Engineering Tripos Part II

Module 4F2 -- NL Systems and control

Nonlinear systems (7 lectures)

Handout 4

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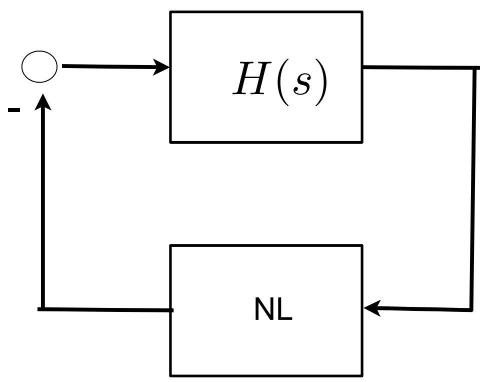
available on RS homepage

Lecture 7

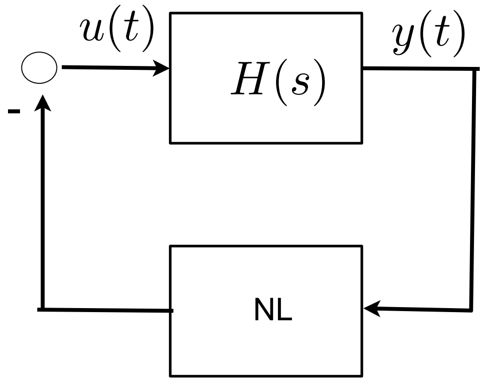
Describing function analysis



Approximating the static nonlinearity by an equivalent frequency-dependent gain (or transfer function)

Using this approximation to predict limit cycles in the feedback loop

The main idea



A limit cycle means that the feedback system admits a periodic behavior (u(t),y(t)).
If H(s) is low-pass, one harmonic might be sufficient to describe y(t):

$$y(t) = A \sin \omega t$$

If $\phi(A \sin \omega t) \approx N(A, \omega) A \sin \omega t$, one should look for a solution of

$$H(j\omega) N(A, \omega) + 1 = 0$$

Computing the equivalent gain

Particular case $\varphi(.)$ odd, memoryless, invariant

$$\Rightarrow N(A, \omega) = \phi(A) = \frac{2}{\pi A} \int_0^{2\pi} \varphi(A \sin \theta) \sin \theta d\theta$$

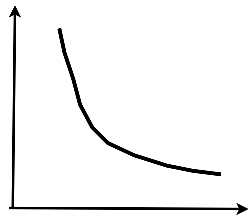
first Fourier coefficient of $\varphi(A \sin \omega t)$

Example: $\varphi(A) = \text{sign}(A)$

$$\begin{aligned} A\,N(A,\omega) &= \frac{1}{\pi}\int_0^{2\pi} \text{sign}(A\sin\theta)\sin\theta\,d\theta, \\ &= \frac{1}{\pi}\int_0^{\pi} \sin\theta\,d\theta - \frac{1}{\pi}\int_\pi^{2\pi} \sin\theta\,d\theta \\ &= \frac{4}{\pi}. \end{aligned}$$

Therefore

$$N(A,\omega) = \frac{4}{\pi A}$$



Example (polynomial nonlinearity): $\phi(A) = A^n,\ \ n$ odd

$$\sin^n\theta = \frac{2}{2^n}\sum_{k=0}^{\frac{n-1}{2}} (-1)^{(\frac{n-1}{2}-k)} \binom{n}{k} \sin([n-2k]\theta)$$

Last term, $k = \frac{n-1}{2}$, gives fundamental component.

Example, $n = 3$: *(Maths Databook)*

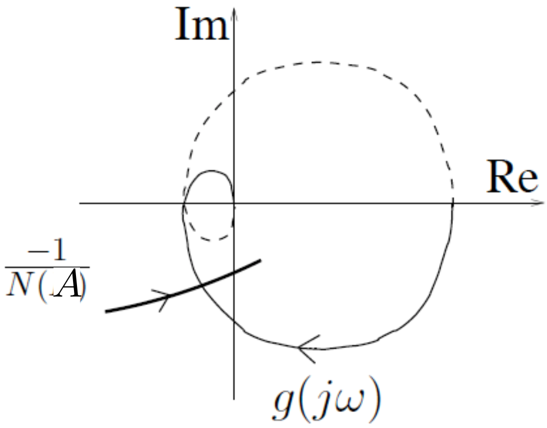
$$(A\sin\theta)^3 = A^3\left(\frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta)\right)$$

$$N(A) = \frac{3A^2}{4}$$

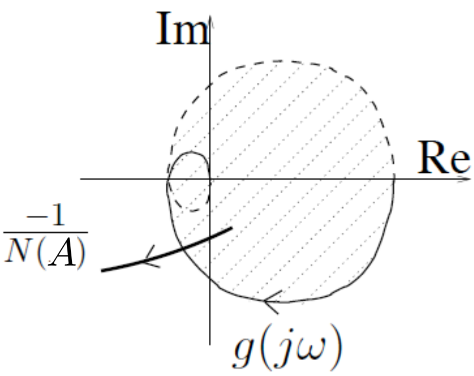
Prediction of a limit cycle

$$H(j\omega)N(A,\omega) + 1 = 0$$

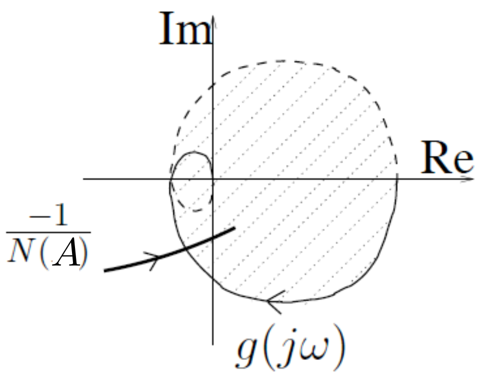
For a frequency independent describing function N(A),
one can graphically determine intersections of the Nyquist plot
with the curve -1/N(A):



Stability of the predicted limit cycle



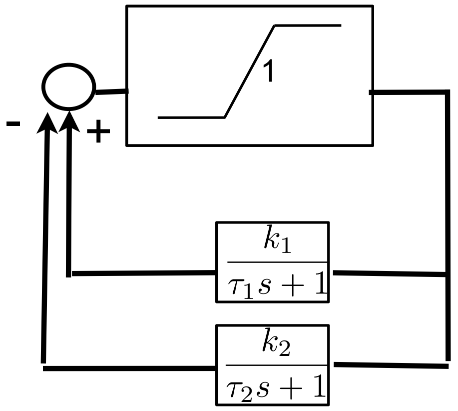
Prediction: STABLE

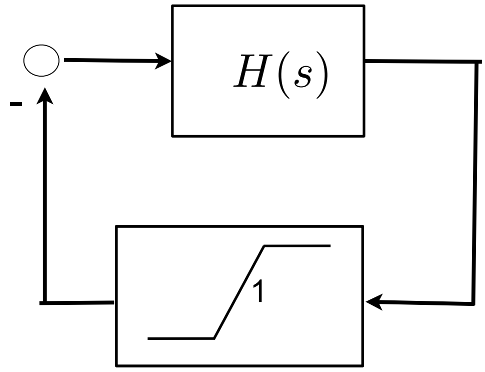


Prediction: UNSTABLE

Stability: greater A should correspond to a point that satisfies the encirclement condition of Nyquist criterion, that is, oscillations of larger amplitude will be attenuated.

Predicting a limit cycle : illustration





$$\tau_1 \dot{x}_1 = -x_1 + S(k_1 x_1 - k_2 x_2)$$
$$\tau_2 \dot{x}_2 = -x_2 + S(k_1 x_1 - k_2 x_2)$$

$$H(s) = \frac{k_2}{\tau_2 s + 1} - \frac{k_1}{\tau_1 s + 1}$$

When does this model admit a limit cycle oscillation?

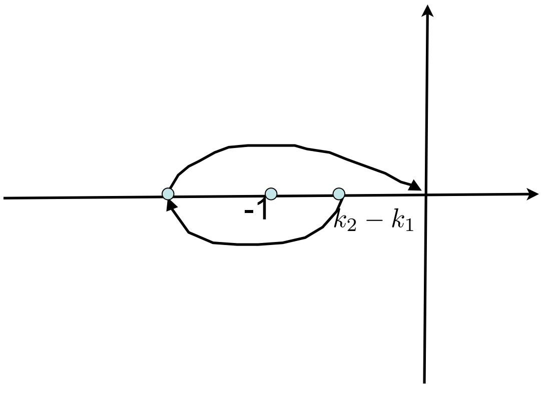
Predicting a limit cycle : illustration

$$H(s) = \frac{k_2 - k_1 + (k_2\tau_1 - k_1\tau_2)s}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

Linearization at the fixed point of the feedback system is unstable if

$$-1 < k_2 - k_1,\ k_2\tau_1 - k_1\tau_2 < -(\tau_1 + \tau_2)$$

Assuming $k_2 < k_1$, we have a second-order non minimum phase system with the Nyquist plot



The describing function predicts a stable limit cycle for any nonlinearity such that N(0)=1 and with a monotone decreasing gain.

Summary of lecture

The describing function method approximates the gain of a static nonlinearity at a given frequency.

Looking for a unity loop gain in the feedback interconnection of a linear system with a static nonlinearity described by its equivalent gain provides a prediction for a limit cycle and its stability.

The prediction is quite accurate when the frequency response of the linear system attenuates higher frequencies and when the periodic solution is well approximated by its first harmonic.