Engineering Tripos Part II

Module 4F2 -- NL Systems and control

Nonlinear systems (7 lectures)

Handout 2

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available on RS homepage

Lecture 3

How to study the asymptotic behavior of NL systems?

Consider the differential equation $\dot{x} = f(x)$ in M and assume that the solution $\phi(t, x_0)$ exists for all $t \geq 0$.

- How to study the limiting behavior of the solution for large t ?
- How complicated can it be?
- Is it stable to perturbations?

The limit set of a solution

The ω-limit set of a (bounded) solution is defined as

$$\omega(x_0) = \{x \in M \mid \exists (t_n)_{n\geq 0} \to +\infty \text{ s.t. } \phi(t_n, x_0) \to x \text{ as } n \to +\infty\}$$

- Examples of limit sets : fixed point, closed orbit, homoclinic orbit, heteroclinic orbit, ...
- Anything else?
- The α-limit set is the same definition for $t_n \to -\infty$.

Poincaré-Bendixson theorem

Limit sets of two-dimensional systems are not arbitrary. They are characterized as follows: .

Suppose that the solution $\phi(t, x_0)$ is confined to an invariant compact set Ω in the plane. If $\omega(x_0)$ does not contain a fixed point, then it is a closed orbit.

- Proof is not trivial.
- Essence of the theorem: trajectories define non-intersecting curves.
- No such restriction in higher dimensions.

A caveat about Poincaré-Bendixson theorem

A false implication of the theorem is as follows: limit sets of planar systems are either equilbria or closed orbits.

A counter-example: heteroclinic orbit

Little is known about limit sets in general.

Limit sets have the following properties: .

- the limit set of a bounded solution is not empty.
- limit sets are closed.
- limit sets are invariant: if $x \in \omega(x_0)$ then $\phi(t,x) \in \omega(x_0)$ for all t.

Limit sets can have a complicated structure. For instance, the limit set of a strange attractor in $\mathbb{R}^3$ is neither a surface nor a curve.

Definition and properties of limit sets extends to periodic differential equations and to (time-invariant or periodic) maps.

Stable and unstable manifolds of a fixed point

Consider $\dot{x} = f(x)$ and assume smoothness.

The stable manifold $W^s(p)$ of a fixed point p is the set

$$W^s(p) = \{x \mid \phi(t,x) \to p \text{ as } t \to +\infty\}$$

The unstable manifold of p is the set

$$W^u(p) = \{x \mid \phi(t,x) \to p \text{ as } t \to -\infty\}$$

Stable manifold theorem: if p is hyperbolic, then $W^s(p)$ is a smooth manifold locally tangent to the stable eigenspace of the linearization at p. Likewise, $W^u(p)$ is a smooth manifold locally tangent to the unstable eigenspace of the linearization.

Stable and unstable manifolds of a bistable system

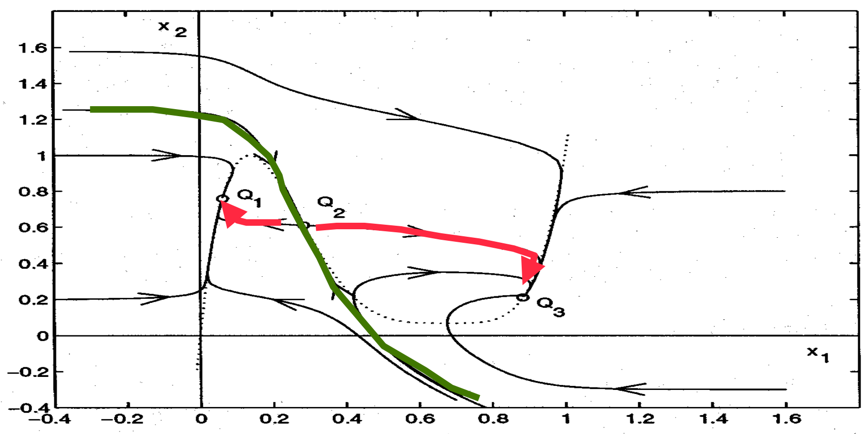


Figure 1.24: Phase portrait of the tunnel diode circuit of Example 1.2.

The green curve is the stable manifold of the saddle point. It is a separatrix of the two basins of attraction. The red curve is the unstable manifold of the saddle. It connects the saddle to the two stable equilibria. A general and important picture.

Lyapunov stability

$$\dot{x} = f(x) \qquad ; \; x \in I\!R^n \; , \; f \in C^1$$

A solution $x^*(t)$ can be

stable : close init. cond. $\Rightarrow$ solutions stay close on $[0, \infty)$ (continuity on the infinite interval)
unstable : not stable
attractive : close init. cond. $\Rightarrow$ same asymptotic behavior
asymptotically stable : stable + attractive
exponentially stable : as. stable + exponential decay estimate of solutions
globally asymptotically stable (GAS) : as. stable and domain of attraction is $I\!R^n$.

Common restriction: $x^*(t)$ is an equilibrium solution, say $x^*(t) \equiv 0$

Checking Lyapunov stability

Lyapunov first method:

Study Jacobian linearization $A = \dfrac{\partial f}{\partial x}(x^*)$

- x^* exp. stable if Jacobian linearization as. stable (i.e. A Hurwitz)

- x^* exp. unstable if Jacobian linearization exp. unstable (i.e. A has one eigenvalue in right-half plane)

- inconclusive otherwise (A has one eigenvalue on the imaginary axis)

Checking Lyapunov stability

Lyapunov second method: Lyapunov function

Find a scalar function $V \in C^1(\mathbb{R}^n, \mathbb{R})$, minimum at the equilibrium, $V(0) = 0$, $V(x) > 0$, and study

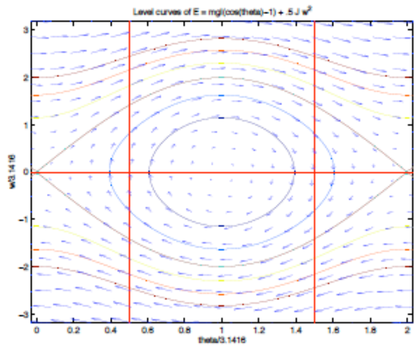
$$\dot{V} = \frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial V}{\partial x} \cdot f(x)$$

- $\dot{V} \leq 0$ near the equilibrium implies Lyapunov stability
- $\dot{V} < 0$ near the equilibrium implies asymptotic stability
- $\dot{V} < 0$ everywhere (except 0) and V proper implies global asymptotic stability

Energy as a Lyapunov function

Remember that the energy $E = \frac{1}{2}v^2 - \frac{g}{l}\cos\theta$

is constant along the solutions of the pendulum equation.



E is minimum at $(0,0)$ and vector field is everywhere tangent to the level curves (conservation of energy) $\Rightarrow$ Lyapunov stability.

Remark: Linearization is inconclusive for stability of $(0,0)$ but proves instability of $(\pi, 0)$.

Checking as stability: an important refinement

Frequent situtation: $\dot{V}(x) \leq -W(x)$ with $W(x) \geq 0$ but not positive definite.

Fact 1 : bounded solutions asymptotically converge to the set $M = \{x \in \mathbb{R}^n : W(x) = 0\}$

Fact 2 : ω-limit sets of $\dot{x} = f(x)$ are invariant

(1)+(2) $\Rightarrow$ bounded solutions converge to the largest invariant set contained in M!

This is LaSalle invariance principle.

Asymptotic stability of the damped pendulum

Pendulum with friction δ:

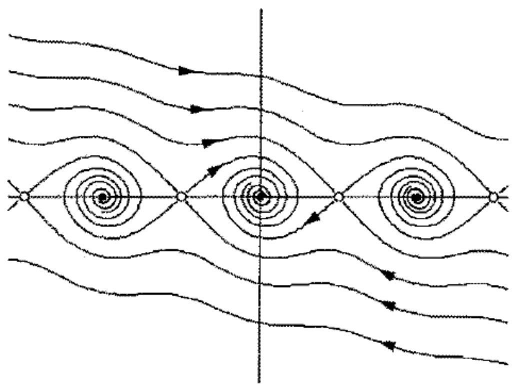
$$\begin{cases} \dot{\theta} = \omega \\ J\dot{\omega} = -mgl\sin\theta - \delta\omega \end{cases}$$

Energy: $E = mgl(1 - \cos\theta) + \frac{1}{2}J\omega^2 \quad \rightarrow \quad \dot{E} = -\delta\omega^2 \leq 0$

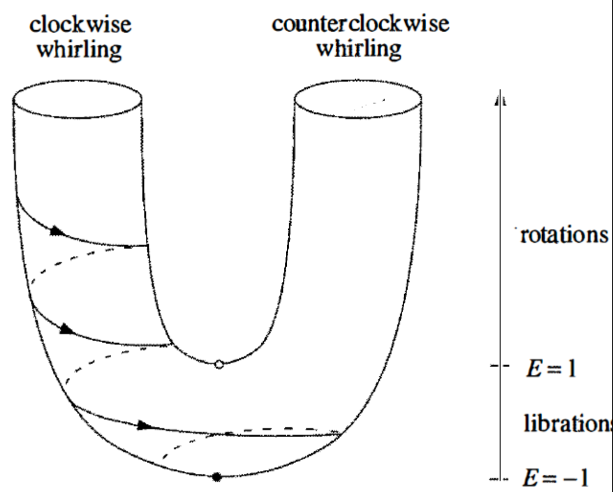
Solutions converge to the largest invariant set with zero velocity: the origin.

If $\delta = \delta(t)$, one only has the weaker conclusion that the velocity asymptotically vanishes along solutions.

Phase portrait of the damped pendulum



Phase portrait



Energy portrait

More on Lyapunov stability

- Mathematical definitions and time-varying case (uniformity !)
- Converse theorems
- Characterizations from estimates
- Weaker regularity conditions on f and V

Standard reference: Khalil

Remark: stability results about equilibria extend almost trivially to stability results about compact sets.

Lyapunov stability for linear systems

$$\dot{x} = Ax$$

$x = 0$ asymptotically stable $\quad\Leftrightarrow\quad$ $x = 0$ exp. stable

$\Leftrightarrow Re\ \lambda(A) < 0$

$\Leftrightarrow \exists\, P = P^T > 0 \ :\ PA + A^T P = -I$ "Lyapunov equation"

$\Leftrightarrow \exists\, \alpha > 0,\ K > 0 \ :\ \|x(t)\| \leq K\|x(0)\|e^{-\alpha t}$

Lyapunov stability for linear systems

- $V(x) = x^T P x$

  $\Rightarrow \dot{V} = -x^T x < 0 \qquad \forall x \neq 0$

  $\Rightarrow \exists\, \alpha_1, \alpha_2 > 0 \ :\ \alpha_1\|x\|^2 \leq x^T P x \leq \alpha_2\|x\|^2$

  $\longrightarrow$ exponential estimate

- Choose

$$P = \int_0^\infty (e^{A\tau})^T (e^{A\tau}) \mathrm{d}\tau$$

$$\left(\Leftrightarrow V(x) = \int_0^\infty \|x\|^2 \mathrm{d}\tau\right)$$

Local Lyapunov stability for nonlinear systems

$$\dot{x} = f(x) \qquad f(0) = 0 \qquad A = \frac{\partial f}{\partial x}(0)$$

- $x = 0$ exponentially stable $\Leftrightarrow$ A Hurwitz

  (use $x^T P x$ as Lyapunov function)

- $x = 0$ asymptotically stable

  $\Leftrightarrow \exists\ \alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) : \dot{V}(x) \leq -\alpha_3(\|x\|)$

  $\Leftrightarrow \|x(t)\| \leq \beta(\|x(0)\|, t)$

α_1 "class K" $\qquad$ β "class KL"

Summary of lecture

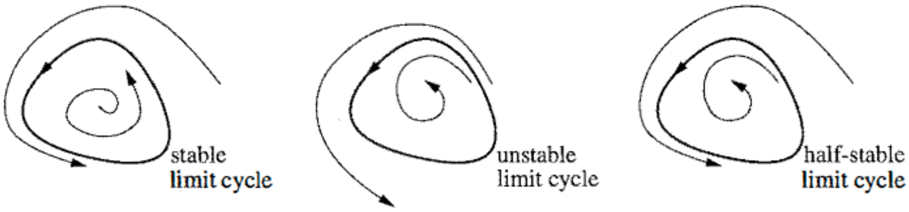
The asymptotic behavior of nonlinear systems is studied through a characterization of its limit sets.

Stable and unstable manifolds of hyperbolic fixed points are important robust geometric objects. Locally characterized by linearization.

Lyapunov stability helps estimating the basin of attraction of a limit set. Quadratic Lyapunov functions can be constructed locally. Energy considerations help the construction.

Lecture 4

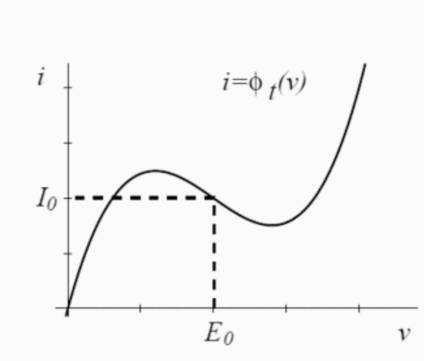
Limit cycles

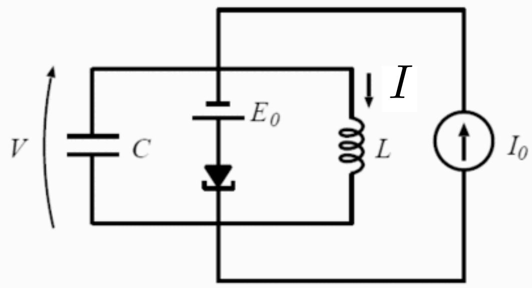


A closed orbit is the trace (in phase space) of a periodic solution with a finite period.

A limit cycle is an isolated closed orbit.

Van der Pol oscillator





$$C\frac{dV}{dt} = -I - \phi(V)$$

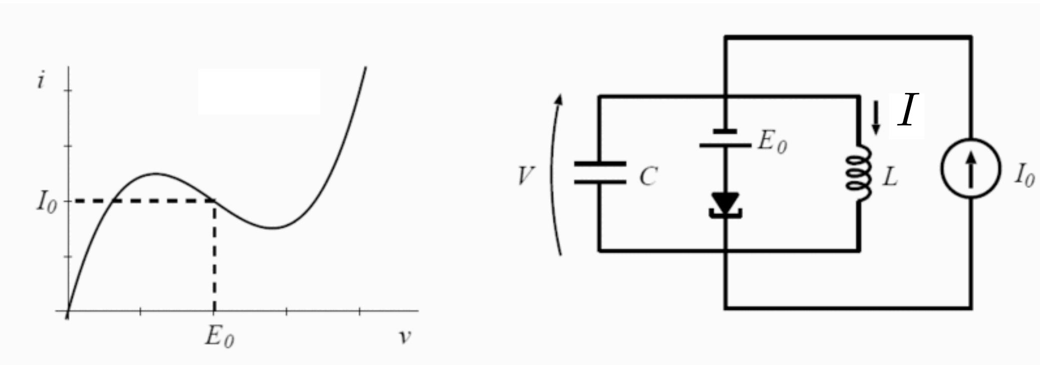
$$L\frac{dI}{dt} = V$$

$$\phi(V) = \frac{1}{3}V^3 - V$$

Van der Pol oscillator: history

- First designed by Dutch electrical engineer Van der Pol while working at Phillips.
- Forcing Van der Pol oscillator with a harmonic signal might lead to chaotic behavior. Accidentally observed experimentally (irregular noise was heard near certain driving frequencies).
- A model for many oscillatory systems, including the two-dimensional reduction of Hodgkin-Huxley model of the action potential (studied by Fitzugh and Nagumo).

A first view: oscillation = hysteresis + adaptation

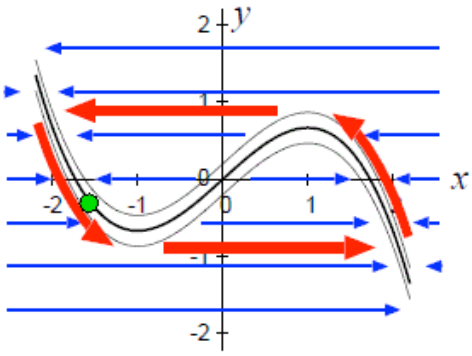


$$C\frac{dV}{dt} = -I - \phi(V)$$ Hysteretic switch studied in first lecture

$$L\frac{dI}{dt} = V$$ Integral negative feedback (= adaptation)

We expect integral action to be slow compared to system dynamics: this mechanism will require L >> C

A first view: oscillation = hysteresis + adaptation



(from Scholarpedia)

$$C\frac{dV}{dt} = -I - \phi(V)$$ Hysteretic switch studied in first lecture

$$L\frac{dI}{dt} = V$$ Integral negative feedback (= adaptation)

We expect integral action to be slow compared to system dynamics: this mechanism will require L >> C

A second view: oscillation = regulated exchange of energy

$$C\frac{dV}{dt} = -I - \phi(V)$$

$$L\frac{dI}{dt} = V$$

rewritten as

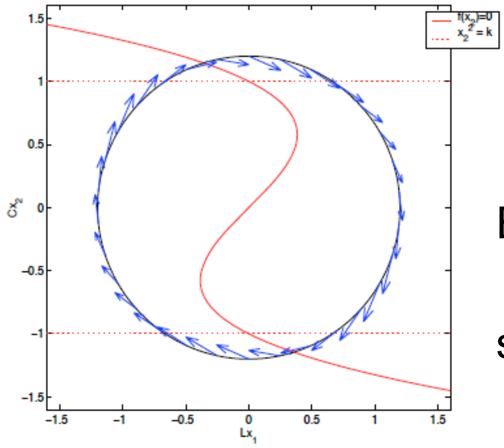
$$LC\,\ddot{V} + V = -L\phi'(V)\dot{V}$$

or, in new time $t' = \frac{t}{\sqrt{LC}}$

$$\ddot{V} + V = -\epsilon\phi'(V)\dot{V}$$ with $\epsilon = \sqrt{\frac{L}{C}}$

Harmonic oscillator with nonlinear damping.
Weakly nonlinear for ϵ small, that is when C>>L

A second view: oscillation = regulated exchange of energy



$$\ddot{V} + V = -\epsilon\phi'(V)\dot{V}$$

Energy $$E = \frac{V^2}{2} + \frac{\dot{V}^2}{2}$$

satisfies $$\dot{E} = -\epsilon\phi'(V)\dot{V}^2$$

If E is small, then $\dot{E} > 0$;

If E is large, then $\dot{E} < 0$ "most of the time"

Sustained oscillation when negative damping balances positive damping on average

Limit cycles: analysis

- Difficult in general because the analysis requires *integrating* the vector field
- Special tools in the plane, e.g. Poincaré-Bendixson
- Asymptotic methods: singular perturbation analysis or averaging

Limit cycles: stability analysis

- stability of limit cycle is different from stability of a periodic solution
- Poincaré idea: convert limit cycle stability analysis to fixed point analysis of the Poincaré map

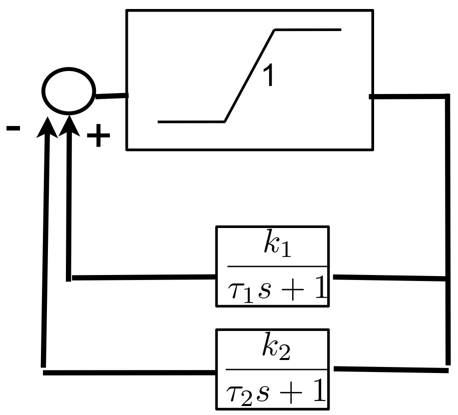
Asymptotic methods and perturbation theory

Introduce a "small" parameter $\varepsilon > 0$

- $\dot{x} = f(x, \varepsilon, t)$: "regular perturbation"
  - series expansion $x(t, \varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \Theta(\varepsilon^3)$ (convergence over *finite* time interval)
  - averaging theory for $\dot{x} = \varepsilon f(t, x, \varepsilon)$ with f T-periodic (conclusion over *infinite* time interval)
- $\begin{cases} \dot{x}_1 &= f_1(x_1, x_2) \\ \varepsilon \dot{x}_2 &= f_2(x_1, x_2) \end{cases}$: "singular perturbation"

(conclusion over *infinite* time interval)

Illustration of Poincaré-Bendixson



A positive feedback loop for hysteresis

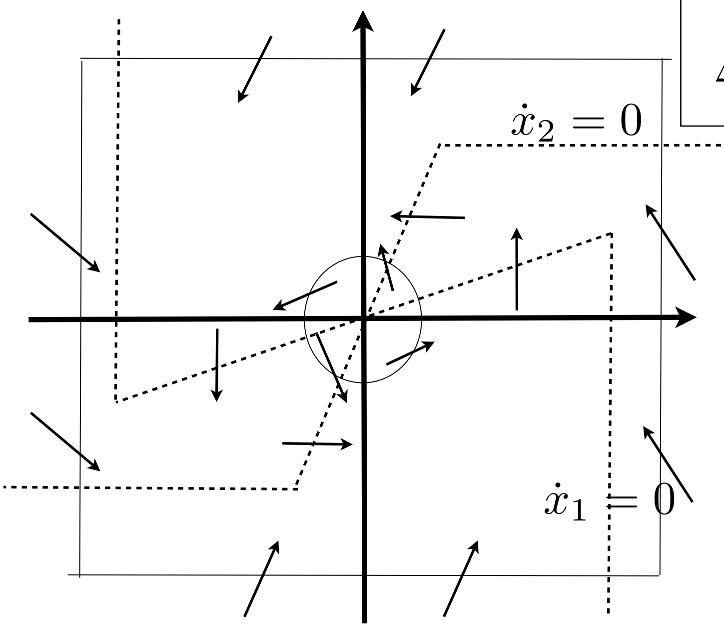
A negative feedback loop for adaptation

$$\tau_1 \dot{x}_1 = -x_1 + S(k_1 x_1 - k_2 x_2)$$
$$\tau_2 \dot{x}_2 = -x_2 + S(k_1 x_1 - k_2 x_2)$$

When does this model admit a limit cycle oscillation?

Illustration of Poincaré-Bendixson

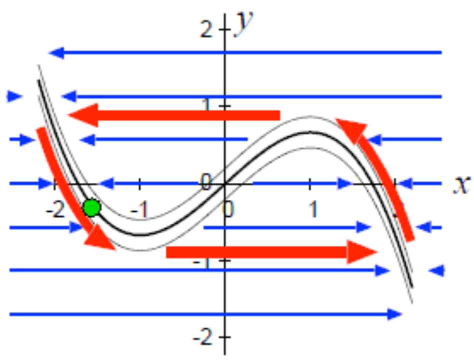
$$Tr > 0 \Leftrightarrow \frac{k_1 - 1}{\tau_1} > \frac{1 - k_2}{\tau_2}$$
$$\Delta > 0 \Leftrightarrow k_1 - k_2 < 1$$



A limit cycle must exist if the origin is a repeller.

Slow-fast analysis (L>>C)

$$\epsilon \frac{dV}{dt} = -I - \phi(V)$$
$$\frac{dI}{dt} = V$$



Fast dynamics: vector field is nearly horizontal away from cubic isocline

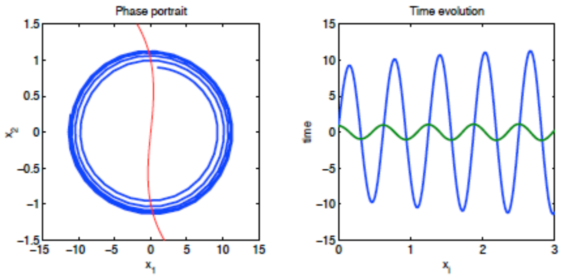
$$\frac{dV}{dt} = -I - \phi(V)$$ (the bistable switch, again)

Slow dynamics: there exists an invariant manifold near the cubic where the local dynamics are roughly

$$\frac{dI}{dt} = V, \; I = -\phi(V) + 0(\epsilon)$$

Averaging analysis (C>>L)

$$\ddot{V} + V = -\epsilon\phi'(V)\dot{V}$$



In polar coordinates $V = r\sin\theta,\ \dot{V} = r\cos\theta$

$$\dot{r} = -\epsilon r\phi'(r\sin\theta)\cos^2(\theta),\ \dot{\theta} = 1 + 0(\epsilon)$$

The averaged system

$$\dot{r} = -\epsilon r\frac{1}{2\pi}\int_0^{2\pi}\phi'(r\sin\theta)\cos^2(\theta)d\theta$$
$$= \frac{\epsilon}{8}(1 - \frac{r^2}{4})$$

has an exponentially stable equilbrium at $r = 2$

Summary of lecture

Limit cycles are the steady-state solutions of nonlinear oscillators.

Two oscillation mechanisms are illustrated by the Van der Pol circuit: (1) hysteresis + adaptation (widespread in biology) and (2) conservative system with nonlinear damping (widespread in electromechanical devices).

Analysis of limit cycles is hard in general. But asymptotic methods are powerful to analyze relaxation oscillators or weakly nonlinear oscillators.

Summary of lecture

Phase portrait: two-dimensional behaviors can be drawed.

Linear phase portraits determine local behavior near hyperbolic fixed points

Tunnel diode: archetype example of bistable behavior

Pendulum: archetype example of conservative mechanical behavior.

The saddle point is an important ‘hidden’ fixed point. A key ruler of nonlinear behaviors.