Engineering Tripos Part IIB

Module 4F2: Nonlinear Systems and Control

Examples Paper 4F2/3

1. Suppose two nonlinear dynamic systems are coupled together, as shown in Fig. 1, and that their defining equations are:

$$\text{System A:} \quad \frac{d^3y(t)}{dt^3} = g[y(t), \dot{y}(t), \ddot{y}(t), u(t), t]$$
$$\text{System B:} \quad \ddot{u} = h[u(t), \dot{u}(t), y(t), t]$$

where g and h are nonlinear functions. Show that they can be represented by a single vector differential equation

$$\dot{x}(t) = f[x(t), t].$$

What is the dimension of x?

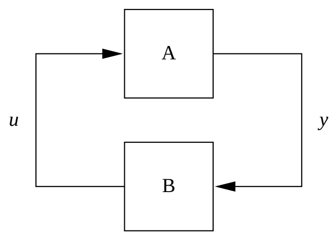

Figure 1: Feedback interconnection

2. Let $S(x) := max(-1, min(1, x))$. Consider the feedback interconnection in Fig. 2.

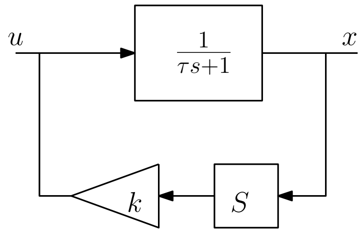

Figure 2: A saturated positive feedback interconnection.

a) Write a state space model of the system;

b) for constant u, find $V_k(x, u)$, such that

$$\dot{x} = -\frac{\partial V_k}{\partial x}(x, u);$$

c) determine for which pairs (u, k) the system is bistable, that is, has 2 stable and 1 unstable fixed point.

3. A phase-locked loop (used in communications networks) is described by the equation

$$\ddot{y}(t) + [a + b\cos y(t)]\dot{y} + c\sin y(t) = 0$$

where $a > b \geq 0$ and $c > 0$. Express this in state-space form. Find the equilibrium points and examine their stability.

4. Find the analytic solution of

$$\dot{x} = -x^3$$

for an arbitrary initial condition x_0 and

a) show that $x(t) \to 0$ as $t \to \infty$ but the decay is not exponential;

b) use MATLAB to compare the solutions of $\dot{x} = -x^3$ and $\dot{x} = -x$ for the initial condition $x_0 = 10$ and $0 \leq t \leq 10$.

5. Consider the two-dimensional system described by the feedback interconnection in Fig. 3, where each block in E is defined by

$$E : \begin{cases} \dot{x} = -x + S(u) \\ y = 2x \end{cases},$$

with $S(\cdot)$ is a smooth sigmoid.

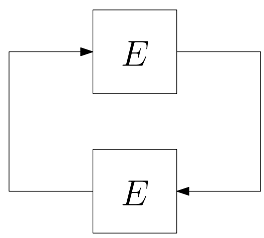

Figure 3: E-E interconnection

a) Draw a phase portrait of the system using MATLAB.

b) Using the change of coordinate $z_i = S(2x_i)$, $i = 1, 2$, show that the system is a gradient system, that is, it can be written as

$$\dot{z} = -Q(z)\frac{\partial V}{\partial z}$$

where $V(\cdot)$ is a scalar potential and the (diagonal) matrix $Q(z)$ is positive definite for all z.

c) Determine the limit sets of the system by using $V(z)$ as a Lyapunov function.

d) Use MATLAB to draw the level curves of V in the phase portrait. Try to visualize the shape of V, in particular in the vicinity of the saddle point.

6. In each of the cases below show that the general solution of the equation $\dot{x} = Ax$ is as given: (Hint: reduce the matrix equation to appropriate scalar equation(s).)

$$\text{(a)}\ A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad x(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\text{(b)}\ A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad x(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 1 \end{pmatrix},$$

(where c_1 and c_2 are arbitrary constants.) Characterise the stability of the equilibrium in each case. Find the eigenvalues of A. Comment.

7. The Rössler system has been used to model a variety of systems to demonstrate evolution from well-behaved limit cycles to chaotic solutions, e.g. dripping water as flow rate is increased, onset of cardiac fibrillation. The system is described by

$$\begin{aligned} \dot{x}_1 &= x_1 - x_1 x_2 - x_3 \\ \dot{x}_2 &= x_1^2 - a x_2 \\ \dot{x}_3 &= b x_1 - c x_3. \end{aligned}$$

Find the equilibrium solutions and investigate their stability as a function of the parameters. (*Note:* The Routh-Hurwitz Stability Criteria given in the Control section of the Electrical and Information Data Book will be useful for part of this question.)

8. Consider the following piecewise-linear system representing a non-linear PD-controller for an inertial mass. The torque input to the inertial mass is provided through a relay with dead-zone: $u = \pm U$ if $|z| > 1$, $u = 0$ otherwise. Let $r = 0$, $k_p = 2$, $k_v = 1$ and $U = 4J$. Obtain a state equation description of the system and find the equilibrium states. (*Hint:* You need a different equation for each condition of the relay.)

Sketch the state-plane trajectories. Notice that, strictly speaking, the vector field for the overall system is discontinuous. Determine the points at which discontinuities occur ("switching surfaces"). Do solutions exist starting at these points? If not, speculate about the possible behaviours of the physical system if it finds itself on one of these switching surfaces.

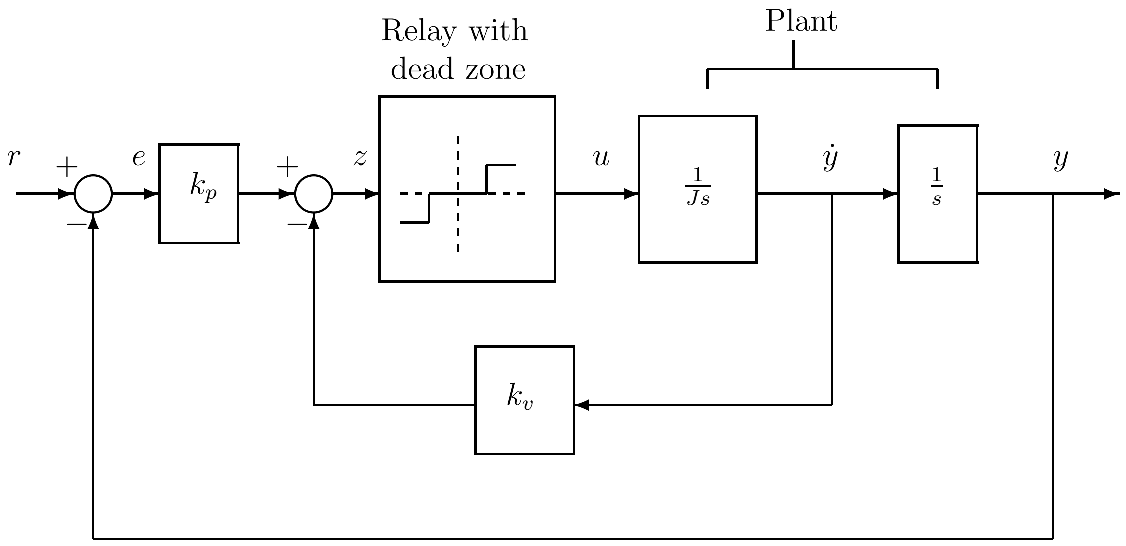

Figure 4: Nonlinear PD controller

9. Prove that limit sets are invariant.

10. Consider the phase-locked loop of question 3.

    (i) By writing the system in the standard state equation form with $x_1 = y$ and $x_2 = \dot{y}$ and using the function
    $$V(x_1, x_2) = c(1 - \cos x_1) + \frac{1}{2}x_2^2$$
    where $c > 0$, show that the origin is a stable equilibrium point.

    (ii) Use LaSalle's theorem to show that the origin is asymptotically stable.

    (iii) Sketch the function $V(x_1, x_2)$ and deduce the stability of each equilibrium.

Answers

3. Equilibrium points: $y = n\pi$ for n an integer. Stable equilibria: n is even or zero.

7. Equilibrium points: origin and, if $d = a(1 - b/c) > 0$, $(\pm\sqrt{d}, 1 - b/c, \pm b\sqrt{d}/c)$. At the origin stability requires $a > 0$, $b > c > 1$. Under the assumption that $d > 0$, stability at other equilibria requires:

$$\begin{aligned} c &> 0 \\ a + c &> b/c \\ a\left((a + c - \frac{b}{c})(c + 2 - \frac{3b}{c}) - 2(c - b)\right) &> 0 \end{aligned}$$

Past papers

The following past Tripos questions are suitable for further practice:
Part IIB **Module 4F2**: 2011, Q. 1. Part IIB **Module 4F3**: 2010, Q. 1. 2009, Q. 1. 2008, Q. 2. 2007, Q. 2. 2006, Q. 4. 2005, Q. 1. 2004, Q. 1. 2003, Q. 2. Part IIB **Module I3**: 2002, Q. 2, 3(b). 2001, Q. 1. 2000, Q. 1(a) and (b). 1999, Q. 1. 1998, Q. 1,4(a). 1997, Q. 1. 1996 Q. 1,2.

R. Sepulchre,
February 2014