The Inerter Concept and Its Application

Malcolm C. Smith
Department of Engineering
University of Cambridge
U.K.

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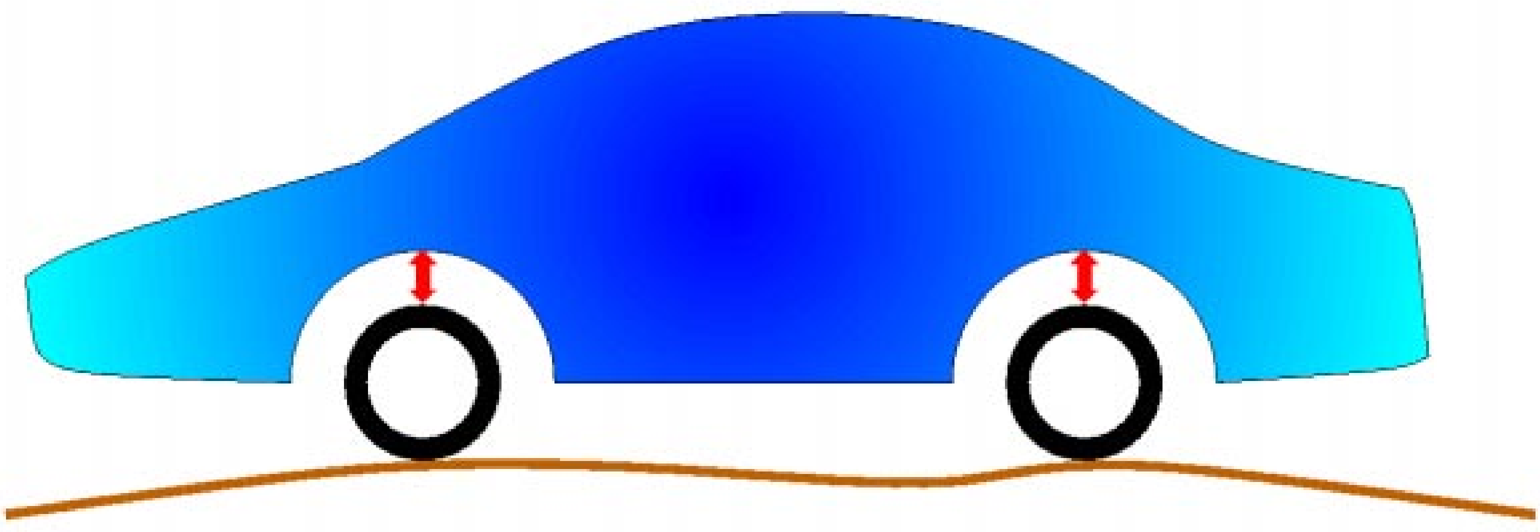
Plenary Lecture

Motivating Example – Vehicle Suspension



Performance objectives

1. Control vehicle body in the face of variable loads.
2. Minimise roll, pitch (dive and squat).
3. Improve ride quality (comfort).
4. Improve tyre grip (road holding).

Types of suspensions

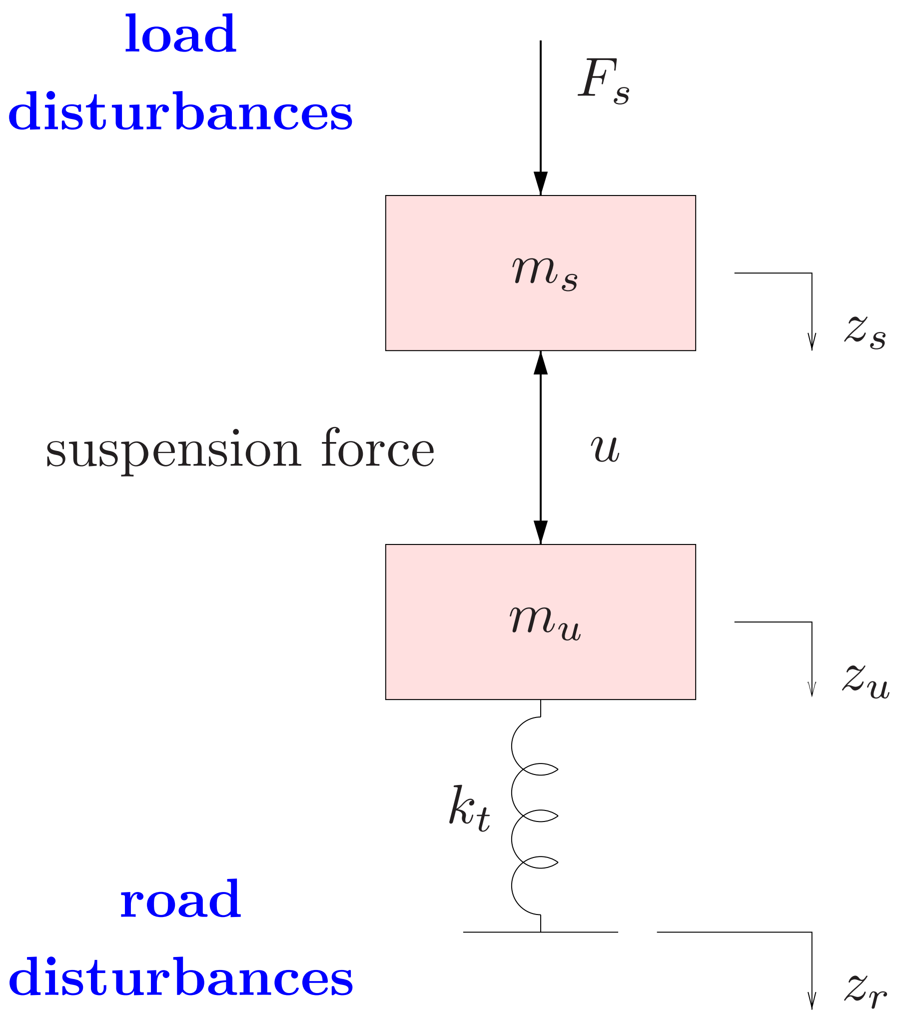
1. Passive.
2. Semi-active.
3. Self-levelling.
4. "Fully active".

Constraints

1. Suspension deflection—hard limit.
2. Actuator constraints (e.g. bandwidth).
3. Difficulty of measurement (e.g. absolute ride-height).

A Challenging set of Problems for the Designer

Quarter-car Vehicle Model



Equations of motion:

$$\begin{aligned} m_s \ddot{z}_s &= F_s - u, \\ m_u \ddot{z}_u &= u + k_t (z_r - z_u). \end{aligned}$$

Invariant Equation

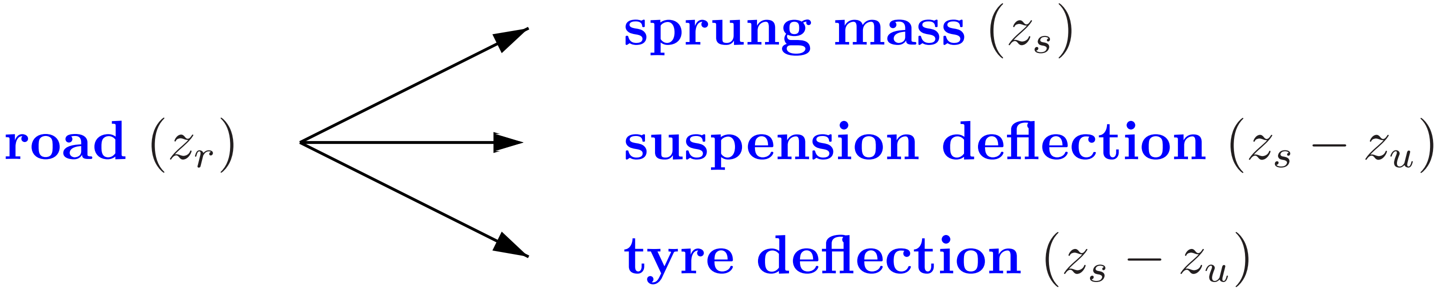
The following equation holds

$$m_s \ddot{z}_s + m_u \ddot{z}_u = F_s + k_t(z_r - z_u)$$

independently of u.

This represents behaviour that the suspension designer cannot influence.

Consequence: any one of the following disturbance transmission paths determines the other two.



J.K. Hedrick and T. Butsuen, Invariant properties of automotive suspensions, *Proc. Instn. Mech. Engrs.*, **204** (1990), pp. 21–27.

Invariant Points

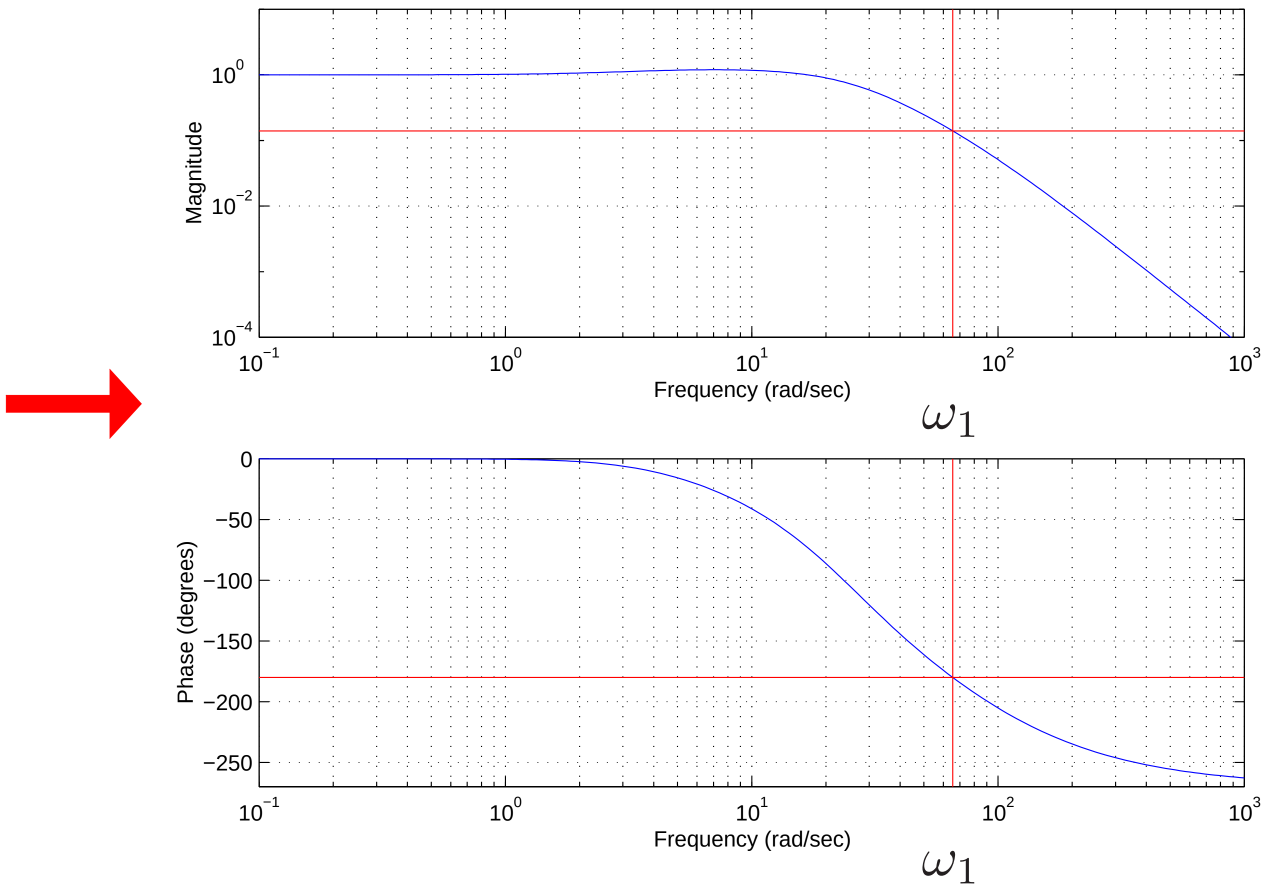
$$\omega_1 = \sqrt{\frac{k_t}{m_u}}$$

"tyre-hop" invariant frequency

For any suspension:

$$\frac{\hat{z}_s}{\hat{z}_r}(j\omega_1) = -\frac{m_u}{m_s}$$

road (z_r) → sprung mass (z_s)



Further Work

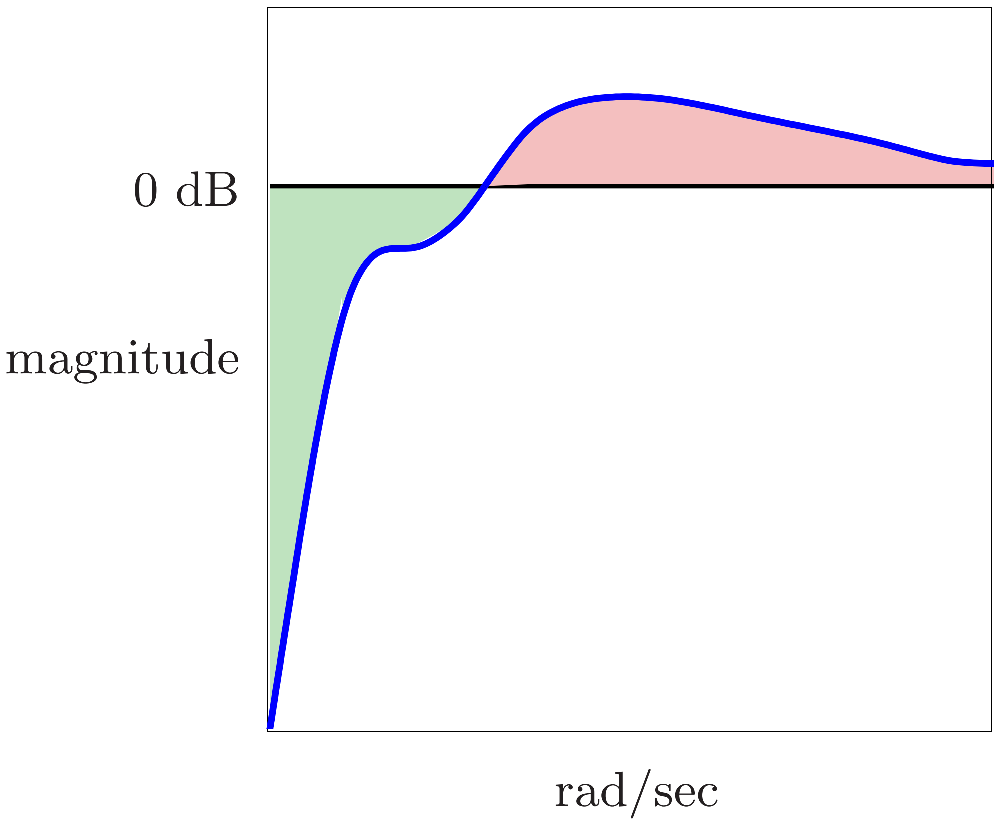
1. What is the **complete** freedom on a given transfer function? [1]
2. What is the minimum number of sensors required to achieve a given behaviour? [1]
3. Are there conservation laws? [1]
4. Can disturbance paths for “ride” and “handling” be adjusted independently? [2]

[1] M.C. Smith, Achievable dynamic response for automotive active suspension, *Vehicle System Dynamics*, **24** (1995), pp. 1–33.

[2] M.C. Smith and G.W. Walker, Performance limitations and constraints for active and passive suspensions: a mechanical multi-port approach, *Vehicle System Dynamics*, **33** (2000), pp. 137–168.

Conservation Laws

grip transfer function (road → tyre deflection)



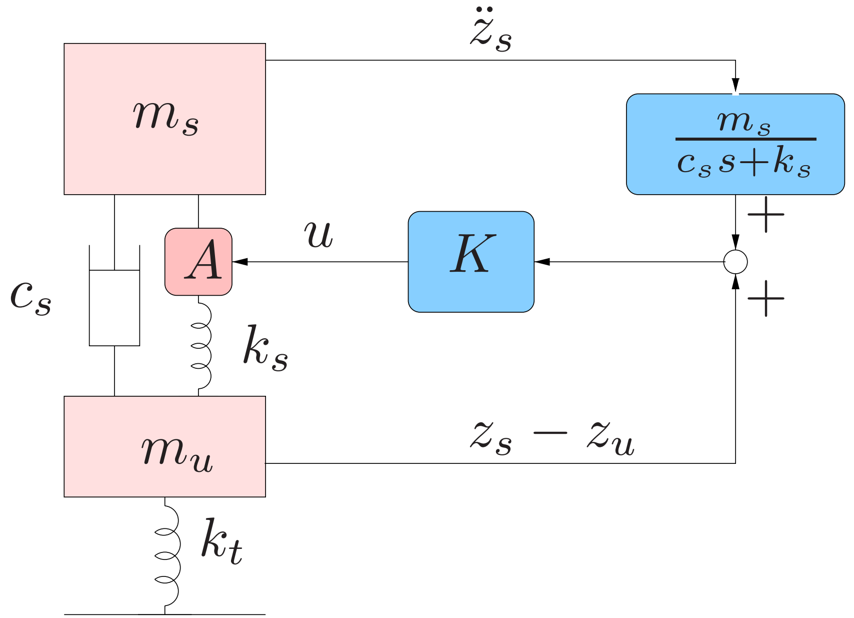
Area formula: Area of amplification is equal to the area of attenuation.

True for any suspension system (active or passive).

Active Suspension Design

(1) Modal Decomposition

(2) Decoupling of Ride and Handling



[1] R.A. Williams, A. Best and I.L. Crawford, "Refined Low Frequency Active Suspension", Int. Conf. on Vehicle Ride and Handling, Nov. 1993, Birmingham, *Proc. ImechE*, 1993-9, C466/028, pp. 285–300, 1993.

[2] K. Hayakawa, K. Matsumoto, M. Yamashita, Y. Suzuki, K. Fujimori, H. Kimura, "Robust H_∞ Feedback Control of Decoupled Automobile Active Suspension Systems", *IEEE Transactions on Automat. Contr.*, **44** (1999), pp. 392–396.

[3] M.C. Smith and F-C. Wang, Controller Parameterization for Disturbance Response Decoupling: Application to Vehicle Active Suspension Control, *IEEE Trans. on Contr. Syst. Tech.*, **10** (2002), pp. 393–407.

Passive Suspensions (Abstract Approach)

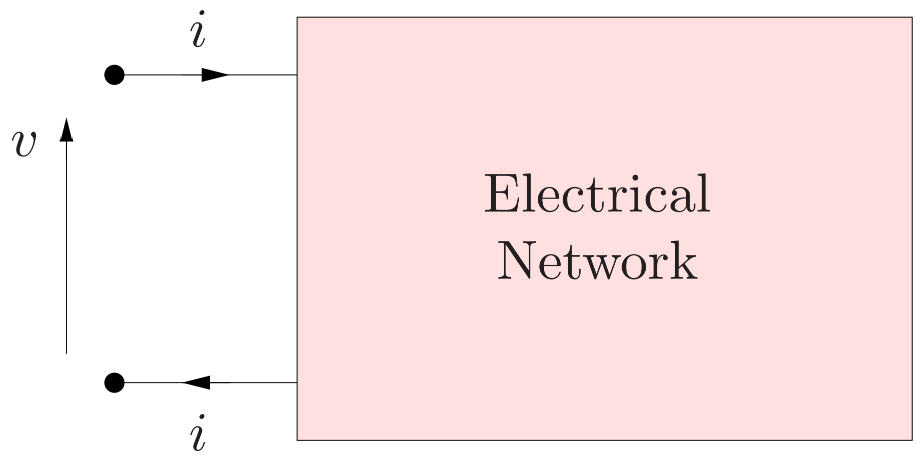
Try to understand which vehicle dynamic behaviours are possible and which are not — without worrying initially **how** the behaviour is realised.

This is a black-box approach.

Classical electrical circuit theory should be applicable.

Driving-Point Impedance

$$Z(s) = \frac{\hat{v}(s)}{\hat{i}(s)}$$

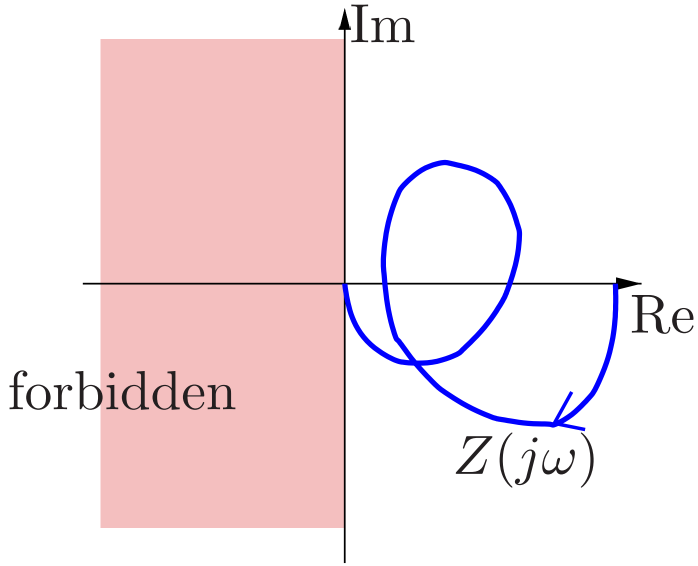


Classical Electrical Network Synthesis

Definition. A network is **passive** if for all admissible v, i which are square integrable on $(-\infty, T]$,

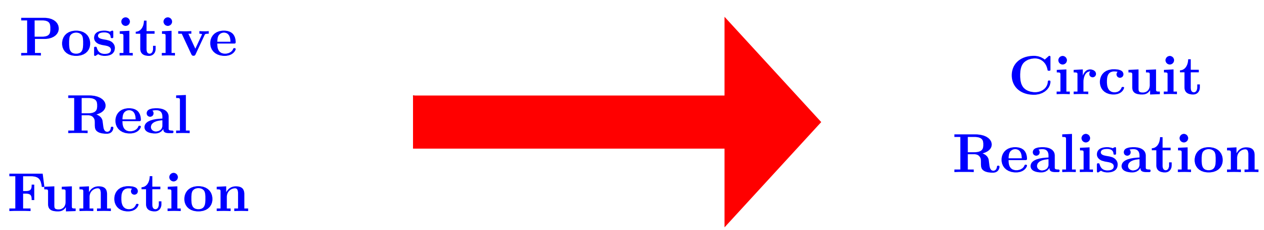
$$\int_{-\infty}^{T} v(t)i(t)\,dt \geq 0.$$

Theorem 1. A network is **passive** if and only if $Z(s)$ is **positive-real**, i.e. $Z(s)$ is analytic and $\mathrm{Re}(Z(s)) \geq 0$ in $\mathrm{Re}(s) > 0$.



Fundamental Theorem of Electrical Network Synthesis

Theorem 2. Brune (1931), Bott-Duffin (1949). Any rational function which is positive-real can be realised as the driving-point impedance of an electrical network consisting of resistors, capacitors and inductors.



Classic reference: E.A. Guillemin, Synthesis of Passive Networks, Wiley, 1957.

Electrical-Mechanical Analogies

1. Force-Voltage Analogy.

voltage	↔	force
current	↔	velocity

Oldest analogy historically, cf. electromotive force.

2. Force-Current Analogy.

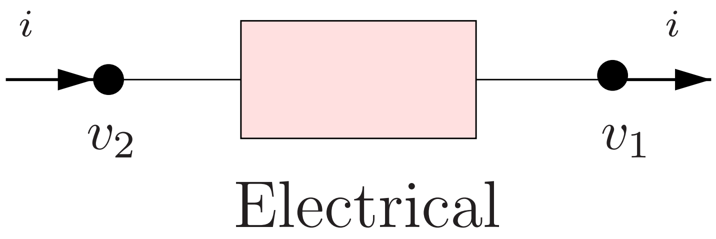
current	↔	force
voltage	↔	velocity
electrical ground	↔	mechanical ground

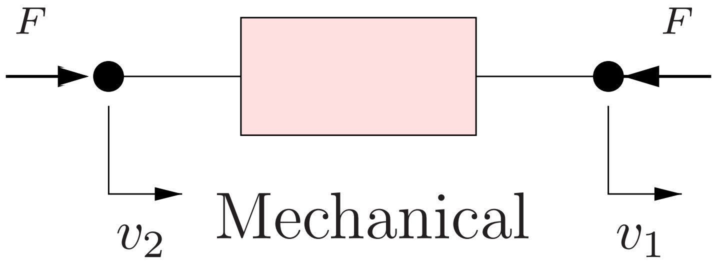
Independently proposed by: Darrieus (1929), Hähnle (1932), Firestone (1933).

Respects circuit "topology", e.g. terminals, through- and across-variables.

Standard Element Correspondences (Force-Current Analogy)

$v = Ri$	resistor	$\leftrightarrow$	damper	$cv = F$
$v = L\frac{di}{dt}$	inductor	$\leftrightarrow$	spring	$kv = \frac{dF}{dt}$
$C\frac{dv}{dt} = i$	capacitor	$\leftrightarrow$	mass	$m\frac{dv}{dt} = F$





What are the **terminals** of the mass element?

The Exceptional Nature of the Mass Element

Newton's Second Law gives the following network interpretation of the mass element:

- One terminal is the centre of mass,
- Other terminal is a fixed point in the inertial frame.

Hence, the mass element is analogous to a **grounded** capacitor.

Standard network symbol for the mass element:

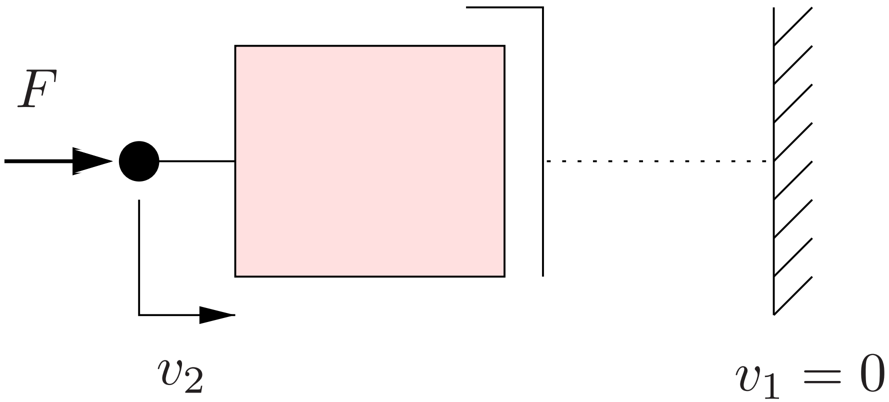
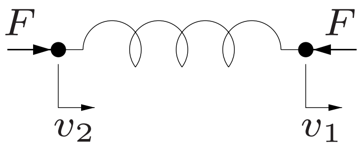
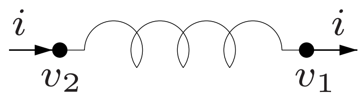
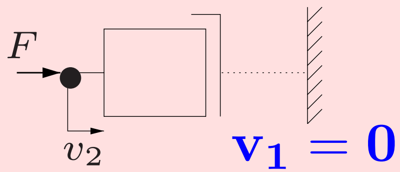
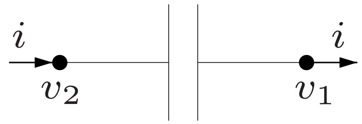
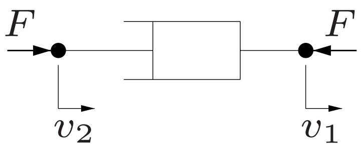
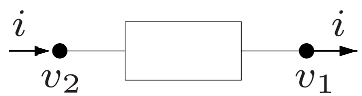


Table of usual correspondences

Mechanical	Electrical
 spring	 inductor
 mass	 capacitor
 damper	 resistor

Consequences for network synthesis

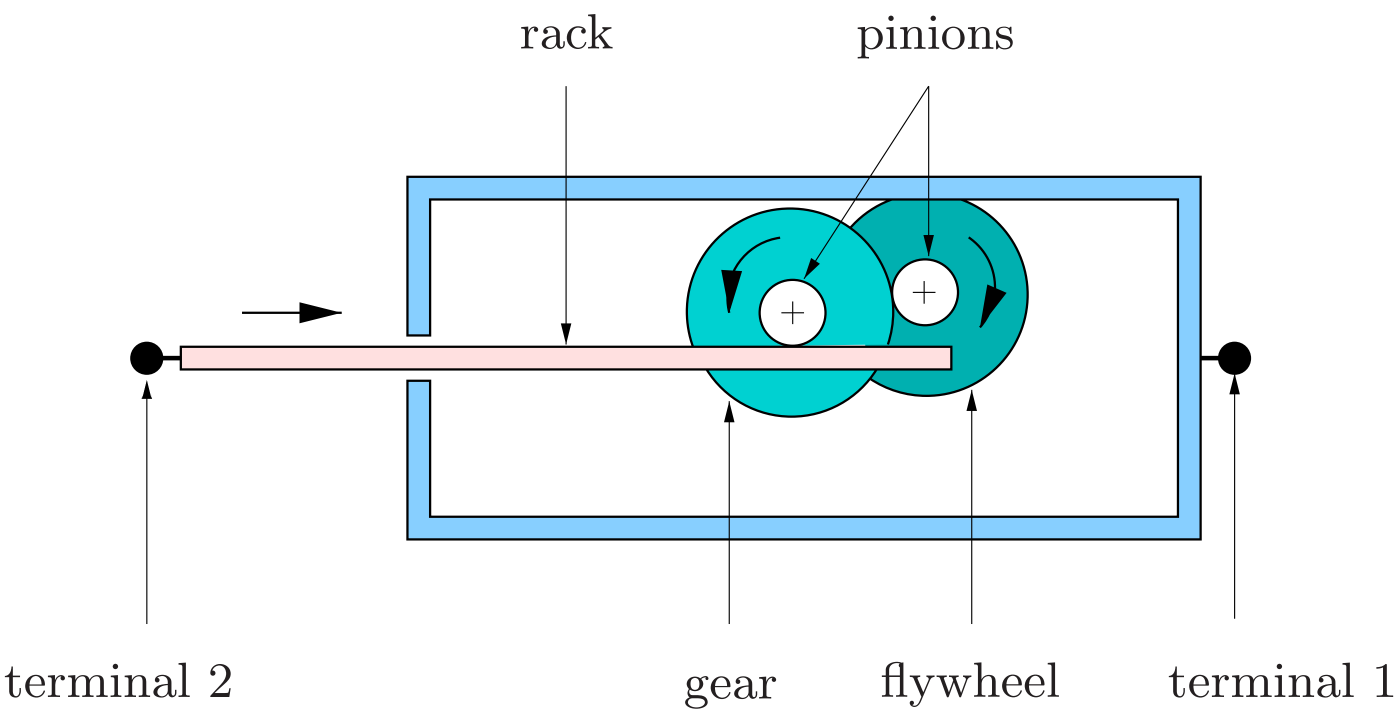
Two major problems with the use of the mass element for **synthesis** of "black-box" mechanical impedences:

- An electrical circuit with ungrounded capacitors will not have a direct mechanical analogue,
- Possibility of unreasonably large masses being required.

Question

Is it possible to construct a physical device such that the relative acceleration between its endpoints is proportional to the applied force?

One method of realisation



r_1	=	radius of rack pinion	γ	=	radius of gyration of flywheel
r_2	=	radius of gear wheel	m	=	mass of the flywheel
r_3	=	radius of flywheel pinion			$\alpha_1 = \gamma / r_3$ and $\alpha_2 = r_2 / r_1$

Equation of motion: $\mathbf{F} = (\mathbf{m}\alpha_1^2\alpha_2^2)\,(\dot{\mathbf{v}}_2 - \dot{\mathbf{v}}_1)$

(Assumes mass of gears, housing etc is negligible.)

The Ideal Inerter

We define the Ideal Inerter to be a mechanical one-port device such that the equal and opposite force applied at the nodes is proportional to the relative acceleration between the nodes, i.e.

$$F = b(\dot{v}_2 - \dot{v}_1).$$

We call the constant b the **inertance** and its units are kilograms.

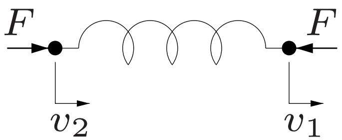
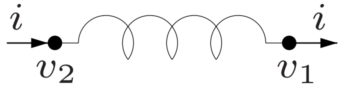
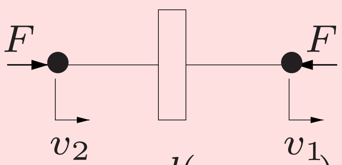
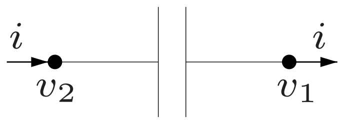
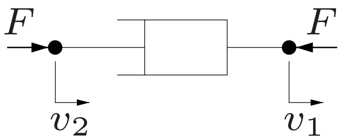
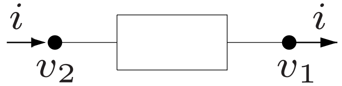
The stored energy in the inerter is equal to $\frac{1}{2}b(v_2 - v_1)^2$.

The ideal inerter can be approximated in the same sense that real springs, dampers, inductors, etc approximate their mathematical ideals.

We can assume its mass is small.

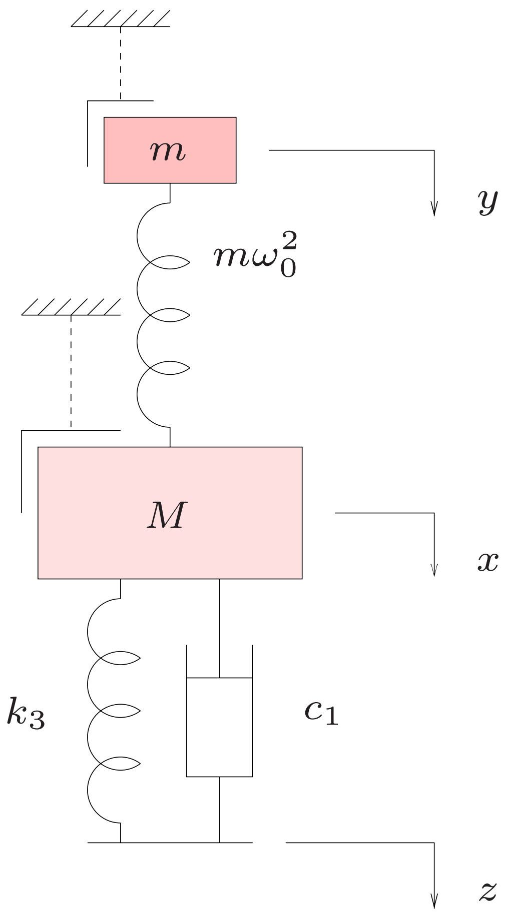
M.C. Smith, Synthesis of Mechanical Networks: The Inerter, *IEEE Trans. on Automat. Contr.*, **47** (2002), pp. 1648–1662.

A new correspondence for network synthesis

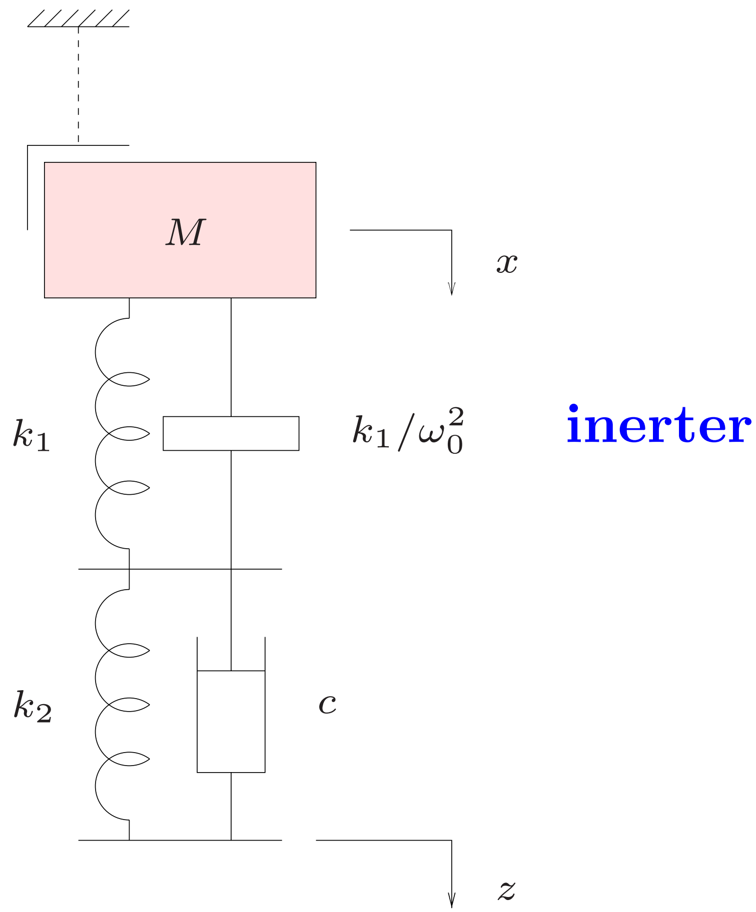
Mechanical	Electrical
 $Y(s) = \frac{k}{s}$ $\frac{dF}{dt} = k(v_2 - v_1)$ spring	 $Y(s) = \frac{1}{Ls}$ $\frac{di}{dt} = \frac{1}{L}(v_2 - v_1)$ inductor
 $Y(s) = bs$ $F = b\frac{d(v_2-v_1)}{dt}$ inerter	 $Y(s) = Cs$ $i = C\frac{d(v_2-v_1)}{dt}$ capacitor
 $Y(s) = c$ $F = c(v_2 - v_1)$ damper	 $Y(s) = \frac{1}{R}$ $i = \frac{1}{R}(v_2 - v_1)$ resistor

$$Y(s) = \text{admittance} = \frac{1}{\text{impedance}}$$

A New Approach to Vibration Absorption

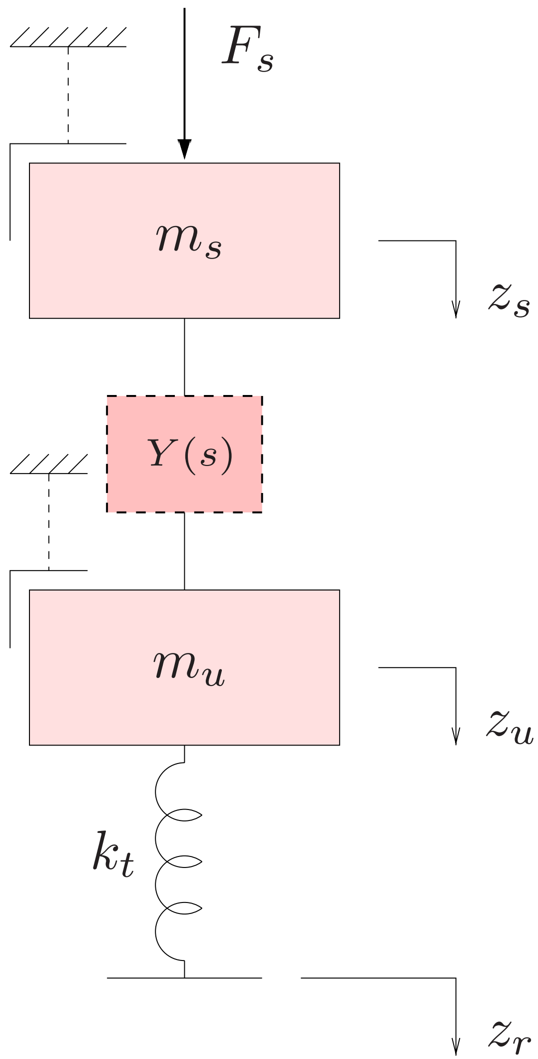


Conventional vibration absorber



Solution using inerters

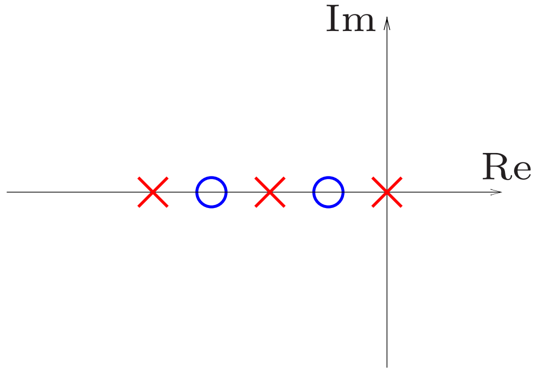
The Inerter applied to Passive Vehicle Suspensions



The design of a *passive* suspension system can be viewed as the search for a suitable *positive-real* admittance $Y(s)$ to optimise desired performance measures.

Traditional Suspension Struts

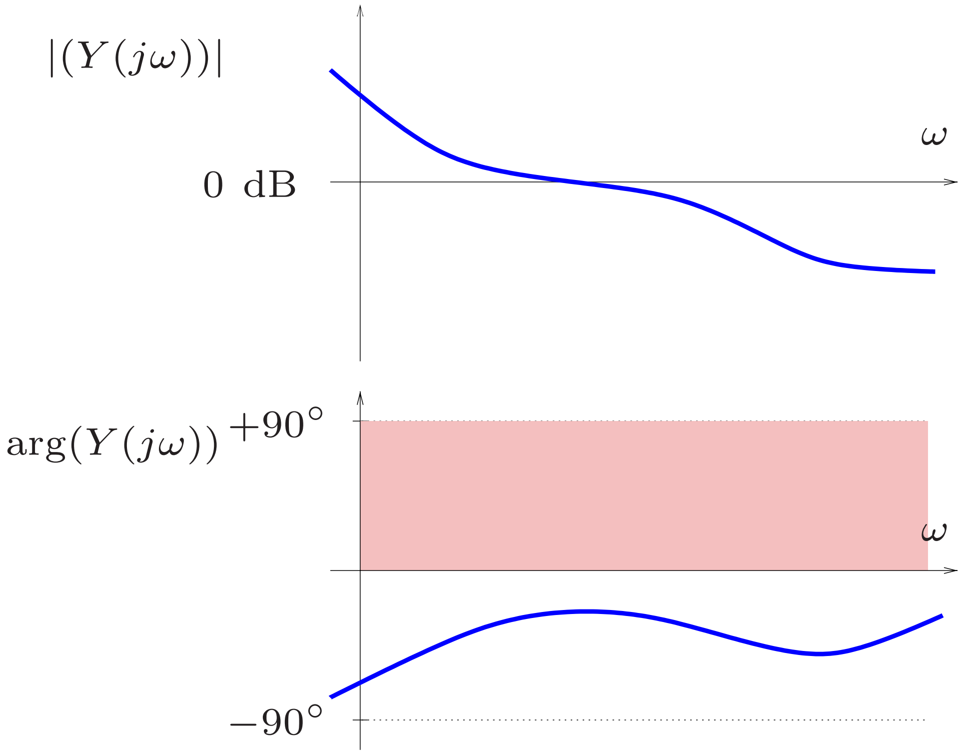
Theorem 3. The driving-point admittance $Y(s)$ of a finite network of **springs and dampers only** has all its poles and zeros simple and alternating on the negative real axis with a pole being rightmost.



Corollary. For $\omega \geq 0$:

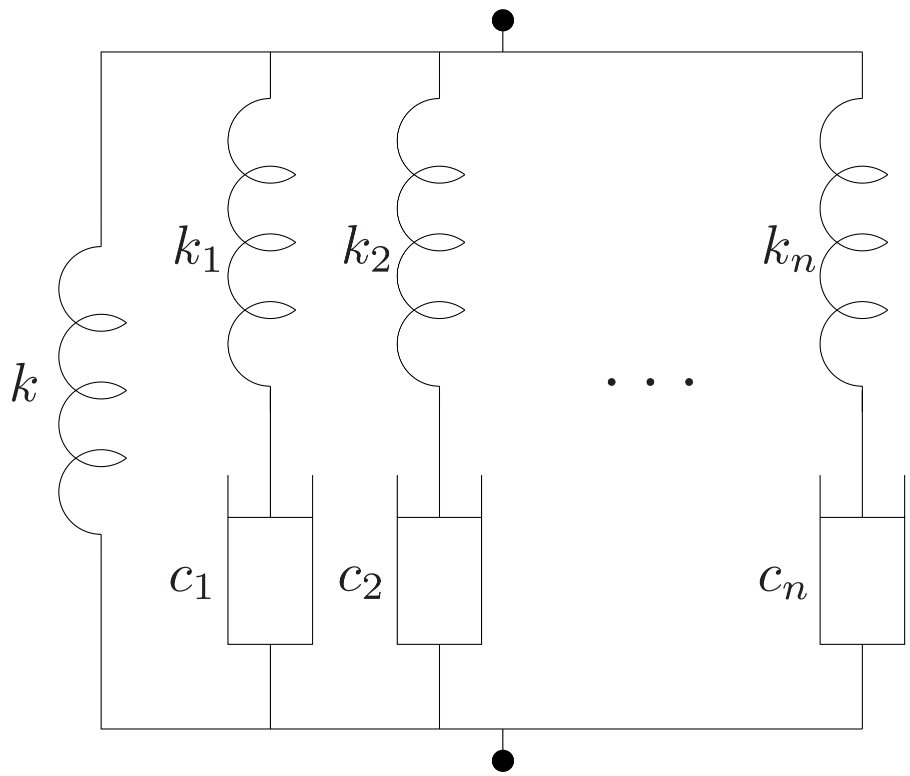
$$-20\,\text{dB} \leq \text{Bode-slope}(|Y(j\omega)|) \leq 0\,\text{dB},$$

$$-90^\circ \leq \arg(Y(j\omega)) \leq 0^\circ.$$



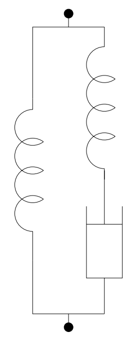
Realisations in Foster form

Theorem 4. Any admittance comprising an arbitrary interconnection of springs, dampers (and levers) can be realised in the following form:



Three Candidate Admittances

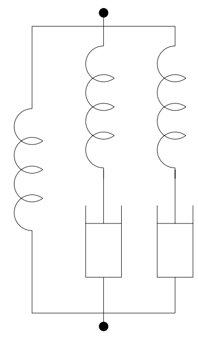
I. One damper:



$$Y_1(s) = k\frac{(T_2 s + 1)}{s(T_1 s + 1)}$$

where $T_2 > T_1 > 0$ and $k > 0$.

II. Two dampers:



$$Y_2(s) = k\frac{(T_4 s + 1)(T_6 s + 1)}{s(T_3 s + 1)(T_5 s + 1)}$$

where $T_6 > T_5 > T_4 > T_3 > 0$ and $k > 0$.

III. Same degree as two damper case **but general positive real**:

$$Y_3(s) = k\frac{a_0 s^2 + a_1 s + 1}{s(d_0 s^2 + d_1 s + 1)}$$

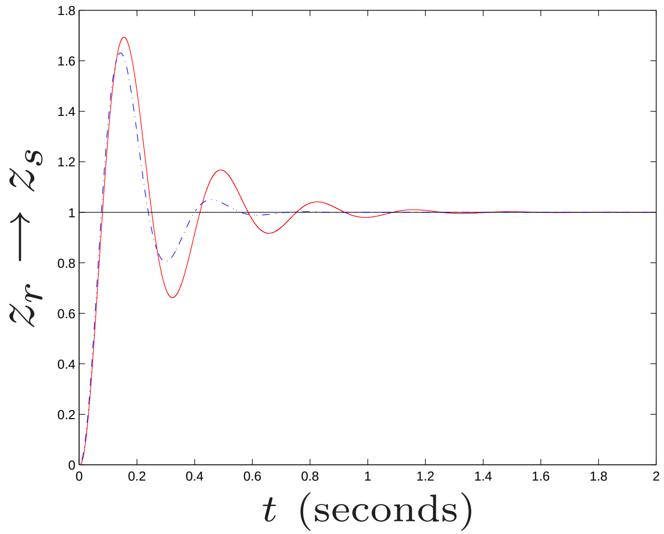
where d_0, $d_1 \geq 0$ and $k > 0$. (Need: $\beta_1 = a_0 d_1 - a_1 d_0 \geq 0$, $\beta_2 := a_0 - d_0 \geq 0$, $\beta_3 := a_1 - d_1 \geq 0$ for positive-realness.)

Design Comparison

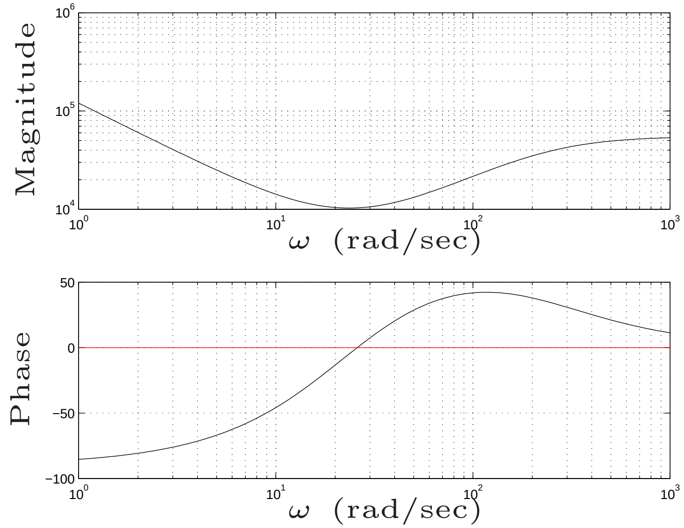
$$m_s = 250 \text{ kg}, \; m_u = 35 \text{ kg}, \; k_t = 150 \text{ kNm}^{-1}$$

Problem: maximise the least damping ratio $\zeta_{\min}$ among all the system poles subject to a static spring stiffness of $k_h = 120 \text{ kNm}^{-1}$.

Results: Y_1 and Y_2: $\zeta_{\min} =$ **0.218**. Y_3: $\zeta_{\min} =$ **0.481**.



Step response: Y_1, Y_2 (**solid**) and Y_3 (**dot-dash**).

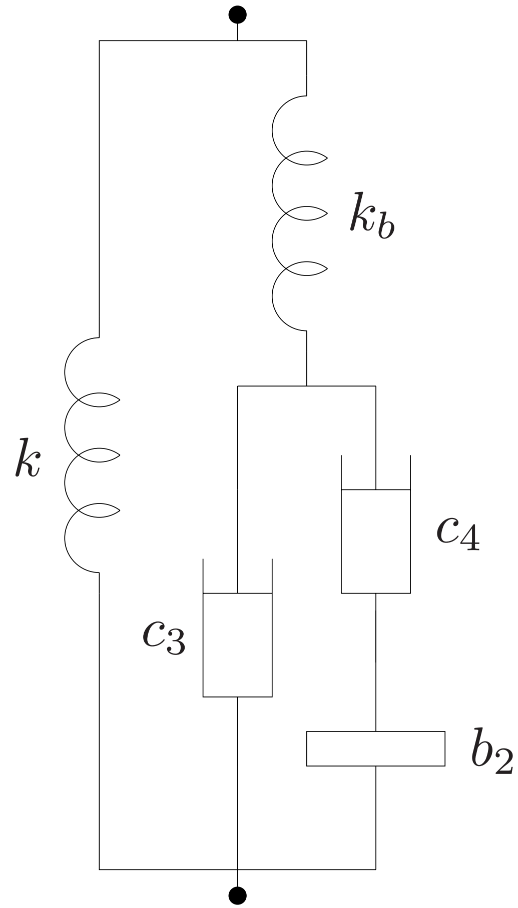


Bode plot of $Y_3(s)$ showing phase lead.

Brune Realisation Procedure for $Y_3(s)$

A continued fraction expansion is obtained:

$$Y_3(s) = k\frac{a_0s^2+a_1s+1}{s(d_0s^2+d_1s+1)} = \frac{k}{s}+\cfrac{1}{\cfrac{s}{k_b}+\cfrac{1}{c_3+\cfrac{1}{\cfrac{1}{c_4}+\cfrac{1}{b_2s}}}}$$



where $k_b = \frac{k\beta_2}{d_0}$, $c_3 = k\beta_3$, $c_4 = \frac{k\beta_4}{\beta_1}$, $b_2 = \frac{k\beta_4}{\beta_2}$ and $\beta_4 := \beta_2^2 - \beta_1\beta_3$.

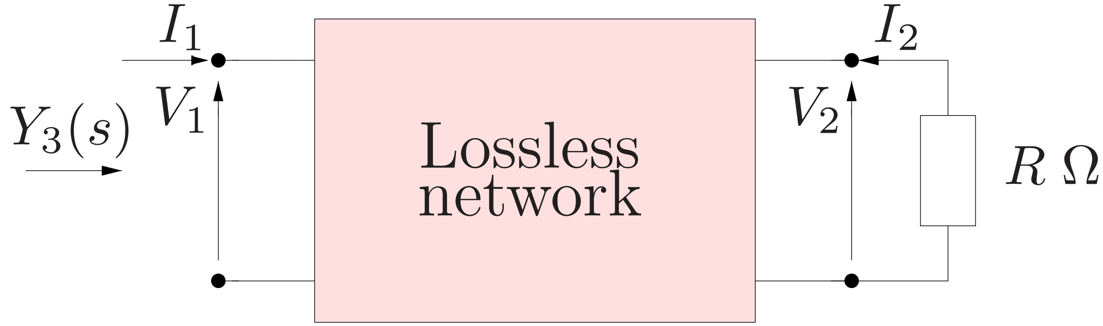
Numerical values in previous design:

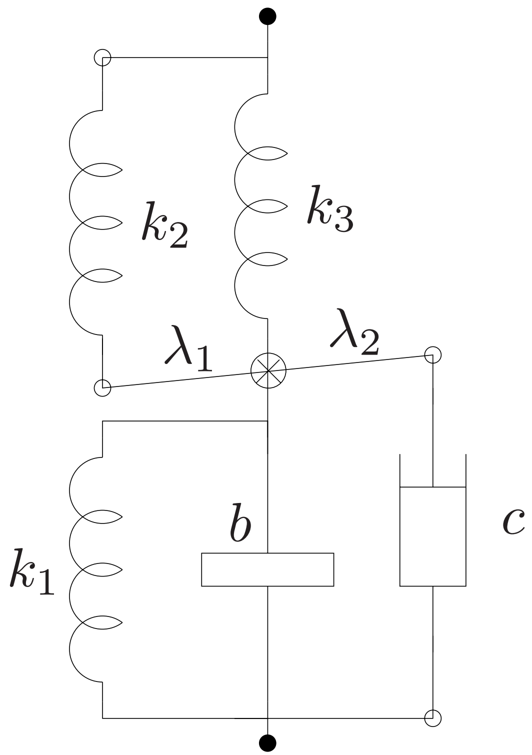
$$k = 120 \text{ kNm}^{-1}, \quad k_b = \infty, \quad \mathbf{b_2 = 181.4 \text{ kg}},$$

$$c_3 = 9.8 \text{ kNsm}^{-1}, \quad c_4 = 45.2 \text{ kNsm}^{-1}$$

Darlington Synthesis

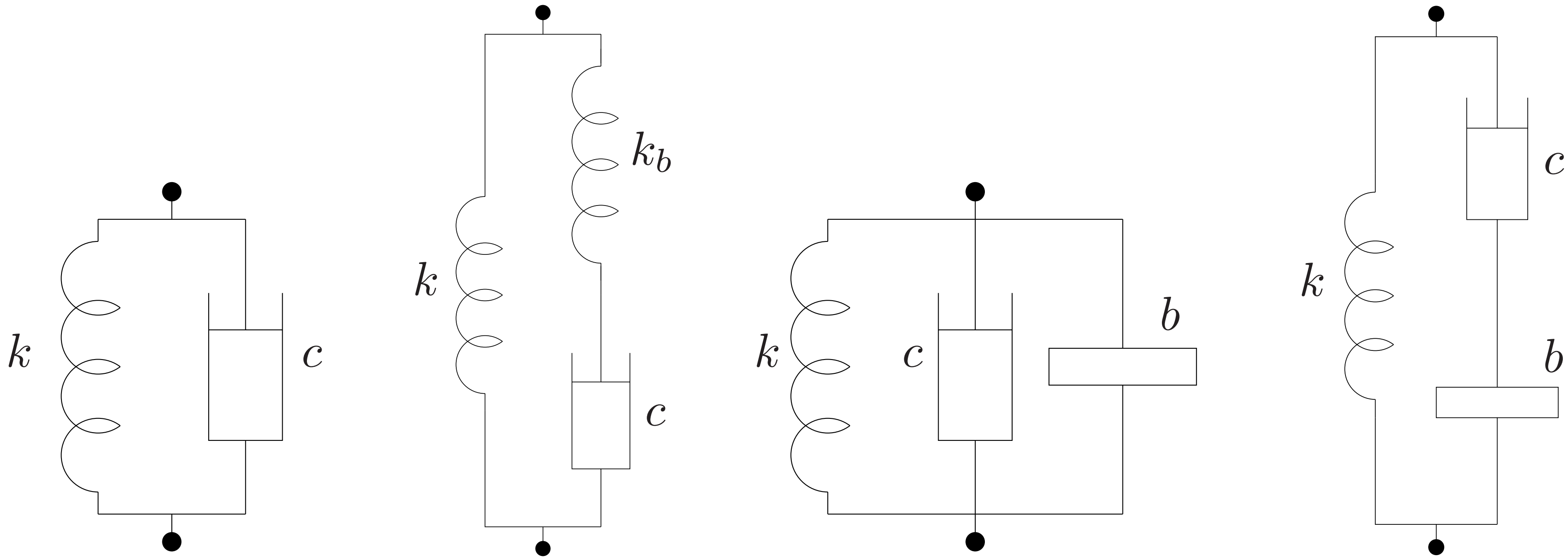
Realisation in Darlington form: a lossless two-port terminated in a single resistor.





Corresponding mechanical network realisation of $Y_3(s)$ has one damper and one inerter but employs a lever.

Simple Suspension Struts

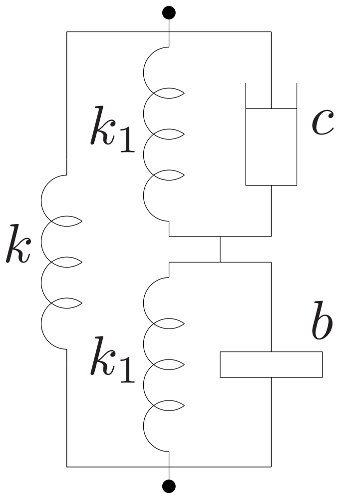


(c) layout S1 (d) layout S2 (e) layout S3 (f) layout S4

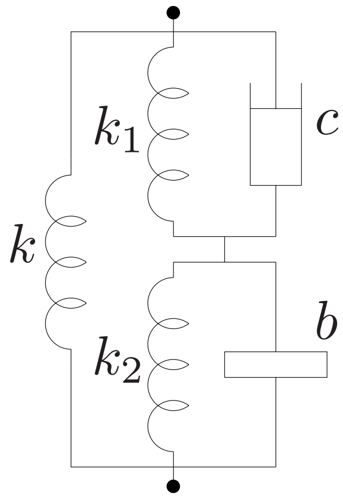
parallel **series**

M.C. Smith and F-C. Wang, Performance Benefits in Passive Vehicle Suspensions Employing Inerters, 42nd IEEE Conference on Decision and Control, December, 2003, Hawaii, to appear.

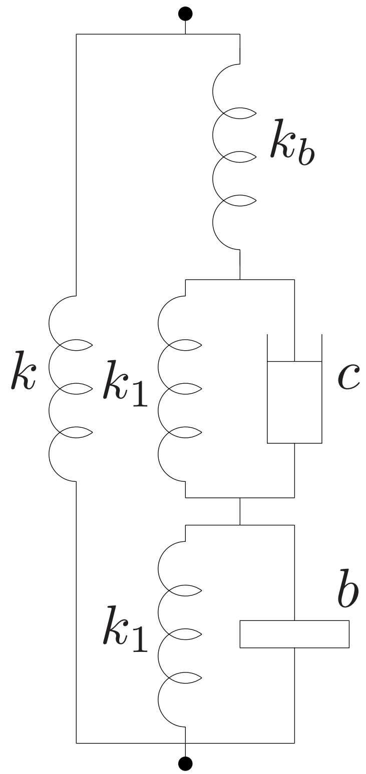
Series Arrangements with Centring Springs



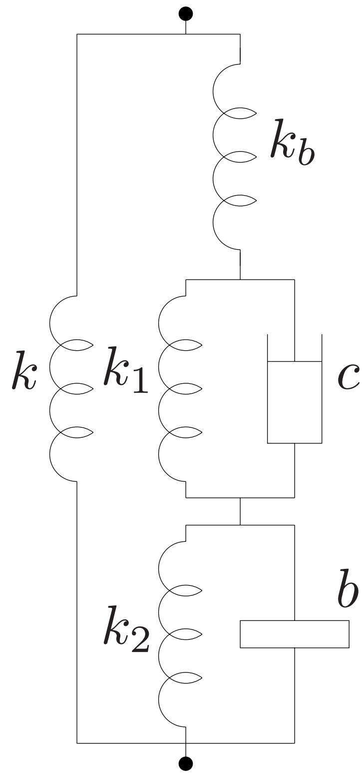
(g) layout S5



(h) layout S6



(i) layout S7



(j) layout S8

Performance Measures

Assume:

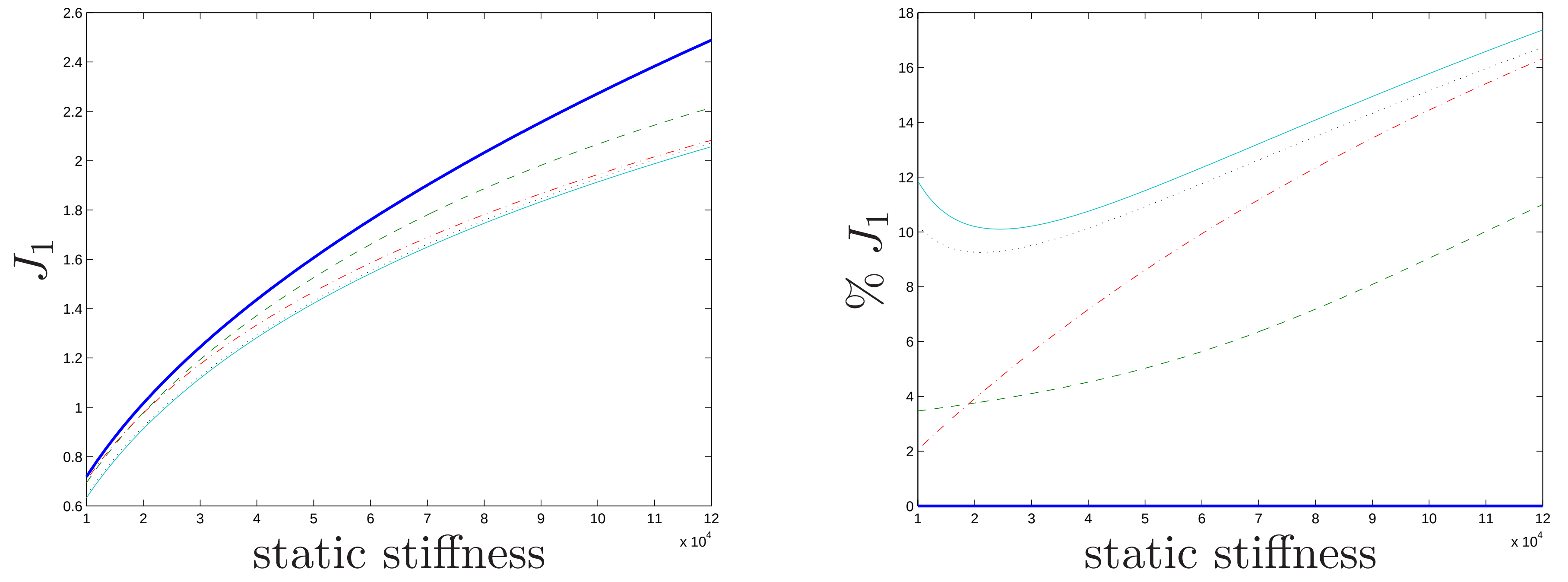
$$\text{Road Profile Spectrum} = \kappa |n|^{-2} \qquad (\text{m}^3/\text{cycle})$$

where $\kappa = 5 \times 10^{-7}$ m^{3}cycle^{-1} = road roughness parameter (typical British principal road) and $V = 25$ ms^{-1}. Define:

$$\begin{aligned} J_1 &= E\left[\ddot{z}_s^2(t)\right] && \textbf{ride comfort} \\ &= \text{r.m.s. body vertical acceleration} \end{aligned}$$

$$\begin{aligned} J_3 &= E\left[(k_t(z_u - z_r))^2\right] && \textbf{grip} \\ &= \text{r.m.s. dynamic tyre load} \end{aligned}$$

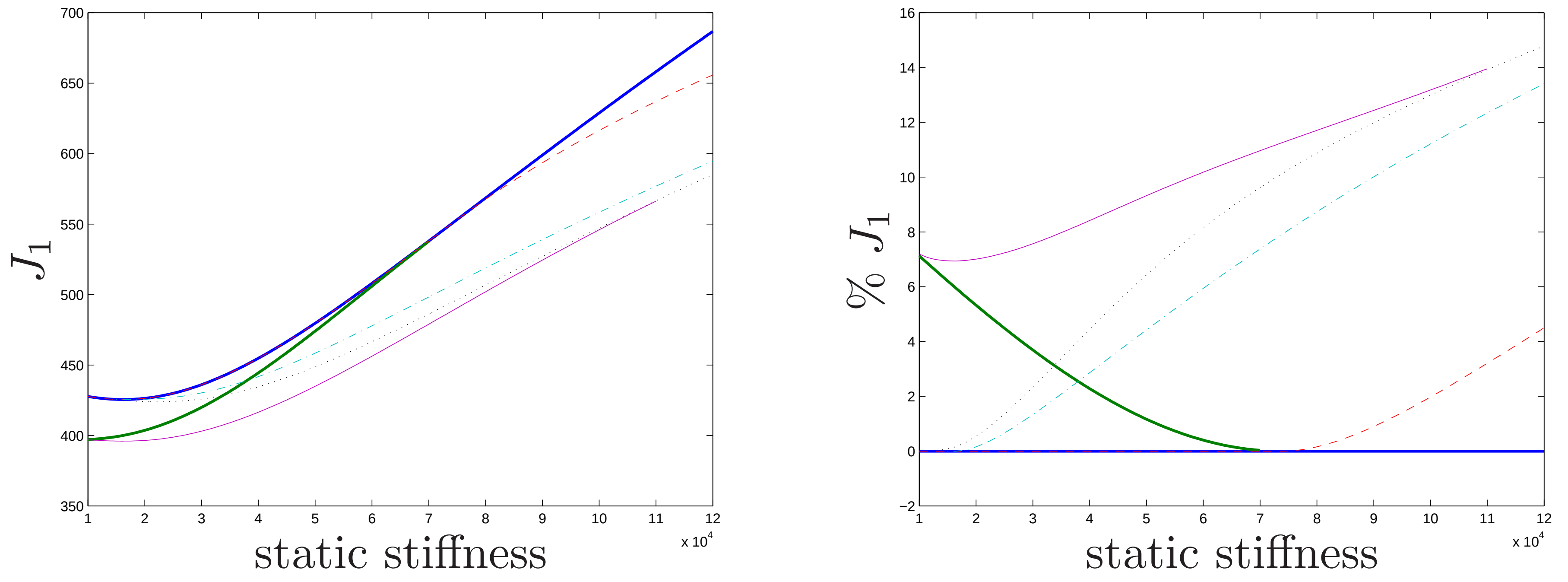
OPTIMISATION OF J_1 (RIDE COMFORT)



(a) Optimal J_1 (b) Percentage improvement in J_1

Key: layout S1 (**bold**), layout S3 (**dashed**), layout S4 (**dot-dashed**), layout S5 (**dotted**) and layout S6 (**solid**)

Optimisation of J_3 (dynamic tyre loads)

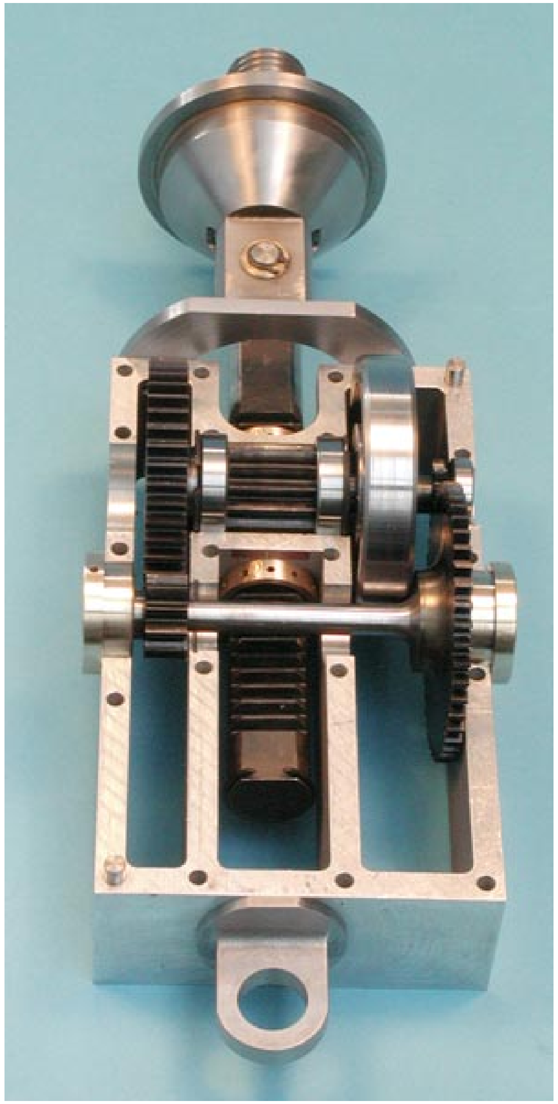


(a) Optimal J_3 (b) Percentage improvement in J_3

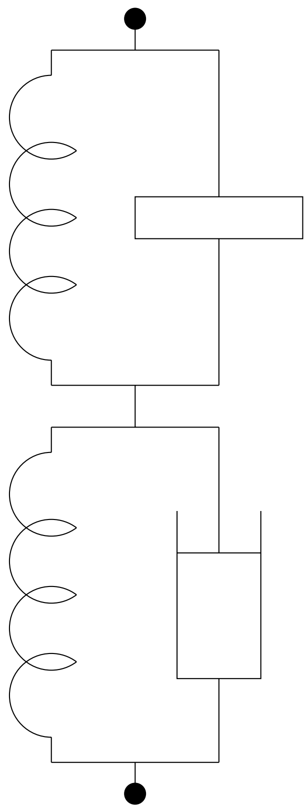
Key: layout S1 (**bold**), layout S2 (**bold**), layout S3 (**dashed**), layout S4 (**dot-dashed**), layout S5 (**dotted**) and layout S7 (**solid**)

Rack and pinion inerter
made at
Cambridge University
Engineering Department

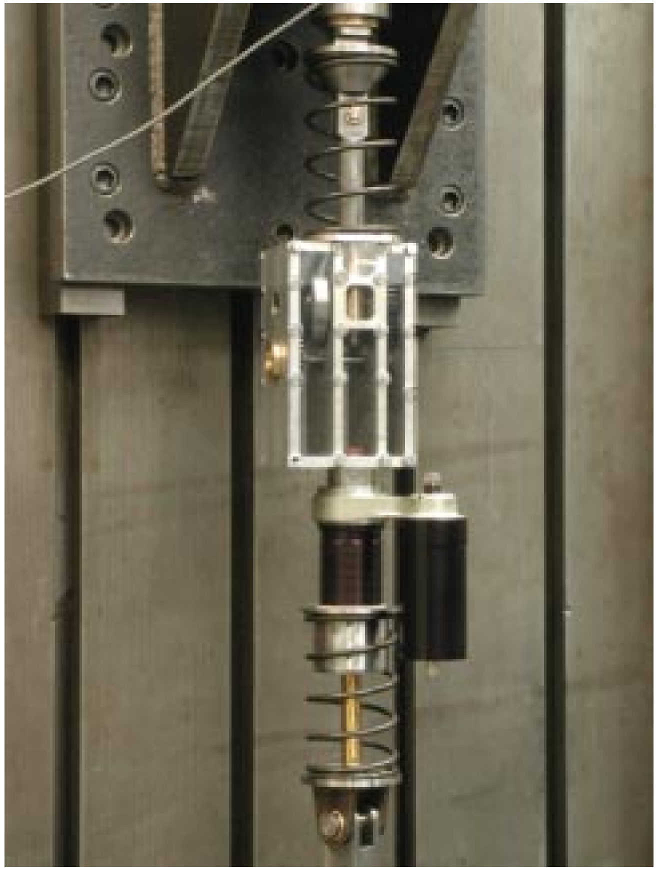
mass ≈ 3.5 kg
inertance ≈ 725 kg
stroke ≈ 80 mm



Damper-inerter series arrangement with centring springs



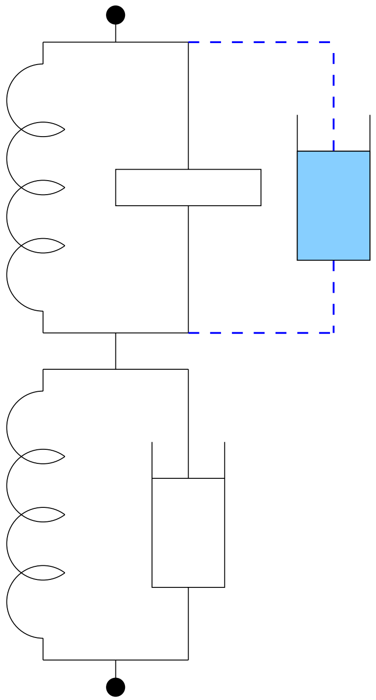
Laboratory Testing

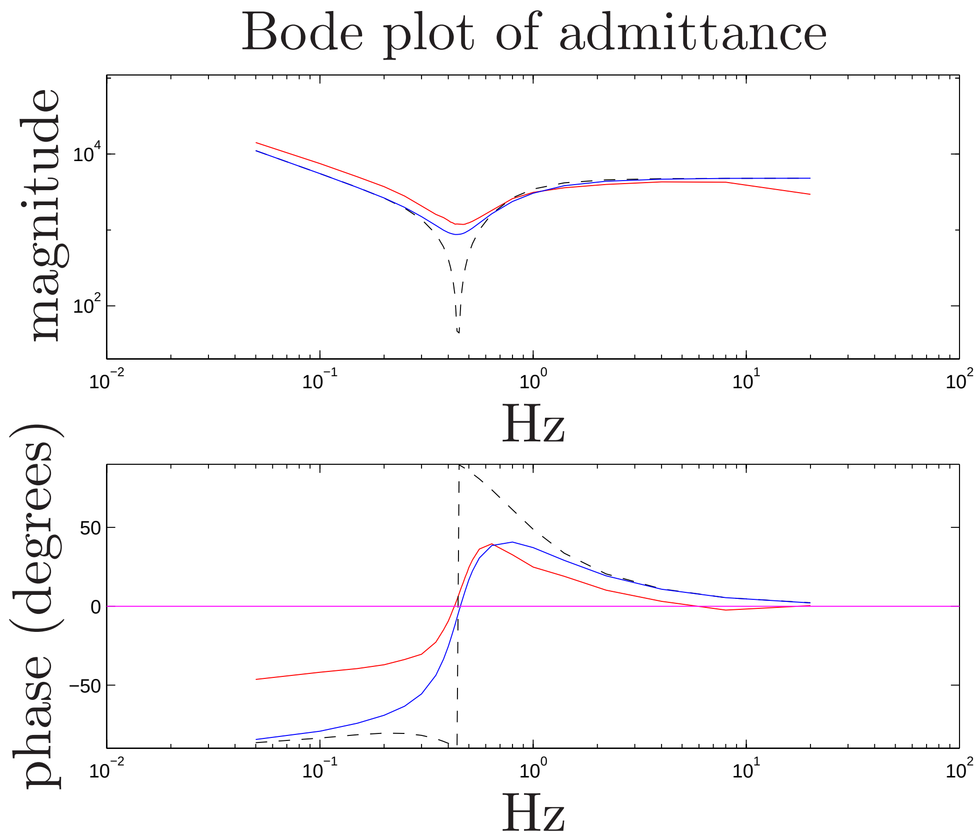


Schenck hydraulic ram

Cambridge University Mechanics Laboratory

Experimental Results



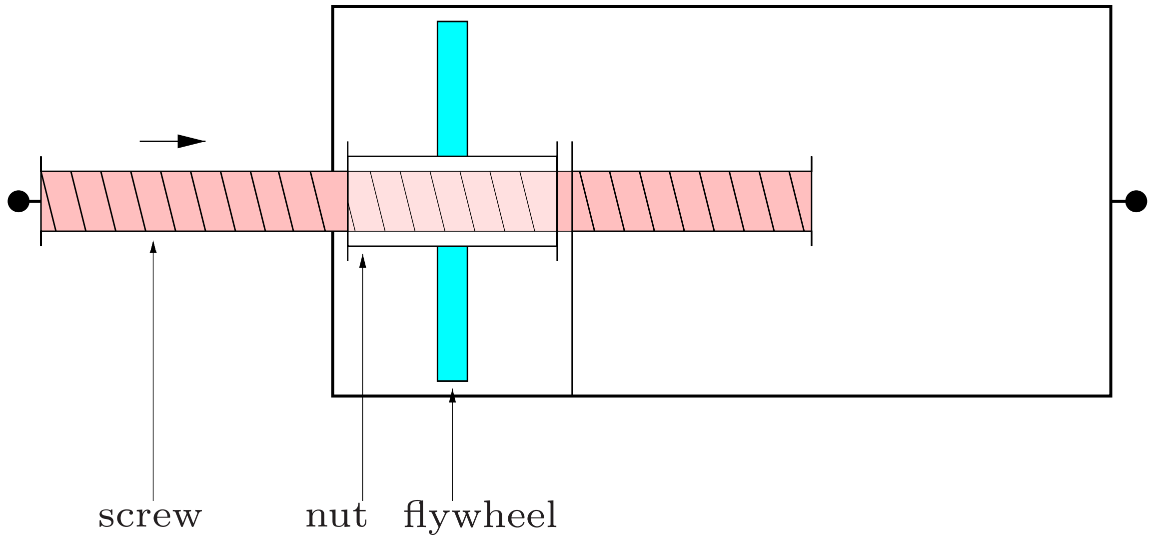


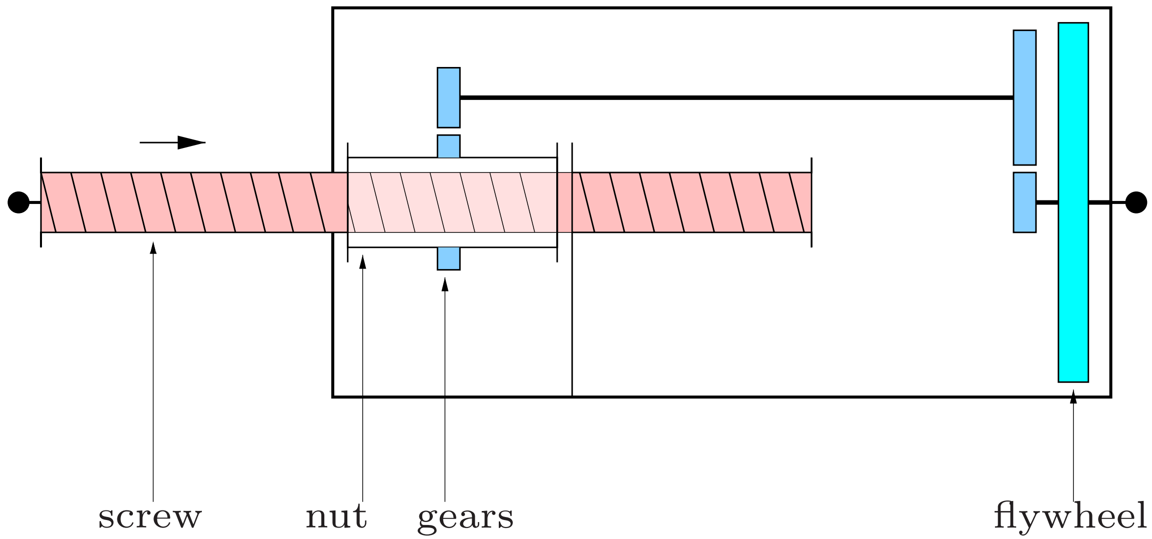
experimental data (——)

theoretical without inerter damping (– –)

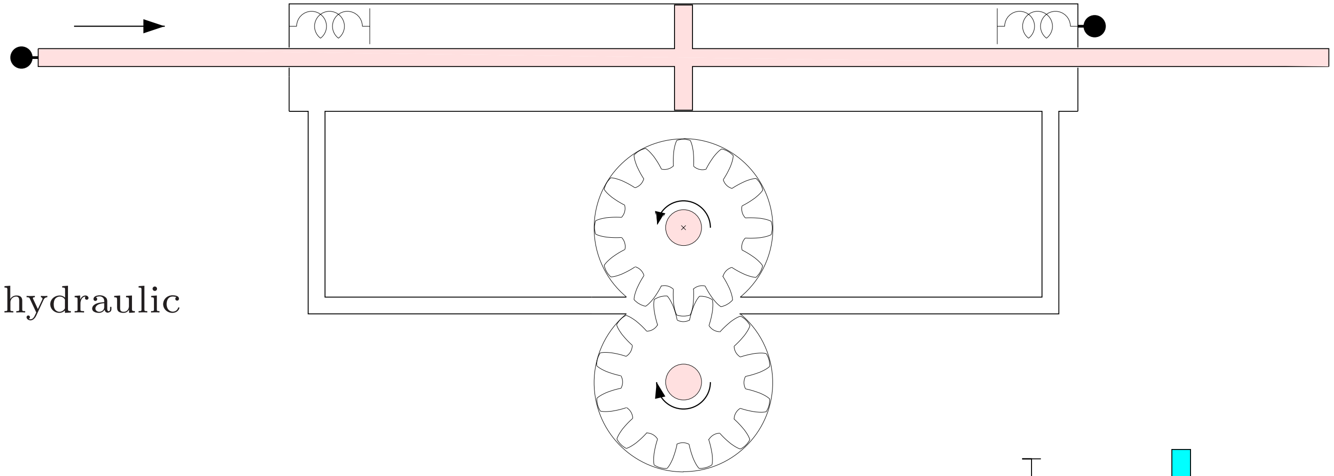
theoretical with inerter damping (——)

Alternative Inerter Embodiments I

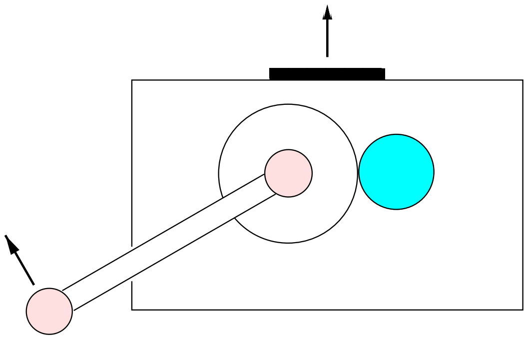




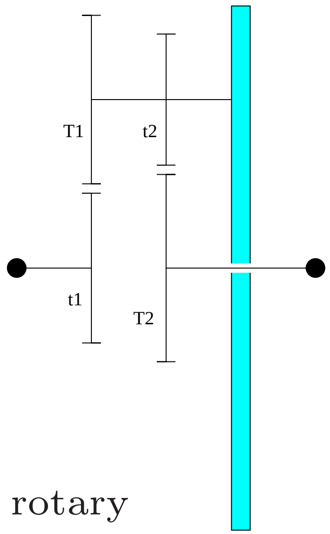
Alternative Inerter Embodiments II



hydraulic



lever arm



rotary

See Cambridge University Technical Services Ltd patent PCT/GB02/03056 for further details.

Conclusion

- A new mechanical element called the "inerter" was introduced which is the true network dual of the spring.
- The inerter allows a complete synthesis theory for passive mechanical networks.
- Performance advantages for problems in mechanical vibrations and suspension systems have been described.
- It is expected that future work will continue to explore both the theoretical and practical advantages as well as its applicability in collaboration with interested commercial partners.