Engineering Tripos Part II

Module 4F2 -- NL Systems and control

Nonlinear systems (7 lectures)

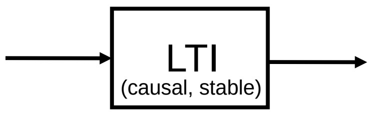
Handout 1

Rodolphe Sepulchre Feb 2014

available on RS homepage

Linear behaviors (LTI systems, mostly 3F1 and 3F2)

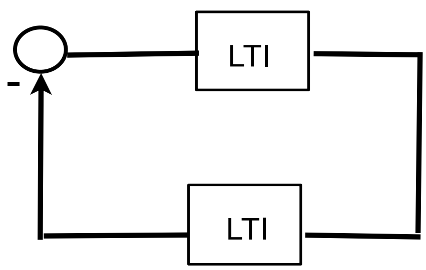
- The transfer function view: sinusoids map to sinusoids



- The state-space view: state parametrizes memory

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \qquad (x, u) \in \mathbb{R}^n \times \mathbb{R}^m$$

- The interconnection view: systems are made of simpler systems



Nonlinear behaviors
(4F2: mostly time-invariant, continuous-time)

• Literally: every behavior that fails the homogeneity and superposition principle

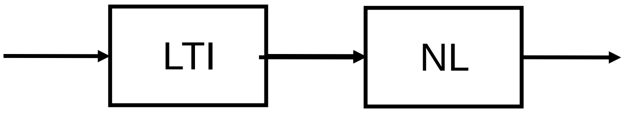
higher harmonics, dependence on amplitude and initial condition, ...

• The state-space view:

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \qquad (x,u) \in M \times U$$

Different sources of nonlinearity: state-space, update equation, output map

• The interconnection view: systems are made of simpler systems



Remark: This 'open systems' viewpoint will not be found in most textbooks on (closed) dynamical systems.

4F2: Main questions of the course

• Nonlinear models versus linear models: why and when?

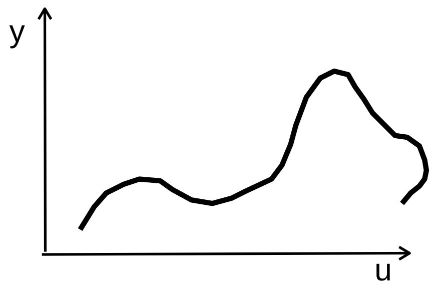
• Basic nonlinear phenomena in engineering: hysteresis, multistability, nonlinear oscillations, ...

• Basic mathematical tools for analysis and design of nonlinear behaviors

Today's lecture: static analysis and one-dimensional state-spaces.

Static behaviors (SISO: one "input", one "output")

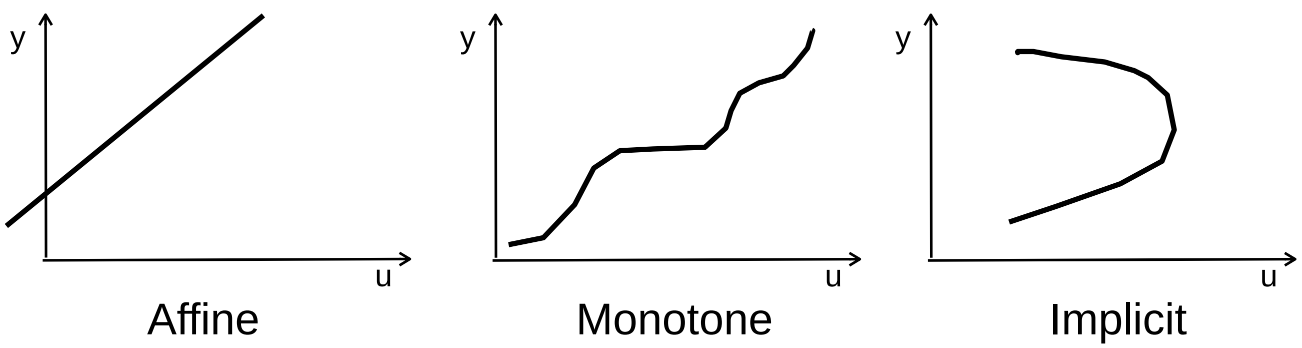
- A static behavior is described by the graph of an algebraic equation F(u,y)=0



NL behavior: every graph that is not a straight line through the origin.

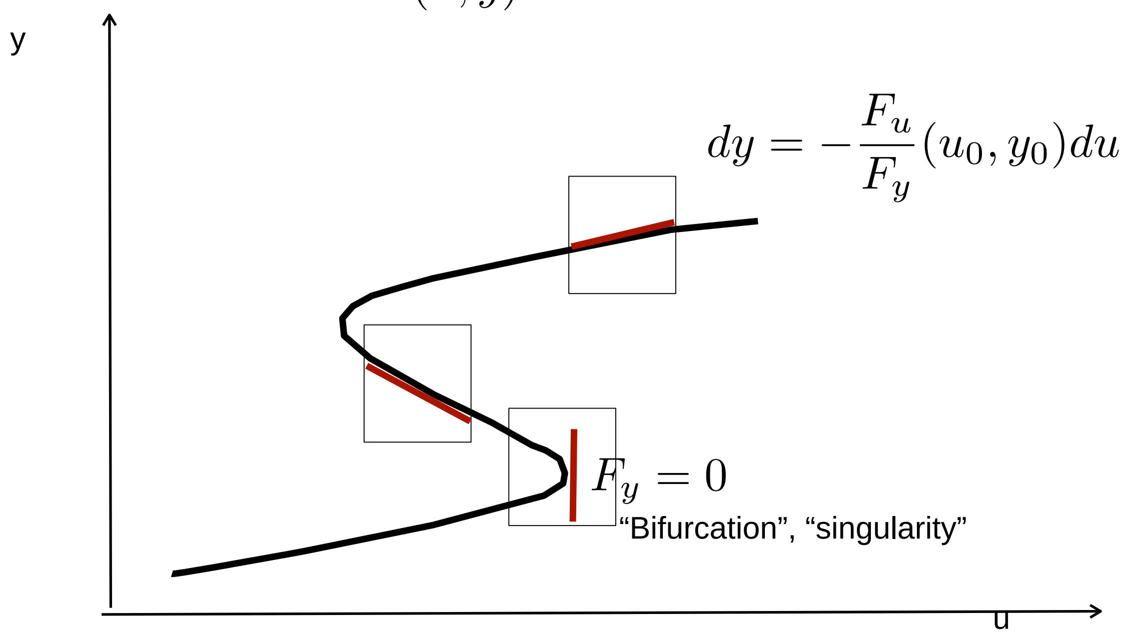
(Unless specified, we will assume differentiability: derivatives exist and are continuous as needed...)

Examples of static behaviors



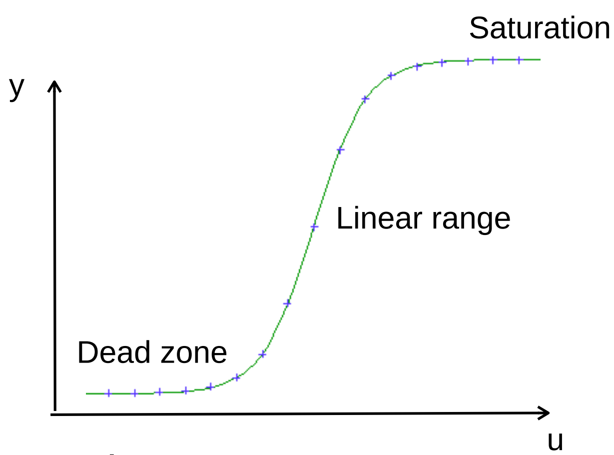
Linear behaviors are local descriptions of global behaviors

$$F(u, y) = 0$$



The sigmoidal behavior

$$y = S(u)$$



Empirically found in many experiments

Sigmoid $y = \frac{1}{1+e^{-u}}$

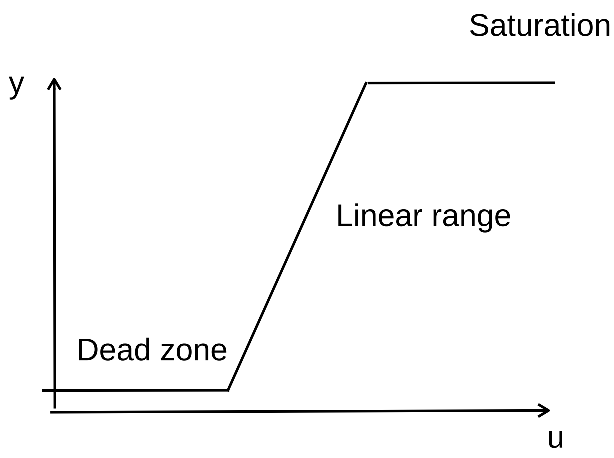
Nearly agrees with cumulative distribution of a gaussian

Kinetics activation $y = \frac{u^N}{M^N + u^N}$

Soft quantization (neural networks) $y = \tanh u$

The saturation behavior

$$y = \mathrm{sat} u$$



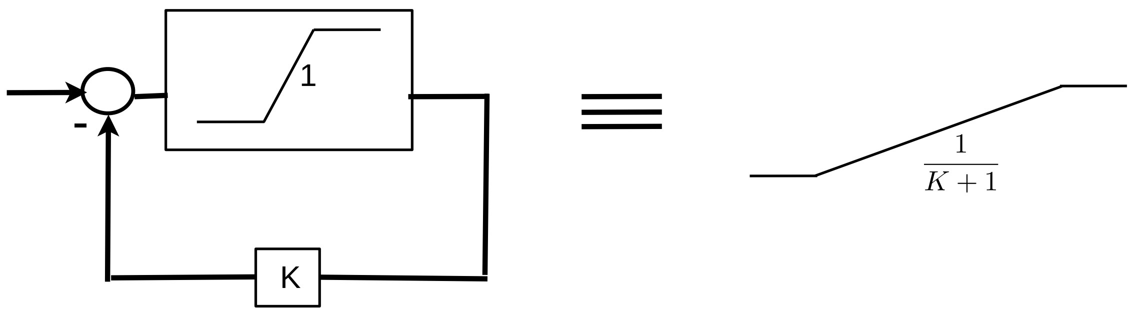
Often found in engineering (control)

Models e.g. finite actuator range

Operational amplifier, transistor, ...

Piecewise linear nature sometimes facilitates analysis

Negative feedback ‘linearizes’

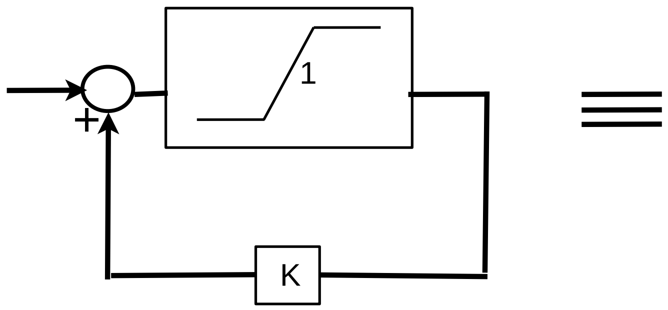


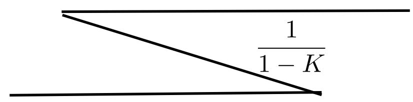
$$y = \mathrm{sat}_1(u - Ky) \quad \equiv \quad y = \mathrm{sat}_{\frac{1}{1+K}}(u)$$

The essence of the feedback amplifier.

The essence of control theory.

Positive feedback 'quantizes'



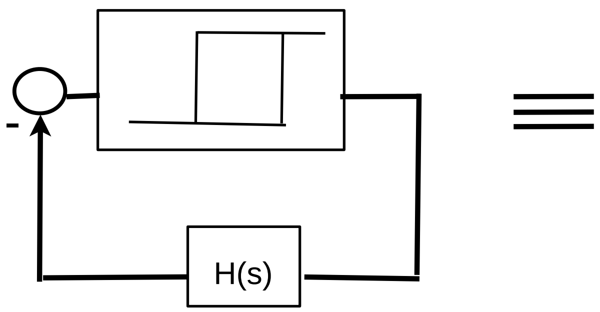


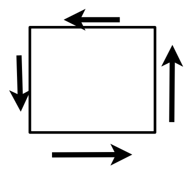
$$y = \mathrm{sat}_1(u + Ky) \quad \equiv \quad y = \begin{cases} +1 & u \geq -1 - K \\ -1 & u \leq K - 1 \end{cases}$$

The essence of switches, ON-OFF devices, boolean behaviors

An essential nonlinear phenomenon

Oscillations = hysteresis + adaptation

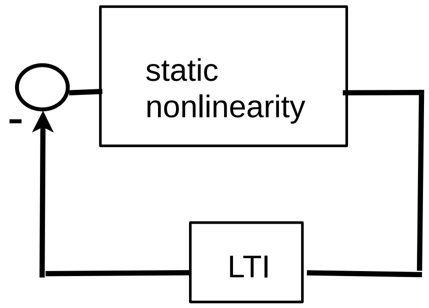




Oscillations arise from hysteresis loops : a combination of positive and negative feedback

An essential nonlinear phenomenon

Lure feedback systems



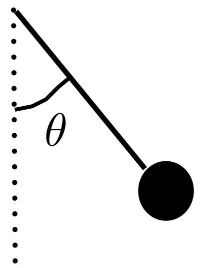
A much studied class of nonlinear behaviors (Lure, Popov, Kalman, Yakubovich, ...).

Aizerman conjecture : stability of the local behavior implies stability of the global behavior?

A frequency characterization of the nonlinear behavior?

The root of dissipativity theory, a pillar of nonlinear systems theory.

Behaviors on nonlinear spaces



What is the state-space of the pendulum ?

Every electromechanical system with an angle among configuration variables defines a behavior on a nonlinear space.

An essential source of nonlinearity.

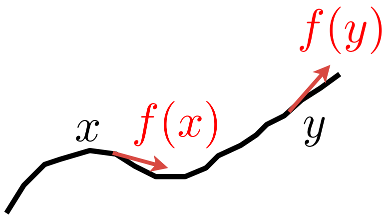
‘Angular’ spaces: circle, rotation group, sphere, ...

Nonlinear state-space models

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$
$$x \in M$$

generalizes

$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$x \in \mathbb{R}^n$$



$f : M \to TM$ is called a vector field. It assigns an arrow to each point in the state space.

The arrow specifies the direction of motion.

A trajectory is an integral curve of the vector field: a curve whose velocity vector is everywhere given by the vector field.

Linear versus nonlinear analysis

In linear systems analysis, the emphasis in on the solution at time t, or the flow:

$$\phi(t, x_0) = e^{At}\, x_0$$

In nonlinear systems analysis, the flow can only be approximated, typically with the help of a numerical integrator.

Therefore, the emphasis is on the vector field and trajectories in the state space.

The central question is : what can be said about the asymptotic behavior (large times) without integrating the vector field?

A problem of historical importance

Newton proposes a general law for motions: F=m a
(1689). Solves the two-body problem and 'proves' Kepler laws for the behavior 'earth+sun'.

(Kepler laws are about integral curves. Newton law is about the vector field). Newton also invents a calculus to approximately solve nonlinear differential equations.

Laplace raises the question of the asymptotic behavior:
Is the solar system stable? Will the observed behavior persist eternally?

Poincaré puts an end to the attempt to answer asymptotic questions from approximate solutions for the three-body problem. He invents the geometric analysis of nonlinear systems, based on a study of the vector field in the state space.

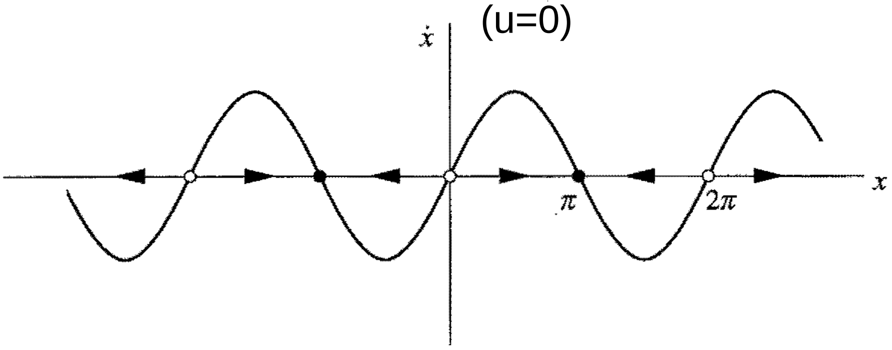
Vector fields on the line

$$\dot{x} = sin(x) + u$$

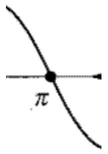
x is a real number, denoting position on the real line.

$\dot{x}$ is a real number, the sign of which indicates the direction of motion

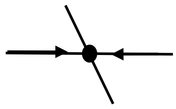
(u=0)



Local analysis near equilibria



$\approx$



$$\dot{x} = \sin(x) + u \qquad\qquad \dot{z} = \sin(\pi)z + v$$

near

$$(\bar{x}, \bar{u}) = (\pi, 0) \qquad\qquad (z, v) \in \mathbb{R} \times \mathbb{R}$$

Rationale: $(\bar{x} + z(t), \bar{u} + v(t))$ approximates $(x(t), u(t))$ for small deviations around $(\bar{x}, \bar{u}) = (\pi, 0)$.

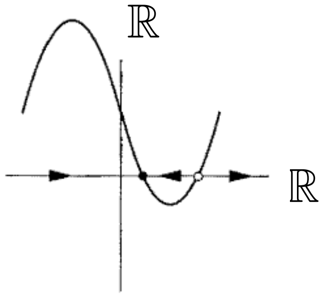
Advantage: linear behaviors can be used to approximate nonlinear behaviors

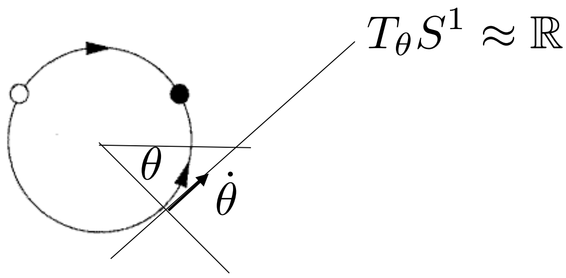
Vector fields on the circle

$$\dot{\theta} = sin(\theta) + u$$

θ is an angle, denoting position on the circle.

$\dot{\theta}$ is a real number, the sign of which indicates the direction of motion in the tangent space $T_\theta S^1 \approx \mathbb{R}$



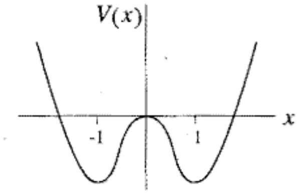


Both the angle and the velocity can be represented by real numbers locally (a coordinate representation) but they are different objects!

Scalar vector fields derive from a potential

$$\dot{x} = x - x^3$$

$$= -\frac{\partial V}{\partial x} \qquad\qquad V(x) = \frac{x^4}{4} - \frac{x^2}{2}$$



By definition, the potential decreases along solutions:

$$\dot{V} = -(\frac{\partial V}{\partial x})^2 \leq 0$$

This means that solutions move ‘downhill’ in the potential landscape.

Existence and uniqueness of solutions

$\dot{x} = x^{1/3}$ several solutions with initial condition

$\dot{x} = 1 + x^2$ solution blows up to infinity in finite time

Existence and Uniqueness Theorem: Consider the initial value problem

$$\dot{x} = f(x), \qquad x(0) = x_0.$$

Suppose that $f(x)$ and $f'(x)$ are continuous on an open interval R of the x-axis, and suppose that x_0 is a point in R. Then the initial value problem has a solution $x(t)$ on some time interval $(-\tau, \tau)$ about $t = 0$, and the solution is unique.

Vector fields versus maps

Discrete-time and continuous-time linear systems are treated on the same foot because of the analogy between the solution of $\dot{x} = Ax$ and the solution of $x_+ = Ax$

The analogy does not extend to nonlinear behaviors.

Trajectories of $\dot{x} = f(x)$ are integral curves of the vector field.

Trajectories of $x_+ = F(x)$ are a sequence of points generated by iterating the map

$$x_0, x_1 = F(x_0), x_2 = F(F(x_0)), \ldots, x_N = F^N(x_0), \ldots$$

This course is primarily about continuous-time nonlinear behaviors.

4F2: Phase portraits, linearization, and saddle points

Past lecture: static analysis and one-dimensional state-spaces.

Today: two-dimensional state-spaces.

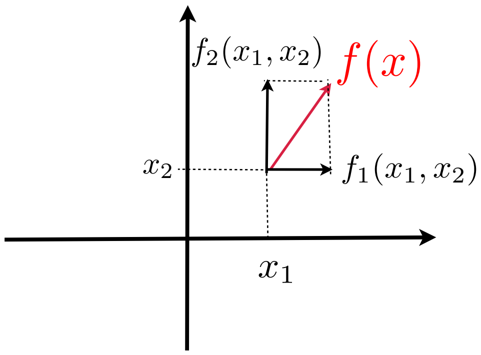
Vector fields and phase portrait

The vector equation $$\dot{x} = f(x)\,, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$$

is equivalent to two coupled scalar equations:

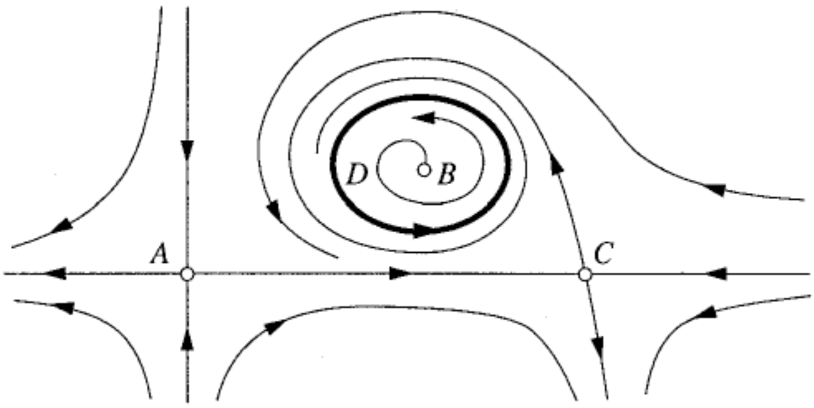
$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

Drawing the phase portrait means attaching an arrow to each point and sketching the integral curves of the vector field.



(Matlab command ‘quiver’ draws f(x))

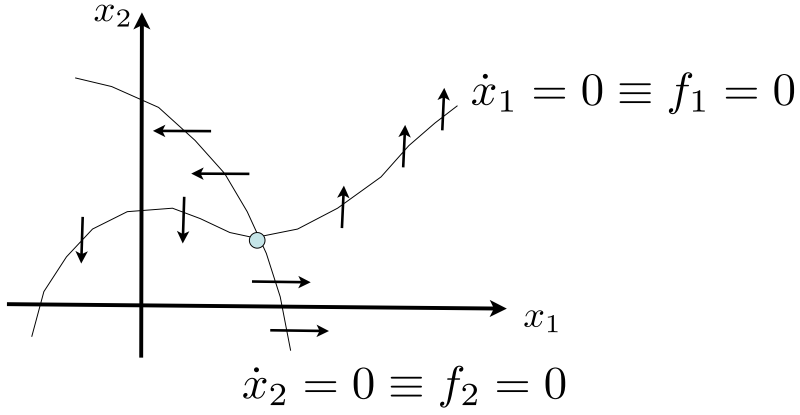
Goal of lecture: understanding the behavior of two-dimensional systems from their phase portrait



Imagine a few possible solutions of this phase portrait.
Where is the time information? What are the asymptotic behaviors?

Observe: trajectories do not intersect !

Isoclines: ‘loci of same slope’



Equilibria or fixed points are solutions of $f(x) = 0$

They are the intersections of the nullclines.

Linear phase portraits

$$\dot{x} = Ax \qquad \equiv \qquad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{At} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

Geometry of solution is determined by the eigenvalues and eigenvectors of A

Linear phase portraits with real eigenvalues

Robust phase portraits:

Stable node | Unstable node | Saddle point

Fragile phase portraits:

repeated eigenvalue | zero eigenvalue

Linear phase portraits with complex eigenvalues

Robust phase portraits:

Stable focus | Unstable focus

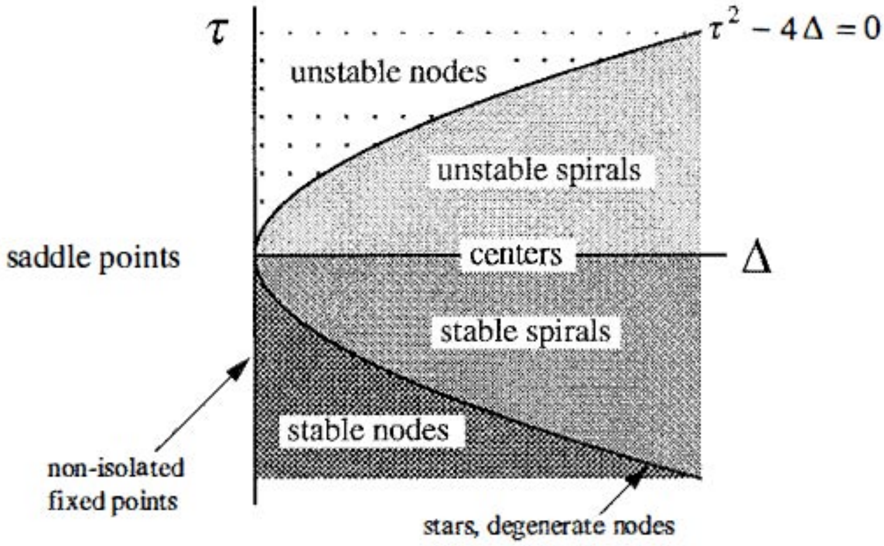
Fragile phase portraits:

center

Classification of linear phase portraits

τ is the trace of A

Δ is the det of A



Linearization

Consider a solution $x^*(t)$ of $\dot{x} = f(x)$

The (Jacobian) linearization (or variational equation) of $\dot{x} = f(x)$ along $x^*(t)$ is the linear system

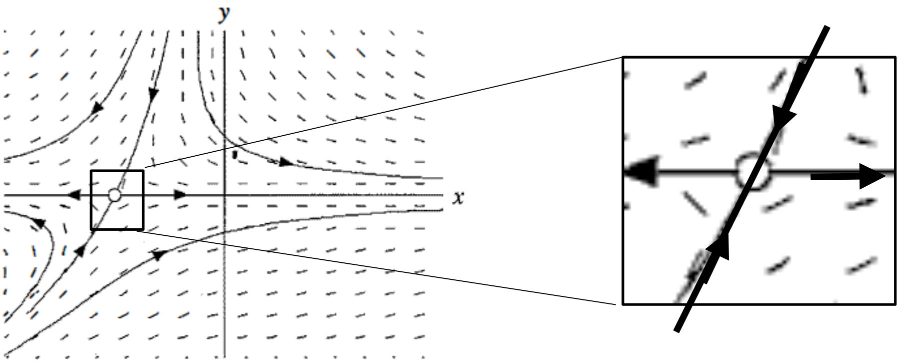
$$\dot{z} = A(t)z, \quad A(t) = \frac{\partial f}{\partial x}(x^*(t)), \; a_{ij}(t) = \frac{\partial f_i}{\partial x_j}(x^*(t))$$

The variational equation is obtained by retaining the first-order terms in the Taylor expansion of $\dot{x} = f(x)$ along $x^*(t)$

Linearization at hyperbolic fixed points

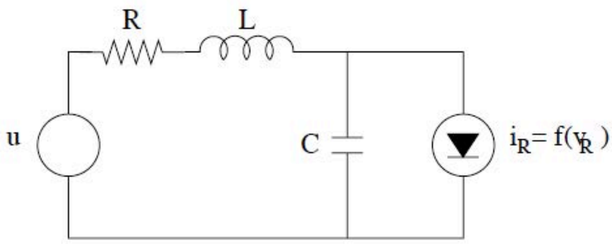
A fixed point is called hyperbolic if the eigenvalues of the linearization lie off the imaginary axis (nodes, foci, saddles).

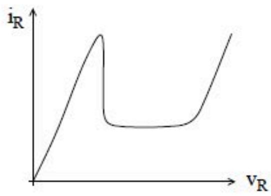
Robust linear phase portraits are important because they capture the local behavior of nonlinear phase portraits near hyperbolic fixed points.



(Hartman Grobam theorem)

A nonlinear electrical circuit: tunnel diode

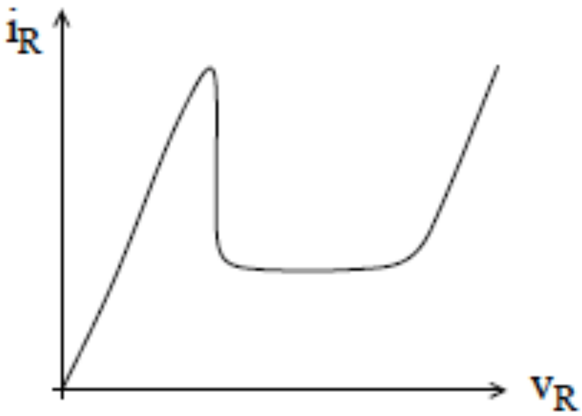




$$\begin{cases} C \cdot \dot{x}_1 = x_2 - f(x_1) \\ L \cdot \dot{x}_2 = -R \cdot x_2 - x_1 + u \end{cases}$$

(Khalil, pp. 7-8)

Phase portrait of the tunnel diode



Reduced modeling

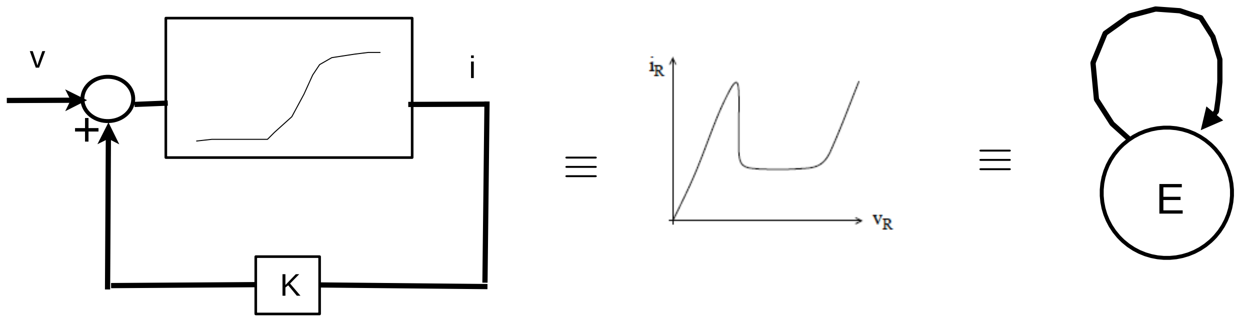
Compare the following three behaviors:

The static model $$0 = \frac{u-x}{R} - f(x)$$

The one-dimensional model $$C\dot{x} = v - f(x)$$ $$v = \frac{u-x}{R}$$

The two-dimensional model $$\begin{cases} C \cdot \dot{x}_1 = x_2 - f(x_1) \\ L \cdot \dot{x}_2 = -R \cdot x_2 - x_1 + u \end{cases}$$

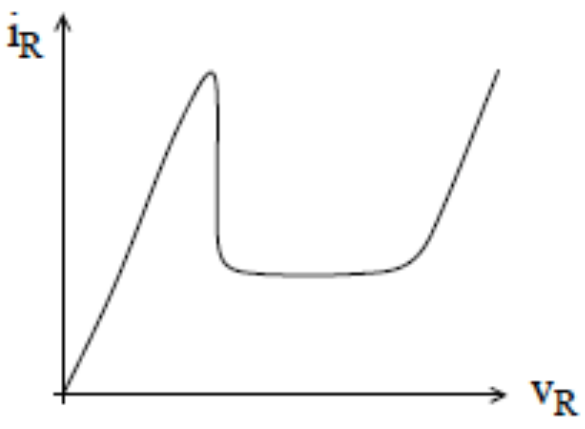
The motif of bistability



An archetype model of switches, on-off behaviors, positive feedback, memory, hysteresis, ...

Widespread in biology and electronics

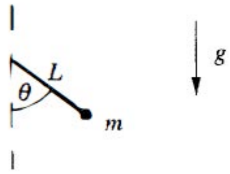
The saddle point organizes the phase portrait



But the saddle point will never be observed experimentally.

Hence the role of modeling!

A nonlinear mechanical system: the pendulum



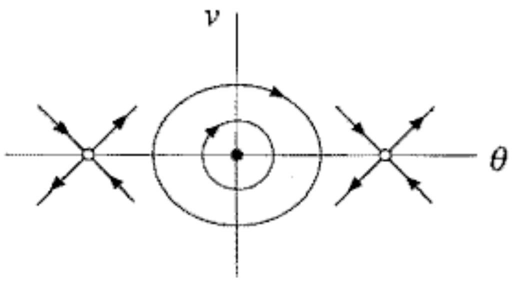
Newton's law: $$\ddot{\theta} = -\frac{g}{l}\sin(\theta)$$

State-space model:

$$\begin{cases} \dot{\theta} &= v \\ \dot{v} &= -\frac{g}{l}\sin\theta \end{cases}, \quad (\theta, v) \in S^1 \times \mathbb{R}$$

The state-space is a cylinder, not a plane !

The pendulum: local analysis near fixed points



The pendulum: energy conservation

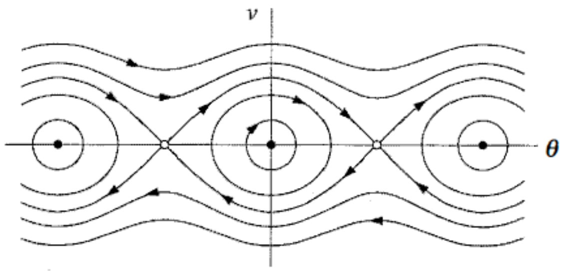
Energy:

$$E = \frac{1}{2}v^2 - \frac{g}{l}\cos\theta$$

Energy conservation:

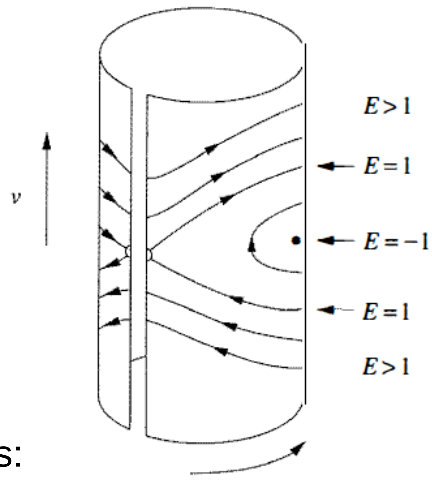
$$\dot{E} = \frac{\partial E}{\partial v}\dot{v} + \frac{\partial E}{\partial \theta}\dot{\theta} = 0$$

Vector field is everywhere tangent to the level curves of E.
This means that the level curves of E are the integral curves of the vector field !



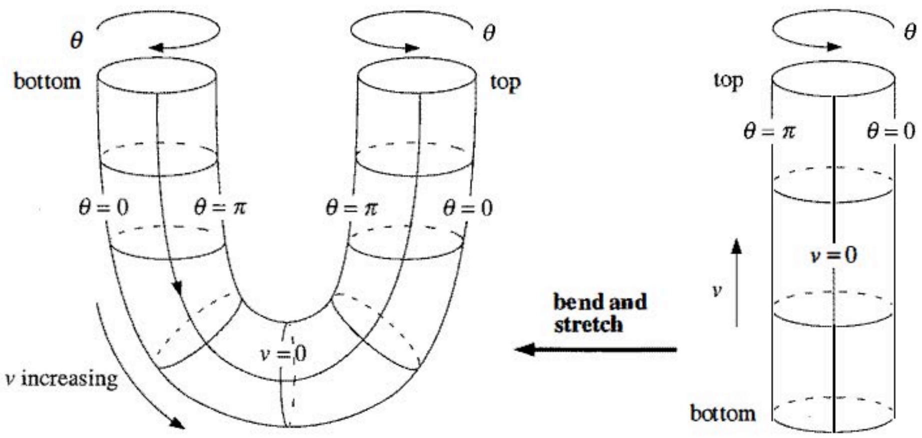
The phase portrait of the pendulum

(Strogatz, pp. 170-171)



Wrapping the plane onto a cylinder:

Plotting E instead of v along the vertical axis:



Summary of lecture

Phase portrait: two-dimensional behaviors can be drawed.

Linear phase portraits determine local behavior near hyperbolic fixed points

Tunnel diode: archetype example of bistable behavior

Pendulum: archetype example of conservative mechanical behavior.

The saddle point is an important ‘hidden’ fixed point. A key ruler of nonlinear behaviors.