Robust Control of Heterogeneous Networks (e.g. congestion control for the Internet)

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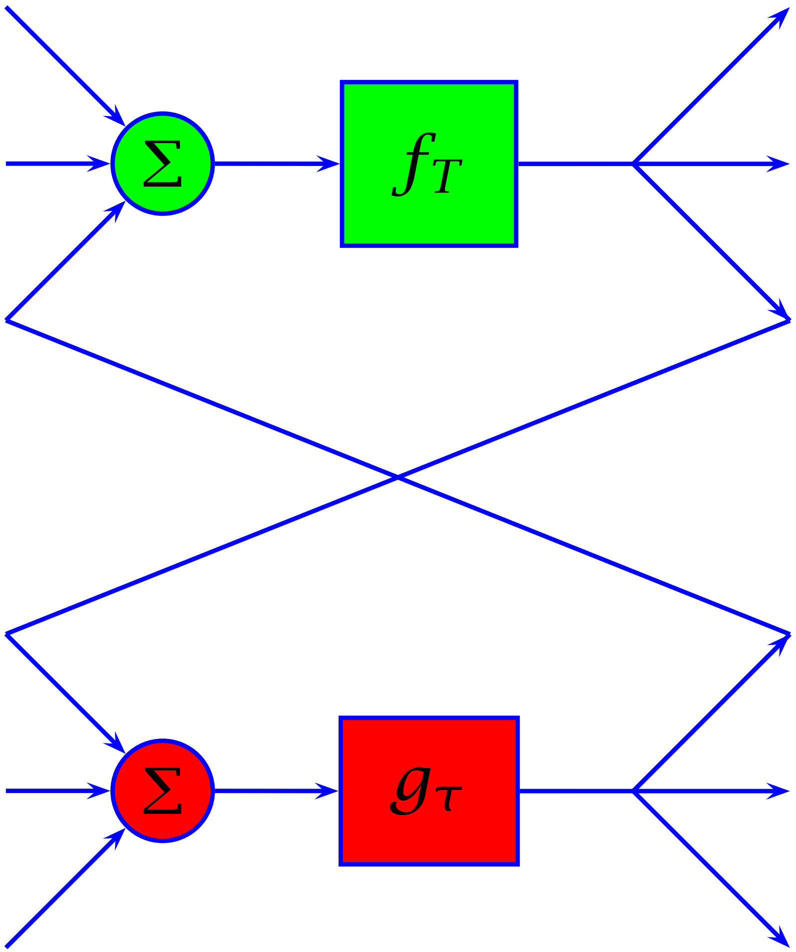
Introduction

- Is it possible to build (locally verifiable) stable and robust large scale feedback interconnections of *linear* dynamical systems?
- Yes, but only limited results so far (e.g. Internet congestion control).

Networks, Modules and Protocols

- We regard a network as being a collection of interconnected *modules*.
- *Modules* are elements of parameterized classes of linear, time-invariant, dynamical systems.
- *Protocols* are the rules by which interconnections between modules be made.
- Will present sufficient conditions for network stability for fairly general classes of asymptotically stable modules but fairly restricted interconnection protocols.

Modules and Protocols



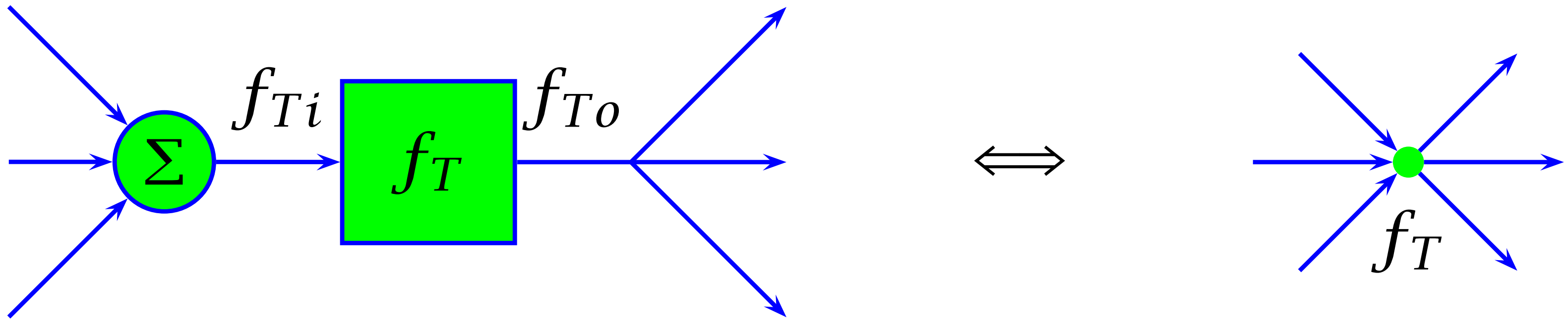
Module:
e.g. $y(t) = \sum u_i(t - T)$

Protocol:
e.g. f/b interconnection
allowed if green's $T <$ red's τ

Module:
e.g. $\tau \dot{y}(t) = \sum u_i(t)$

Note: this protocol still allows each red module to be connected to many green modules and each green module to be connected to many red modules.

Signal Flow Graphs (Mason '53)

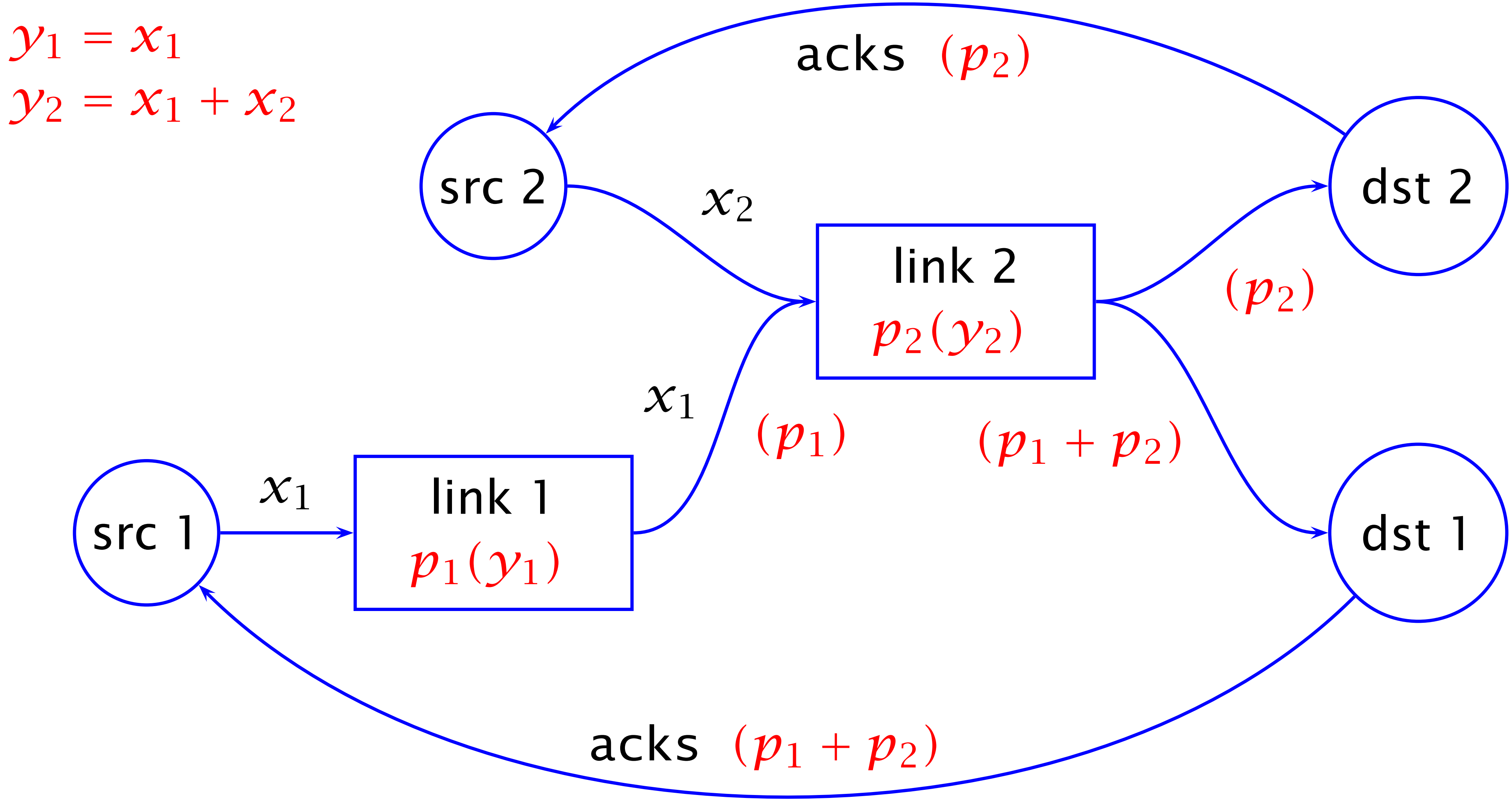


Assume linearity, and that Laplace transforms have been taken everywhere, so

$$f_{To}(s) = f_T(s) f_{Ti}(s)$$

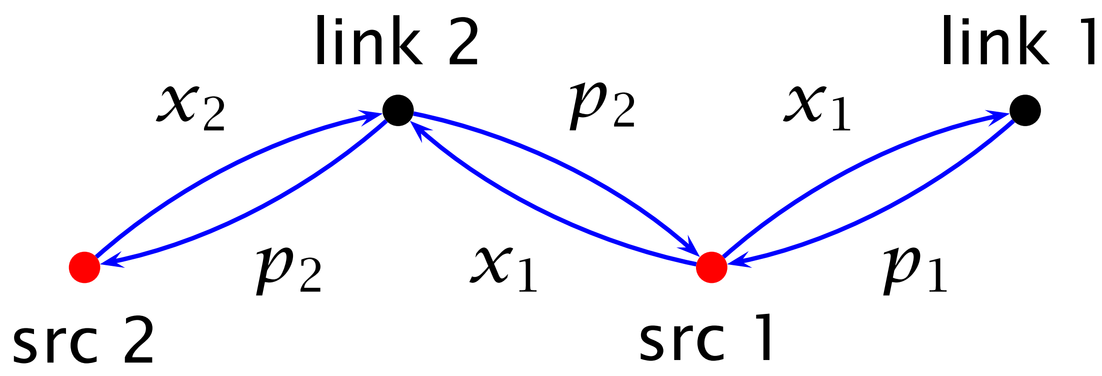
Example: Congestion Control

Network Paradigm



Important: Feedback delayed by a “Round Trip Time”

Dynamic fluid model (links)



- Flow through each link given by:

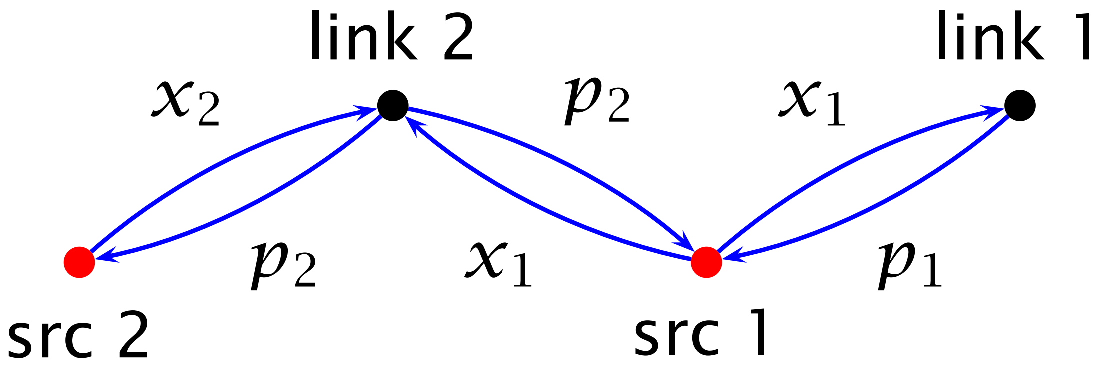
$$y_j(t) = \sum_{r:r \text{ uses } j} x_r(t - \vec{\tau}_{jr}).$$

  $\vec{\tau}_{jr}$ is the propagation delay from source r to link j.

- Links set their "prices" according to:

$$p_j(t) = G\big(\{y_j(\tau) : \tau \le t\}\big)(t)$$

Dynamic fluid model (sources)



- Aggregate price at a source given by:

$$q_r(t) = \sum_{j:j \text{ used by } r} p_j(t - \overleftarrow{\tau}_{jr})$$

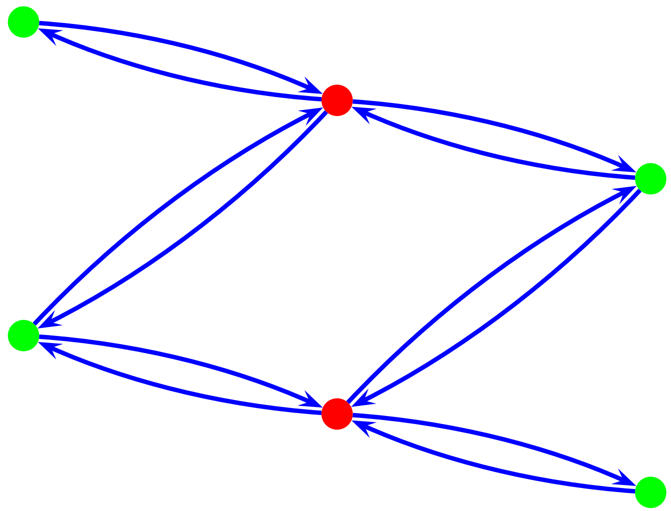
where $\overleftarrow{\tau}_{jr}$ is the delay from link j back to source r.

$$T_r = \overleftarrow{\tau}_{jr} + \overleftarrow{\tau}_{jr}(\text{RTT})$$

- The sources set their rates according to:

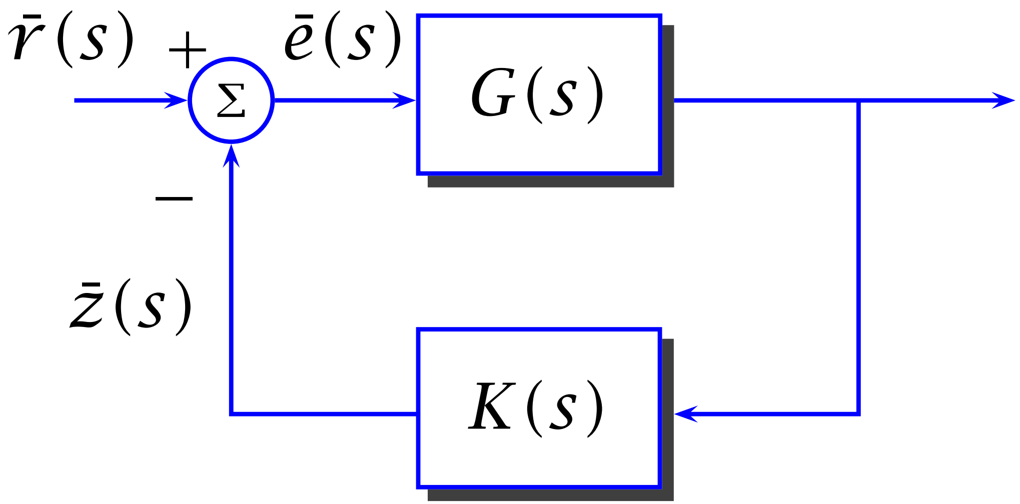
$$x_r(t) = F\big(\{q_r(\tau) : \tau \le t\}\big)(t)$$

Stability of Networks



- We shall consider networks where all connections are symmetric, and all cycles have even length.
- $\Longrightarrow$ <u>graph is bipartite.</u>
- This structure is clearly guaranteed by the protocol described.

Nyquist stability criterion



represents the simultaneous equations:

$$e(t) = r(t) - z(t)$$

and $\bar{z}(s) = K(s)G(s)\bar{e}(s)$ (i.e. $z(t) = k * g * e$)

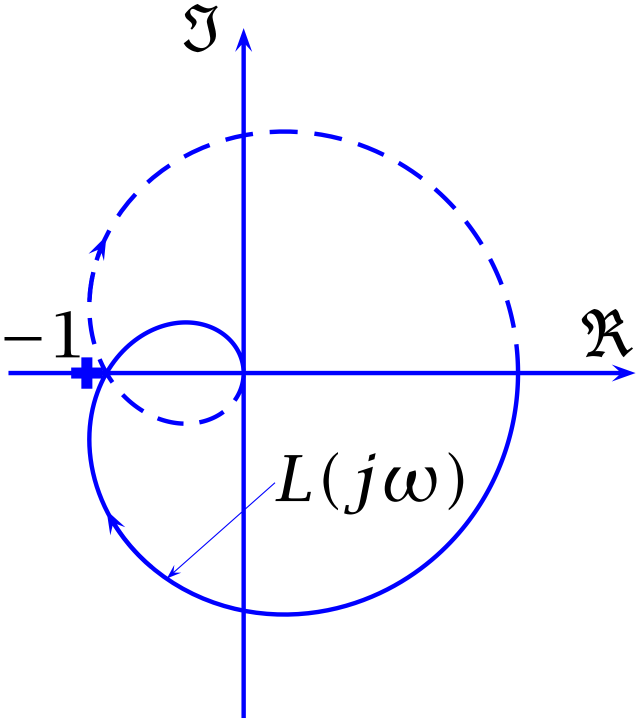
$$\implies \bar{y}(s) = \frac{K(s)G(s)}{1 + K(s)G(s)}\bar{r}(s)$$

We write

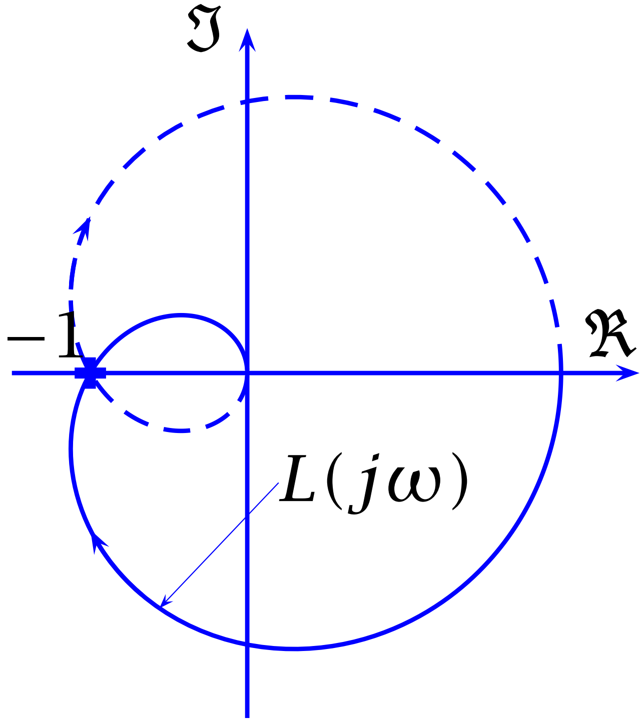
$L(s) = G(s)K(s)$ (the *Return Ratio*)

Nyquist stability criterion

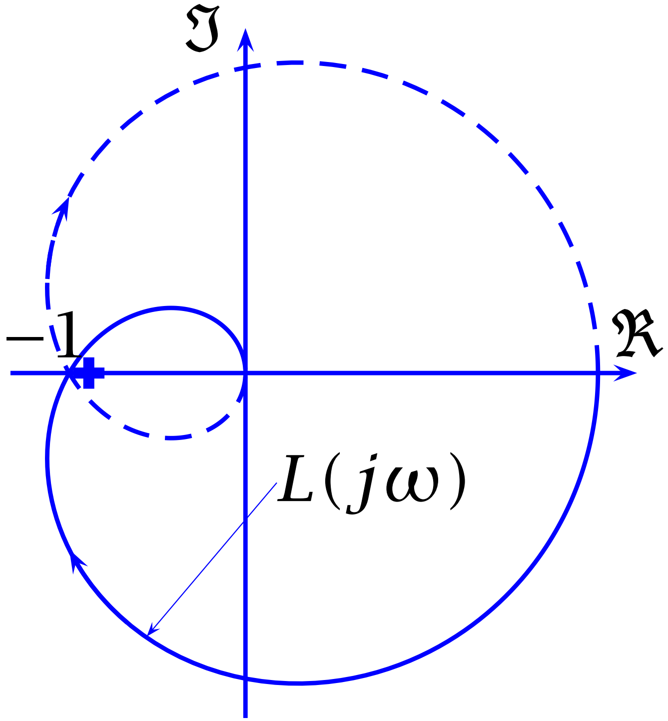
If $L(s)$ is stable, then



$\implies \dfrac{L(s)}{1+L(s)}$ is asymptotically stable

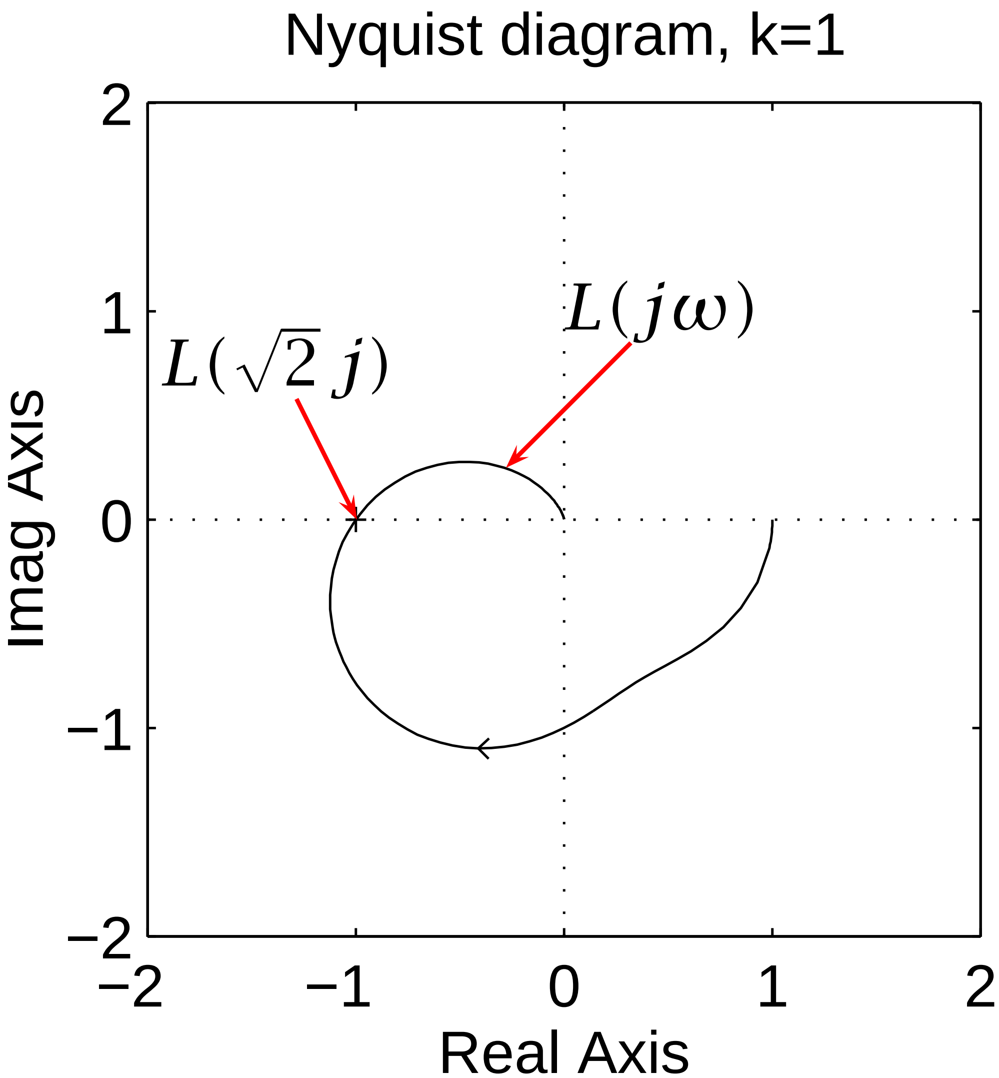


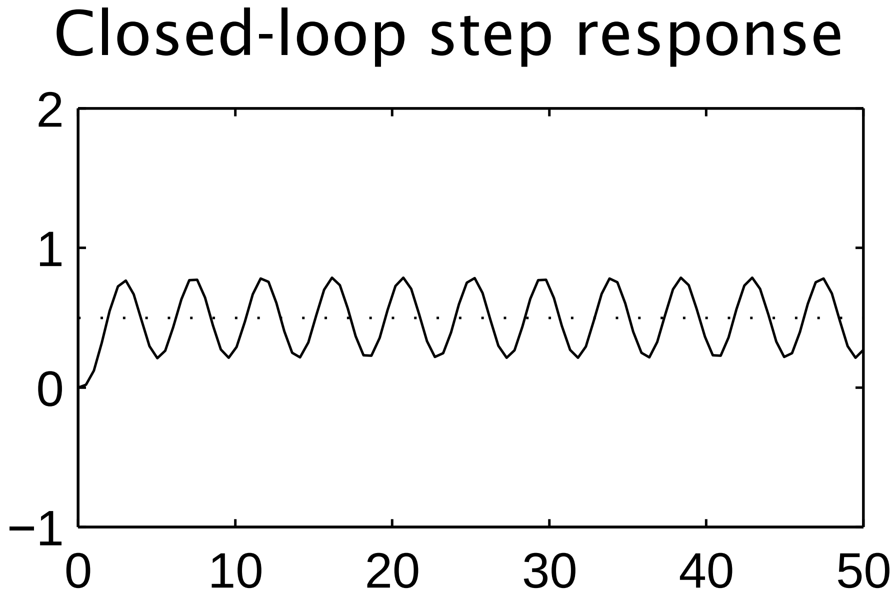
$\implies \dfrac{L(s)}{1+L(s)}$ is marginally stable



$\implies \dfrac{L(s)}{1+L(s)}$ is unstable

$k = 1$

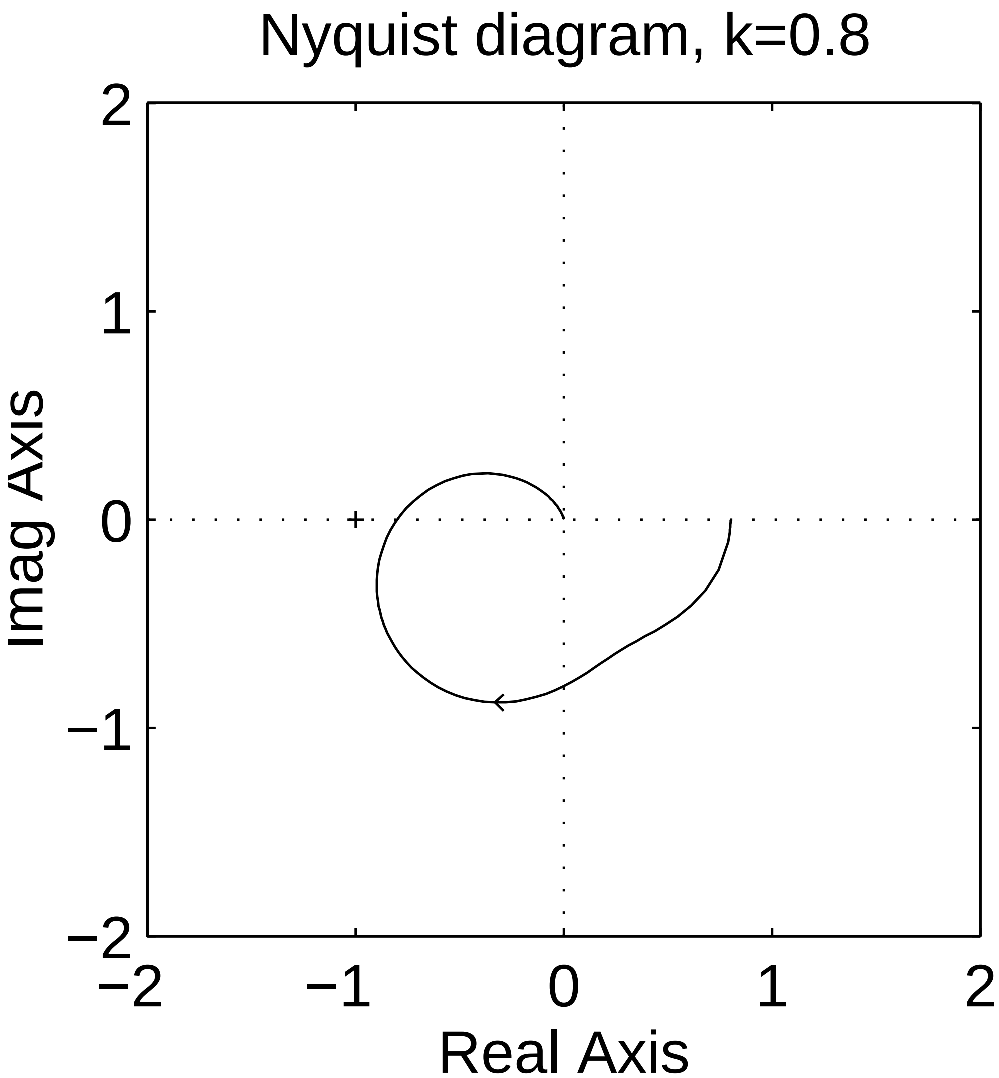


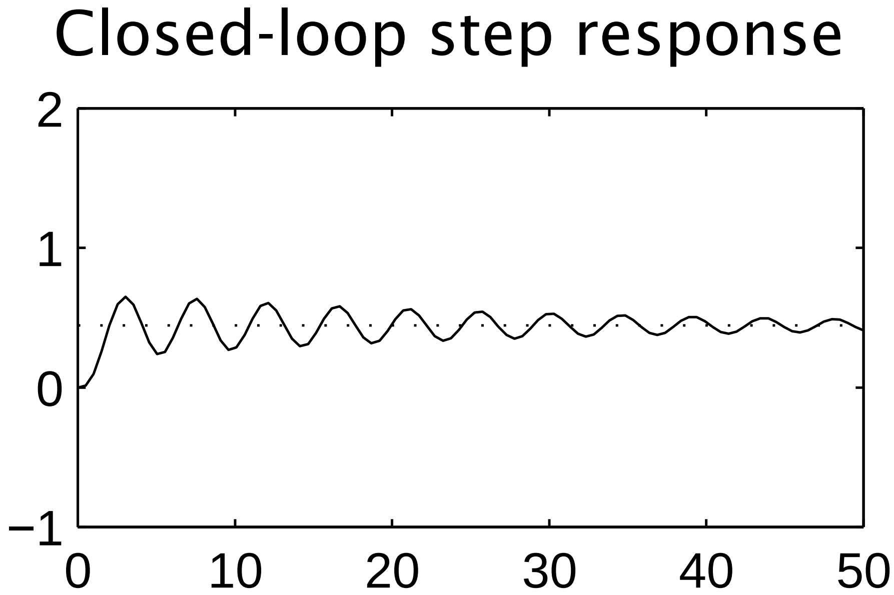


Closed-loop is marginally stable

Closed-loop poles are at the roots of $s^3 + s^2 + 2s + 2 = 0$, i.e.
$s = -1$ and $0 \pm 1.4142j$ (since $1 + L(1.4142j) = 0$)

$k = 0.8$



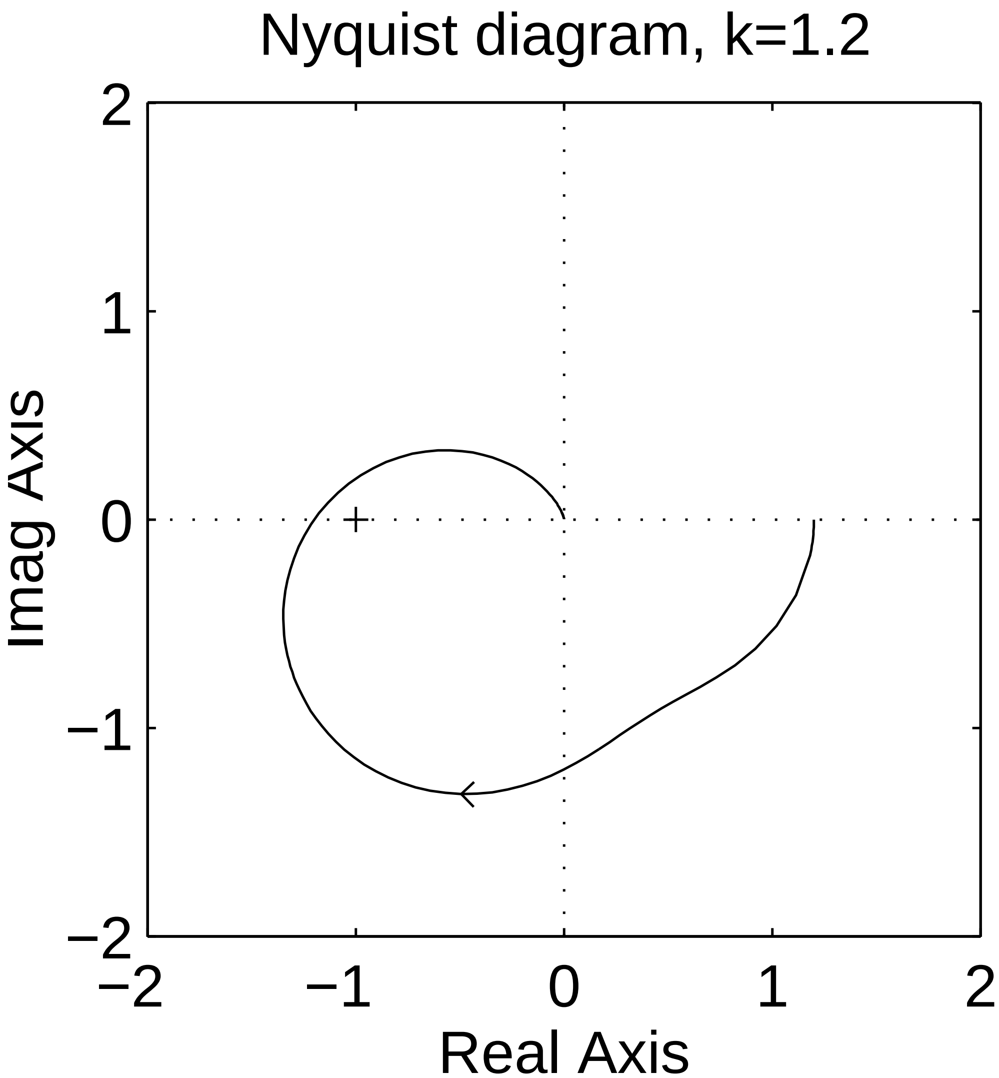


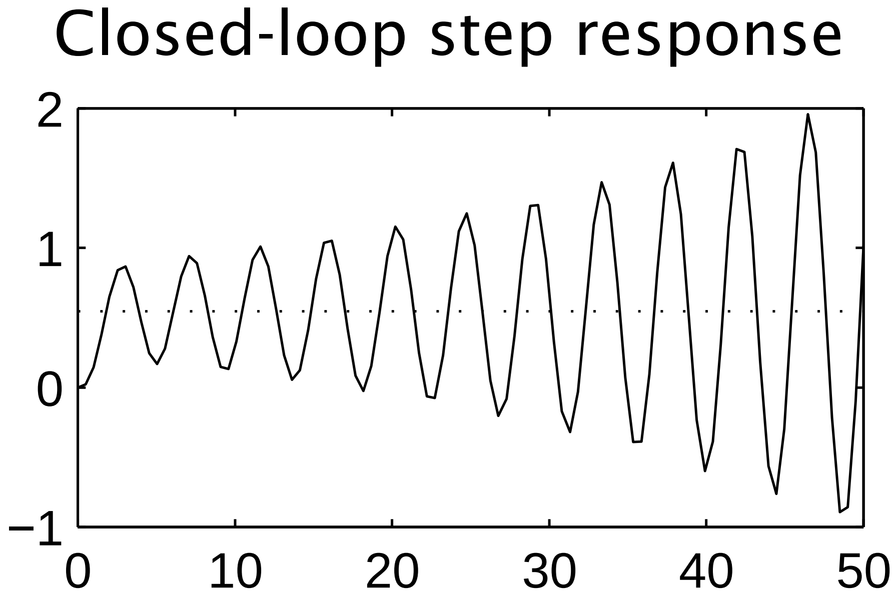
Closed-loop is asymptotically stable

Closed-loop poles are at the roots of $s^3 + s^2 + 2s + 1.8 = 0$, i.e.

$$s = \ -0.0349 + 1.3906j, \ \ -0.0349 - 1.3906j, \ \ -0.9302$$

$k = 1.2$



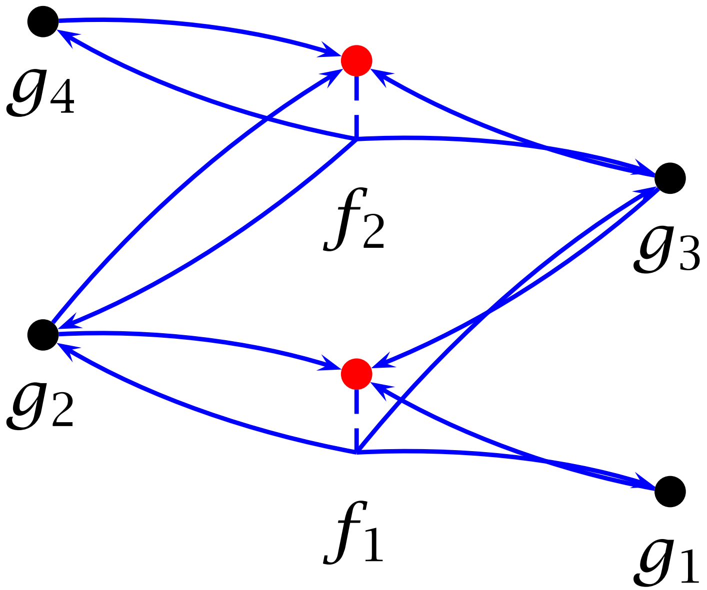


Closed-loop is unstable

Closed-loop poles are at the roots of $s^3 + s^2 + 2s + 2.2 = 0$, i.e.

$$s = \ 0.0319 + 1.4377j, \ \ 0.0319 - 1.4377j, \ \ -1.0639$$

Breaking the loop



$f_{1o}(s) = f_1(s) f_{1i}(s)$ etc

Open the loop at the "red" systems, to get

$$\begin{bmatrix} f_{1i} \\ f_{2i} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}}_{R} \begin{bmatrix} g_1 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 \\ 0 & 0 & g_3 & 0 \\ 0 & 0 & 0 & g_4 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}}_{R^T} \begin{bmatrix} f_{1o} \\ f_{2o} \end{bmatrix}$$

$R_{ij} = 1$ if f_i is connected to g_j

Multivariable Nyquist Criterion

"return ratio" $\begin{bmatrix} f_{1o} \\ f_{2o} \end{bmatrix} \to \begin{bmatrix} f_{1o} \\ f_{2o} \end{bmatrix}$ is given by

$$\underbrace{\begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}}_{F(s)} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} g_1 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 \\ 0 & 0 & g_3 & 0 \\ 0 & 0 & 0 & g_4 \end{bmatrix}}_{G(s)} \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}}_{R^T}$$

Assume each $f_i(s)$, $g_j(s)$ stable (i.e. analytic and bounded in $\Re(s) > 0$) and continuous on $\Re(s) = 0$ (i.e. approximable by rationals). Then

$$\text{Closed-loop stable} \iff \begin{array}{c} \{\lambda_i(F(j\omega)RG(j\omega)R^T) : \omega \in \Re\} \\ \text{do not encircle } +1 \end{array}$$

Aside: a graph viewpoint?

- Stability can also be related to the non-singularity of

$$I - \begin{bmatrix} f_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & f_{n_f} & 0 & 0 & 0 \\ 0 & 0 & 0 & g_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{n_g} \end{bmatrix} \underbrace{\begin{bmatrix} 0 & R \\ R^T & 0 \end{bmatrix}}_{A}$$

  (as $\det(\cdot) = \det(I - FRGR^T)$)

- $A = A^T$ is the adjacency matrix of an undirected bipartite graph.
- Some sort of weighted Laplacian (note: f_i, g_j complex)?
- Non-symmetric A (directed graphs)?
- Non-bipartite graphs?

Simple case (G real)

Assume all g's real and positive, and let $Q = RGR^T$.

LEMMA:
Let $Q = Q^* \geq 0$, $f_i \in \mathbb{C}$, then

$$\sigma\left(\operatorname{diag}(f_1, f_2, \ldots, f_n)Q\right) \subset \operatorname{Co}(0 \cup \{f_i\})\rho(Q)$$

PROOF:

$$\begin{aligned}\sigma\left(\operatorname{diag}(f_1, f_2, \ldots, f_n)Q\right) &= \sigma\left(Q^{1/2}\operatorname{diag}(f_1, f_2, \ldots, f_n)Q^{1/2}\right)\\ &\subset \{v^*Q^{1/2}\operatorname{diag}(f_1, f_2, \ldots, f_n)Q^{1/2}v : \|v\| = 1\}\\ &\subset \rho(Q)\{w^*\operatorname{diag}(f_1, f_2, \ldots, f_n)w : \|w\| \leq 1\}\end{aligned}$$

(since $\|Q^{1/2}v\| \leq \|Q^{1/2}\| = \sqrt{\rho(Q)}$)

$$= \rho(Q)\{\sum_i |w_i|^2 f_i : \sum |w_i^2| \leq 1\}$$

Bounding $\rho(Q)$

- Let $y = Rx$, $q = R^T p$ for real and positive x, p.
  e.g. $x = 1_{n_g}$, $p = 1_{n_f} \implies y$ is vector of degree of in-degrees of each f, (d_i), and q is vector of in-degrees of each g, (e_j).
- Then can rescale, and bound spectral radius in terms of row sums, to get
  $$\sigma(FRGR^T) \subset \mathrm{Co}\left(\left\{f_i \frac{y_i}{p_i} g_j \frac{q_j}{x_j} : i, j, R_{ij} \neq 0\right\}\right)$$
  e.g.
  $$\sigma(F(j\omega)RGR^T) \subset \mathrm{Co}\left(\left\{f_i(j\omega) d_i g_j e_j : i, j, R_{ij} \neq 0\right\}\right)$$
- Hence,
  $$1 \notin \mathrm{Co}\left(\left\{f_i(j\omega) d_i g_j e_j : i, j, R_{ij} \neq 0,\ \omega > 0\right\}\right) \implies \text{stability}$$
- **Note: this is a local condition**

Example I

For the source law

$$\dot{x}_r(t) = k_r\Big(w_r - \overbrace{x(t - T_r)q_r(t)}^{\text{price/sec}}\Big)$$

willingness to pay

and link law

$$p_j(t) = g_j(y_j(t))$$

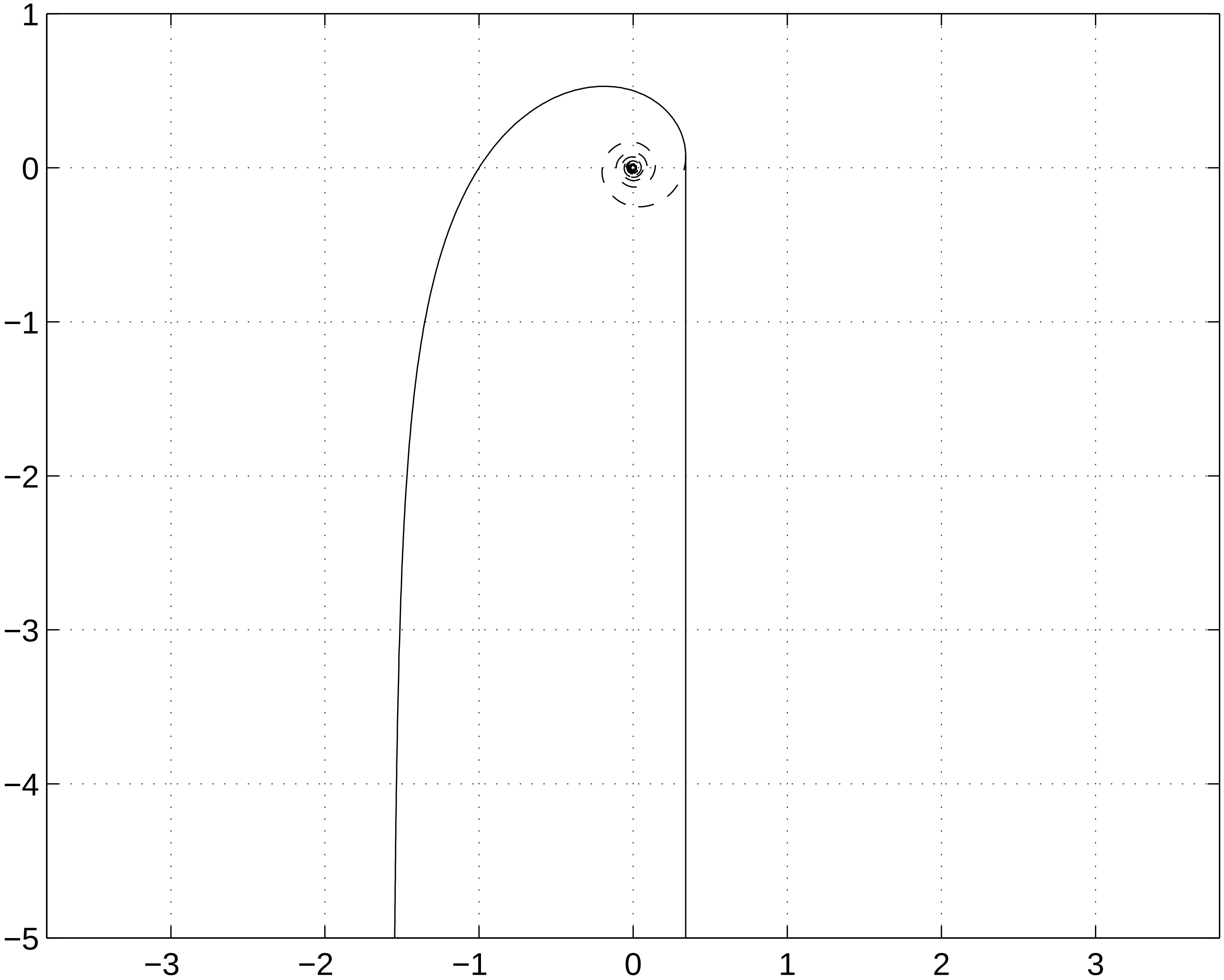
sufficient condition for (local) stability is

$$k_r T_r \sum_{j \text{ used by } r} \left(g_j + g_j' y_j\right) \le \frac{\pi}{2} \; \forall r$$

as conjectured by Johari & Tan [2000],
(proof: Massoulie [2000] (for 1 on RHS), Vinnicombe [2000]).
since all loops have a return ratio of the form

$$k\frac{e^{-sT}}{sT + \epsilon}$$

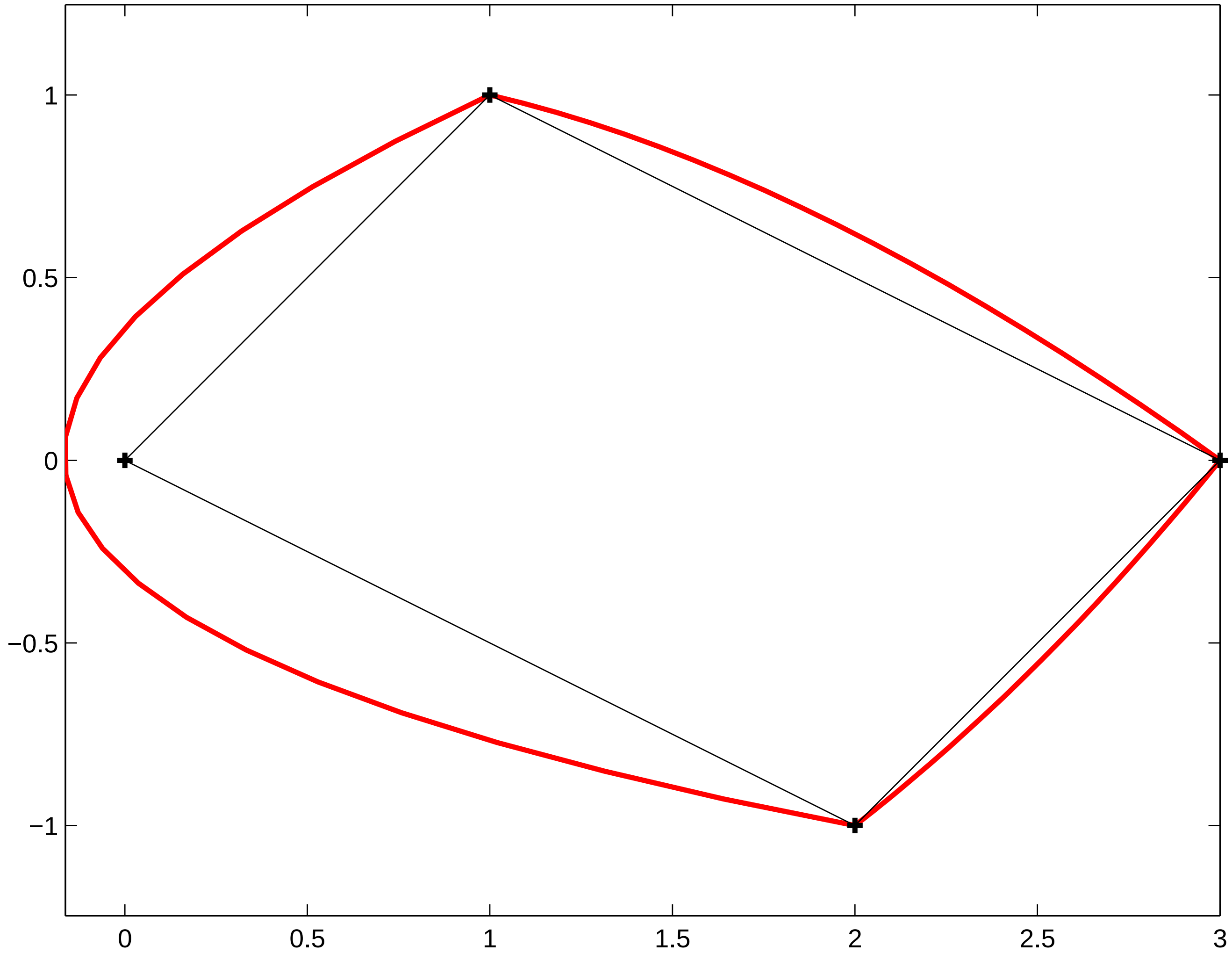
Example I



$\Longrightarrow$ OK

(More) General case: Dynamic G

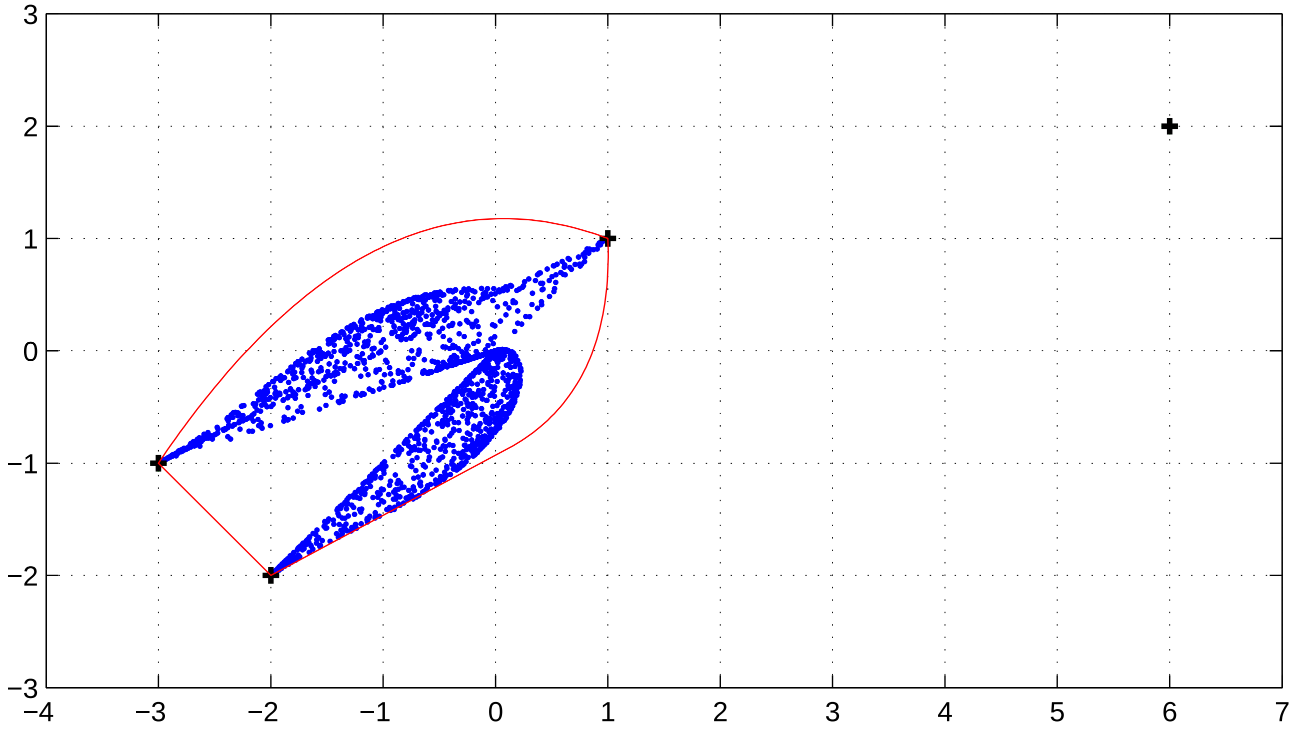
For $X \subset \mathbb{C}$ let $S(X) = \left(\mathrm{Co}\left(\sqrt{X}\right)\right)^2$, where $\sqrt{X} := \{y : y^2 \in X\}$.
Note that $S(X)$ is a little larger that $\mathrm{Co}(0 \cup X)$.



(More) General case: Dynamic G

LEMMA: (f_i, $g_j \in \mathbb{C}$)

$$\sigma(\mathrm{diag}\{f_i\} R\, \mathrm{diag}\{g_j\} R^T) \subset \rho(R^T R)\, \mathrm{Co}\{f_i S\{g_j : R_{ij} \neq 0\} : i\} \quad \text{etc}$$



$\sigma(GRFR^*)$

$f = (1 + j, -2 - 2j)$, $g = (1, -2 + j)$, $\|R\| \leq 1$, $R_{22} = 0$

A proposed scheme

- The sources each have a utility function $U_r(x_r)$ and set their rates according to:

$$T_r \dot{x}_r(t) = \frac{\kappa}{B} x_r(t) \left(1 - \frac{q_r(t)}{U_r'(x_r(t))}\right).$$

- Links set prices (marking probabilities) according to:

$$p_j(z_j) = \left(\frac{z_j}{C_j}\right)^B, \quad \beta_j \dot{z}_j + z_j = y_j$$

capacity

with $\beta_j < K \min_{r:r \text{ uses } j} T_r$.

A proposed scheme

- Equilibrium is unique maximum of

Kelly, Maulloo & Tan [1998]

$$\sum_r U_r(x_r) - \sum_j \int_0^{y_j} p_j(v)\, dv$$

(and, in the absence of delays, stable for all $k > 0$).

- Source law linearizes to

$$\delta\bar{x}_r = -\frac{\kappa}{B}\frac{x_r}{q_r}\frac{1}{sT_r + \alpha_r}\delta\bar{q}_r, \quad \alpha_r > 0$$

- Link law linearizes to

$$\delta\bar{p}_j = B\frac{p_j}{y_j}\frac{1}{\beta_j s + 1}\delta\bar{y}_j$$

Stability Proof

- Including delay:

$$f_r(s) = -\frac{\kappa}{B}\frac{x_r}{q_r}\frac{e^{-sT_r}}{sT_r + \alpha_r}, \quad g_j(s) = B\frac{p_j}{y_j}\frac{1}{\beta_j s + 1}$$

  Interconnection permitted only if $\beta_j \le KT_r$.

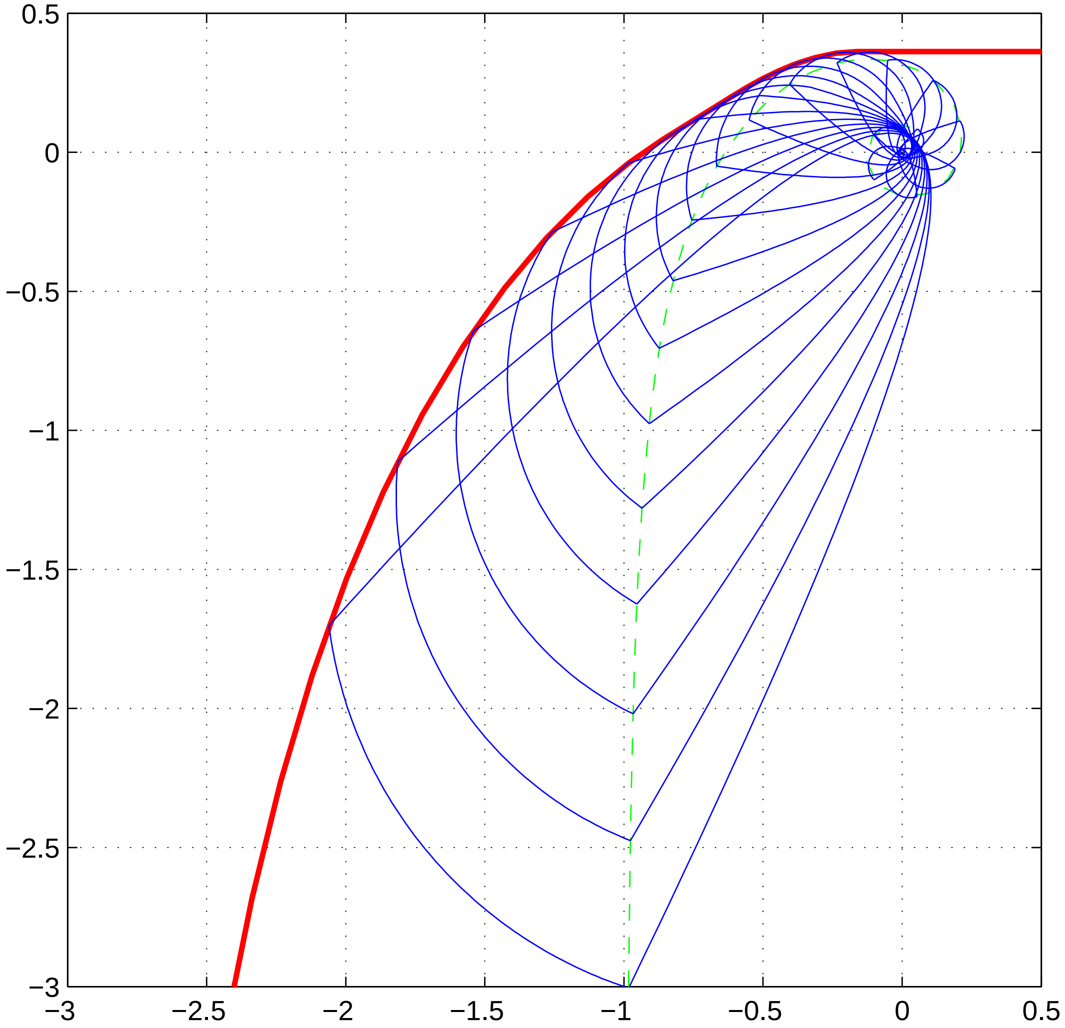
- Note that $y = Rx$, $q = R^T p$ at equilibrium

$$\Longrightarrow \; -\sigma(FR^T GR)(j\omega) \subset \kappa\,\mathrm{Co}\left\{f_r(j\omega) S\left\{g_j(j\omega) : R_{rj} \neq 0\right\}\right\}$$

$$\subset \kappa\,\mathrm{Co}\left\{\frac{e^{-jx}}{jx} S\left\{\frac{1}{jy+1} : 0 \le y \le Kx\right\} : x > 0\right\}$$

- Scheme is (locally) stable if $-1 \notin$ this set.

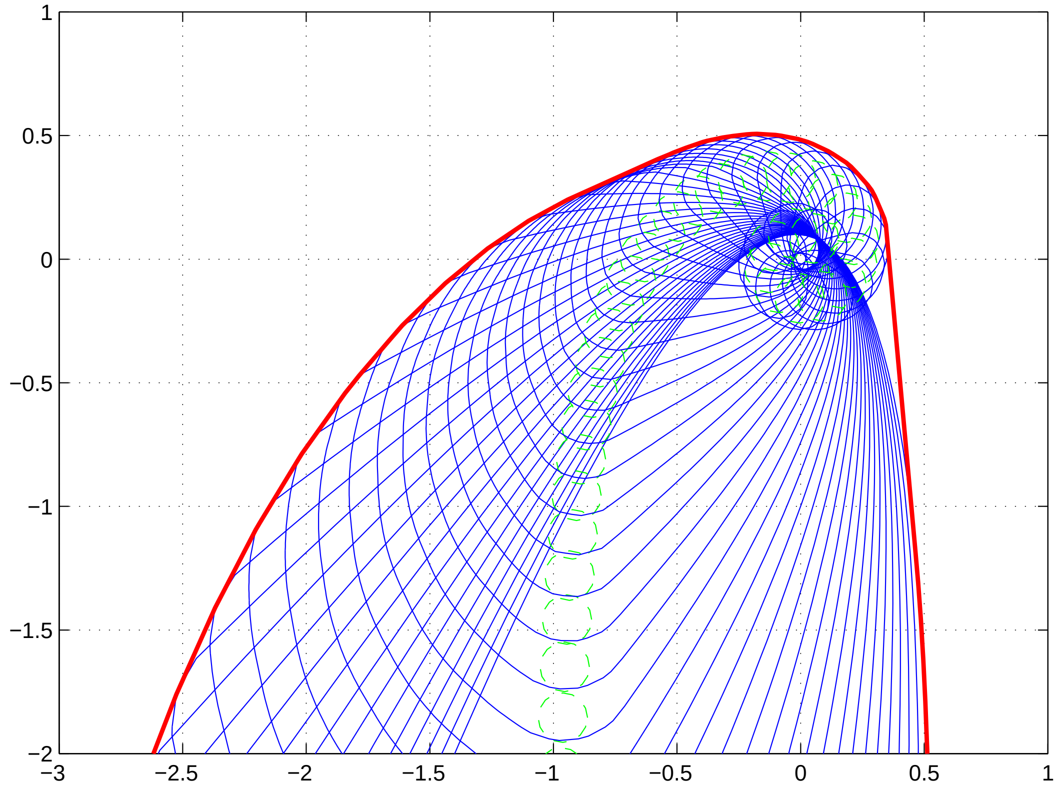
Example: $K = 2$



$\Longrightarrow$ closed-loop stability for $\kappa < 1$.

- Note: this provides a stability certificate for the *scheme*.

Robust stability: $K = 2$, $\delta = 0.1$



$\Longrightarrow$ if $\kappa < \frac{1}{1.3}$ then closed-loop stability is maintained for all source/delay and link dynamics within 0.1 of nominal.