Engineering Tripos Part II

Module 4F2 -- NL Systems and control

Nonlinear systems (7 lectures)

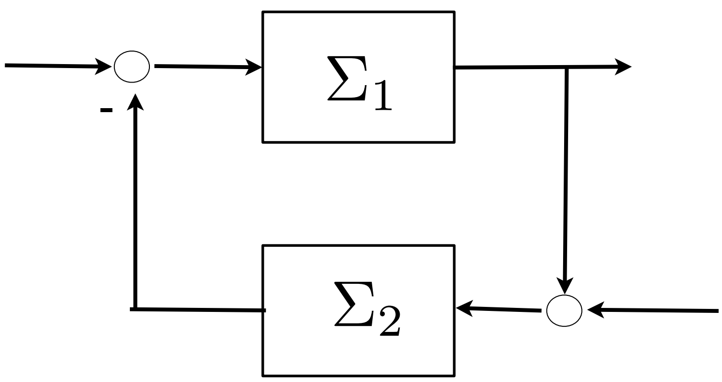
Handout 3

Rodolphe Sepulchre Feb 2014

available on RS homepage

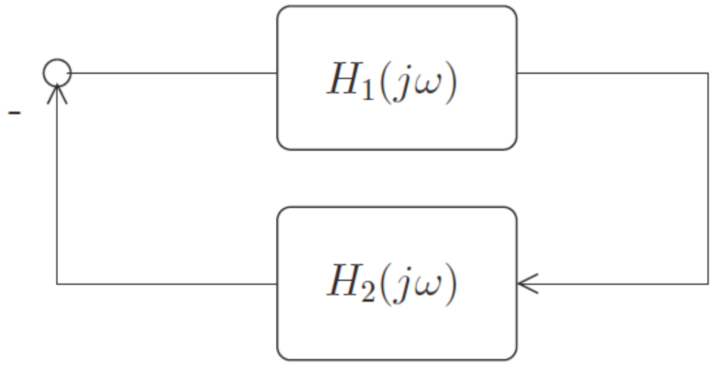
Lecture 5

Feedback interconnections and passivity



Sufficient conditions for stability of the interconnection?

Linear analysis



How do sinusoïdal signals propagate through the feeback loop ?

- condition for oscillation : $1 + \overbrace{H_1(j\omega)H_2(j\omega)}^{L(j\omega)} = 0$
- stability $\Leftrightarrow$ $1 + L(s) = 0$ has no root in RHP.

Nyquist Criterion :

Number of encirclements of $(-1, 0)$ by the Nyquist curve $L(j\omega)$

$$=$$

Number of RHP roots of $1 + L(s) = 0$ minus number of RHP poles of $L(s)$

If $H_1(s)$ and $H_2(s)$ are stable transfer functions, we have the following (sufficient) conditions :

(i) "gain condition" : $|H_1(j\omega)|\,|H_2(j\omega)| < 1 \quad \forall\, \omega$

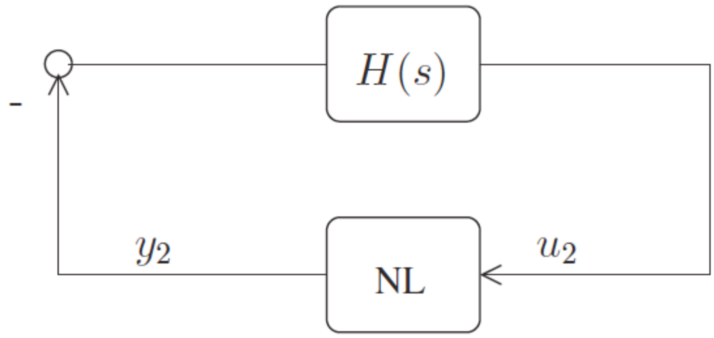
(ii) "phase condition" : $\arg H_1(j\omega) + \arg H_2(j\omega) > -180^o \quad \forall\, \omega$

Nonlinear analogs

gain condition $\longleftrightarrow$ small-gain theorem

phase condition $\longleftrightarrow$ passivity theorem

History: Lure systems



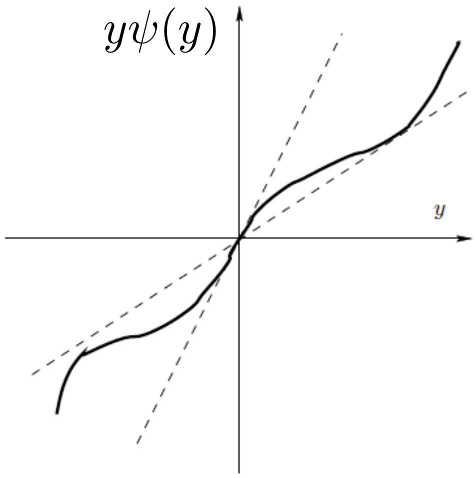
$H(s)$ = transfer function of a linear system

NL is a static nonlinearity $y_2 = \varphi(u_2)$

$H(s) = c^T(sI - A)^{-1}b$

(Assume SISO and minimal realization for simplicity)

Sector nonlinearity and absolute stability



$$k_1 \leq \psi(y) \leq k_2$$

Absolute stability: stability of Lure feedback loop for arbitray nonlinear gain in the sector $[k_1, k_2]$

A necessary condition (and Aizerman conjecture)

A necessary condition for absolute stability is that the feedback system is stable for every constant gain $k \in [k_1, k_2]$

(For a stable system H(s), this means that the Nyquist curve of H(s) does not intersect the interval

$$[-\frac{1}{k_1} + j0, -\frac{1}{k_2} + j0]$$

in the complex plane.

Aizerman conjectured that this condition is also sufficient (1949). Several counterexamples were constructed in the following years (1952-1953).

A sufficient condition

A sufficient condition for absolute stability in the sector $(0, +\infty)$ is that there exists a matrix $P = P^T \succ 0$ such that

$$PA + A^T P \preceq 0$$
$$PB = C^T$$

Proof: Lyapunov analysis with the quadratic Lyapunov function

$$V(x) = x^T P x$$

Time derivative is

$$\dot{V} = x^T (PA + A^T P)x - c^T x \, x^T Pb \, \psi(c^T x)$$

Kalman-Yakubovich-Popov Lemma

The state-space conditions

$$PA + A^T P \preceq 0$$
$$PB = C^T$$

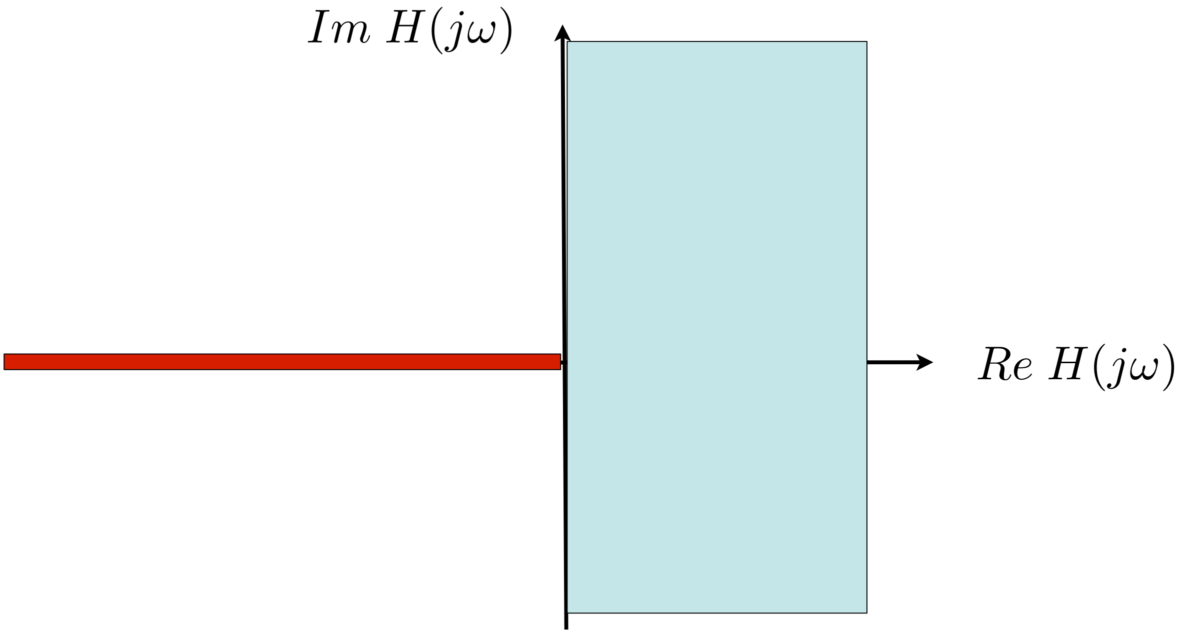
are equivalent (for a minimal realization) to the frequency-domain condition that $H(s) = C^T(sI - A)^{-1}B$ is positive real:

(i) $Re(\lambda_i(A)) \leq 0, \;\; 1 \leq i \leq n$;

(ii) $H(j\omega) + H^T(-j\omega) \geq 0$ for all $\omega \in I\!R$, $j\omega \neq \lambda_i(A)$;

(iii) the eigenvalues of A on the imaginary axis are simple and the corresponding residues $\lim_{s \to s_0}(s - s_0)H(s)$, are Hermitian and nonnegative definite matrices.

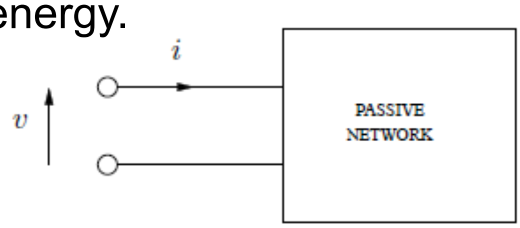
Positive realness is a phase condition:



Gap between the necessary condition:
the Nyquist plot does not intersect the negative real axis
and the sufficient condition:
the Nyquist plot is entirely in the right half plane

Circuit theory: positive realness is a passivity condition.

A passive network can only absorb energy.

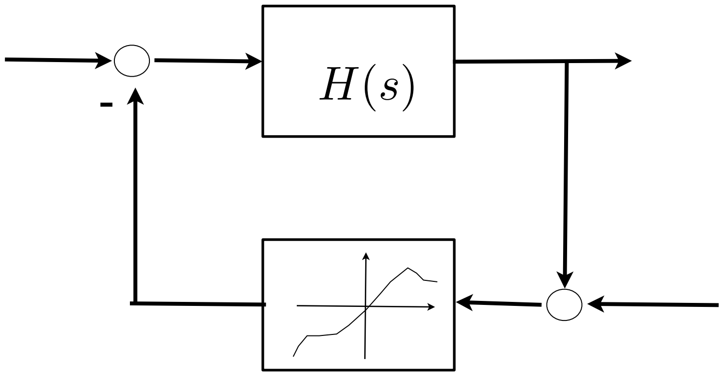


Definition. A network is **passive** if for all admissible v, i which are square integrable on $(-\infty, T]$,

$$\int_{-\infty}^{T} v(t)i(t)\,dt \geq 0.$$

Proposition. Consider a one-port electrical network for which the impedance $Z(s)$ exists and is real-rational. The network is passive if and only if $Z(s)$ is positive-real.

Absolute stability in the sector $(0, +\infty)$

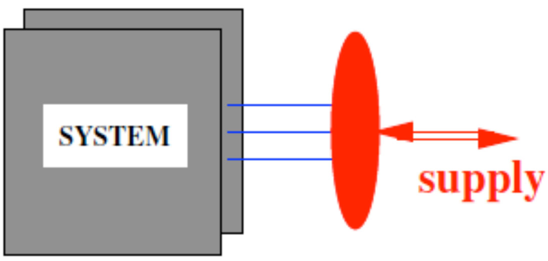


Positive realness: a frequency-domain (phase) condition with energy interpretation from circuit theory.

Equivalence to a quadratic Lyapunov function in the state-space domain thanks to KYP lemma.

Willems dissipativity theory: the linking concept

A dissipative system absorbs supply.



Lyapunov functions quantify the internal dissipation.

The supply rate quantifies the external supply of energy.

Dissipativity theory is a Lyapunov theory for open systems.

Dissipativity, storage, supply

$$(H) \quad \begin{cases} \dot{x} = f(x,u), & x \in I\!R^n \\ y = h(x,u), & u, y \in I\!R^m \end{cases}$$

Assume that associated with the system H is a function $w : I\!R^m \times I\!R^m \to I\!R$, called the *supply rate*, which is locally integrable for every $u \in U$, that is, it satisfies $\int_{t_0}^{t_1} |w(u(t), y(t))| \, dt < \infty$ for all $t_0 \le t_1$. Let X be a connected subset of $I\!R^n$ containing the origin. We say that the system H is *dissipative* in X with the supply rate $w(u,y)$ if there exists a function $S(x)$, $S(0) = 0$, such that for all $x \in X$

$$S(x) \ge 0 \;\text{ and }\; S(x(T)) - S(x(0)) \le \int_0^T w(u(t), y(t)) \, dt \qquad (2.1.5)$$

for all $u \in U$ and all $T \ge 0$ such that $x(t) \in X$ for all $t \in [0, T]$. The function $S(x)$ is then called a *storage function*. □

Dissipativity, passivity, contractivity

A quadratic supply rate is a quadratic form in u and y.

Two important special cases:

Contractivity: $w(u,y) = \| u \|^2 - \| y \|^2$

Passivity: $w(u,y) = u^T y$

Passivity and physics

The importance of passivity stems from its physical interpretation:

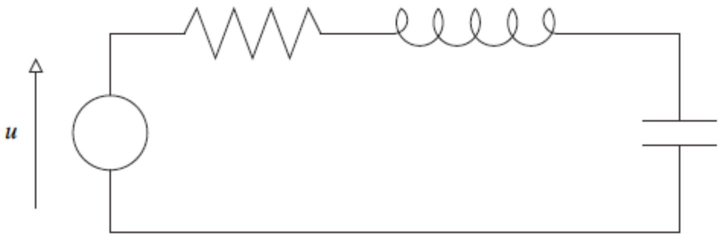
supply = power

electrical circuits: voltage. current

mechanics: force. velocity

...

Passivity of an RLC circuit



$$\left| \begin{array}{lcl} C\,\dot{v}_c & = & i_L \\ L\,\dot{i}_L & = & -R\,i_L - v_C + u \end{array} \right.$$

$$S = \frac{1}{2} L\,{i_L}^2 + \frac{1}{2} C\,{v_C}^2$$

$$\dot{S} = \underbrace{u\,i_L}_{\text{power}} - \underbrace{R\,{i_L}^2}_{\text{dissipation}}$$

Link with the circuit concept of passivity

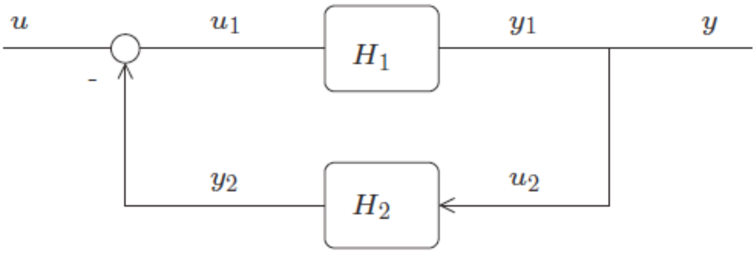
$$S(x(t)) - S(x(0)) \leq \int_0^t y^T(\tau)u(\tau)d\tau$$

or

$$\int_0^t y^T(\tau)u(\tau)d\tau \geq \beta \qquad \forall t$$

(Passivity can be established without the explicit knowledge of the storage function)

The key interconnection property of passive systems



The feedback interconnection of two passive systems is passive

Proof :

$$S = S_1 + S_2 \quad \Rightarrow \quad \dot{S} \leq u_1 y_1 + u_2 y_2 = u^T y$$

Passivity for LTI systems

Let $u(t) = \cos \omega t \quad \Rightarrow \quad y(t) = \Re H(j\omega) \cos \omega t + \Im H(j\omega) \sin \omega t$

$$\Rightarrow \int_0^t u\, y \; d\tau \geq \Re H(j\omega) \int_0^t \cos^2 \omega\tau + \beta$$

Passivity $\Leftrightarrow \quad \Re H(j\omega) \geq 0 \quad \forall\, \omega$

$\Leftrightarrow$ positive real transfer functions : the Nyquist curve is entirely in RHP

$\Leftrightarrow \quad |\arg H(j\omega)| \leq 90^o$

Passivity: a LMI characterization

Passivity of (A, B, C) means the existence of $P = P^T \geq 0$ such that

$$x^T(PA + A^T P)x + 2x^T PBu \leq x^T C^T u$$

or equivalently

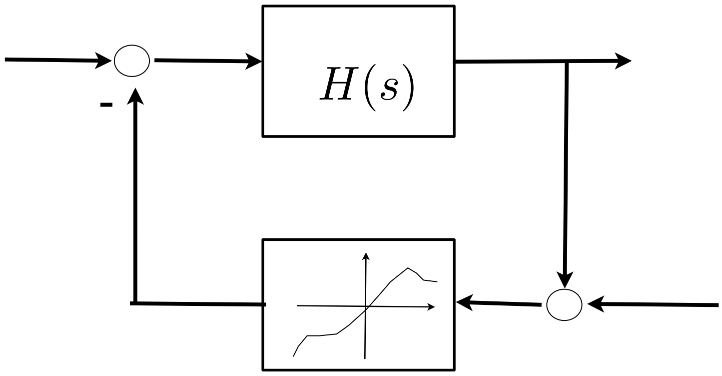
$$\begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq 0$$

Finding $P = P^T$ such that

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & 0 \end{bmatrix} \leq 0$$

is a Linear Matrix Inequality (LMI) in the unknown P.

Absolute stability in the sector $(0, +\infty)$



The static nonlinearity is passive : $y\phi(y) \geq 0$

If H(s) is passive, then the feedback interconnection is passive.

The Lyapunov function is the internal storage.

Passivity and Lyapunov stability

Lyapunov stability is an *internal* concept. Passivity is an *external* concept.

The linking property is zero-state detectability (weaker than observability): the equilibrium $x = 0$ of $\dot{x} = f(x, 0)$ is asymptotically stable in the largest positive invariant set contained in $\{x \in \mathbb{R}^n | \; y = h(x, 0) = 0\}$.

Basic result: If H is passive with C^1 storage function S then the equilibrium $x = 0$ of H with $u = 0$ is stable if either S is positive definite or H is zero-state detectable

Important application: nonlinear PI control

A PI controller is passive for any choice of the proportional and integral gain.

Consequence: PI control cannot destabilize a passive system.

Widely appreciated in industrial control.

Strict passivity and asymptotic stability

Output strict passivity means $\dot{S} \leq u^T y - \delta |y|^2$

(1) With $u = 0$, output strict passivity implies $\dot{S} \leq -\delta |y|^2$
$\Rightarrow$ bounded solutions converge to $\{x \in I\!R^n |\ y = h(x, 0) = 0\}$.

(2) Invariance principle: ω-limit sets of $\dot{x} = f(x)$ are invariant.

(1)+(2) $\Rightarrow$ bounded solutions converge to the largest invariant set in $\{x \in I\!R^n |\ y = h(x, 0) = 0\}$.

This set is the origin $x = 0$ if (and only if) H is zero-state detectable.

Summary of lecture

Stability analysis of feedback interconnections is a central theme of system theory.

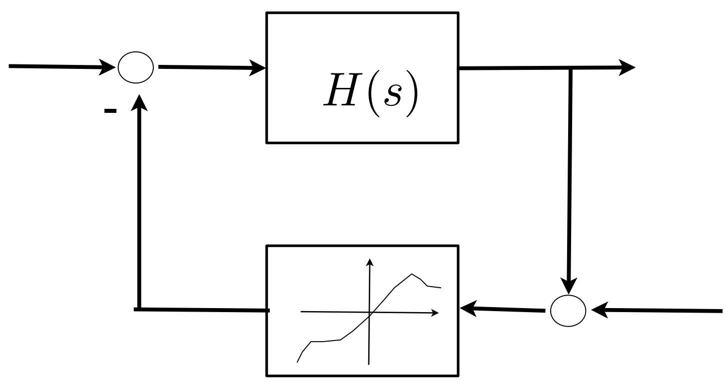
The history of absolute stability illustrates the importance of linking internal and external concepts.

Dissipativity theory is Lyapunov theory for open systems.

Passivity and small gain are two fundamental interconnections theorems. Passivity has an edge because of its physical meaning.

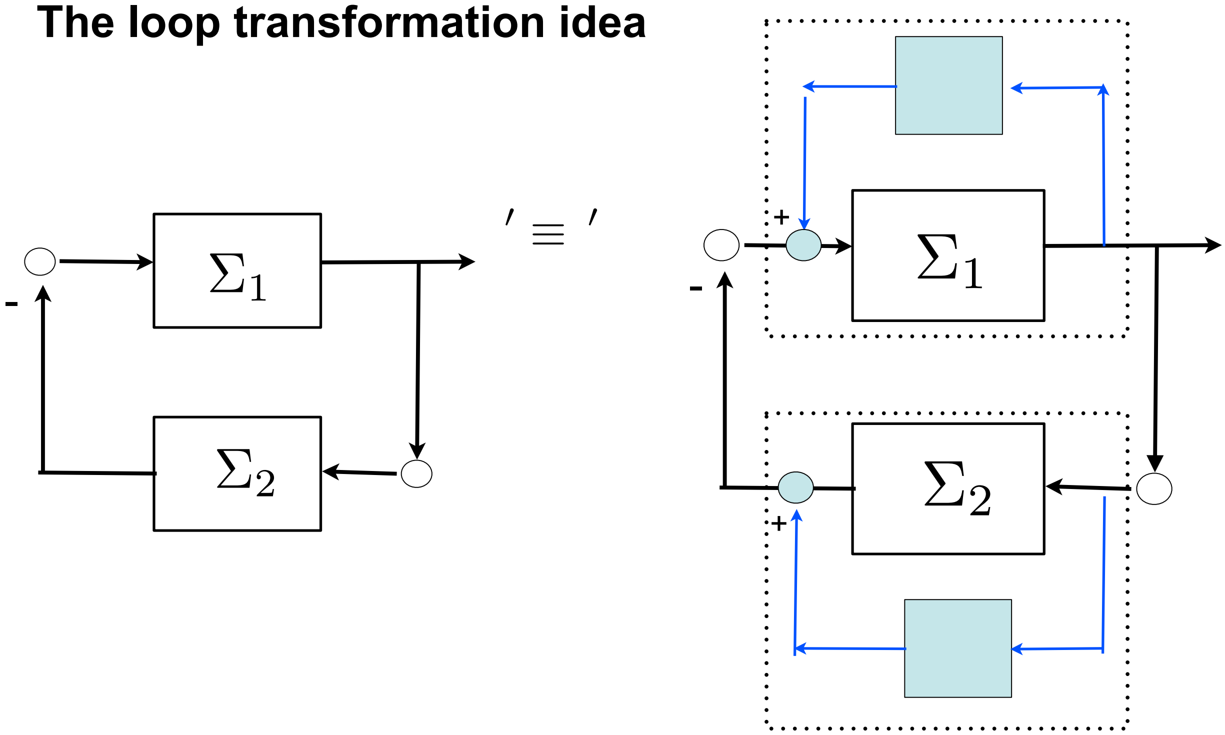
Lecture 6

Circle criterion and Popov criterion



Further exploiting the passivity theorem

The loop transformation idea



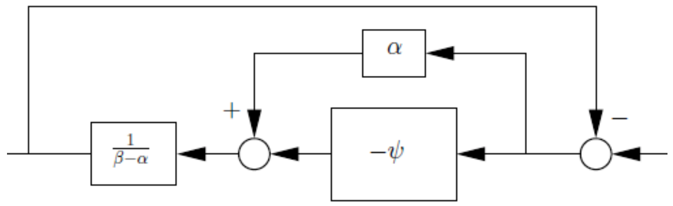
Loop transformations do not change the interconnection but do change each of the subsystems.

The loop transformation idea

- Loop transformations broaden the applicability of passivity theorems.
- Interpretation: *excess* of passivity in one channel may compensate for *shortage* of passivity in the other channel.
- Classical theorems: circle criterion and Popov criterion
- Modern theory: state-space formulation + search for multipliers recast as convex optimization problem.

Excess of passivity of a sector nonlinearity

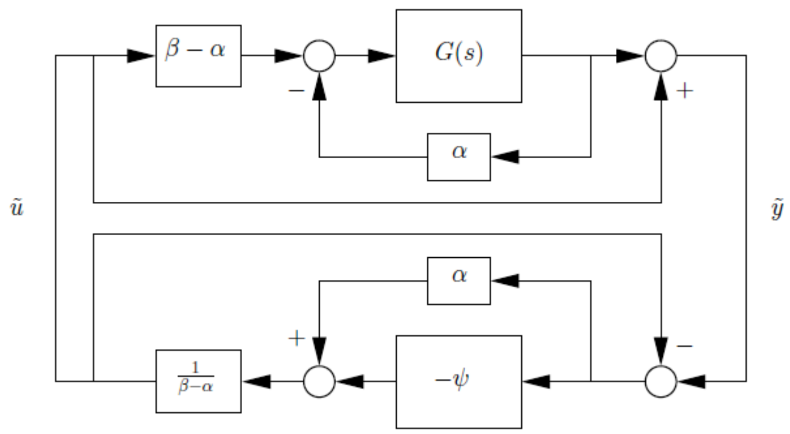
$$\alpha \leq \psi(y,t) \leq \beta \qquad \text{means} \qquad \tilde{\psi} = \frac{\psi - \alpha}{\beta - \psi} \geq 0$$



Excess of passivity:

$$uy - \alpha u^2 \geq 0 \quad \text{and} \quad uy - \tfrac{1}{\beta}y^2 \geq 0$$

Loop transformation for a sector nonlinearity (α, β)

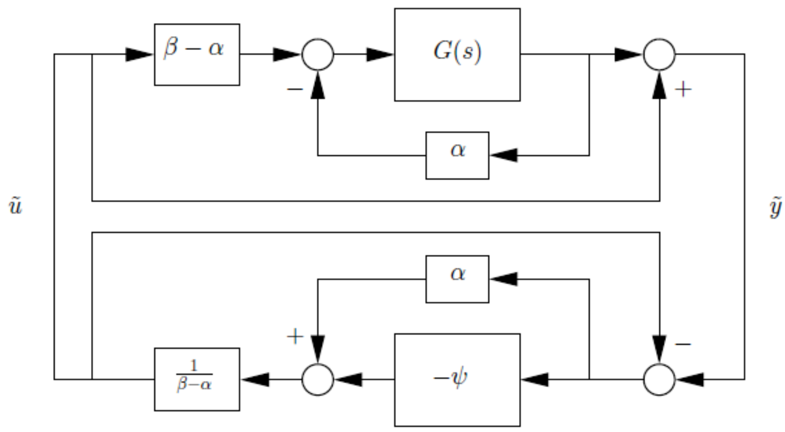


$$\tilde{u} = \frac{1}{\beta - \alpha}(\alpha - \psi)(\tilde{y} - \tilde{u})$$

$$\Rightarrow \quad (\beta - \psi)\tilde{u} = (\alpha - \psi)\tilde{y}$$

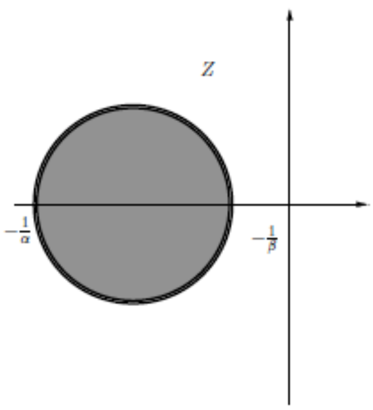
$$\Rightarrow \quad \tilde{u} = -\tilde{\psi}\tilde{y}.$$

Loop transformation of the LTI system



$$\begin{aligned}\tilde{y} &= \tilde{u} + \frac{G}{1+\alpha G}(\beta-\alpha)\tilde{u} \\ &= \frac{1+\beta G}{1+\alpha G} = \tilde{G}\tilde{u}.\end{aligned}$$

Interpretation in the frequency domain



The transformation

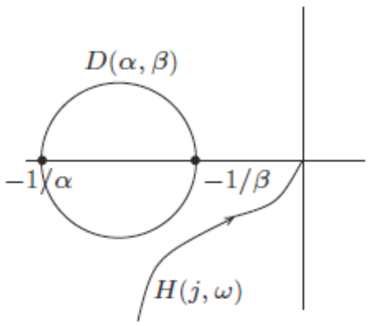
$$Z = \frac{\tilde{Z}-1}{\beta-\alpha\tilde{Z}}$$

maps the RHP to a disk.

Positive realness of $\tilde{Z}$ means that the Nyquist plot if Z does not intersect the disk.

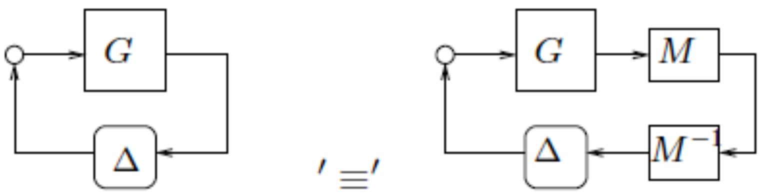
Circle criterion: quadratic storage for absolute stability

Graphical interpretation : Nyquist encirclement condition with the disk $D(\alpha, \beta)$ playing the role of the point $(-1, 0)$



Compare with the necessary condition: the disk replaced by a segment.

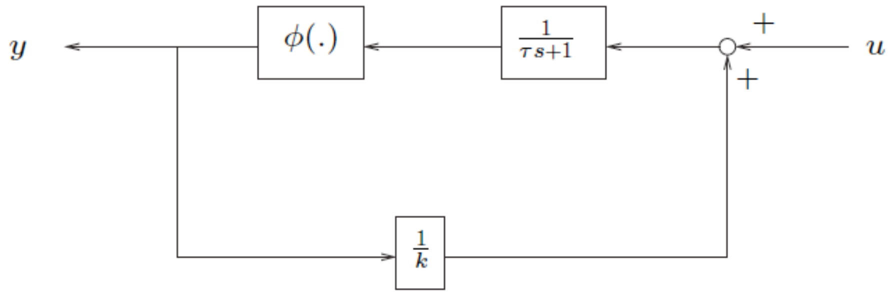
The multiplier idea



Similar idea as the idea of loop transformations: do not change the interconnection but do change each of the subsystems.

Popov multiplier

$\varphi \in [0, k]$ implies passivity of



$$\tau \dot{x} = -x + u + \frac{1}{k}\varphi(x)$$

Storage: $S = \int_0^x \varphi(s)ds$

Popov criterion: Lure-type storage for absolute stability

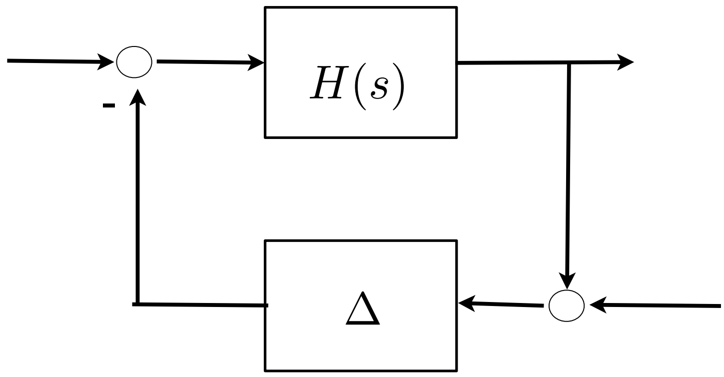
Absolute stability if $\frac{1}{k} + H(s)(\tau s + 1)$ strictly passive.

Graphical interpretation: Popov contour to the right of line through $-k^{-1} + j0$ with slope τ^{-1}.

Note: $V(x) = x^T Px + \int_0^y \phi(s)ds$ is a Lure-type Lyapunov function.

Popov criterion is restricted to time-invariant NL.

Feedback interconnections through convex optimization



Characterize the non LTI part with integral quadratic constraints,

$$\left\langle \begin{bmatrix} u \\ y \end{bmatrix}, \Pi \begin{bmatrix} u \\ y \end{bmatrix} \right\rangle_{\mathcal{H}} \geq 0 \; \forall u, y \in \mathcal{H}$$ (e.g. passivity: $\Pi = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$)

Express the search of a quadratic storage via LMIs using KYP lemma.

(Megretski and Rantzer, 1997)

Summary of lecture

The application of the passivity and small gain theorems is considerably broadened with the use of loop transformations and multipliers.

The circle criterion illustrates loop transformations. Popov criterion illustrates the use of multipliers.

There is a general theory that formulates the search of a quadratic Lyapunov function as a convex optimization problem starting from an IQC characterization of the non LTI part of the interconnection.

Studying stability from an interconnection viewpoint is a fundamental component of system theory.