Positive-real functions: regularity, essential regularity, and a structure-immittance format

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with

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Workshop on Networks and Control
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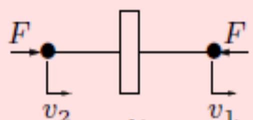
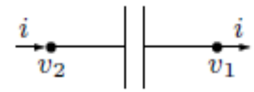
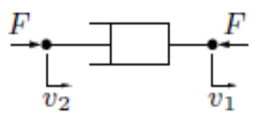
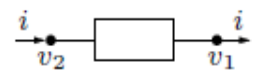
In honor of Malcolm C. Smith on the occasion of his sixtieth birthday

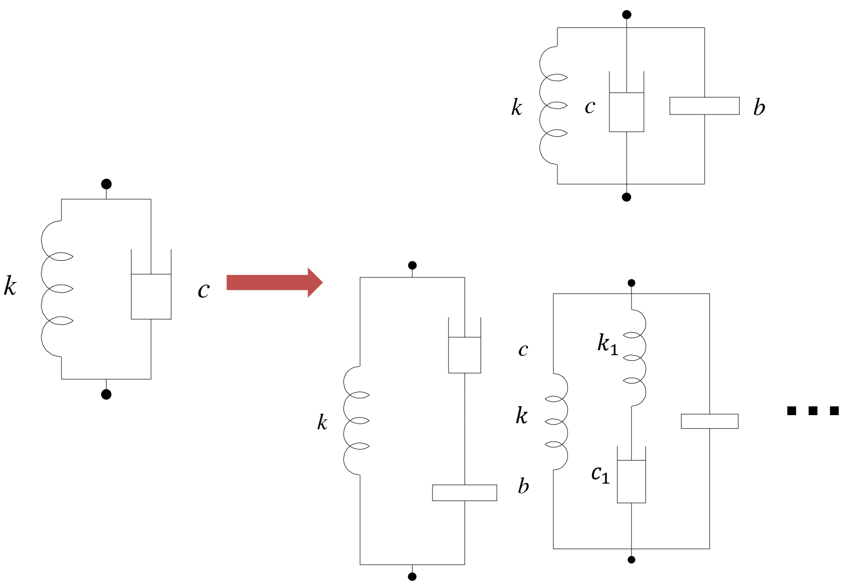
Inerter

Mechanical	Electrical
 $Y(s) = \frac{k}{s}$ $\frac{dF}{dt} = k(v_2 - v_1)$ spring	 $Y(s) = \frac{1}{Ls}$ $\frac{di}{dt} = \frac{1}{L}(v_2 - v_1)$ inductor
 $Y(s) = bs$ $F = b\frac{d(v_2 - v_1)}{dt}$ inerter	 $Y(s) = Cs$ $i = C\frac{d(v_2 - v_1)}{dt}$ capacitor
 $Y(s) = c$ $F = c(v_2 - v_1)$ damper	 $Y(s) = \frac{1}{R}$ $i = \frac{1}{R}(v_2 - v_1)$ resistor

M.C. Smith, Synthesis of Mechanical Networks: The Inerter, IEEE Trans. on Automat. Contr., 2002.

Design of passive vibration absorbers

Structure-based



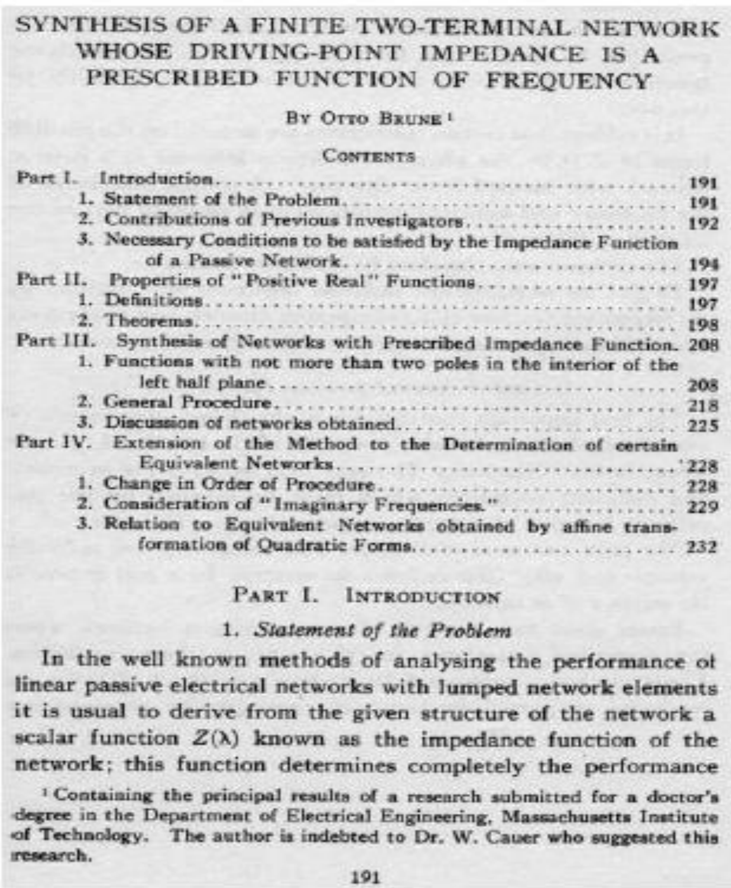
Immittance-based

Step 1: Identify *positive-real* functions, eg:

$$Z_1(s) = \frac{As^2 + Bs + C}{Ds^2 + Es + F}$$

Step 2: Synthesis the functions by network structures

Positive-real functions



SYNTHESIS OF A FINITE TWO-TERMINAL NETWORK WHOSE DRIVING-POINT IMPEDANCE IS A PRESCRIBED FUNCTION OF FREQUENCY

By Otto Brune [1]

Contents

Part I. Introduction

1. *Statement of the Problem*

In the well known methods of analysing the performance of linear passive electrical networks with lumped network elements it is usual to derive from the given structure of the network a scalar function $Z(\lambda)$ known as the impedance function of the network; this function determines completely the performance

[1] Containing the principal results of a research submitted for a doctor's degree in the Department of Electrical Engineering, Massachusetts Institute of Technology. The author is indebted to Dr. W. Cauer who suggested this research.

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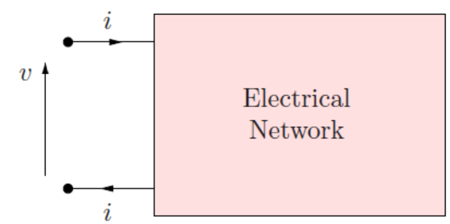


O. Brune showed that any positive-real function could be realised as the impedance or admittance of a network Comprising resistors, capacitors, inductors and *transformers*. (1931)

Positive-real functions

Letters to the Editor

The Ordering Reaction in Co-Pt Alloys†

J. B. Newkirk,* A. H. Geisler,* and D. L. Martin**

March 2, 1949

An ordering reaction can occur in binary alloys of cobalt and platinum whose composition is near 50 atomic percent. The maximum temperature of order is about 825°C for the 50 atomic percent alloy and lower for those off this composition. No other reaction occurs below the maximum temperature of order. The unit cell is face-centered cubic above this temperature and ordered face-centered tetragonal below. In some respects this reaction has for its prototype the one found in the CuAu alloy.

Evidence is given which indicates that at certain temperatures and compositions the ordering reaction proceeds through a two-phase stage that by holding within a measurable temperature range discreet regions of order and of disorder may be caused to exist together in equilibrium.

On the basis of preliminary evidence, it appears that at an early stage of the ordering process, coherency between regions of order and of disorder may exist. Lattice straining, induced as a consequence of this, may account for the unusual physical properties which develop during the course of the ordering process. Thus, the process may resemble that of solid solution percipitation (aging) in its effect on certain physical properties.

Further study of the alloy is in progress.

† This letter is part of the Special Section on the Pittsburgh X-Ray and Electron Diffraction Conference which appears on pages 725–746 of this issue.

* Graduate Student, Department of Metallurgical Engineering, Carnegie Institute of Technology, Pittsburgh 13, Pennsylvania.

** Research Associate, Research Laboratory, General Electric Company, Schenectady, New York.

Impedance Synthesis without Use of Transformers

R. Bott and R. J. Duffin

Department of Mathematics, Carnegie Institute of Technology, Pittsburgh, Pennsylvania

December 13, 1948

Let $Z(s)$ be termed a B(rune) function if: (1) it is a rational function; (2) it is real for real s; and (3) the real part of Z is positive when the real part of s is positive. In his significant thesis, O. Brune[1] shows that the driving-point impedance of a passive network is a B function of the complex frequency variable s. Conversely, he shows that any B function can be realized by some passive network and gives rules for constructing such a network. In this synthesis he is forced to employ transformers with perfect coupling. It is recognized by Brune and others that the introduction of perfect transformers is objectionable from an engineering point of view. Prior to Brune, R. M. Foster[2] had shown how to synthesize the driving-point impedance of networks containing no resistors by simple series-parallel combinations of inductors and capacitors. This note gives a similar synthesis of an arbitrary impedance by series-parallel combinations of inductors, resistors, and capacitors.

A B function can be expressed as the ratio of two polynomials without common factor. Let the "rank" be the sum of the degrees of these polynomials. Obviously any B function of rank O can be synthesized. Suppose, then, it has been shown that all B functions of rank lower than n can be synthesized, and let $Z(s)$ be a B function of rank n. Brune gives four rules for carrying out a mathematical induction to a B function of lower rank:

(a) If Z has a pole on the imaginary axis, then Z can be synthesized by a parallel resonant element in series with an impedance Z' of lower rank; $Z=1/(cs+1/ls)+Z'$ where $l^{-1}, c\geq 0$.

(b) If Z has a zero on the imaginary axis, then Z can be synthesized by a series resonant element in parallel with an impedance Z' of lower rank; $1/Z=1/(ls+1/cs)+1/Z'$ where $l, c^{-1}\geq 0$.

(c) If the real part of Z does not vanish on the imaginary axis, $Z=r+Z_0$ where r is a positive constant (to be interpreted as resistance) and Z_0 is a B function of no greater rank than Z.

Brune's fourth rule, (d), which employs the perfect transformer, we replace by the following procedure:

(d′) If none of these reductions are possible, there exists a $w>0$ such that $Z(iw)$ is purely imaginary. First assume that $Z(iw)=iwL$ with $L>0$. We now make use of a key theorem discovered by P. I. Richards.[3] Let k be a positive number, and let

$$R(s)=\frac{kZ(s)-sZ(k)}{kZ(k)-sZ(s)}. \tag{1}$$

Then $R(s)$ is a B function whose rank does not exceed the rank of $Z(s)$. Richards states this theorem for $k=1$; the above form is an obvious modification, because $Z(ks)$ is also a B function. Let k satisfy the equation $L=Z(k)/k$. This is clearly always possible, because the function on the right varies from ∞ to 0 as k varies from 0 to ∞. With this choice of k, clearly $R(iw)=0$. Solving (1) for Z gives

$$\begin{aligned} Z(s)&=(1/Z(k)R(s)+s/kZ(k))^{-1}+(k/Z(k)s+R(s)/Z(k))^{-1} \\ &=(1/Z_1(s)+Cs)^{-1}+(1/Ls+1/Z_2)^{-1}. \end{aligned} \tag{2}$$

Here $Z_1(s)=kLR(s)$, $Z_2(s)=kL/R(s)$, $C=1/k^2L$. Since Z_1 is a B function with a zero on the imaginary axis, it can be synthesized. Likewise, Z_2 is a B function with a pole on the imaginary axis and can be synthesized. $Z(s)$ is therefore synthesized by two networks in series. The first network consists of the impedance Z_1 in parallel with a capacitor C, and the second network consists of the impedance Z_2 in parallel with an inductor L. In the case that $Z(iw)=-iwL$, similar considerations applied to the function $1/Z$ show that Z is synthesized by two networks in parallel. The synthesized network finally resulting has the configuration of a tree whose branches are ladder networks.

Richards[4] has sought necessary and sufficient conditions for the driving-point impedance of resistor-transmission-line circuits by means of an ingenious transformation of the Brune theory. The perfect transformers, which are again found to be objectionable, may be dispensed with by the above procedure.

[1] O. Brune, J. Math. and Phys. 10, 191–236 (1931).
[2] R. M. Foster, Bell Syst. Tech. J. 3, 259 (1924).
[3] P. I. Richards, Duke Math. J. 14, 777–786 (1947).
[4] P. I. Richards, Proc. I.R.E. 36, 217–220 (1948).

An Improvement in the Shadow-Cast Replica Technique

S. J. Singer* and R. F. Petzold

*Gates and Crellin Laboratories of Chemistry, California Institute of Technology, Pasadena, California***

May 6, 1949

Williams and Backus[1] have recently discussed in full detail the shadow-cast replica technique of electron microscopy. In the course of an investigation of the reactions of proteins in thin films,[2,3] we have performed some experiments with this technique embodying an improvement which we wish to report.

In this technique, a thin film of a metal such as chromium or uranium is deposited at an oblique angle onto the surface to be examined, by evaporation in a high vacuum. One method of removing this replica from the surface involves first, the deposition of a thin film (about 1000A) of parlodion on top of the metal film,

Bott and Duffin showed that the transformers were unnecessary in the synthesis of positive-real functions. (1949)

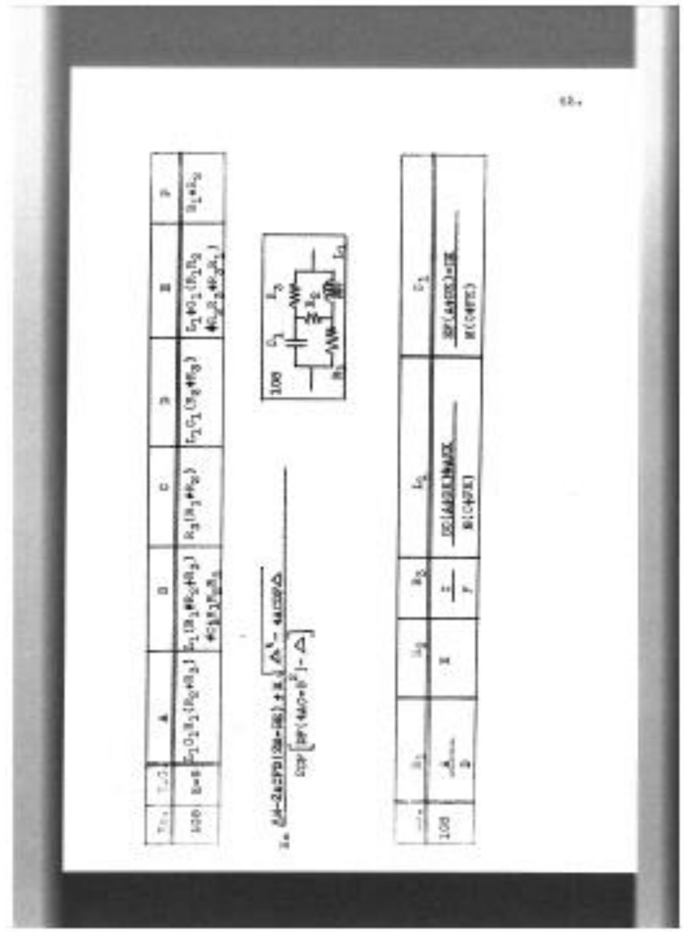
Question: what is the minimal realisation of the positive-real functions?

Ladenheim’s master thesis (1948)

Ladenheim considered all networks with at most five elements and at most two reactive elements, and reduced the whole set to *108* networks (1948).

Questions not answered:

- What is the totality of biquadratics which may be realised?
- How many different networks are needed?

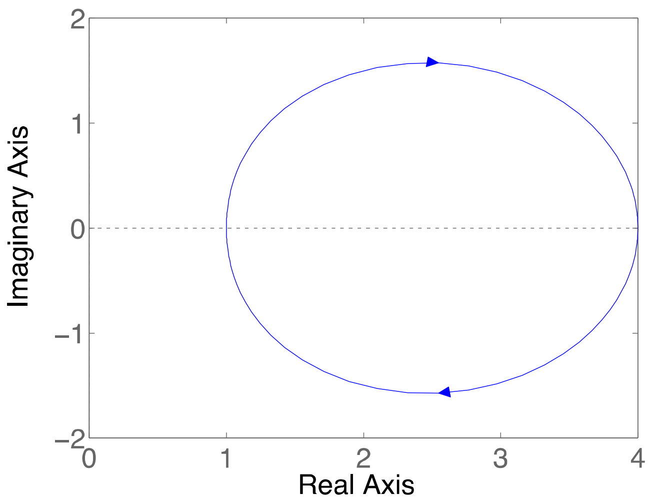


Regularity and Essential Regularity

Regular positive-real functions

A positive-real function $Z(s)$ is defined to be *regular* if the smallest value of $\mathrm{Re}\left(Z(j\omega)\right)$ or $\mathrm{Re}\left(Z^{-1}(j\omega)\right)$ occurs at $\omega = 0$ or $\omega = \infty$.

Example $Z_1(s) = \left(\frac{s+2}{s+1}\right)^2$

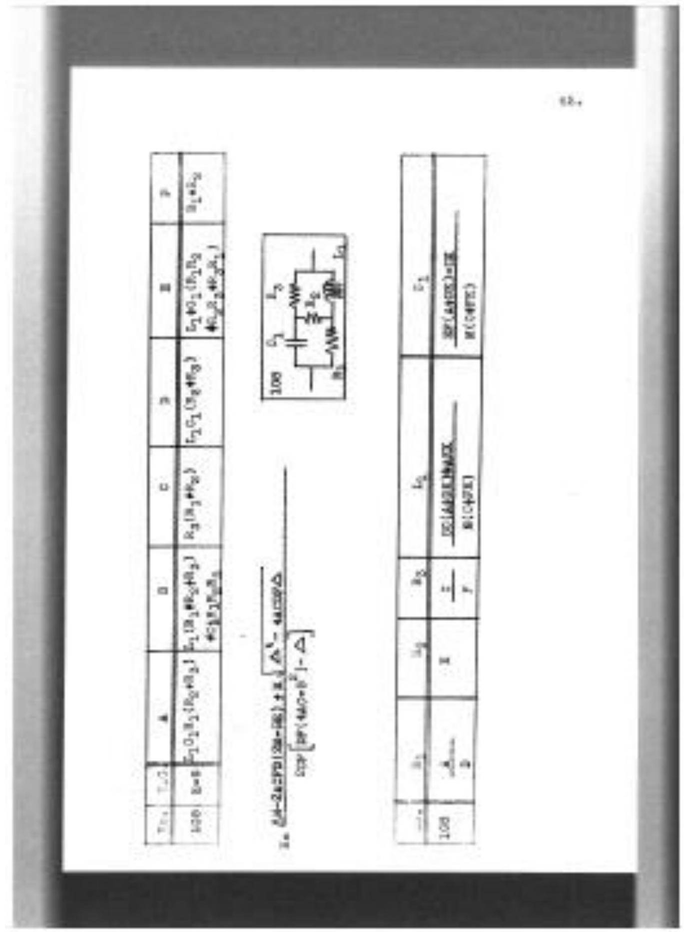


Smallest value of $\mathrm{Re}\left(Z_1(j\omega)\right)$ occurs at $\omega = \infty$, hence $Z_1(s)$ is regular.

J.Z. Jiang and M.C. Smith, Regular positive-real functions and five-element network synthesis for electrical and mechanical networks, IEEE Trans. On Automat. Contr. 2011.

Questions not answered:

- What is the totality of biquadratics which may be realised?
- How many different networks are needed?



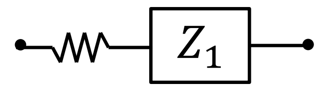
Answers:

- All 103 series parallel networks are regular.
- Six networks can cover all regular biquadratics.
- All bridge networks are regular apart from two networks.

Foster preamble for a positive-real $Z(s)$

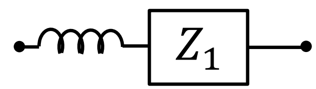
Subtract minimum real part

$$Z = R + Z_1$$



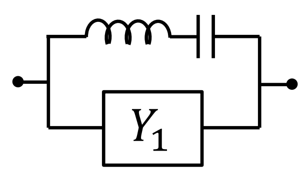
Removal of poles/zeros on $\{0\} \cup \{\infty\}$

$$Z = Ls + Z_1$$



Removal of poles/zeros on jR

$$Z = \left(\frac{As}{s^2+\omega^2} + Y_1\right)^{-1}$$



Regular positive-real functions $Z(s)$

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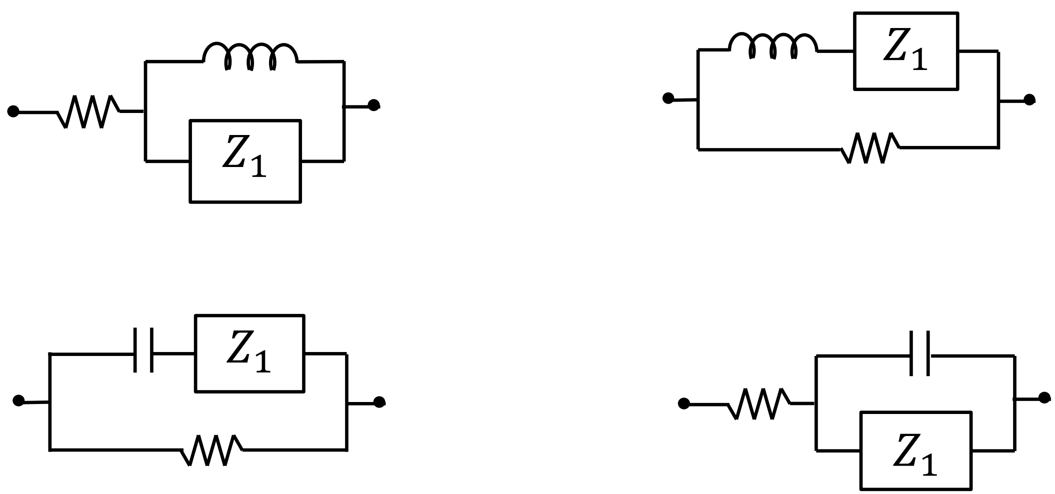
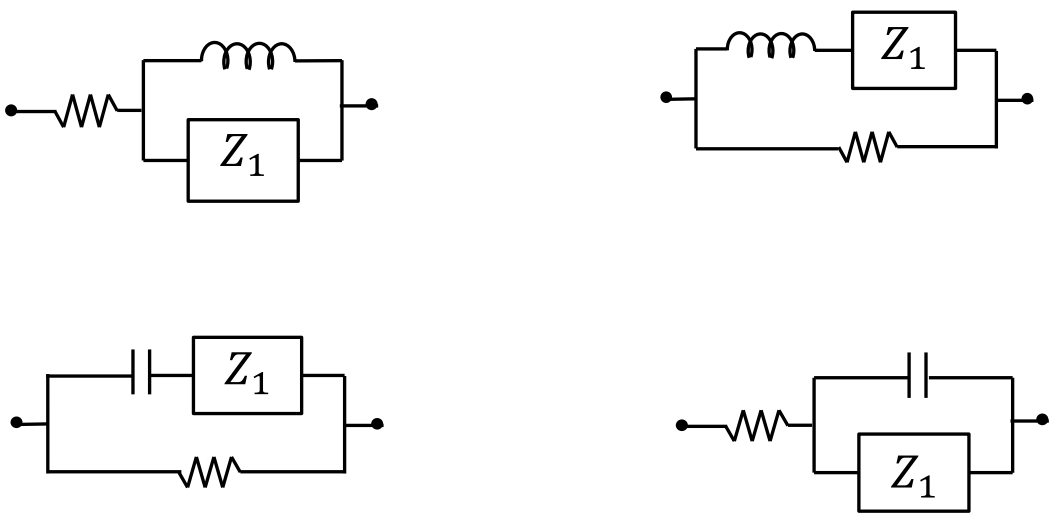


where Z_1 is a positive-real function with one McMillan degree less than Z.

Essential-regular positive-real functions $Z(s)$

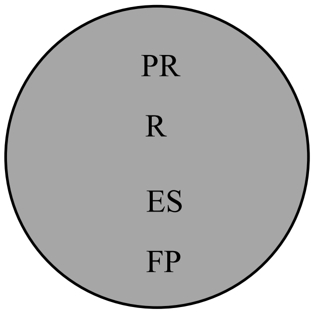


where Z_1 is a regular positive-real function with one McMillan degree less than Z, and this procedure can be executed till the McMillan degree equals 0.

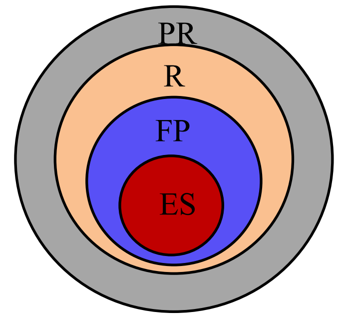
S.Y. Zhang, J. Z. Jiang, H. L. Wang and S.A. Neild, Synthesis of essential-regular bicubic impedances, International Journal of circuit theory and applications, 2017.

Relations amongst regular (R), essential-regular (ES), foster preamble realisable (FP) and positive-real (PR) functions with arbitrary McMillian degree N

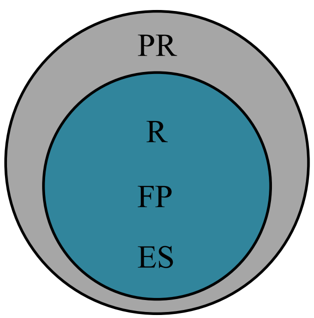
$N = 1$



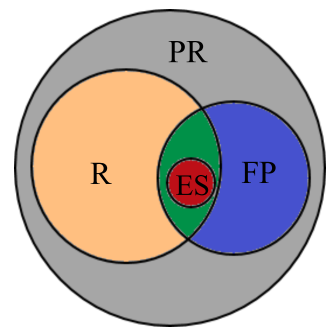
$N = 3$



$N = 2$

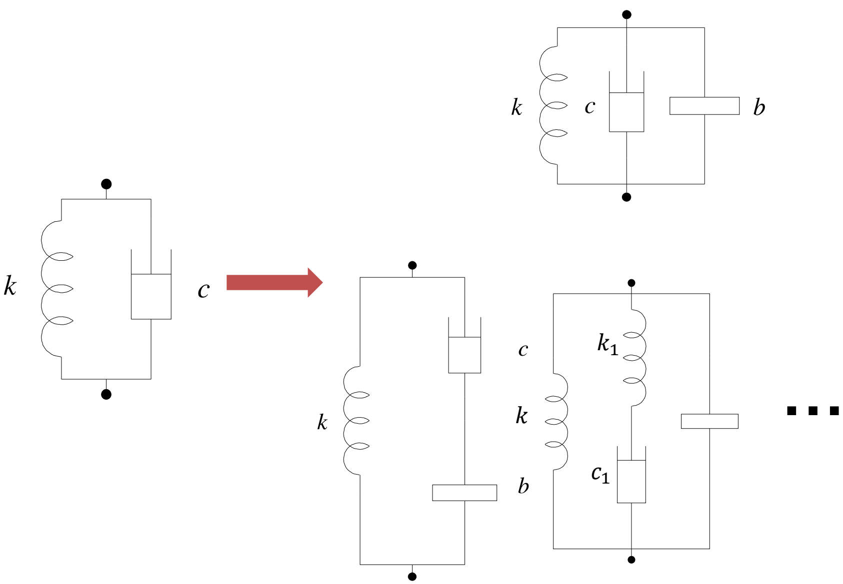


$N \geq 4$



Design of passive vibration absorbers

Structure-based



Immittance-based

Step 1: Identify positive-real functions, eg:

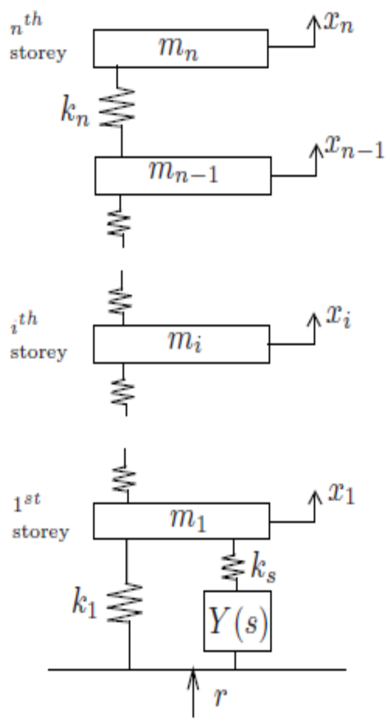
$$Z_1(s) = \frac{As^2 + Bs + C}{Ds^2 + Es + F}$$

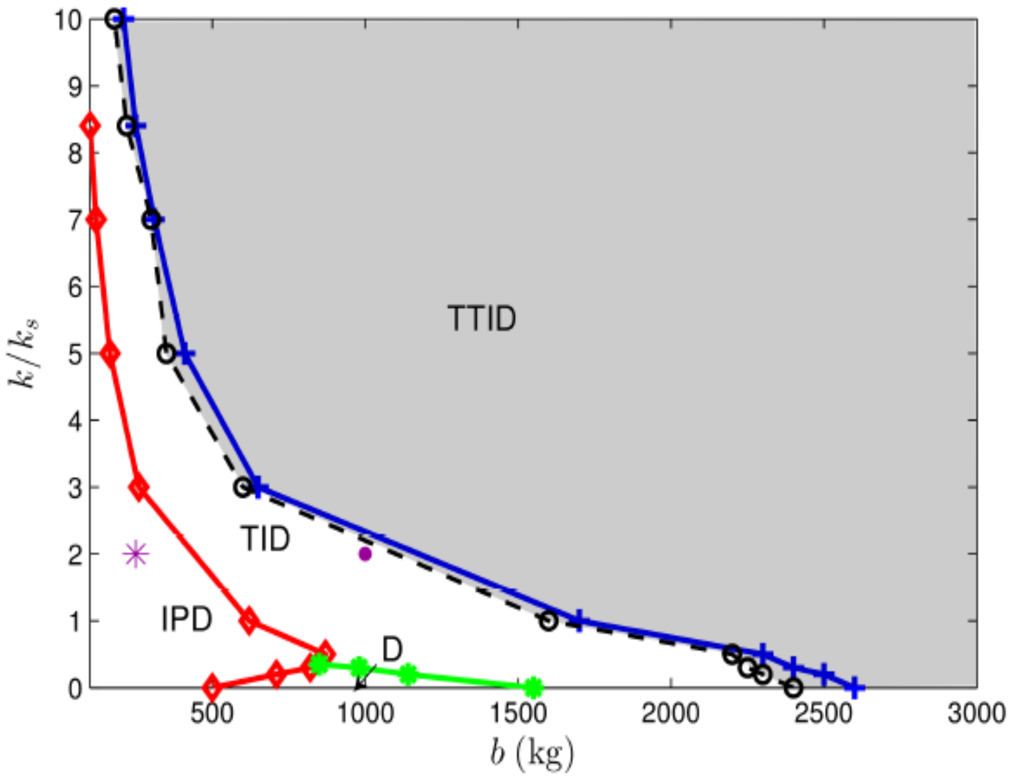
Step 2: Synthesis the functions by network structures

Comparison of the two approaches

	Structure-based	Immittance-based
Ability to cover a wide range of layout possibilities		

Element values are important



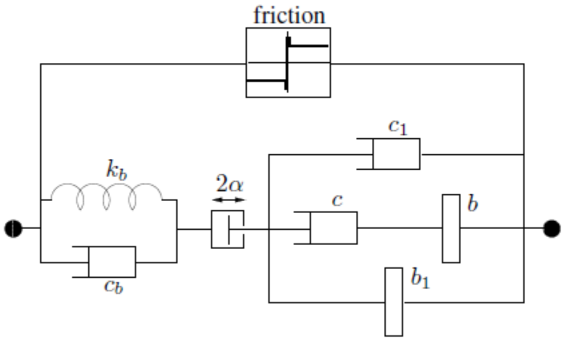


S.Y. Zhang, J.Z. Jiang and S.A. Neild, Optimal Configurations for a Linear Vibration Suppression Device in a Multi-Storey Building, Structural Control and Health Monitoring, 2016.

Comparison of the two approaches

	Structure-based	Immittance-based
Ability to cover a wide range of layout possibilities		
Ability to fix and/or constraint element values and network complexities		

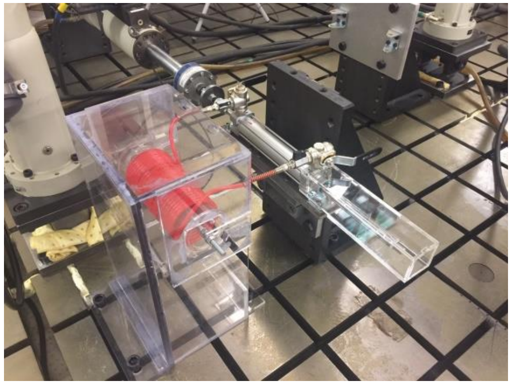
Topological information is important

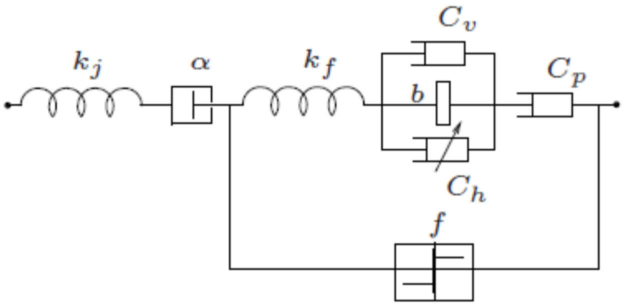


A prototype steering compensator device, University of Cambridge

S. Evangelou, D.J.N. Limebeer, R.S. Sharp and M.C. Smith, Steering compensation for high-performance motorcycles, Transactions of ASME, Journal of Applied Mechanics, 2017.

Topological information is important





A prototype fluid inerter built and tested at the University of Bristol

Comparison of the two approaches

	Structure-based	Immittance-based
Ability to cover a wide range of layout possibilities		
Ability to fix and/or constraint element values and network complexities		
Ability to fix and/or constraint network topologies		

Comparison of the two approaches

	Structure-based	Immittance-based
Ability to cover a wide range of layout possibilities		
Ability to fix and/or constraint element values and network complexities		
Ability to fix and/or constraint network topologies		

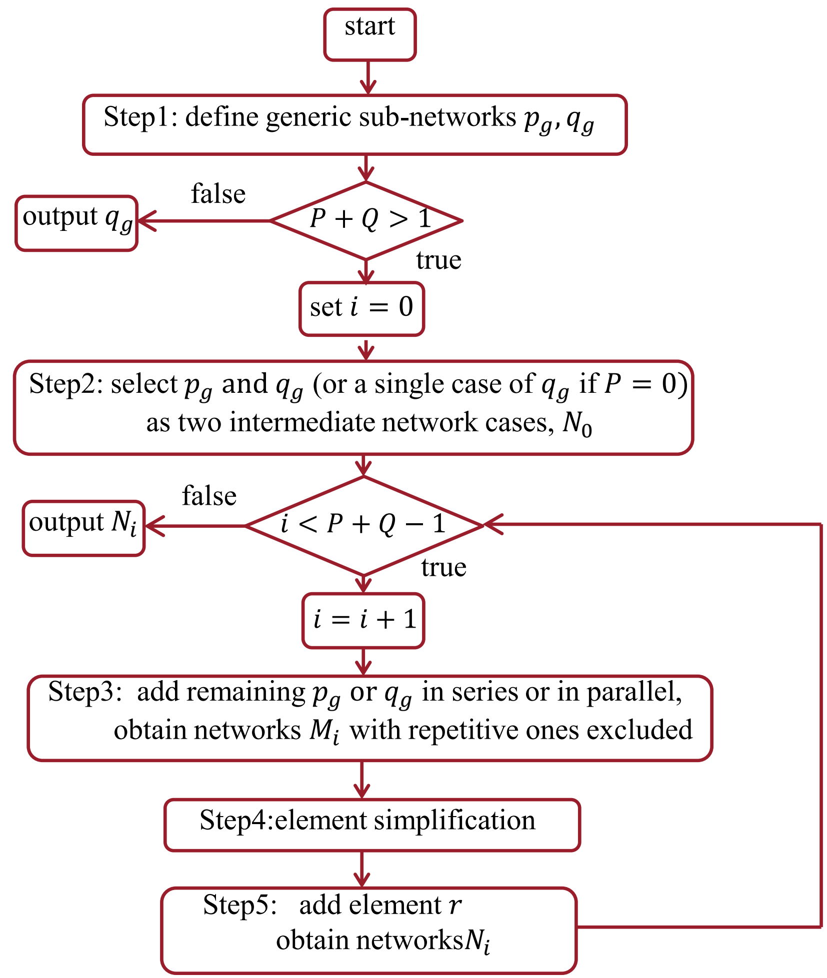
? Is there an alternative approach which has all the advantages

A structure-immittance format

- The structural immittances can be obtained based on generic networks

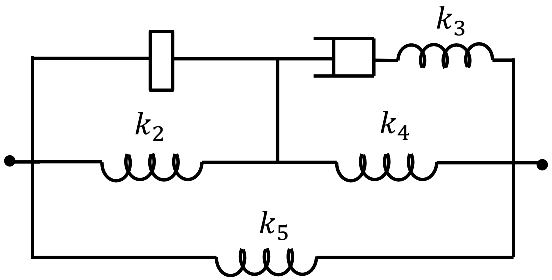
- A general formulation of generic network has been established for networks with three elements P p, Q q, R r where $P \leq Q \leq R$
  - ✓ P, Q are base elements and R is added element

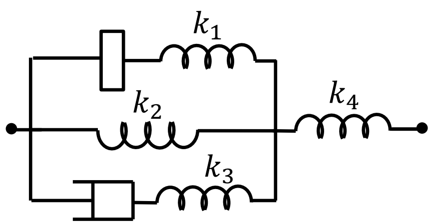
Sara Ying Zhang, Jason Zheng Jiang and Simon Neild, Passive Vibration Control: A Structure-immittance approach, Proceedings of Royal Society, 2017



Case demonstration: networks with one damper, one inerter and one spring

Generic networks





Structural immittances

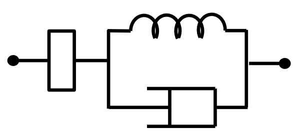
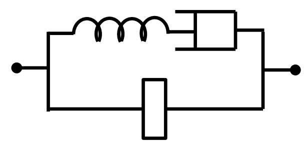
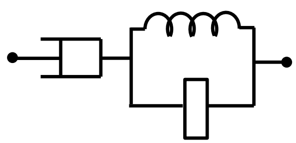
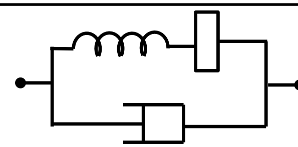
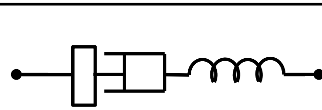
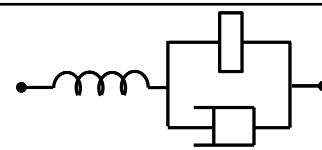
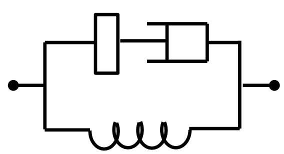
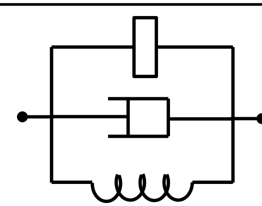
$$Y_1(s) = \frac{bcs^2 + b(k_4 + k_5)s + c(k_2 + k_5)}{bc(1/k_3)s^3 + bs^2 + cs + k_2 + k_4}$$

$$Y_2(s) = \frac{bc(1/k_1 + 1/k_3)s^3 + bs^2 + cs + k_2}{b(1/k_1 + 1/k_4)s^3 + c(1/k_3 + 1/k_4)s^2 + s}$$

Conditions: for $Y_1(s)$, at most one of the parameters $k_2, 1/k_3, k_4, k_5$ is positive and the others equal 0,
for $Y_2(s)$, at most one of the parameters $1/k_1, k_2, 1/k_3, 1/k_4$ is positive and the others equal 0

Case demonstration: networks with one damper, one inerter and one spring

- Full set of networks

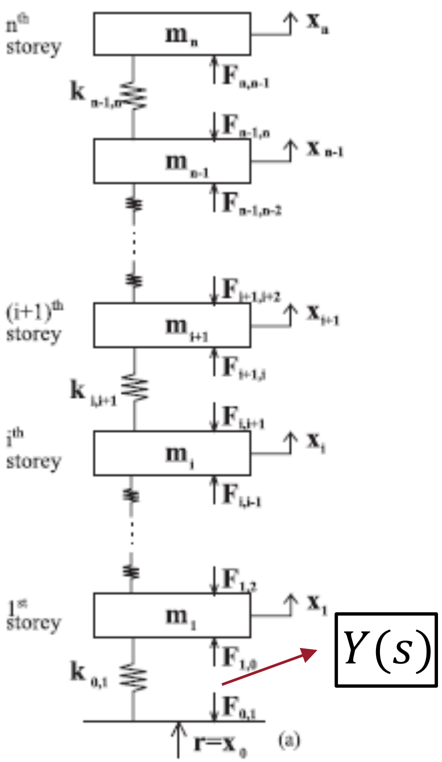
$Y_1(s)$	$Y_2(s)$
	
	
	
	

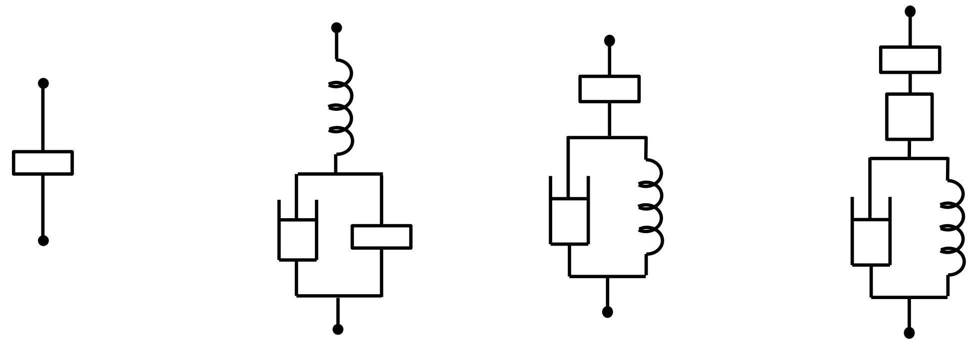
Structure-immittance approach

Structural immittances can be obtained based on *generic networks*, which

- cover a full set of series-parallel networks with predetermined numbers of each element type,
- make use of element values as variables,
- contain explicit topological information for each network layout possibility

Civil engineering case study





- Performance index:

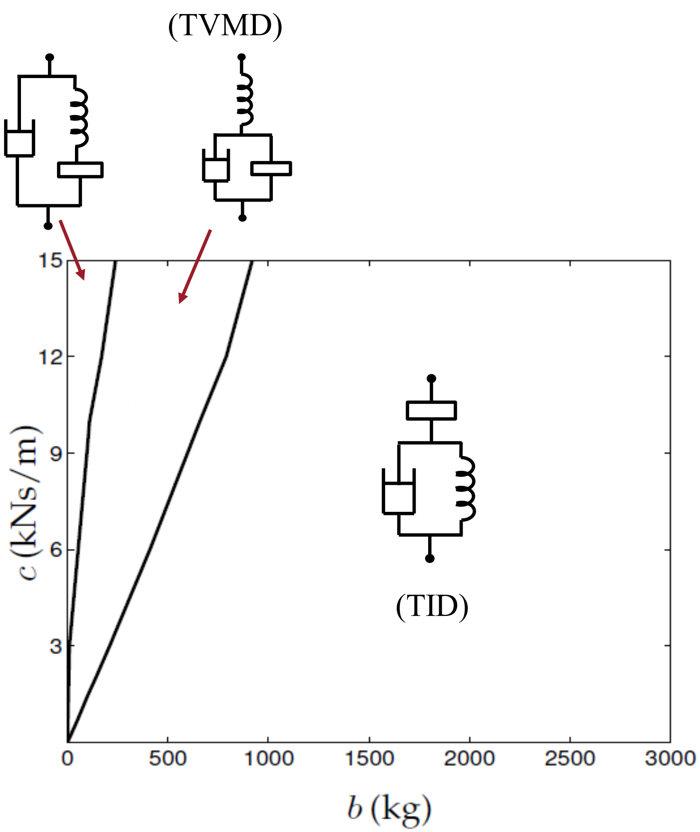
  Relative displacements of the building storeys to the base

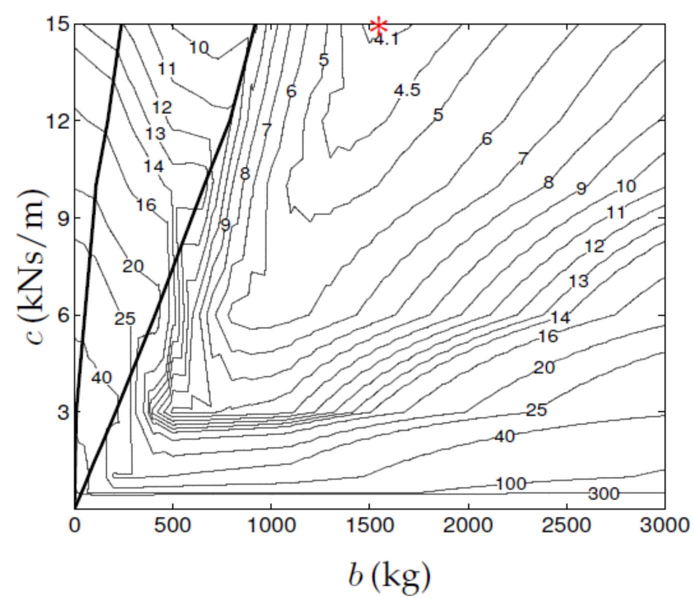
$$J_\infty = \max_{i=1:3}\left(\left\| T_{\hat{R}\to\hat{Z}_i} \right\|_\infty \right)$$

- Constraints:

  $0\ \text{kg} \leq b \leq 3000\ \text{kg},\ 0.01\ \text{kNs/m} \leq c \leq 15\ \text{kNs/m}$

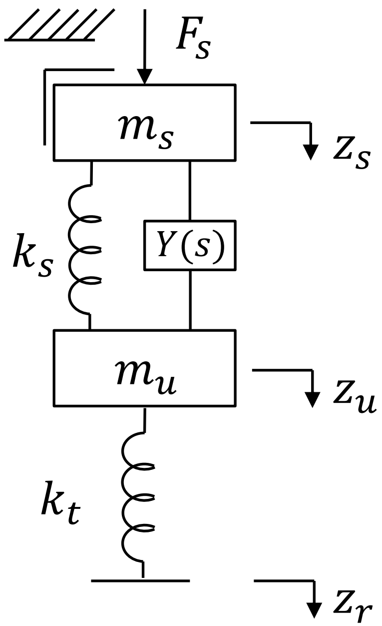
Optimisation results for the $1k$ case





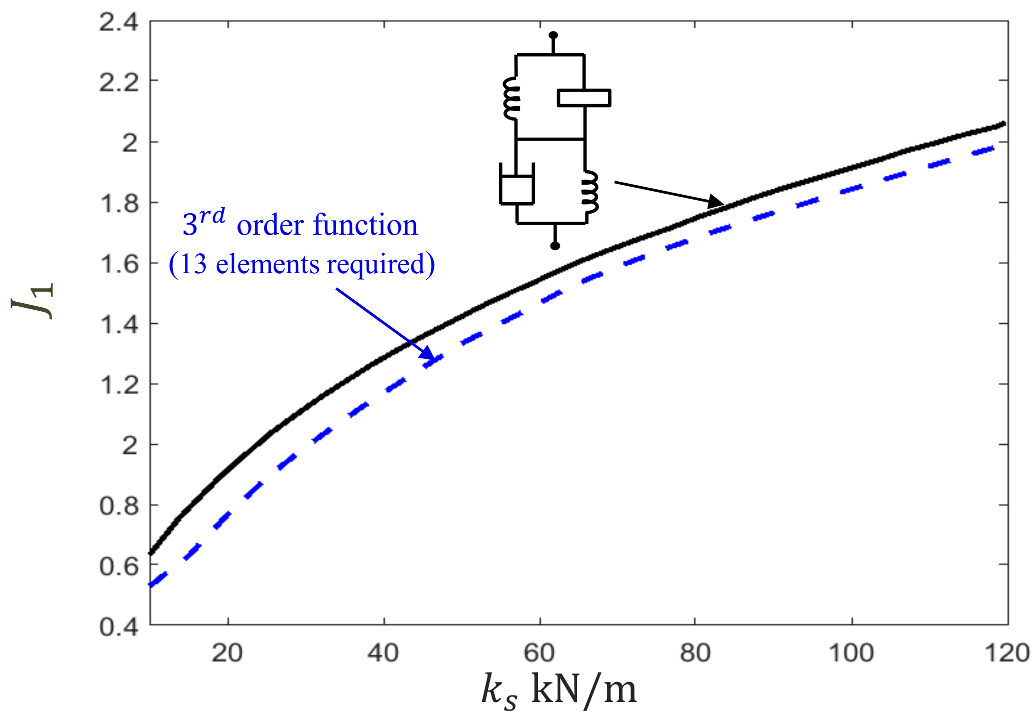
$1\ b, 1\ c, 1\ k$

Automotive case study



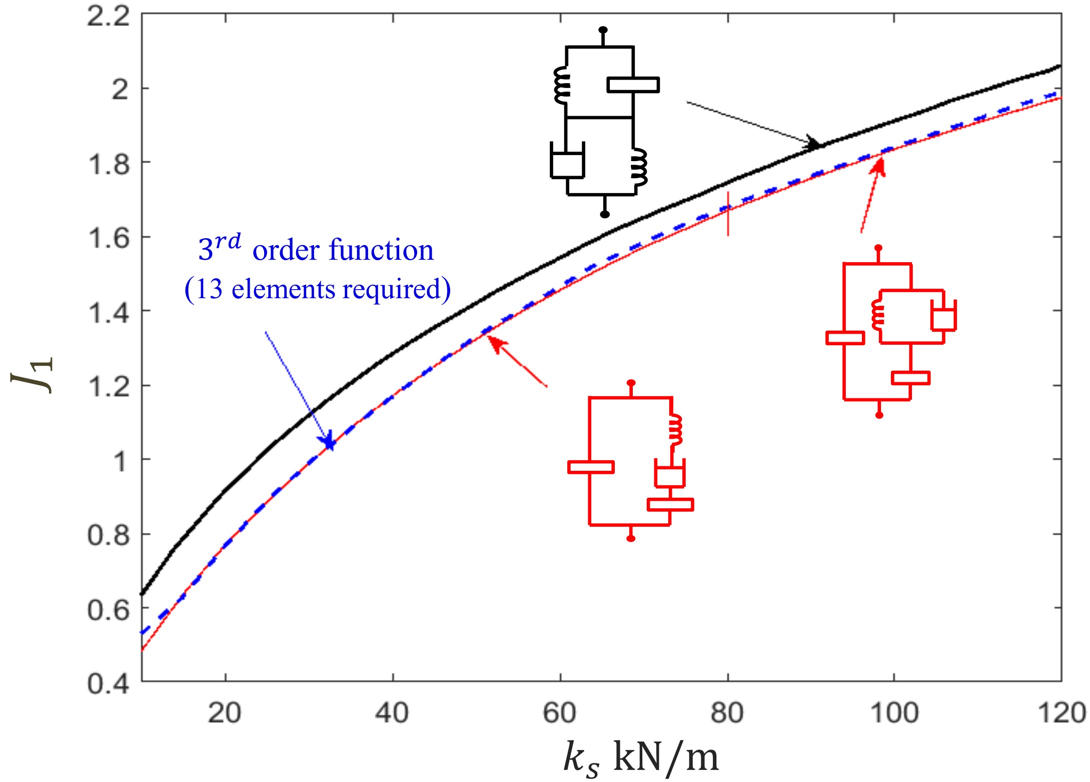
- Performance index:

  Ride comfort $J_1 = 2\pi(V\kappa)^{1/2}\left\|sT_{\hat{z}_r \to \hat{z}_s}\right\|_2$



Automotive case study

- Optimal configurations with one damper, one spring and at most two inerters



Happy birthday, Malcolm

European Control Conference, Budapest, 2009

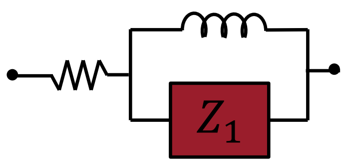
Appendix-Relations

- Essential-regular functions are all regular, (ES ⟹ R).
- Essential-regular functions can always be realised via Foster Preamble, (ES ⟹ FP).
- Relation of regular and Foster preamble is dependant on the McMillan degree of functions (N).

For bilinear and biquadratic functions, $N = 1, 2$: R ⟺ ES

Example:

Regular realisation:



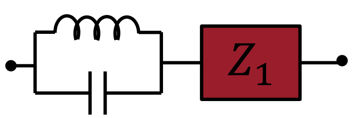
If $N = 1$, Z_1 is a constant.

If $N = 2$, Z_1 is a bilinear function.

For bilinear and biquadratic functions, $N = 1, 2$: FP ⟺ ES

Example:

Foster preamble:



$N = 2$, Z_1 is a constant.

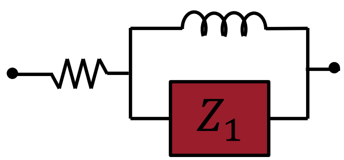
Hence, R ⟺ ES ⟺ FP

Appendix-Relations

For bicubic functions, $N = 3$: ES $\Rightarrow$ R $\nRightarrow$ FP

Example:

Regular realisation:

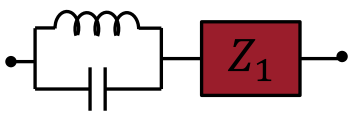


Z_1 is a positive-real biquadratic function, where the minimum positive-real function cannot be realised by Foster preamble.

For bicubic functions, $N = 3$: ES $\Rightarrow$ FP $\Rightarrow$ R

Example:

Foster preamble:



Z_1 is a bilinear function, and this bicubic function is regular.

Hence, ES $\Rightarrow$ FP $\Rightarrow$ R

Appendix-Relations

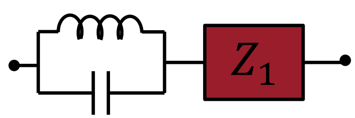
For functions $Z(s)$ with McMillan degree $N \geq 4$:

Similar to bicubic functions, ES $\Rightarrow$ R $\nRightarrow$ FP

For functions with McMillan degree $N \geq 4$: ES $\Rightarrow$ FP $\nRightarrow$ R

Example:

Foster preamble:

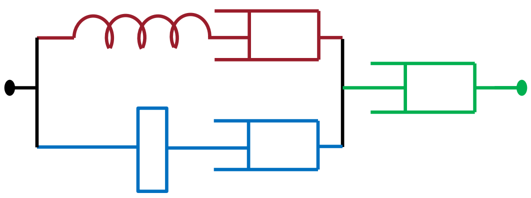


If $N = 4$, Z_1 is a biquadratic function realisable via foster preamble, then $Z(s)$ can be non-regular. For higher degrees of functions, similar results can be obtained.

Appendix-Origin of the idea of generic network formulation

Stage A: Two reactive elements belong to different networks

Stage B: These two networks are combined together



Stage C: Remaining resistors/dampers are connected in series or in parallel

Jason Zheng Jiang and Malcolm C. Smith. Regular positive-real functions and passive networks comprising two reactive elements. In Proc. of the European Control Conf., Budapest, Hungary. 2009.

Appendix-Origin of the idea of generic network formulation

Stage A

Stage B

Stage C

generalisation →

Stage A: The base element P p and Q q are in separate generic sub-networks

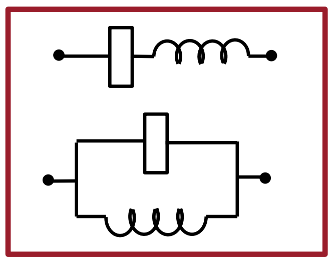
Stage B: These sub-networks are connected in all possible sequences and arrangements

Stage C: The remaining added elements r are then included in series or in parallel

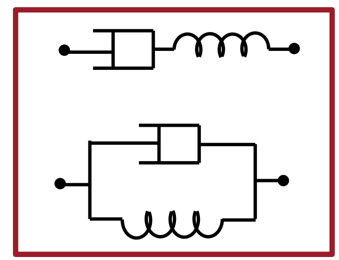
Appendix-Case demonstration: networks with one damper, one inerter and one spring

Stage A: the damper and the inerter belong to separate generic sub-networks;

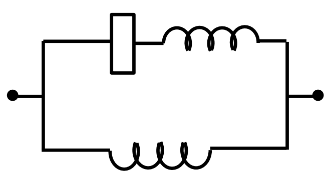
Inerter sub-network

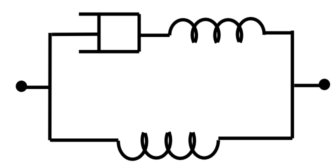


Damper sub-network



generic sub-network

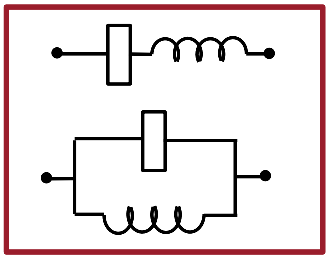




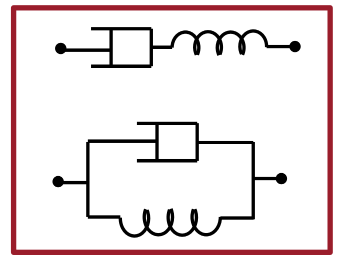
Appendix-Case demonstration: networks with one damper, one inerter and one spring

Stage A: the damper and the inerter belong to separate generic sub-networks;

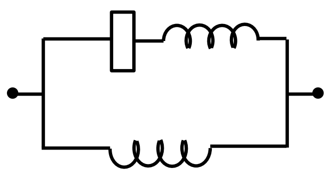
Inerter sub-network

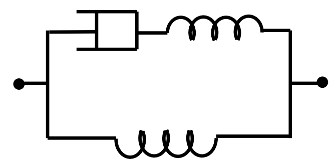


Damper sub-network



generic sub-network





Condition: at most one spring is *present* and the others are *removed*

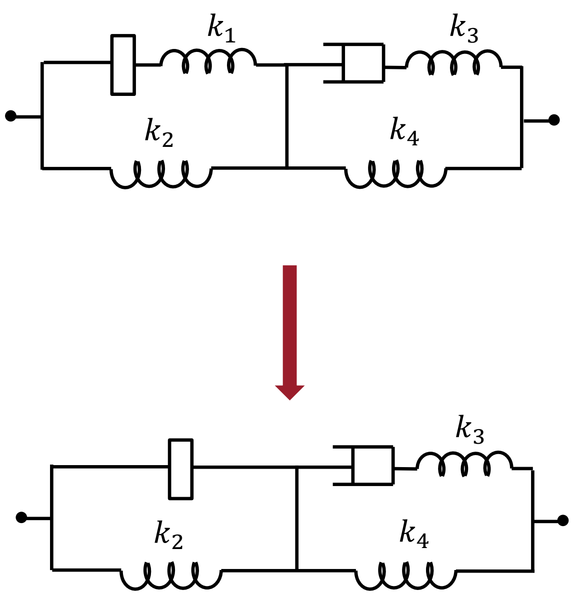
Present: an element has positive and finite value

Removed: an element takes the value of 0 or ∞ – ensure that no other elements are locked rigid and the terminals are not disconnected;

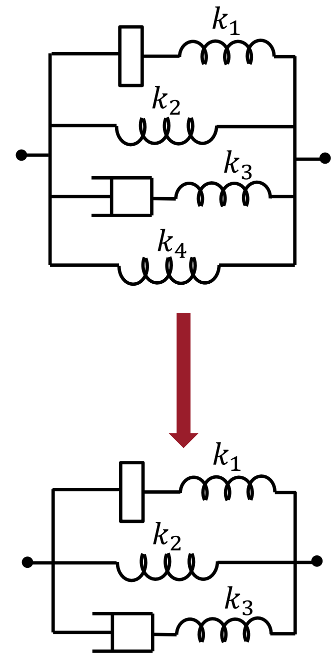
Appendix-Case demonstration: networks with one damper, one inerter and one spring

Stage B: the damper and inerter generic sub-networks are connected either in series or in parallel;

Series connection



parallel connection

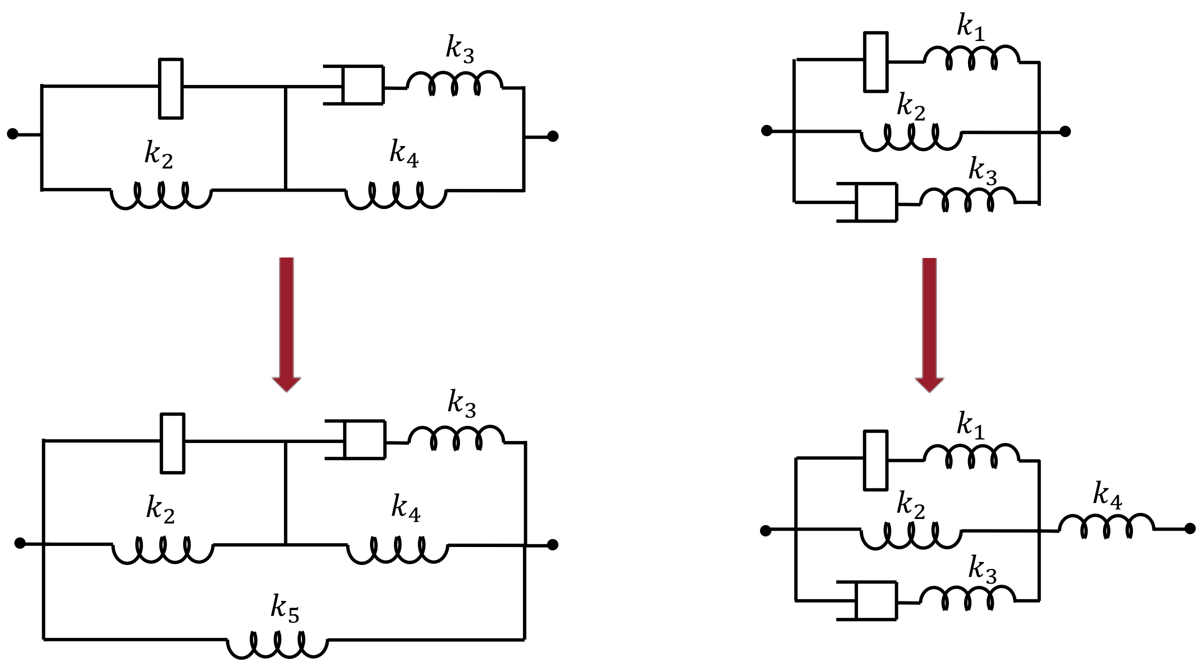


Condition: at most one spring is present

Condition: at most one spring is present

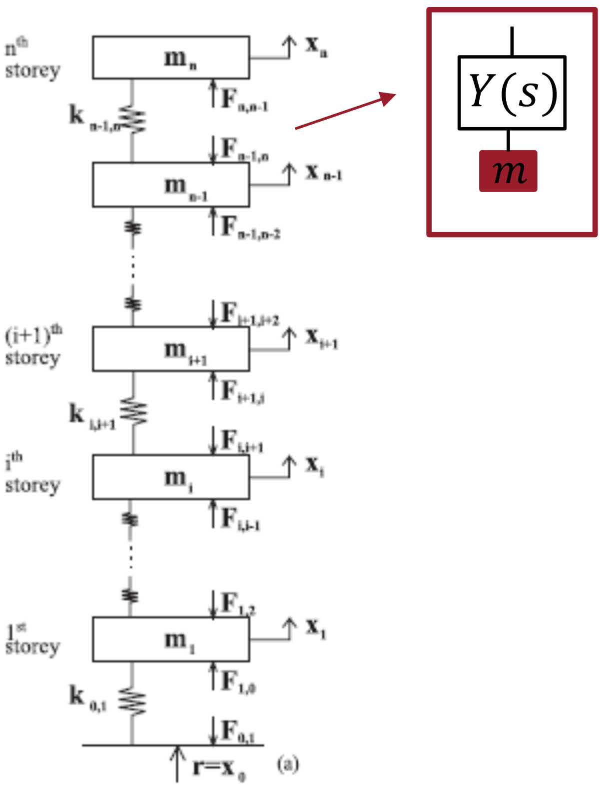
Appendix-Case demonstration: networks with one damper, one inerter and one spring

Stage C: the remaining springs are added in series or in parallel.

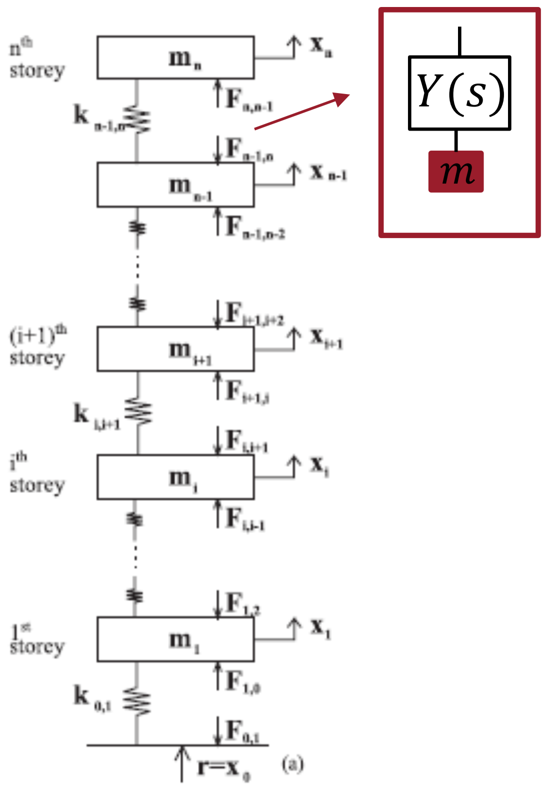


Condition: at most one spring is present

Appendix-Inclusion of the mass



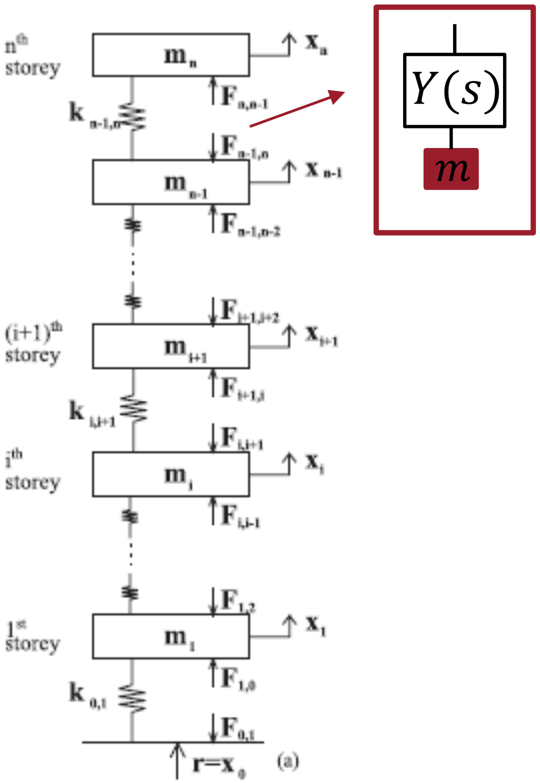
Appendix-Inclusion of the mass



- ✓ Treating the attached mass as an optimisable part of the structure

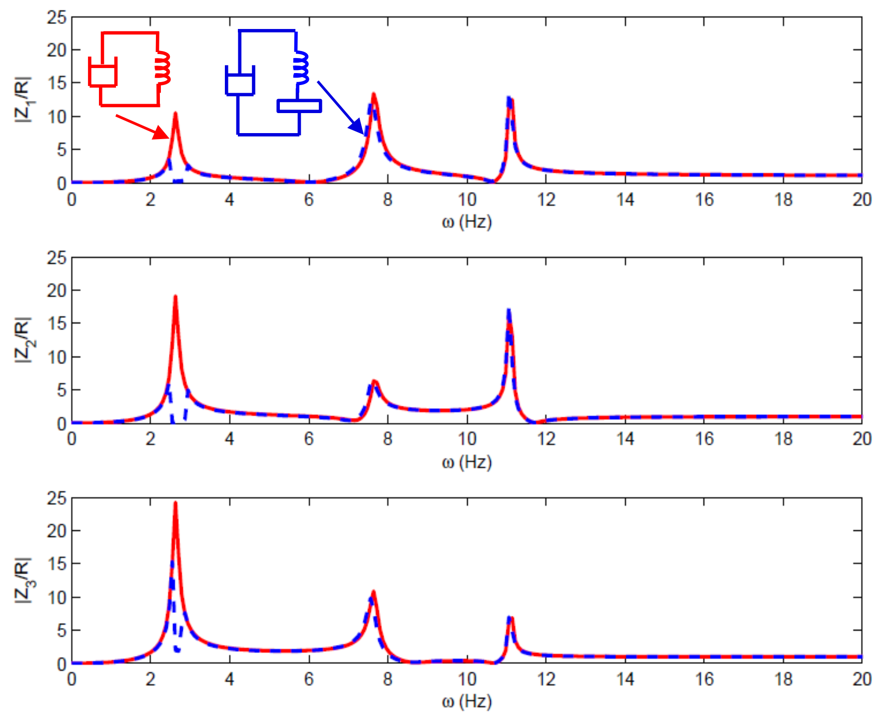
Example: consider $Y(s)$ has one inerter, one damper and one spring

Appendix-Inclusion of the mass



✓ Treating the attached mass as an optimisable part of the structure

Example: consider $Y(s)$ has one inerter, one damper and one spring



✓ The obtained structure outperform the TMD

Appendix-Automotive case study

- Candidate layouts: structures with one damper, one spring and at most two inerters

Generic networks