Module 3F2: Systems and Control

LECTURE NOTES 1: STATE-SPACE SYSTEMS

Contents

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(building on previous versions by Keith Glover and Jan Maciejowski)

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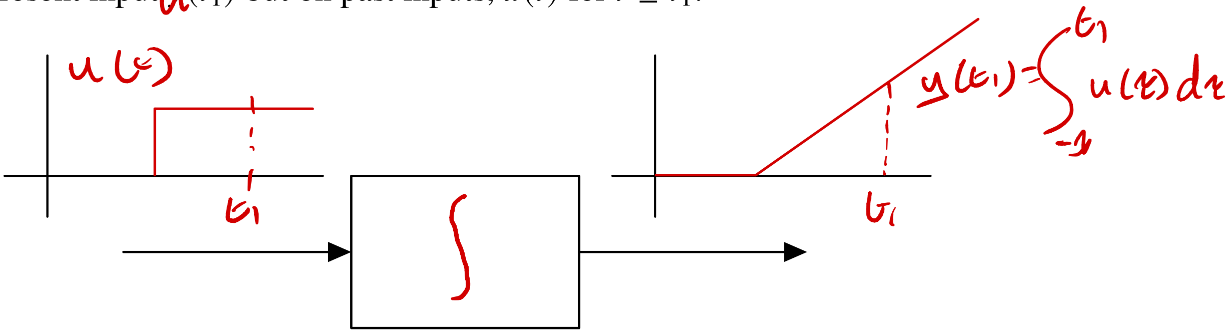
1 State Space Descriptions Of Dynamical Systems

1.1 What are States?

The essence of a (causal) dynamical system is its memory, i.e. the present output, $y(t_1)$ say, depends not just on the present input $u(t_1)$ but on past inputs, $u(t)$ for $t \leq t_1$:

eg Integrator



The *states* are a set of system variables chosen to represent this memory. The choice is not unique, but for $\underline{x}()$ to be a valid state for a dynamical system it must satisfy the following two properties:

1. for any $t_o < t_1$, $\underline{x}(t_1)$ can always be determined from $\underline{x}(t_o)$ and $\{\underline{u}(\tau), t_o \leq \tau \leq t_1\}$.
2. The outputs $\underline{y}$ at time t_1, is a *memoryless* function of $\underline{x}(t_1)$ and $\underline{u}(t_1)$ (that is, $\underline{y}(t_1)$ depends only on $\underline{x}(t_1)$ and $\underline{u}(t_1)$).

- State properties

That is, $\underline{x}(t_o)$ completely summarizes the effect on the future of inputs and states prior to t_o.

eg Integrator

$x(t) = 3\,y(t)$ (Say)

$$x(t_1) = 3\int_{-\infty}^{t_1} u(\tau)\,d\tau = \underbrace{3\int_{-\infty}^{t_0} u(\tau)\,d\tau}_{3y(t_0) = x(t_0)} + \underbrace{3\int_{t_0}^{t_1} u(\tau)\,d\tau}_{f(\{u(\tau):\, t_0 \leq \tau \leq t_1\})}$$

Also, $y(t) = \frac{1}{3}\,x(t)$

$\Rightarrow$ $x(t)$ is a valid choice of state

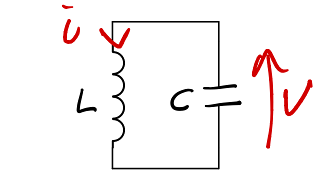
A **dynamical system** thus relates three sets of variables:

The **inputs**, $\underline{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix} \in \mathbb{R}^m$, the **states**, $\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^n$, the **outputs**, $\underline{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix} \in \mathbb{R}^p$.

1.2 How do you choose the states?

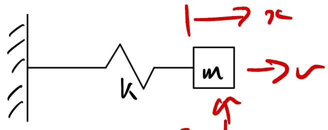
Method 1: Choose system states, $x_1, x_2, \ldots x_n$ by considering the independent 'energy storage devices' or 'memory elements',

$E = \frac{1}{2}Li^2$ $\quad$ $E = \frac{1}{2}CV^2$



e.g.

> electrical circuits - current in L or voltage on C:
>
> mechanics - positions and velocities of masses (linear or angular):
>
> chemical engineering - temperature, pressure, volume, concentration.



$E = \frac{1}{2}kx^2$

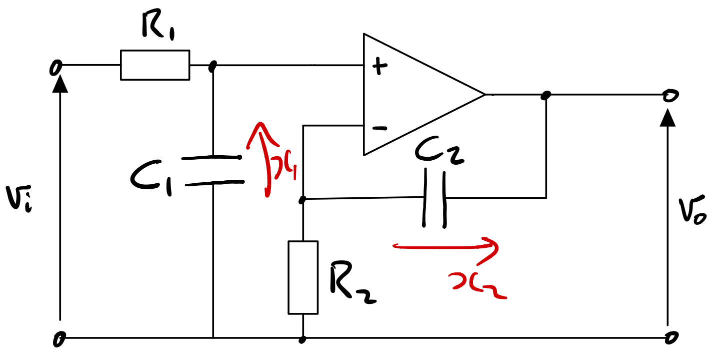
$E = \frac{1}{2}mv^2$

Then use the basic physical laws derive expressions involving $\dot{x}_i$, e.g.

- $\frac{dv}{dt} = -\frac{i}{C}, \quad \frac{di}{dt} = \frac{v}{L}$
- $\frac{dv}{dt} = \frac{F}{m} = -\frac{kx}{m}, \quad \frac{dx}{dt} = v$
- $\frac{d(\text{volume})}{dt} = \text{flow} = f(\text{pressure})$

Write equations as $\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}, t)$.

Example: **Ideal Operational Amplifier Circuits**



In a negative feedback configuration the amplifier acts so as to make the +ve and -ve inputs have essentially equal voltages.

$v_+ = v_- = x_1$

$$\begin{cases} C_1\dot{x}_1 = (V_i - x_1)/R_1; & C_2\dot{x}_2 = x_1/R_2 \\ V_o = x_1 + x_2 \end{cases}$$

"A" $\quad$ "B"

$$\dot{\underline{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1/R_1C_1 & 0 \\ 1/R_2C_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/R_1C_1 \\ 0 \end{bmatrix} V_i$$

"C"

$$V_o = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Method 2: Given a set of differential equations, choose state variables as successive time derivatives (not including the highest derivative of each variable) i.e. the variables for which it would be necessary to specify initial conditions in order to solve the differential equations.
Example 1:

$$\dddot{y} + 6y\ddot{y} + 5(\dot{y})^3 + 12\sin(y) = \cos(t)$$

a state variable is given by: $x_1 = y,\ \ x_2 = \dot{y},\ \ x_3 = \ddot{y},$ giving

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -6x_1x_3 - 5x_2^3 - 12\sin(x_1) + \cos(t) \end{cases}$$

which is in the form

$$\underline{\dot{x}}(t) = \underline{f}(\underline{x}(t), t)$$

where $\underline{f}$ is a vector valued function of a vector, i.e.

$$\underline{f}(\underline{x}, t) = \begin{bmatrix} x_2 \\ x_3 \\ -6x_1x_3 - 5x_2^3 - 12\sin(x_1) + \cos(t) \end{bmatrix}$$

The most general class of dynamical system that we will consider is described by the set of first order ordinary differential equations -

$$\mathcal{S} \begin{cases} \dfrac{dx_i}{dt} &= f_i(x_1(t), x_2(t), \ldots x_n(t), u_1(t), \ldots u_m(t), t), \quad i = 1, 2, \ldots n. \\ y_j(t) &= g_j(x_1(t), \ldots x_n(t), u_1(t), \ldots u_m(t), t) \qquad j = 1, 2, \ldots p. \end{cases}$$

$\mathcal{S}$ is the standard form for a **state-space dynamical system model**. Or a vector form -

$$\mathcal{S} \begin{cases} \underline{\dot{x}}(t) &= \underline{f}(\underline{x}(t), \underline{u}(t), t) \\ \underline{y}(t) &= \underline{g}(\underline{x}(t), \underline{u}(t), t) \end{cases}$$

2 Linearizing Nonlinear Dynamical Systems

2.1 Linearizing the standard form

Suppose we have a nonlinear (time invariant) dynamical system in state space form -

$$\mathcal{S}\left\{\begin{array}{rcl} \dot{\underline{x}} & = & \underline{f}(\underline{x},\underline{u}) \\ \underline{y} & = & \underline{g}(\underline{x},\underline{u}) \end{array}\right.$$

Let $\underline{x}_e$ be an equilibrium state for the system when $\underline{u}(t) = \underline{u}_e$ (constant)

i.e. $\underline{f}(\underline{x}_e, \underline{u}_e) = \underline{0}$ and also let $\underline{y}_e = \underline{g}(\underline{x}_e, \underline{u}_e)$.

Now consider small perturbations from this equilibrium, let

$$\underline{x}(t) = \underline{x}_e + \delta\underline{x}(t), \quad \underline{u}(t) = \underline{u}_e + \delta\underline{u}(t), \quad \underline{y}(t) = \underline{y}_e + \delta\underline{y}(t)$$

A Taylor Series expansion of the i-th equation of $\dot{\underline{x}} = \underline{f}(\underline{x},\underline{u})$ gives

$$\begin{aligned}
\dot{x}_i &= \cancelto{0}{\dot{x}_{e_i}} + \delta\dot{x}_i \quad = f_i\left(x_{e_1}+\delta x_1,\ x_{e_2}+\delta x_2,\ \ldots,\ u_{e_1}+\delta u_1,\ \ldots\right) \\
&= \cancelto{0}{f_i(\underline{x}_e,\underline{u}_e)} + \left.\frac{\partial f_i}{\partial x_1}\right|_{\underline{x}_e,\underline{u}_e} \delta x_1 + \left.\frac{\partial f_i}{\partial x_2}\right|_{\underline{x}_e,\underline{u}_e} \delta x_2 + \ldots + \left.\frac{\partial f_i}{\partial x_n}\right|_{\underline{x}_e,\underline{u}_e} \delta x_n \\
&\quad + \left.\frac{\partial f_i}{\partial u_1}\right|_{\underline{x}_e,\underline{u}_e} \delta u_1 + \ldots + \left.\frac{\partial f_i}{\partial u_m}\right|_{\underline{x}_e,\underline{u}_e} \delta u_m \ + \ \text{Remainder}
\end{aligned}$$

(higher order terms)

Thus for small $\delta\underline{x}$ and $\delta\underline{u}$ ($\Rightarrow$ a v. small remainder) we get

$$\delta\dot{x}_i = \left[\frac{\partial f_i}{\partial x_1} \ \ \frac{\partial f_i}{\partial x_2} \ \cdots \right]_e \delta\underline{x} + \left[\frac{\partial f_i}{\partial u_1} \ \cdots \right]_e \delta\underline{u}$$

$$\delta\dot{\underline{x}} \cong A\delta\underline{x} + B\delta\underline{u}$$

where

$$A = \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\underline{x}=\underline{x}_e,\underline{u}=\underline{u}_e} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\underline{x}=\underline{x}_e,\,\underline{u}=\underline{u}_e} \leftarrow \text{Jacobian}$$

$$\boxed{\begin{aligned} &\underline{f}(\underline{x}+\delta\underline{x}, \underline{u}+\delta\underline{u}) \\ &= \underline{f}(\underline{x},\underline{u}) + \frac{\partial \underline{f}}{\partial \underline{x}}\delta\underline{x} \\ &\quad + \frac{\partial \underline{f}}{\partial \underline{u}}\delta\underline{u} \end{aligned}}$$

+ h.o. terms

$$B = \left.\frac{\partial \underline{f}}{\partial \underline{u}}\right|_{\underline{x}=\underline{x}_e,\underline{u}=\underline{u}_e} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & & \vdots \\ \frac{\partial f_n}{\partial u_1} & & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{\underline{x} = \underline{x}_e, \underline{u} = \underline{u}_e}$$

Similarly $\delta \underline{y} = C\delta \underline{x} + D\delta \underline{u}$ where $C = \left.\frac{\partial \underline{g}}{\partial \underline{x}}\right|_{\underline{x}=\underline{x}_e,\underline{u}=\underline{u}_e}$ and $D = \left.\frac{\partial \underline{g}}{\partial \underline{u}}\right|_{\underline{x}=\underline{x}_e,\underline{u}=\underline{u}_e}$

The linearized system equations are thus

$$\begin{cases} \delta\dot{\underline{x}} = A\delta\underline{x} + B\delta\underline{u} \\ \delta\underline{y} = C\delta\underline{x} + D\delta\underline{u} \end{cases}$$

which will accurately predict the system behaviour for $\underline{x}$ and $\underline{u}$ close to $\underline{x}_e$ and $\underline{u}_e$ respectively.

For systems which are both linear and time-invariant to start with we simply use the standard form:

$$\mathcal{S} \begin{cases} \dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t) \\ \underline{y}(t) = C\underline{x}(t) + D\underline{u}(t) \end{cases}$$

2.2 Linearizing when the State Equations are Implicit

Quite often a system's equations cannot easily be written as $\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u})$ but can be written as

$\underline{F}(\dot{\underline{x}}, \underline{x}, \underline{u}) = \underline{0}$ (n equations in the n unknowns $\dot{x}_1 \ldots \dot{x}_n$.)

The linearized model can be derived without solving for $\dot{\underline{x}}$ as follows. Since $(\underline{x}_e, \underline{u}_e)$ gives an equilibrium we have,

$\underline{F}(\underline{0}, \underline{x}_e, \underline{u}_e) = \underline{0}$

Now linearize $\underline{F}$ about $(\underline{0}, \underline{x}_e, \underline{u}_e)$ to get

$$\underbrace{\left.\frac{\partial \underline{F}}{\partial \dot{\underline{x}}}\right|_{(\underline{0},\underline{x}_e,\underline{u}_e)}}_{L} \delta\dot{\underline{x}} + \underbrace{\left.\frac{\partial \underline{F}}{\partial \underline{x}}\right|_{(\underline{0},\underline{x}_e,\underline{u}_e)}}_{M} \delta\underline{x} + \underbrace{\left.\frac{\partial \underline{F}}{\partial \underline{u}}\right|_{(\underline{0},\underline{x}_e,\underline{u}_e)}}_{N} \delta\underline{u} \simeq \underline{0}$$

$$L\delta\dot{\underline{x}} + M\delta\underline{x} + N\delta\underline{u} \simeq \underline{0} \implies \delta\dot{\underline{x}} \simeq -L^{-1}M\delta\underline{x} - L^{-1}N\delta\underline{u}$$

Is L is invertible

(N.B. no nonlinear equations to solve except to obtain the equilibrium point)