A generating set for the class of series-parallel minimally reactive bicubic impedances

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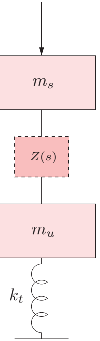
Cambridge University Engineering Department

Motivation (courtesy of Malcolm)

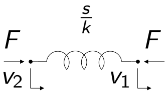
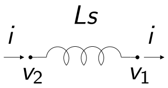
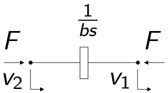
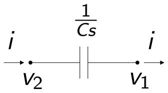
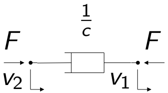
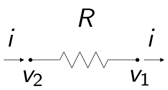
THE MOST GENERAL PASSIVE VEHICLE SUSPENSION



Replace the spring and damper with a general positive-real impedance $Z(s)$.

But is $Z(s)$ physically realisable?

Motivation (courtesy of Malcolm)

Mechanical		Electrical	
	$\frac{dF}{dt} = k(v_2 - v_1)$ $k > 0$ spring		$\frac{di}{dt} = \frac{1}{L}(v_2 - v_1)$ $L > 0$ inductor
	$F = b\frac{d(v_2 - v_1)}{dt}$ $b > 0$ inerter		$i = C\frac{d(v_2 - v_1)}{dt}$ $C > 0$ capacitor
	$F = c(v_2 - v_1)$ $c > 0$ damper		$i = \frac{1}{R}(v_2 - v_1)$ $R > 0$ resistor

Minimal realisations

MINIMAL: least possible number of elements
MINIMALLY REACTIVE: impedance degree = no. reactive elements

Minimal and minimally reactive		
Degree	Series-parallel	General
Bilinear		
Biquadratic		

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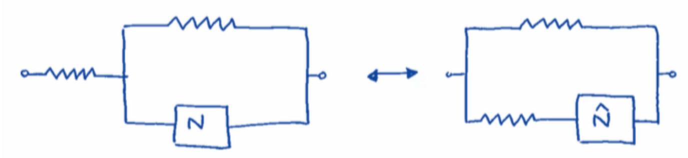
CONJECTURE: minimal and minimally reactive realisations of the impedance $\frac{p_n s^n + p_{n-1} s^{n-1} + \ldots + p_0}{q_n s^n + q_{n-1} s^{n-1} + \ldots + q_0}$ (degree n) contain (at most) n+1 resistors

A (very) useful result

The Zobel/Norton transformation:



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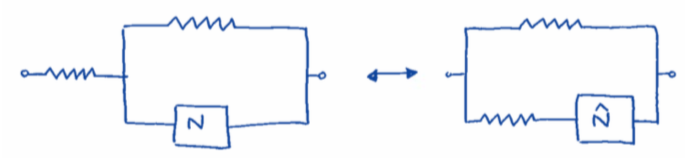
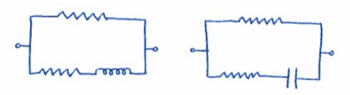
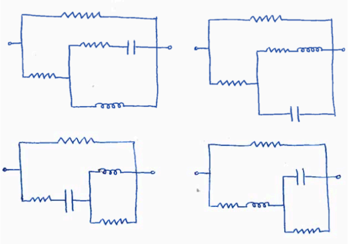
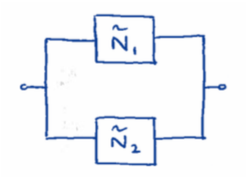






A (nonminimal) generating set



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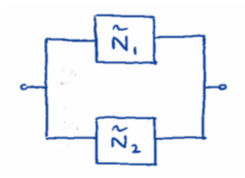

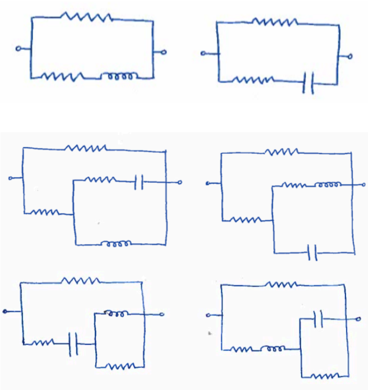

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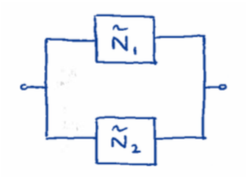
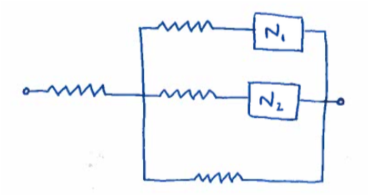
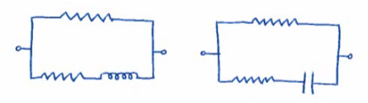
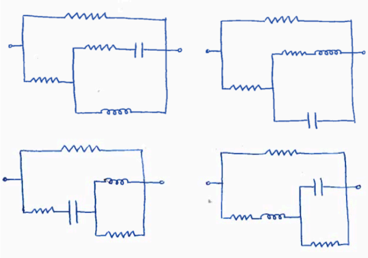
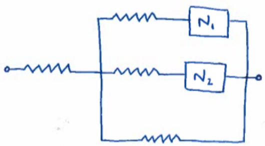








A minimal generating set



- 3 inductors or 3 capacitors: SOLVED
- Consider 2 inductors + 1 capacitor
- 1 inductor + 2 capacitor by *freq. inversion*
- other cases by duality

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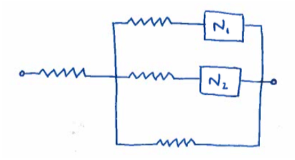

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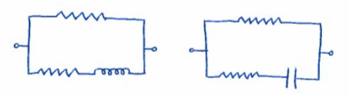

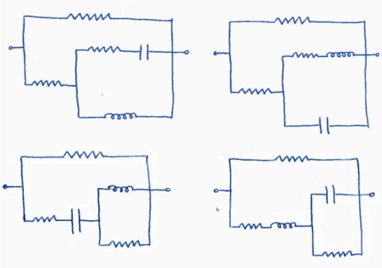

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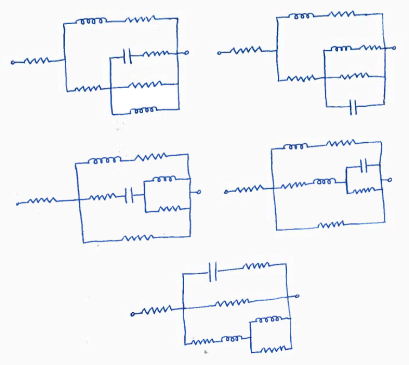

- 5 networks with 3 reactive elements and (at most) 5 resistors

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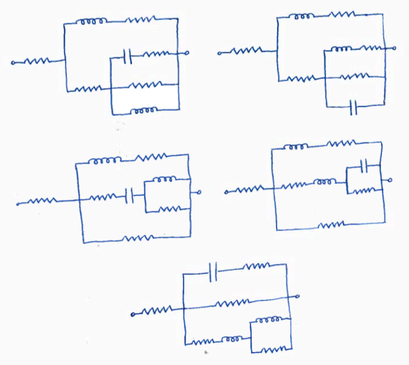
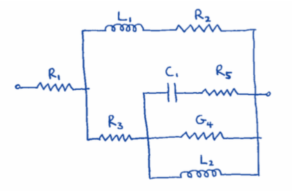


- 5 networks with 3 reactive elements and (at most) 5 resistors
- Set each of the series resistances/ parallel conductances to zero in turn:
- 25 networks with 3 reactive elements and 4 resistors (some duplicates)

A minimal generating set



$$Z(s) = \frac{a(s)}{b(s)} = \frac{a_3 s^3 + a_2 s^2 + a_1 s + a_0}{b_3 s^3 + b_2 s^2 + b_1 s + b_0}$$

$$R_1 = \frac{\hat{a}}{\hat{b}}, \quad R_2 = \frac{\gamma_1 x}{\hat{b}^2}, \quad R_3 = \frac{\gamma_1 c_0}{\hat{b} d_0}, \quad G_4 = \frac{\hat{b} d_0 \gamma_7}{\gamma_4^2 \gamma_1},$$

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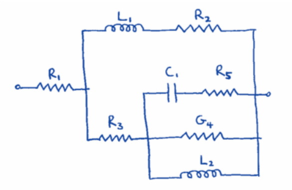

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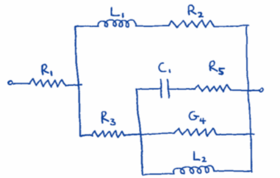
$$\hat{a}(x) = a(-x), \ \hat{b}(x) = b(-x), \ \gamma_1 = \hat{a}\frac{d\hat{b}}{dx} - \hat{b}\frac{d\hat{a}}{dx}$$

$$S = \begin{vmatrix} b_3 & b_2 & b_1 & b_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & b_3 & b_2 & b_1 & b_0 \\ 0 & 0 & a_3 & a_2 & a_1 & a_0 \end{vmatrix}, \ d_2 = \begin{vmatrix} 0 & 1 & 2x & x^2 & 0 \\ 0 & 0 & 1 & 2x & x^2 \\ b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & a_3 & a_2 & a_1 & a_0 \\ 0 & b_3 & b_2 & b_1 & b_0 \end{vmatrix}, \ d_1 = -\begin{vmatrix} 1 & 2x & x^2 & 0 & 0 \\ 0 & 0 & 1 & 2x & x^2 \\ b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & a_3 & a_2 & a_1 & a_0 \\ 0 & b_3 & b_2 & b_1 & b_0 \end{vmatrix}, \ d_0 = \begin{vmatrix} 1 & 2x & x^2 & 0 & 0 \\ 0 & 1 & 2x & x^2 & 0 \\ b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & a_3 & a_2 & a_1 & a_0 \\ 0 & b_3 & b_2 & b_1 & b_0 \end{vmatrix},$$

$$c_2 = \begin{vmatrix} 0 & 1 & x & 0 \\ 0 & 0 & 1 & x \\ a_3 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 \end{vmatrix}, \quad c_1 = -\begin{vmatrix} 1 & x & 0 & 0 \\ 0 & 0 & 1 & x \\ a_3 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 \end{vmatrix}, \quad c_0 = \begin{vmatrix} 1 & x & 0 & 0 \\ 0 & 1 & x & 0 \\ a_3 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 \end{vmatrix}, \quad \gamma_2 = \begin{vmatrix} d_2 & d_1 \\ c_2 & c_1 \end{vmatrix}, \quad \gamma_3 = \begin{vmatrix} d_2 & d_0 \\ c_2 & c_0 \end{vmatrix},$$

$$\gamma_4 = \begin{vmatrix} d_1 & d_0 \\ c_1 & c_0 \end{vmatrix}, \quad \gamma_5 = \begin{vmatrix} d_2 & d_1 & d_0 \\ c_2 & c_1 & c_0 \\ 0 & d_2 & d_1 \end{vmatrix}, \quad \gamma_6 = \begin{vmatrix} d_2 & d_1 & d_0 \\ c_2 & c_1 & c_0 \\ 0 & c_2 & c_1 \end{vmatrix}, \quad \gamma_7 = \begin{vmatrix} d_1 & d_0 & 0 \\ d_2 & d_1 & d_0 \\ c_2 & c_1 & c_0 \end{vmatrix}, \quad \gamma_8 = \begin{vmatrix} c_1 & c_0 & 0 \\ d_2 & d_1 & d_0 \\ c_2 & c_1 & c_0 \end{vmatrix}$$

A minimal generating set



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$\exists x \geq 0$ such that one of the following holds:

(a) $\hat{b} > 0,\ \gamma_7 \geq 0,\ \gamma_1 > 0,\ S > 0,\ \hat{a} \geq 0,\ \gamma_4 < 0,\ \gamma_3 \leq 0,\ c_0 \leq 0,\ d_0 < 0$;

(b) $\hat{b} > 0,\ \gamma_7 \leq 0,\ \gamma_1 > 0,\ S > 0,\ \hat{a} \geq 0,\ \gamma_4 < 0,\ \gamma_3 \leq 0,\ c_0 \geq 0,\ d_0 > 0$;

(c) $\hat{b} < 0,\ \gamma_7 \geq 0,\ \gamma_1 > 0,\ S > 0,\ \hat{a} \leq 0,\ \gamma_4 > 0,\ \gamma_3 \geq 0,\ c_0 \leq 0,\ d_0 > 0$;

(d) $\hat{b} < 0,\ \gamma_7 \leq 0,\ \gamma_1 > 0,\ S > 0,\ \hat{a} \leq 0,\ \gamma_4 > 0,\ \gamma_3 \geq 0,\ c_0 \geq 0,\ d_0 < 0$;

Show that if any one of conditions (a)–(d) hold, then x can be altered to make either $\hat{a} = 0$, $c_0 = 0$, $\gamma_7 = 0$, $\gamma_3 = 0$ or $x = 0$ (without violating any of the other inequalities)

Higher degree

Minimally reactive		
Degree	Series-parallel	General
Bilinear	2 resistors	2 resistors
Biquadratic	3 resistors	3 resistors
Bicubic	4 resistors	
Degree n		

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	Minimally reactive	
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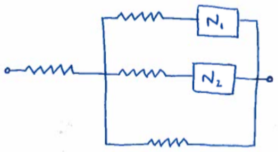

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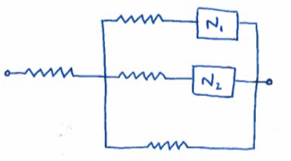

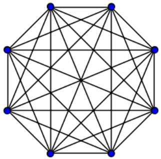

Note on mechanical design

- Considered problem of realising bicubic impedance with a series-parallel network containing 3 reactive elements (no limit on no. of resistors/dampers)
- Case of 3 inductors or 3 capacitors is known. Considered 2 inductors + 1 capacitor (case of 1 inductor + 2 capacitors is similar)
- Showed impedance is always realised by one of < 50 series-parallel networks, each containing 3 reactive elements and 4 resistors
- Realisability conditions of form: $\exists x \geq 0$ such that $f_1(x) = 0$, $f_2(x) \geq 0, \ldots, f_m(x) > 0, \ldots$
- Quantifier elimination can remove x, but not pretty!
- In practice: root finding algorithms give (approximate) realisations