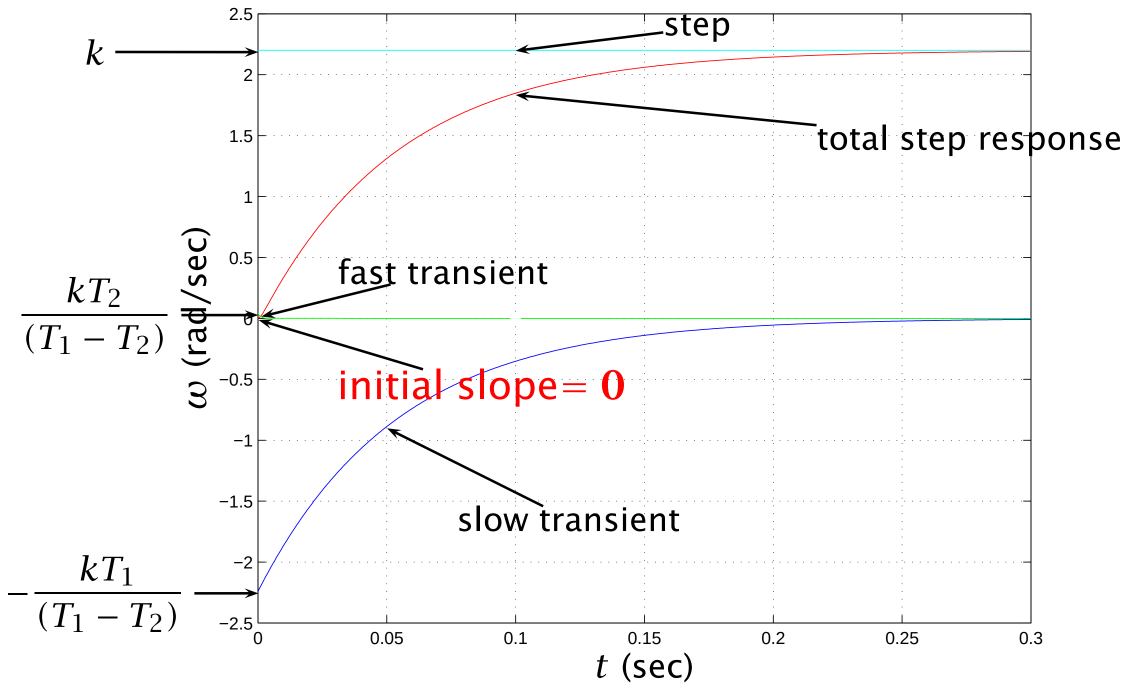
$$\bar{\omega}(s) = \frac{k}{(T_2 s + 1)(T_1 s + 1)} \times \mathbf{\frac{1}{s}}$$

Split into partial fractions:

$$\bar{\omega}(s) = k\left[\frac{1}{s} + \frac{T_2}{T_1 - T_2} \times \frac{1}{s + \frac{1}{T_2}} - \frac{T_1}{T_1 - T_2} \times \frac{1}{s + \frac{1}{T_1}}\right]$$

Hence,

$$\omega(t) = k\Bigg[H(t) + \underbrace{\frac{T_2}{T_1 - T_2} \times e^{-t/T_2}}_{\text{Fast Transient}} - \underbrace{\frac{T_1}{T_1 - T_2} \times e^{-t/T_1}}_{\text{Slow Transient}}\Bigg]$$



2.5.3 Deriving the step response from the impulse response

For a system with impulse response $g(t)$, the step response is given by

$$y(t) = \int_{0^-}^{t} g(\tau)H(t-\tau)\,d\tau = \int_{0^-}^{t} g(\tau)\,d\tau$$

i.e.

$$\boxed{\text{step response = integral of impulse response}}$$

Check this on the example. Make sure you understand where the term $H(t)$ in the step response comes from.

2.6 Transforms of signals vs Transfer functions of systems

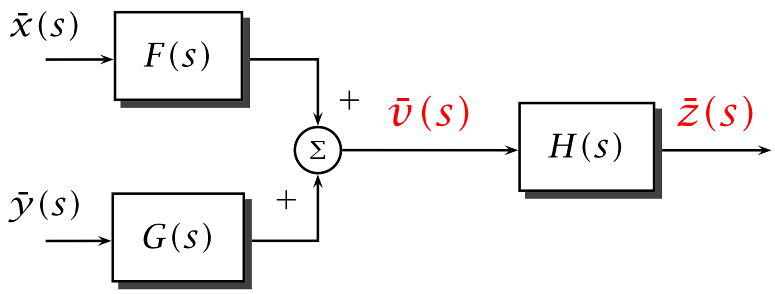
Mathematically there is no distinction, but in practice they have different interpretations

$G(s)$	Signals $\mathcal{L}^{-1}G(s)$	Systems $\bar{y}(s) = G(s) \cdot \bar{u}(s)$ $\Updownarrow$
1	$\delta(t)$	$y(t) = u(t)$
$\frac{1}{s}$	$H(t)$	$y(t) = \int_0^t u(\tau)d\tau$ (integrator)
$\frac{1}{s+a}$	e^{-at}	$\dot{y}(t) + ay(t) = u(t)$ (lag)
$\frac{\omega}{s^2+\omega^2}$	$\sin(\omega t)$	$\ddot{y}(t) + \omega^2 y(t) = \omega u(t)$
e^{-sT}	$\delta(t-T)$	$y(t) = u(t-T)$ (delay)

In each case, the "signal" is the impulse response of the "system".

2.7 Interconnections of LTI systems

a)



represents the equations:

$$\bar{v}(s) = F(s)\bar{x}(s) + G(s)\bar{y}(s)$$

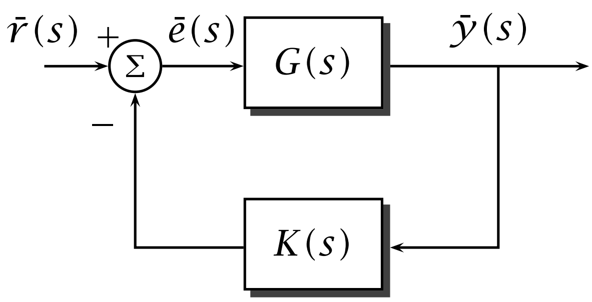
and

$$\bar{z}(s) = H(s)\bar{v}(s)$$

$$\Longrightarrow \bar{z}(s) = H(s)\big(F(s)\bar{x}(s) + G(s)\bar{y}(s)\big)$$

$$= \underbrace{H(s)F(s)}_{\substack{\text{transfer function} \\ \text{from } \bar{x}(s) \text{ to } \bar{z}(s)}} \bar{x}(s) + \underbrace{H(s)G(s)}_{\substack{\text{transfer function} \\ \text{from } \bar{y}(s) \text{ to } \bar{z}(s)}} \bar{y}(s)$$

b)



represents the simultaneous equations:

$$\bar{e}(s) = \bar{r}(s) - K(s)\bar{y}(s)$$

and

$$\bar{y}(s) = G(s)\bar{e}(s)$$

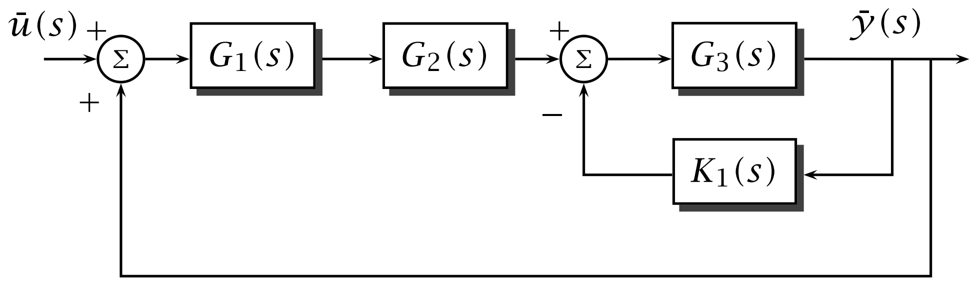
$$\Longrightarrow \bar{y}(s) = G(s)\bar{r}(s) - G(s)K(s)\bar{y}(s)$$

$$\Longrightarrow (1 + G(s)K(s))\bar{y}(s) = G(s)\bar{r}(s)$$

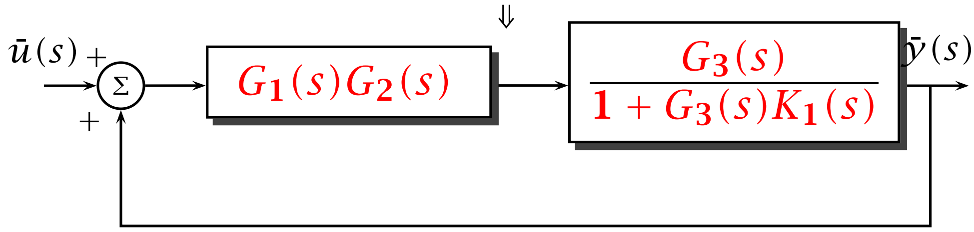
$$\boxed{\Longrightarrow \bar{y}(s) = \frac{G(s)}{1 + G(s)K(s)}\bar{r}(s)}$$

2.7.1 "Simplification" of block diagrams

Recognizing that blocks represent multiplications, and using the above formulae, it is often easier to rearrange block diagrams to determine overall transfer functions, e.g. from $\bar{u}(s)$ to $\bar{y}(s)$ below.



$$\Downarrow$$



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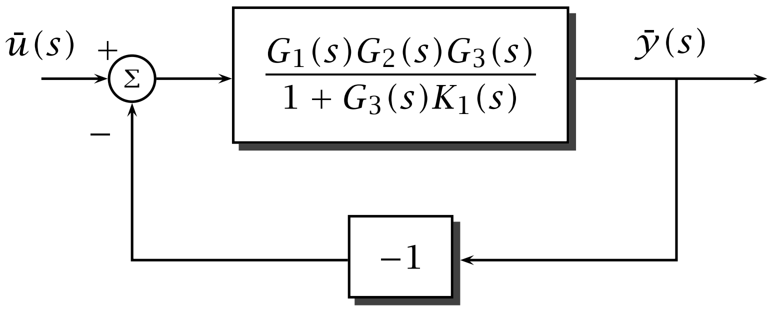


$$\Downarrow$$

$$\bar{y}(s) = \frac{\dfrac{G_1(s)G_2(s)G_3(s)}{1+G_3(s)K_1(s)}}{1 - \dfrac{G_1(s)G_2(s)G_3(s)}{1+G_3(s)K_1(s)}}\bar{u}(s)$$

or

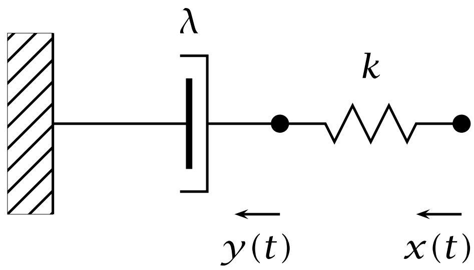
$$y(s) = \frac{G_1(s)G_2(s)G_3(s)}{1+G_3(s)K_1(s) - G_1(s)G_2(s)G_3(s)}\bar{u}(s)$$

2.8 More transfer function examples

To obtain the transfer function, in each case, we take all initial conditions to be zero.

The following three systems all have the same transfer function (a *1st order lag*)

1) Spring/damper system



$$\lambda \dot{y} = k(x - y)$$

$$\Longrightarrow \quad \frac{\lambda}{k}\dot{y} + y = x$$

$$\Longrightarrow \quad \frac{\lambda}{k}s\bar{y}(s) + \bar{y}(s) = \bar{x}(s)$$

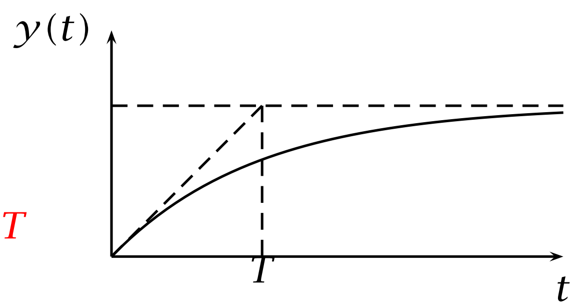
$$\Longrightarrow \quad \left(\frac{\lambda}{k}s + 1\right)\bar{y}(s) = \bar{x}(s)$$

$$\Longrightarrow \quad \boxed{\bar{y}(s) = \frac{1}{Ts+1}\bar{x}(s)} \qquad T = \lambda/k$$

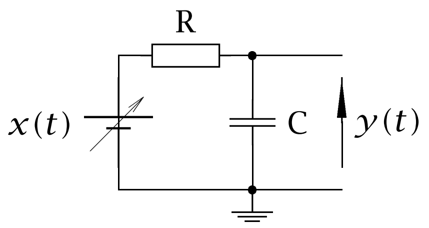
Step Response:

$$\bar{x}(s) = \frac{1}{s} \Longrightarrow \bar{y}(s) = \frac{1}{s(sT+1)} = \frac{1}{s} - \frac{T}{sT+1}$$

$$\Longrightarrow \; y(t) = H(t) - e^{-t/T}$$



2) RC network



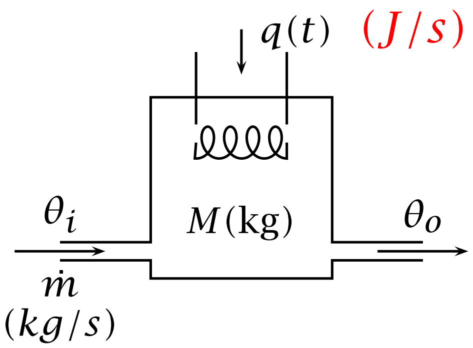
$$i = C\frac{dy}{dt} = \frac{x-y}{R}$$

$$\Longrightarrow \qquad RC\dot{y} + y = x$$

$$\Longrightarrow \qquad (RCs+1)\bar{y}(s) = \bar{x}(s)$$

$$\Longrightarrow \qquad \boxed{\bar{y}(s) = \frac{1}{Ts+1}\bar{x}(s)} \qquad T = RC$$

3) Water Heater



(assuming perfect mixing – i.e. all the water in the tank is at the same temperature, θ_o)

specific
heat capacity

$$q = Mc\dot{\theta}_0 + \dot{m}c(\theta_o - \theta_i)$$

$$\Longrightarrow \qquad (Mcs + \dot{m}c)\bar{\theta}_o(s) = \bar{q}(s) + \dot{m}c\bar{\theta}_i(s)$$

$$\Longrightarrow \qquad \boxed{\bar{\theta}_o(s) = \frac{1}{\dot{m}c}\frac{1}{Ts+1}\bar{q}(s) \;+\; \frac{1}{Ts+1}\bar{\theta}_i(s)} \qquad T = M/\dot{m}$$

input

disturbance

2.9 Key Points

- The step response is the integral (w.r.t time) of the impulse response.
- The transfer function is the Laplace transform of the impulse response.
- By using transfer functions, block diagrams represent simple algebraic relationships (all blocks become multiplications).