Engineering Tripos Part IIB **Module 4F2**

Nonlinear Systems and Control

Examples Paper 4F2/4

1. Consider the system

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -a_1 x_2 - a_2 x_1 - (b_1 x_2 + b_2 x_1)^2 x_2 \end{aligned}$$

Using $V(\mathbf{x}) = a_2 x_1^2 + x_2^2$, show that this system is globally asymptotically stable if $a_1 > 0$ and $a_2 > 0$.

2. Suppose that $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has an equilibrium point at $\mathbf{x} = 0$. Define the matrix

$$J(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ & \cdots & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}.$$

If there is a region Δ, which includes $\mathbf{0}$, and a matrix $P = P^T > 0$, such that

$$-(J^T(\mathbf{x})P + PJ(\mathbf{x})) > 0$$

for all $\mathbf{x}$ in Δ, and $\mathbf{0}$ is the only equilibrium state in Δ, prove that $\mathbf{x} = \mathbf{0}$ is asymptotically stable. (Hint: Use $V(\mathbf{x}) = \mathbf{f}^T(\mathbf{x})P\mathbf{f}(\mathbf{x})$.)

3. Recall that a matrix $P = P^T$ is said to be positive-definite if $\mathbf{x}^T P \mathbf{x} > 0$ for all $\mathbf{x} \neq 0$ and we write $P > 0$. If $P = P^T > 0$ and $Q = Q^T \geq 0$ show that

   (i) $P^{-1} > 0$,

   (ii) $P + Q > 0$,

   (iii) $PQP > 0$ if $Q > 0$,

   (iv) $R^T P R \geq 0$ for arbitrary R (of appropriate dimensions),

   (v)

$$\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \geq 0.$$

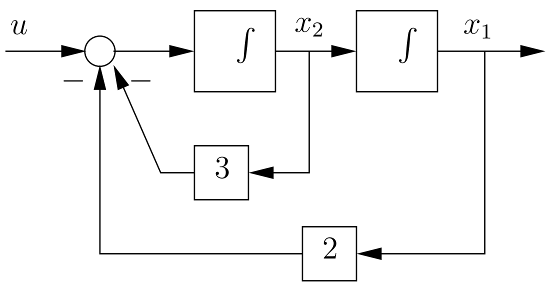

Figure 1: Linear system.

4. The linear system shown in Fig. 1 is to have feedback placed around it to speed up its reponse and return to equilibrium from non-zero initial conditions. The input signal $u(t)$ is constrained in amplitude: $|u(t)| \leq U$. Write the system's state equation in the form

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{b}u(t)$$

and find the symmetric matrix P such that

$$A^T P + PA = -I.$$

Note that $P > 0$; why? Show that if the feedback law is chosen as

$$u(t) = -U\text{sign}(\mathbf{b}^T P\mathbf{x}(t))$$

then the closed-loop system is globally asymptotically stable.

5. Which of the following transfer functions are positive real or strictly positive real (a transfer function $H(s)$ is strictly positive real is there exists $\epsilon > 0$ such that $H(s-\epsilon)$ is positive real)?

(a) $\frac{1}{1+sT}$ $(T > 0)$ (b) $\frac{1}{1-sT}$ $(T > 0)$

(c) $\frac{1}{s}$ (d) $\frac{100(s+1)}{(s+10)^2}$

(e) $\frac{s\omega_n^2}{s^2+\omega_n^2}$ (ω_n real)

6. Figure 2 shows the attitude control system of a satellite. The attitude sensor has a nonlinear gain $\psi(y)$, where

$$0.25 \leq \psi(y) < \beta$$

and β can be chosen by the system designer. Use the Circle and Popov criteria to predict the maximum value of β for which the system is globally asymptotically stable. (Sketch the appropriate loci accurately for $1 \leq \omega \leq 5$).

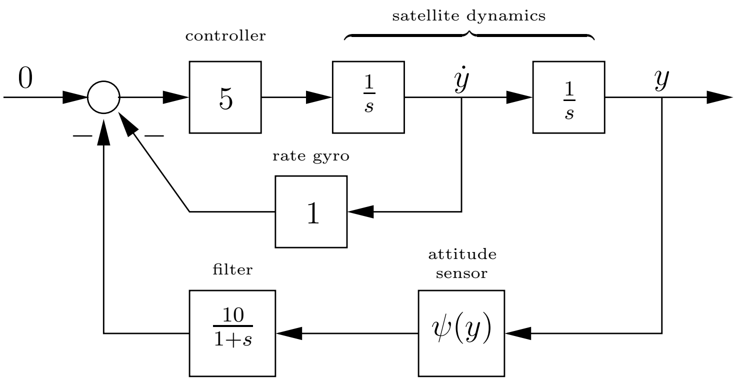

Figure 2: Feedback system.

Would your answer change if you knew that the sensor characteristic 'drifted' with time, although it remained within the limits $0.25 \leq \psi(y,t) < \beta$?

7. What form does the Circle Criterion take when

   (i) $\alpha = 0$?

   (ii) α becomes negative?

8. Find the describing function $N(E)$ of a "relay with dead-zone":

$$f(e) = \begin{cases} -R & \text{if} \quad e < -\delta \\ 0 & \text{if} \quad |e| \leq \delta \\ +R & \text{if} \quad e > \delta \end{cases}$$

   and sketch its variation with E.

9. (a) If $u = f_1(e)$ is single-valued, and

$$f_1(e) = f_2(e) + f_3(e),$$

   show that

$$N_1(E) = N_2(E) + N_3(E)$$

   where $N_i(E)$ is the describing function of $f_i(e)$.

   (b) Figure 3 shows the charactersitic of an A-D converter. Using (a), find its describing function. Assume that $|f(e)| = N\delta$ if $|e| \geq \frac{2N-1}{2}\delta$.

10. (a) Find the describing function of the polynomial nonlinearity $f(e) = e^5$.

    (b) Describing functions of nonlinearities such as e^3, e^5, etc, seem to be of little use. Consider how they could be used to find approximate describing functions of arbitrary (smooth) nonlinearities.

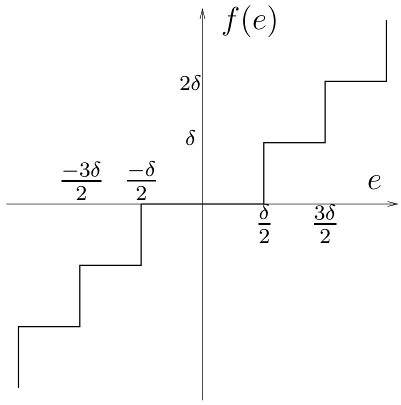

Figure 3: A-D converter.

11. Figure 4 shows a feedback system containing one nonlinear and one linear element. The transfer function of the linear element is

$$g(s) = \frac{20}{s(s+1)(s+2)}.$$

The nonlinear element is a 'relay with dead-zone', as defined in question 5, with $\delta = 0.25$ and $R = 1$. Use the describing function method to predict whether any stable limit cycle exists and if so predict its frequency and amplitude.

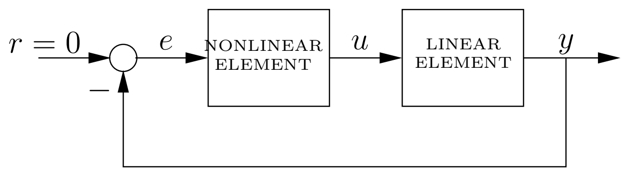

Figure 4: Feedback system.

12 Show that if the conditions of the Circle Criterion are satisfied, the nonlinear gain does not vary with time, and is an odd function ($\psi(-y) = -\psi(y)$), then the describing function $N(E)$ satisfies

$$\alpha \leq N(E) \leq \beta$$

and hence that the describing function predicts that no limit cycle will exist.

Answers

4.

$$\begin{bmatrix} \frac{5}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

6.
Circle criterion: $\beta \leq 0.392$ approx.
Popov criterion: $\beta \leq 0.6$

8.

$$N(E) = \begin{cases} 0 \text{ if } E \leq \delta \\ \frac{4R}{\pi E}\sqrt{1-(\frac{\delta}{E})^2} \text{ if } E > \delta \end{cases}$$

9.

$$N(E) = \begin{cases} 0 \text{ if } E \leq \delta/2 \\ \frac{4\delta}{\pi E}\sum_{i=1}^{k}\left\{1-[\frac{(2i-1)\delta}{2E}]^2\right\}^{1/2} \text{ if } \frac{(2k-1)\delta}{2} < E \leq \frac{(2k+1)\delta}{2} \\ \frac{4\delta}{\pi E}\sum_{i=1}^{N}\left\{1-[\frac{(2i-1)\delta}{2E}]^2\right\}^{1/2} \text{ if } E > \frac{(2N-1)\delta}{2}. \end{cases}$$

10. (a) $N(E) = 5E^4/8$.

11. Frequency $\sqrt{2}$ rad/sec, Amplitude 4.237.

Past Papers

The following past Tripos questions are suitable for further practice:
Part IIB **Module 4F2**. 2011 Q.2. Part IIB **Module 4F3**. 2010 Q.2(a)–(b), 2009 Q.2(a)–(d)(i), 2008 Q.1(a)–(c), 2007 Q.1, 2006 Q.3, 2005 Q.2, 2004 Q.2 (a)–(d). 2003 Q.1.
Part IIB Module I3. 2002 Q.1. 2001 Q.2(a), 3(a),(b). 2000 Q.1(c), 2. 1999 Q.2.
1998 Q.1, 2(a)(b), 4(b). 1997 Q.2. 1996 Q.1,2.
EIST Paper E3. 1994 Q.5(b),6. 1993 Q.4. 1992 Q.5(b),6.

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