Engineering Tripos Part IIA THIRD YEAR

Module 3F1 – Signals and Systems

Examples Paper 3F1/1 – Discrete Time Systems

Straightforward questions are marked with a +. Questions 4, 5 and 11 illustrate how the techniques are applicable in three applications but are not essential to understand the basic material (marked with a †).
Note: this example sheet covers lectures 1–6, and hence is about 50% longer than a standard one.

(The z-transform table in the Electrical and Information Data Book will be helpful).

+1. Find from first principles the z-transforms of the sequences obtained by sampling, (with uniform sampling period T), the continuous-time waveforms whose Laplace transforms are

(i) $\frac{1}{s}$; (ii) $\frac{1}{(s+a)}$; (iii) $\frac{1}{(s+a)(s+b)}$; (iv) $\frac{s+a}{(s+a)^2+b^2}$.

Note that the first sample is taken at $t = 0^+$.

+2. A sequence is given by $u_k = 1$ for $k = 0, 1$ and zero otherwise. Find the z-transform of this sequence and hence solve the following difference equations for y_k, using the z-transform technique.

(i) $y_k = u_k + u_{k-1} + u_{k-2}$, (FIR filter).

(ii) $y_k = 0.8\ y_{k-1} + 0.2\ u_k,\ y_{-1} = 0$.(exponential smoother)

(iii) $y_k = 0.98\ y_{k-1} - 0.9604\ y_{k-2} + u_k,\ y_{-1} = 0,\ y_{-2} = 1$. (IIR filter).

3. For the three difference equations of question 2, write down the z-plane transfer function relating $\{y_k\}$ to $\{u_k\}$.

(a) Evaluate their poles and zeros, and calculate and sketch the response to a unit step on u_k. (assume $y_k = 0,\ k < 0$).

(b) Assuming the time between samples is T, calculate and sketch the **steady state** response to $u_k = \cos(\omega kT)$ for $\omega T = 0$ for case (i), $\omega T = \pi$ for case(ii) and $\omega T = \pi/3$ for case (iii).

4.† *Supply and demand cycles.* In agriculture a one season prediction of price is required by the farmer to determine how much product to produce. The price at harvesting will then depend on the supply via the demand curve. Assume that the demand at time k is given by

$$d_k = d_e - ap_k$$

where p_k is the price at time k. Now assume that the supply is given by

$$s_k = s_e + b\hat{p}_k \ ,$$

where $\hat{p}_k$ is the predicted price at time k. (d_e and s_e are constants). The price then adjusts to equate supply and demand at time k (i.e. $s_k = d_k$). Let $c = b/a$ and determine conditions on c for the stability of this system (i.e. the stability of the difference equation determining p_k), for

(i) $\hat{p}_k = p_{k-1}$ and

(ii) $\hat{p}_k = 2p_{k-1} - p_{k-2}$ (i.e. a linear extrapolation through the last two prices).

5.† *Numerical solutions of differential equations*

(a) Euler's method for solving the differential equation

$$\frac{dx(t)}{dt} = f(x(t)) \tag{1}$$

is to make the approximation,

$$x((k+1)T) \ \cong \ x(kT) + Tf(x(kT)) \tag{2}$$

for $k = 0, 1, 2 \ldots$, where T is the step length. Assuming $f(x) = ax$ for $a < 0$, what range of values of T in (2) will ensure that $x(kT) \to 0$ as $k \to \infty$?

(b) Euler's method is inaccurate unless T is very small and an alternative is to consider higher-order extrapolation based on $x(kT)$ and $x((k-1)T)$ and $\dot{x}(kT) = f(x(kT))$. The function

$$g(t) = \{c - b + mT\}\frac{t^2}{T^2} + mt + b$$

is the quadratic function that satisfies

$$g(0) = b, \quad \dot{g}(0) = m, \quad g(-T) = c$$

Now if we let

$$b = x(kT), \quad m = f(x(kT)), \quad c = x((k-1)T)$$

then $x((k+1)T) = g(T) = x((k-1)T) + 2Tf(x(kT))$ is an extrapolation of the next value of x based on this quadratic approximation. If $f(x) = ax$ with $a < 0$ show that the method would be unstable for any $T > 0$! What would be the nature of the instability? Note that this apparently sensible modification to Euler's method gives a hopeless method, but there exist many other higher-order methods with good stability properties. (e.g. Runge-Kutta methods).

6. A discrete time system with input $\{u_k\}$ and output $\{y_k\}$, has pulse response $\{g_k\}$ and transfer function, $G(z)$, given by,

$$G(z) = \frac{b(z^{-1})}{(1-p_1 z^{-1})^{n_1}(1-p_2 z^{-1})^{n_2}\dots(1-p_r z^{-1})^{n_r}}$$

where $b(z^{-1}) = b_0 + b_1 z^{-1} + \dots + b_m z^{-m}$ and there are no pole/zero cancellations.

(a) Show that if $|p_i| < 1$ for all i then $\sum_{k=0}^{\infty} |g_k|$ is finite.

(b) Suppose that $G(z)$ has at least one pole, p_1, with $|p_1| > 1$. Show that $\{u_k\}$ with $U(z) = (1-p_1 z^{-1})^{n_1-1}(1-p_2 z^{-1})^{n_2}\cdots(1-p_r z^{-1})^{n_r}$ will be bounded and

$$Y(z) = G(z)U(z) = r(z^{-1}) + \frac{A_1}{(1-p_1 z^{-1})}$$

where r is a polynomial, and hence that $\{y_k\}$ is not bounded.

(c) Suppose that $|p_1| = 1$. Show that $\{u_k\}$ with $U(z) = (1-p_1 z^{-1})^{n_1-2}(1-p_2 z^{-1})^{n_2}\cdots(1-p_r z^{-1})^{n_r}$ will be bounded and $\{y_k\}$ is not bounded.

(d) Hence deduce that the system is stable if and only if $|p_i| < 1$ for all i.

7. For a sampling period of 35 milliseconds the Bode plots of the following transfer functions were plotted (see figure). As usual, however, it was forgotten to label the graphs. Can you help?

(1) $\dfrac{z+2}{z-1}$

(2) $\dfrac{2z+1}{z-1}$

(3) $\dfrac{1}{z^2-0.5z+0.9}$

(4) $\dfrac{1}{z(z^2-0.5z+0.9)}$

(5) $\dfrac{(z+1)^2}{4(z+3)(z+0.5)}$

(6) $\dfrac{3z+1}{4z+2}$

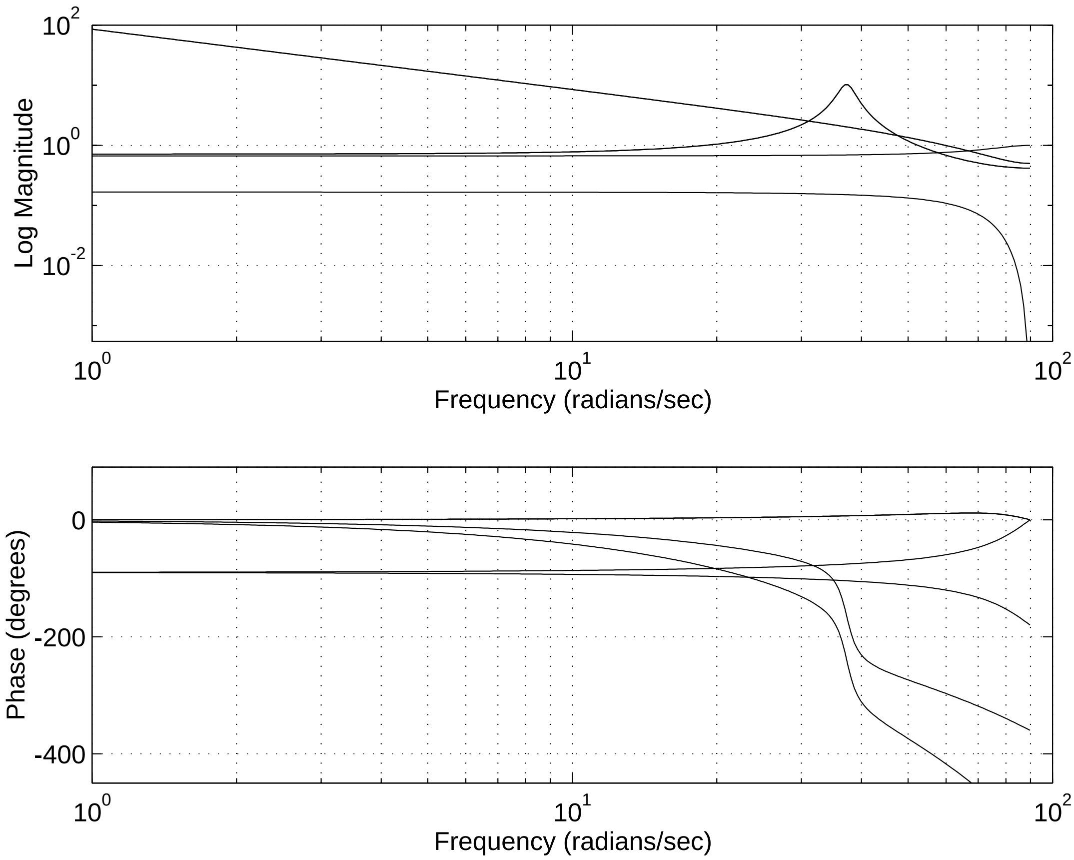


Figure 1: Bode diagrams for question 7

8. Find the forward difference, backward difference and Tustin transformation of the analog low-pass filter

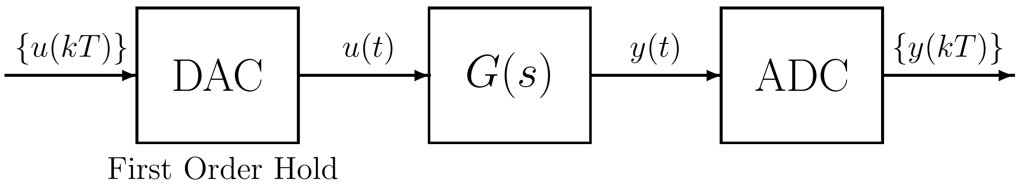
$$G(s) = \frac{a}{s+a}$$

($a > 0$) assuming a sampling period of T seconds. What conditions must aT satisfy for these digital filters to be stable? For what range of values of aT would these filters be reasonable approximations of $G(s)$?

9. A motor driving a rotating inertia has transfer function $1/s(s+1)$ from the motor current input to the position output. The output is sampled with period T, and the current input is held constant between sampling points. Show that the equivalent discrete-time system, from the sequence of current inputs to the sampled outputs, has the z-plane transfer function,

$$G(z) = \frac{(e^{-T} - 1 + T)z^{-1} + [1 - (1+T)e^{-T}]z^{-2}}{(1 - z^{-1})(1 - e^{-T}z^{-1})}.$$

10. Consider a continuous time system with transfer function $G(s)$ connected as shown below. The output of the first order hold is the linear extrapolation through the last two discrete inputs.



Explain why the discrete system taking $\{u(kT)\}$ to $\{y(kT)\}$ has a z-transfer function. Show that the transfer function is given by the expression:

$$H(z) = \frac{(z-1)^2}{Tz^2}\mathcal{Z}\left(\mathcal{L}^{-1}\left(G(s)\frac{Ts+1}{s^2}\right)\Big|_{t=kT}\right).$$

11.† An economic indicator is measured once every quarter and it is desired to estimate the underlying trend in the face of seasonal fluctuations. Assume that the indicator at the k^{th} quarter, v_k, is related to the underlying trend, u_k, by

$$v_k = u_k + w_k, \qquad k \geq 0,$$

where $u_k = a + bk$, and the seasonal variation, w_k, is a periodic function with period 4 and zero mean value, (i.e. $w_{4k+i} = w_i$ and $w_0 + w_1 + w_2 + w_3 = 0$).

(a) Show that

$$\begin{aligned} W(z) = \mathcal{Z}(\{w_k\}) &= \frac{w_0 + w_1 z^{-1} + w_2 z^{-2} + w_3 z^{-3}}{1 - z^{-4}} \\ &= \frac{w_0 + (w_0 + w_1)z^{-1} - w_3 z^{-2}}{(1 + z^{-1})(1 + z^{-2})} \end{aligned}$$

(b) The underlying trend is estimated by passing $\{v_k\}$ through a FIR filter giving

$$y_k = \frac{1}{8}\{v_k + 2v_{k-1} + 2v_{k-2} + 2v_{k-3} + v_{k-4}\}$$

with error given by

$$e_k = y_k - u_{k-2}$$

(i.e. y_k is supposed to be an estimate of u_{k-2}).

(i) Find the transfer function, $G(z)$, of this filter, and its zeros, and show that the poles of $W(z)$ are cancelled by zeros of $G(z)$.

(ii) Show that $(G(z) - z^{-2}) = \frac{1}{8}(1 - z^{-1})^2(1 + 4z^{-1} + z^{-2})$, and hence that its zeros cancel the poles of $U(z)$.

(iii) Hence show that the error, e_k, will be zero for $k \geq 4$, and hence this filter accurately extracts the trend under these assumptions.

(iv) Show that

$$G(e^{j\theta}) = \cos\theta\cos^2(\theta/2)e^{-j2\theta}$$

and sketch the Bode diagram. Comment on the frequencies at which $G(e^{j\theta}) = 0$.

12. The plots of $G(e^{j\theta})$ for θ ranging from 0^+ to π are shown on page 7 (diagrams are in random order) for the following transfer functions:

$$\text{(i)}\frac{1}{z^2(z-1)},\ \text{(ii)}\frac{4z-2}{3(z-1)^2},\ \text{(iii)}\frac{4}{(z-1)^3}.$$

Sketch the complete Nyquist diagrams for each transfer function and use the Nyquist criterion to determine for what values of gain (if any) closed loop stability will be achieved when constant gain negative feedback is connected around them.

Relevant Tripos questions from 3F1 papers: 2003–2012: Q1, Q2 (a).

Answers:

1. $1/(1-z^{-1}),\ 1/(1-e^{-aT}z^{-1})$,

$$\frac{z^{-1}(e^{-aT}-e^{-bT})}{(b-a)(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})},\ \frac{(1-e^{-aT}\cos(bT)z^{-1})}{(1-2e^{-aT}\cos(bT).z^{-1}+e^{-2aT}z^{-2})}$$

2. $\mathcal{Z}\{u_k\} = 1+z^{-1}$. (i) $1,2,2,1,0,0,\ldots$
   (ii) $0.2, 0.36, \ldots, 0.36(0.8)^{k-1}$
   (iii) $(0.98)^{k-1}\{0.0448 \sin[(k+1)\pi/3] + 1.1547 \sin[k\pi/3]\}$.

3. (a) Poles and zeros:

	Poles	Zeros
(i)	$0, 0$	$-\frac{1}{2} \pm j\sqrt{3}/2$
(ii)	0.8	0
(iii)	$0.98e^{\pm j\pi/3}$	$0, 0$

Step responses:
(i) $1,2,3,3,3,\ldots$
(ii) $1-(0.8)^{k+1}$
(iii) $1.02 + (0.98)^k\{-0.02\ \cos(k\pi/3) + 1.1427\ \sin(k\pi/3)\}$

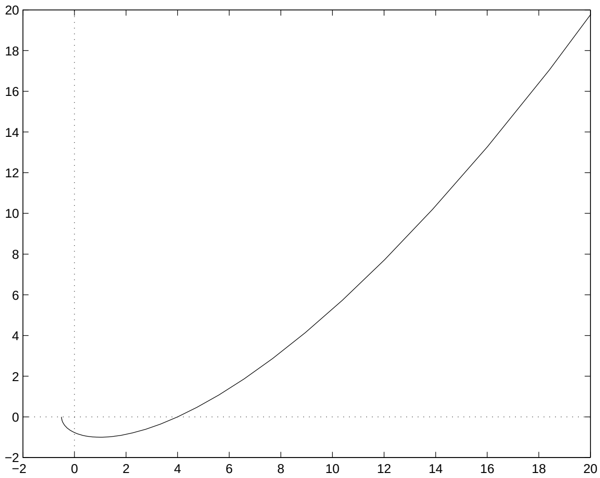
(b)

$$\begin{aligned}\omega T = 0 &: \quad \text{(i) } 3;\\ \omega T = \pi &: \quad A(-1)^k \text{ for (ii) } A = 1/9,\\ \omega T = \pi/3 &: \quad A\ \cos(k\pi/3+\theta) \text{ for (iii) } (A,\theta) = (29.16, -0.518).\end{aligned}$$

4. $c < 1,\ c < 1/3$.

11. (b)(i) $G(z) = \frac{1}{8}(1+z^{-1})^2(1+z^{-2})$. Zeros at -1, $\pm j$.

KG October 2012
RS October 2013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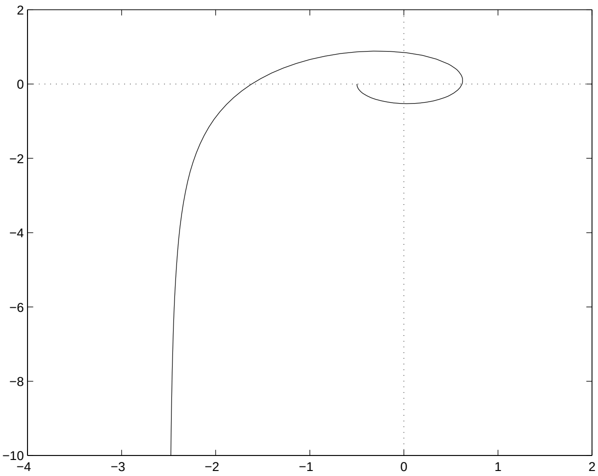
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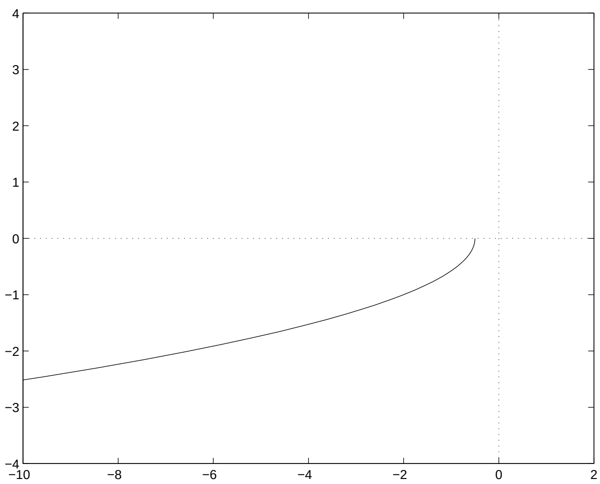
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